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Quantum Mechanics in Curved Space-Time

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A set of equations for elementary particles when the space is curved is obtained by substituting the curved space metric for the flat space one in the algebraic relationship that determines the matrices which describe these particles. It is observed that for the Dirac or Duffin-Kemmer matrices, the same set of equations are obtained if we assume that the flat space equations are true in the local frame of reference at any point. The above is taken as a fundamental postulate. The approximate corrections to the different fields when the curvature is small have been obtained. It is also observed that this procedure means a redefinition of the affine relationship. It is throughout assumed that the curvature is introduced in the space-time world due to the presence of matter by the standard relationship of general relativity.

§ 1. Introduction

Many interesting approaches have been made to explain the gravitational phenomena within the framework of special theory of relativity with the help of spin 2 particles.^{1), 2), 3)} There have been also some different approaches to explain the same with the help of functional treatment⁴⁾ or by a drastic revision of our basic concepts⁵⁾. But, since the success in these cases has not been outstanding, we have here attempted an alternative and more conventional procedure to consider the same problem. We deduce the equations of elementary particles and their interactions in curved space-time very much within the framework of general relativity.

To obtain these equations, it seems helpful to postulate that our quantum mechanical equations and expressions are given at any point by what we already have in flat space-time in terms of the local geodesic coordinates at that point. This method, however, defines an affine relationship different from what we usually have.

It is here assumed that the curvature is introduced by the energy momentum tensor by the equation

$$G_{\mu\nu} = \kappa \mathcal{T}_{\mu\nu} \quad (1)$$

as in the general theory of relativity.

It has not been possible to solve these equations explicitly, but in certain specific cases such as that of a space conformal to a flat space-time, the form of

these equations is interesting in the sense that the mass term is replaced by an invariant space-time function.

However, in the linear approximation when we assume that these corrections are small, it is possible to give explicitly the way in which they occur, but since these do not throw any new light about the exact effects of the gravitational fields at very small distances at which they are expected to be important, the solutions in these approximations have not been calculated.

§ 2. General theory

We shall take $\vec{x} = (x^1, x^2, x^3)$ to represent the space coordinates and x^0 to represent the time coordinate in curved space-time, and we throughout take natural units so that $\hbar = c = 1$. The metric of the flat Minkowski space is taken as $\delta_{\mu\nu}$ where $\delta_{11} = \delta_{22} = \delta_{33} = -\delta_{00} = 1$ and $\delta_{\mu\nu} = 0$ when $\mu \neq \nu$. We shall first consider the case when the space is almost flat, and take the metric of the curved space-time as

$$g^{\mu\nu} = \delta^{\mu\nu} + \theta^{\mu\nu} \quad (2a)$$

$$g_{\mu\nu} = \delta_{\mu\nu} - \theta_{\mu\nu} \quad (2b)$$

where $\theta_{\mu\nu}$ are small, and the above result corresponds to the linear approximation. Clearly, in this approximation

$$\theta_{\mu\nu} = \delta_{\mu\lambda} \delta_{\nu\kappa} \theta^{\lambda\kappa}$$

and thus $\theta_{\mu\nu}$ behave as tensors in raising and lowering indices in flat space-time. We shall always raise and lower the indices with the metric of the flat space-time unless otherwise stated.

In the approximation when the coordinate condition

$$\frac{\partial \theta_{\mu\nu}}{\partial x_\nu} - \frac{1}{2} \frac{\partial \theta}{\partial x^\mu} = 0 \quad (3)$$

is satisfied, we have,

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{1}{2} (\square \theta_{\mu\nu} - \frac{1}{2} \delta_{\mu\nu} \square \theta), \quad (4)$$

where we have used the notation $\theta = \delta_{\mu\nu} \theta^{\mu\nu}$ and $\square = \delta^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu}$. Hence we obtain from Equation (1),

$$\square \theta_{\mu\nu} - \frac{1}{2} \delta_{\mu\nu} \square \theta = -2\kappa \mathcal{J}_{\mu\nu}. \quad (5)$$

Equations (3) and (5) will give us the value of $\theta_{\mu\nu}$ when we know the value of $\mathcal{J}_{\mu\nu}$ for any classical matter field.

When we assume that the quantized fields are included in our description, we replace Equation (5) by

$$\square \theta_{\mu\nu} - \frac{1}{2} \delta_{\mu\nu} \square \theta = -2\kappa \langle \mathcal{J}_{\mu\nu} \rangle, \quad (5')$$

where $\langle \mathcal{J}_{\mu\nu} \rangle$ is the expectation value of the energy-momentum operator for our dynamical system.

We now assume that $\theta_{\mu\nu}$ are known, and consider the equations for elementary particles in such a case. Bhabha⁶⁾ has given a set of linear equations for elementary particle-fields of nonzero rest-mass as

$$\left(\alpha^\mu \frac{\partial}{\partial x^\mu} + m \right) \Psi = 0 \quad (6)$$

where the α -matrices satisfy different algebraic rules for different particles in flat space-time. We assume tentatively that in this relationship of these matrices, the $\delta^{\mu\nu}$ of flat space is changed to $g^{\mu\nu}$, and the equations (6) remain the same in form. A justification of this procedure will be seen later on.

Thus for a Dirac particle, we should have, replacing α -matrices by γ -s,

$$\gamma'^\mu \gamma'^\nu + \gamma'^\nu \gamma'^\mu = 2g^{\mu\nu}, \quad (7)$$

where the primed indices here and henceforward indicate the *corresponding* changed quantities in curved space-time. Clearly, the relationship (7) is satisfied in the first order in the $\theta^{\mu\nu}$ if we take

$$\gamma'^\mu = \gamma^\mu + \frac{1}{2} \theta^{\mu\nu} \gamma_\nu \quad (8)$$

where

$$\gamma_\nu = \delta_{\nu\lambda} \gamma^\lambda. \quad (8')$$

Again, we can see by direct verification that to the same order in $\theta^{\mu\nu}$, the changed Duffin-Kemmer relationship,

$$\beta'^\mu \beta'^\nu \beta'^\lambda + \beta'^\lambda \beta'^\nu \beta'^\mu = g^{\mu\nu} \beta'^\lambda + g^{\nu\lambda} \beta'^\mu, \quad (9)$$

is satisfied when we define β'^μ in terms of the Duffin-Kemmer matrices in a way similar to Equation (8) for γ'^μ in terms of γ^μ . This fact is also obvious when we adopt for the β -matrices the representation⁷⁾

$$\beta^\mu = \frac{1}{2} (I \times \gamma^\mu + \gamma^\mu \times I)$$

where γ -s are the above-mentioned Dirac matrices, and \times denotes a direct product.

The discussion above includes particles of spin $\frac{1}{2}$, 0 and 1, which are physically most important. Hence, in Equations (6) we can take the changed α -matrices as

$$\alpha'^\mu = \alpha^\mu + \frac{1}{2} \theta^{\mu\nu} \alpha_\nu, \quad (10)$$

a form justifiable for at least the above set of particles, and Equation (6) changes to

$$\left(\alpha'^\mu \frac{\partial}{\partial x^\mu} + m \right) \Psi'(x) = 0. \quad (11)$$

In Equation (11) we substitute $\Psi'(x) = \Psi(x) + \delta\Psi(x)$ where $\Psi(x)$ is the solution of Equation (6) and $\delta\Psi(x)$ is a small correction to this field. Then we obtain the equation for this correction as

$$\left(\alpha^\mu \frac{\partial}{\partial x^\mu} + m\right) \delta \mathcal{F}(x) = -\frac{1}{2} \theta^{\mu\nu} \frac{\partial \mathcal{F}}{\partial x^\nu}. \quad (12)$$

In order to solve Equation (12), we require Green's function $G(x, x')$ satisfying the equation

$$\left(\alpha^\mu \frac{\partial}{\partial x^\mu} + m\right) G(x, x') = -\delta_4(x - x'), \quad (13)$$

in which case we obtain

$$\delta \mathcal{F}(x) = \frac{1}{2} \int G(x, x') \theta^{\mu\nu}(x') \alpha_\nu \frac{\partial \mathcal{F}(x')}{\partial x'^\mu} d^4 x'. \quad (14)$$

The relationship (10) for curved space-time is seen to satisfy the requirements of the algebra of the α -matrices in a formal way for the Dirac and Duffin-Kemmer cases. We can obtain to the same approximation the set of equations (11) from an entirely different consideration that is physically more satisfactory and hence may be taken to embrace all other equations of the same type.

At any point 0 in space-time having coordinates x_0 we choose local coordinate system X_0 such that we have, at 0,

$$\partial_{\mu\nu} dX_0^\mu dX_0^\nu = g_{\mu\nu}(x_0) dx^\mu dx^\nu, \quad (15)$$

which gives

$$\partial_{\mu\nu} = t^\lambda_{\mu\nu}(0) t^\mu_\nu(0) g_{\lambda\kappa}(x_0) \quad (15')$$

where

$$\begin{aligned} dX_0^\mu &= T^\mu_\nu(0) dx^\nu, \\ dx^\mu &= t^\mu_\nu(0) dX_0^\nu \end{aligned} \quad (15'')$$

give us the relationship between the lengths of the infinitesimal measuring rods and intervals of time as viewed by an observer in the general frame of reference and an observer in the local frame of reference at 0. It is easily seen that Equations (15') are satisfied when we take

$$t^\lambda_\mu(0) = \delta^\lambda_\mu + \frac{1}{2} \theta^\lambda_\mu(x_0) \quad (16)$$

in the linear approximation.

Now we assume that the original set of equations (6) are satisfied at the point 0 in terms of the local coordinates at 0. Thus at this point we have

$$\left(\alpha^\mu \frac{\partial}{\partial X_0^\mu} + m\right) \mathcal{F}' = 0 \quad (17)$$

which in terms of the original coordinates becomes

$$\left(\alpha^\mu t^\lambda_\mu(0) \frac{\partial}{\partial x^\lambda} + m\right) \mathcal{F}' = 0. \quad (18)$$

Substituting the value of $t^\lambda_\mu(0)$ from Equation (16), we obtain the equation identical with (11) when we note that

$$\theta^{\lambda}_{\mu}(x_0)\alpha^{\mu}=\theta^{\lambda\mu}(x_0)\alpha_{\mu}.$$

Since 0 is any point in space-time, the above physical assumption gives us the same set of equations as obtained earlier in a formal manner. This leads us to the statement of a general postulate:

Postulate A. Our quantum mechanical equations and expressions at any point are given by what we already have in flat space-time in terms of the local coordinates at that point.

The above postulate corresponds to the fact that in general relativity the space is locally flat.

§ 3. Equations in a space conformal to a flat space-time

In this case we take the metric tensor as⁸⁾

$$g^{\mu\nu}(x)=\delta^{\mu\nu}\Lambda(x). \quad (19)$$

Hence

$$g_{\mu\nu}(x)=\delta_{\mu\nu}(\Lambda(x))^{-1},$$

where $\Lambda(x)=1+\lambda(x)$ is an invariant space-time function and $\lambda(x)$ can be taken as small when we take the linear approximation. Then we can take for Equations (15') and (15''),

$$t^{\mu}_{\nu}(x)=\delta^{\mu}_{\nu}(\Lambda(x))^{1/2}, \quad (20)$$

and

$$T^{\mu}_{\nu}(x)=\delta^{\mu}_{\nu}(\Lambda(x))^{-1/2}$$

such that Equation (17) becomes

$$\left(\alpha^{\nu}t^{\mu}_{\nu}(x)\frac{\partial}{\partial x^{\mu}}+m\right)\Psi'=0,$$

which reduces to

$$\left(\alpha^{\mu}\frac{\partial}{\partial x^{\mu}}+\frac{m}{\sqrt{\Lambda(x)}}\right)\Psi'(x)=0. \quad (21)$$

Equation (21) has the interesting feature that in such a space the equation for the elementary particle is slightly changed in the term involving the rest-mass of the particle, which is now replaced by a space-time function.

In order to solve for $\Lambda(x)$, we note that here the coordinate condition (3) is not satisfied. A direct evaluation of $G_{\mu\nu}\equiv R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R$ gives us the result that

$$\begin{aligned} G_{\mu\nu} &= (\Lambda(x))^{-1} \left(-\frac{\partial^2 \Lambda(x)}{\partial x^{\mu} \partial x^{\nu}} + \delta_{\mu\nu} \square \Lambda(x) \right) \\ &+ (\Lambda(x))^{-2} \left(\frac{1}{2} \frac{\partial \Lambda}{\partial x^{\mu}} \frac{\partial \Lambda}{\partial x^{\nu}} - (5/4) \delta_{\mu\nu} \delta^{\lambda\kappa} \frac{\partial \Lambda}{\partial x^{\lambda}} \frac{\partial \Lambda}{\partial x^{\kappa}} \right). \end{aligned} \quad (22)$$

Here we are interested in a dynamical system that gives rise to a curved space conformal to a flat one. This is possible only if Equations (1)

$$G_{\mu\nu} = \kappa \mathcal{J}_{\mu\nu}$$

are satisfied. We wish to find out $\Lambda(x)$ in this particular case, and for this purpose it is sufficient to evaluate

$$g^{\mu\nu} G_{\mu\nu} = -R = \kappa g^{\mu\nu} \mathcal{J}_{\mu\nu},$$

which gives rise to the equation

$$3\Box\Lambda(x) - (9/2)(\Lambda(x))^{-1} \frac{\partial\Lambda}{\partial x^\lambda} \frac{\partial\Lambda}{\partial x^\kappa} \delta^{\lambda\kappa} = \kappa g^{\mu\nu} \mathcal{J}_{\mu\nu}. \quad (23)$$

It may be noted here that $\mathcal{J}_{\mu\nu}$ must have certain symmetry so that when we obtain the solution of Equation (23), each one of Equations (1) should be also satisfied.

Equation (23) can be written down in a slightly better form by the substitution $\Lambda(x) = \exp(\alpha(x))$. On the right-hand side of the above equation, since $g^{\mu\nu} \mathcal{J}_{\mu\nu}$ is a scalar, at any point we can take this as equal to $\delta^{\mu\nu} \mathcal{J}_{\mu\nu}(0; \text{local})$, and by postulate A we can write down the usual expression for $\mathcal{J}_{\mu\nu}(0; \text{local})$ of flat space. Hence Equation (23) after simplification becomes

$$\exp(\alpha(x)) \delta^{\mu\nu} \left(\frac{\partial^2 \alpha(x)}{\partial x^\mu \partial x^\nu} - (1/2) \frac{\partial \alpha}{\partial x^\mu} \frac{\partial \alpha}{\partial x^\nu} \right) = (\kappa/3) \delta^{\mu\nu} \mathcal{J}_{\mu\nu}(0; \text{local}). \quad (24)$$

In the linear approximation, however, the above equation attains the very simple form

$$\Box \lambda(x) = (\kappa/3) \delta^{\mu\nu} \mathcal{J}_{\mu\nu}(\text{local}) \quad (25)$$

which can be solved easily when the value of $\mathcal{J}_{\mu\nu}$ is known in terms of the local coordinates for any particle-field.

In this approximation Equation (21) simplifies to

$$\left(\alpha^\mu \frac{\partial}{\partial x^\mu} + m \right) \Psi'(x) = \frac{1}{2} m \Psi'(x) \lambda(x)$$

and for the correction term, Equation (12) becomes

$$\left(\alpha^\mu \frac{\partial}{\partial x^\mu} + m \right) \delta \Psi(x) = \frac{1}{2} m \lambda(x) \Psi(x), \quad (26)$$

which does not involve the derivative of flat space-time wave-functions, as was the case for the more general type of space.

In particular, for the Dirac particle, we can take in the local coordinates

$$\mathcal{J}^{\mu\nu}(0; \text{local}) = -\frac{1}{2} \left(\bar{\psi} \gamma^\nu \frac{\partial \psi}{\partial X_{0\mu}} - \frac{\partial \bar{\psi}}{\partial X_{0\mu}} \gamma^\nu \psi \right)$$

such that, applying equations of motion, we get

$$\delta_{\mu\nu} \mathcal{J}^{\mu\nu}(0; \text{local}) = m\bar{\psi}\psi. \quad (27)$$

On the right-hand side of the above equation, $\bar{\psi}\psi$ can be interpreted as the probability of finding an electron in the space-time volume element d^4X_0 and representing this function by $\rho(x_0)$ ($=\rho(X_0)$, since this is an invariant function), we may solve the exact equation (24) or the approximate one (25). Thus, Equation (25) becomes

$$\square \lambda(x) = (m\kappa/3)\rho(x), \quad (28)$$

where $\rho(x)\sqrt{-\det|g_{\mu\nu}(x)|}d^4x$ indicates the probability of finding the electron in the space-time volume d^4x .

In the above, we have assumed throughout that the fields are not second quantized. In the latter case, however, we have to take $\langle \mathcal{J}_{\mu\nu} \rangle$ wherever we have $\mathcal{J}_{\mu\nu}$.

§ 4. Conditions of covariance

The method of deducing Equation (11) from Equation (17) with the help of the postulate stated is very interesting because it gives us a method for writing down the equations in an exact manner and solving them in some special cases. But, for the purposes of covariance, we must rather take normal coordinates⁸⁾ in which case not only is the metric of the form (15) at the point, but the ordinary derivatives are identical with the covariant derivatives, so that Equation (17) or any similar equation or expression is covariant under a coordinate transformations when it is covariant under rotations in a flat space.

Following the notations of Eisenhart⁹⁾ for transformations to normal coordinates, we first make the coordinate transformation from x^i to \bar{x}_0^i such that

$$x^i = x_0^i + \bar{x}_0^i - \frac{1}{2!} \left\{ \begin{matrix} i \\ \alpha_1 \alpha_2 \end{matrix} \right\}_0 \bar{x}_0^{\alpha_1} \bar{x}_0^{\alpha_2} - \frac{1}{3!} (I^i_{\alpha_1 \alpha_2 \alpha_3})_0 \bar{x}_0^{\alpha_1} \bar{x}_0^{\alpha_2} \bar{x}_0^{\alpha_3} - \dots \quad (29)$$

where $\bar{x}_0^i = (dx^i/ds)_0 s$, and s is the arc length parameter of the (non-null) geodesic passing through 0, and thus \bar{x}_0^i are the Riemannian coordinates at the point 0 corresponding to the original coordinates x^i . For these coordinates, we know that

$$\left\{ \begin{matrix} i \\ \alpha_1 \alpha_2 \end{matrix} \right\}_0 = (\bar{I}^i_{\alpha_1 \alpha_2 \alpha_3})_0 = \dots = 0$$

where the bar denotes the corresponding quantities in the Riemannian coordinate system.

The coordinate transformation (29) has a unique inverse so long as we confine our attention to a domain characterised by the fact that through any point of the domain and through the point 0 only a single geodesic can pass. Hence if the above transformation is a bi-unique transformation throughout space-time, we must have as necessary condition that space-time world must have infinite extension with absolute past and absolute future. However, when we are interested in this transformation for deducing properties at 0 or in its immediate neighbourhood, the

above assumption of Equation (29) always having an inverse may be taken for granted.

After obtaining the Riemannian coordinates, we have further to make a linear transformation with constant coefficients to obtain the normal (local) coordinates at that point, the differential invariant form for which will be the same as that of flat space-time. These must be the coordinates to be used in the differential equation (17) at that point.

Instead of Riemannian coordinates above, we could also maintain covariance of the equations by taking the geodesic coordinates⁸⁾ at the point 0, and then obtaining as before the local geodesic coordinates. The transformation to geodesic coordinates is more general and is given by (with the prime denoting the geodesic coordinates)

$$x^i = x_0^i + x_0'^i - (1/2!) \left\{ \begin{matrix} i \\ \alpha_1 \alpha_2 \end{matrix} \right\} x_0'^{\alpha_1} x_0'^{\alpha_2} - (1/3!) c_{\alpha_1 \alpha_2 \alpha_3}^i x_0'^{\alpha_1} x_0'^{\alpha_2} x_0'^{\alpha_3} - \dots \quad (30)$$

where the coefficients $c_{\alpha_1 \alpha_2 \alpha_3}^i$, etc., are arbitrary. Here also

$$\left\{ \begin{matrix} i \\ \alpha_1 \alpha_2 \end{matrix} \right\}' = 0,$$

hence, after obtaining the local coordinates as before, we can take the ordinary derivatives as identical with the covariant derivatives.

Thus in the postulate A we must always take the local normal coordinates or the local geodesic coordinates at any point, in which case not only is the metric of the form required, but the Christoffel symbols vanish at that point in this particular coordinate system. With this restriction (*Postulate A'*) the covariance of postulate A for any equation, where covariance under flat space transformations already exists, is established.

Let us assume that at the point 0 we have obtained the local coordinates satisfying the above requirements such that Equations (15'') are satisfied. Now, considering Equation (1) at the point 0, we can adopt the usual value of $\mathcal{J}_{\mu\nu}$ in the local coordinate system by the postulate A' in terms of the wave-function and the derivatives of the field in the same coordinate system,

$$\mathcal{J}_{\mu\nu}(x_0) = T_{\mu}^{\lambda}(0) T_{\nu}^{\kappa}(0) \mathcal{J}_{\lambda\kappa}(0; \text{local geodesic}) \quad (31)$$

and thus we obtain the equation

$$G_{\mu\nu}(x_0) = \kappa T_{\mu}^{\lambda}(x_0) T_{\nu}^{\kappa}(x_0) \mathcal{J}_{\lambda\kappa}(0; \text{local geodesic}). \quad (32)$$

In the above equation, we may regard $G_{\mu\nu}(x_0)$ as a function of the matrix $\|T_{\mu}^{\lambda}(x_0)\|$ by Equation (15'), or otherwise, may regard as a function of the metric through the Christoffel symbols by Equation (34) or any of the earlier ones, and thus know their values when the metric caused by the matter field is known. The equations thus obtained are a set of coupled differential equations and are almost impossible to take exactly beyond the linear approximation.

Thus, when we know the value of $\|T(x_0)\|$ and hence of $\|t(x_0)\|$, the exact equations for studying the behaviour of the field for which we take $\mathcal{J}_{\mu\nu}(0; \text{local$

geodesic) in Equation (32) can be written down by replacing the usual differentiation operator by

$$\frac{\partial}{\partial X_0^\mu} \Big|_0 \doteq t^\lambda_{\mu}(x_0) \frac{\partial}{\partial x^\lambda} \Big|_0. \quad (33)$$

Also, if we have any integral of a tensor occurring in the flat space-time relationship, in this case we are to take the integrand in the local coordinates at that point, and then convert it to the general frame of reference by $T^\lambda_{\mu}(x_0)$ for the covariant index or by $t^\lambda_{\mu}(x_0)$ for the contravariant index, and integrate with the invariant space-time volume element $\sqrt{-\det |g_{\mu\nu}(x_0)|} d^4x_0$ at that point.

However, the situation here is always complicated, and even for the simple transformation

$$x^i = x_0^i + x_0'^k - \frac{1}{2} \left\{ \begin{matrix} i \\ j \ k \end{matrix} \right\}_0 x_0'^j x_0'^k \quad (34)$$

it is not possible to write down the inverse transformation explicitly. Also, when we take into account the necessity of covariance, the above treatment for the space conformal to a flat one is seen to be invalid, and a more detailed analysis in the line mentioned is necessary.

§ 5. Second quantization

Previously, in the linear approximation, we have expressed the change in the wave-function of any particle-field when the metric is already known, in Equation (14). We shall assume that the quantization of the flat space wave-function $\Psi(x)$, remains the same, and that the quantization that this will induce in the total wave-function will be maintained. This procedure is unique only when we can express the curved space field operator in terms of the flat space one unambiguously. In the linear approximation we can write the commutator or the anti-commutator of the fields as

$$iS'(x, y) \equiv [\Psi'(x), \bar{\Psi}'(y)] = [\Psi(x), \bar{\Psi}(y)] + [\partial\Psi(x), \Psi(y)] + [\Psi(x), \partial\Psi(y)] \quad (35)$$

where we take the value of

$$[\Psi(x), \bar{\Psi}(y)] = iS(x-y) \quad (36)$$

according to Udgaonkar⁹⁾ and remember the expressions (14) for $\partial\Psi(x)$ which thus introduces the derivatives of the flat space commutator or anti-commutator in the correction terms of Equation (35).

When the flat space fields are second quantized, Green's functions of Equation (13) for the general particle-fields have been given by Gupta¹⁰⁾ and in a more elegant manner, by Schwinger¹¹⁾ for the fields in which we may be interested. Similar to Equation (35), we can write down the expression for the propagator as

$$\dot{\Psi}'(x)\dot{\bar{\Psi}}'(y) \equiv K'(x-y) = \dot{\bar{\Psi}}(x)\dot{\bar{\Psi}}(y) + \delta\dot{\bar{\Psi}}(x)\dot{\bar{\Psi}}(y) + \dot{\bar{\Psi}}(x)\delta\dot{\bar{\Psi}}(y), \quad (37)$$

where again we shall need Equation (14). In the above formula, the dots denote contraction in the sense of Wick.¹²⁾

When the metric is known and is introduced by some external matter and we neglect the gravitational effects due to the particles themselves, the above equations will give us the description of a free quantized field. But, on the other hand, if the metric introduced by the field itself is to be taken into account, in all our previous discussions, we must take $\mathcal{H}_{\mu\nu}$ for the physical system we are considering (e.g. how many real particles are present and in what state) and the equations can be written down in the manner described after knowing the value of this metric.

§ 6. Interacting fields

To consider interactions with this system of quantization, we first note that the Tomonaga equation has to be now taken as

$$i \frac{\delta \Psi'(\sigma)}{\delta \sigma(x)} \frac{1}{\sqrt{-\det|g_{\mu\nu}(x)|}} = H_I(x) \Psi'(\sigma). \quad (38)$$

In the above equation, we have used that $\sqrt{-\det|g_{\mu\nu}(x)|}d^4x$ is an invariant element in the curved space, $H_I(x)$ is the invariant interaction hamiltonian density operator and $\Psi'(\sigma)$ is the state-vector functional on the space-like surface σ . This equation is consistent only if the integrability condition is satisfied. For example, in absence of derivatives in $H_I(x)$ of the field operators, we must have

$$[H_I(x), H_I(x')] = 0$$

whenever x and x' are separated by a space-like interval.

Writing $\Psi'(\sigma) = U'(\sigma) \Psi'(-\infty)$, and proceeding according to the standard method of Dyson, we obtain

$$U'(\infty) = 1 + \sum_{n=1}^{\infty} S_n' \quad (39)$$

where

$$S_n' = (-i)^n / (n!) \int d^4x_1 \cdots \int d^4x_n P(\sqrt{-g(x_1)} H_I(x_1) \cdots \sqrt{-g(x_n)} H_I(x_n)), \quad (40)$$

and $g(x) = \det||g_{\mu\nu}(x)||$.

Thus, while evaluating the P -bracket above, besides the usual contractions, there will be extra terms arising out of $\delta\Psi(x)$ or $\delta\bar{\Psi}(x)$ of Equation (14) which will thus involve the integrals of the differential coefficients of the flat space propagators. However, we can use this formula in the linear approximation only if the integrability condition is satisfied.

§ 7. Gravitational corrections for meson and electromagnetic fields

We shall first consider the corrections to a meson field since here the equations are explicitly in the form (6), whereas for the electromagnetic field we are to replace m in Equation (6) by a suitable singular matrix. We shall take the case of a scalar meson, since this case is simple and since we can directly go over to pseudo-scalar mesons that are important for the nuclear forces. The meson field will be described by a 5×5 irreducible representation of the β -matrices satisfying Duffin-Kemmer commutation rules. The representation we choose is¹³⁾

$$\begin{aligned} \beta^1 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, & \beta^2 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \\ \beta^3 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, & \beta^0 &= \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, & \chi &= \begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \\ \chi_5 \end{pmatrix}, \end{aligned}$$

where β^μ are Duffin-Kemmer matrices of flat space-time and χ is the corresponding wave-function. The changed wave function thus becomes

$$\chi' = \chi + \frac{1}{2} \int g(x, x') \theta^{\mu\nu}(x') \beta_\nu \frac{\partial \chi(x')}{\partial x'^\mu} d^4 x', \quad (41)$$

where $g(x, x')$ is given by Gupta.¹⁰⁾

We know that in our representation χ_5 is a scalar and satisfies the Klein-Gordon equation. We shall explicitly deduce this well-known result as follows: We first note that

$$(\beta^\mu \beta^\nu)^\beta_\alpha = \delta^{\mu\nu} \delta^\beta_\alpha. \quad (42)$$

Let us now consider the equation

$$(\beta^\mu)^\lambda_\tau \frac{\partial \chi_\tau}{\partial x^\mu} + m \chi_\lambda = 0 \quad (43)$$

which also means

$$(\beta^\mu)^\beta_\lambda \frac{\partial \chi_\lambda}{\partial x^\mu} + m \chi_\beta = 0. \quad (44)$$

Substituting the value of $\partial \chi_\lambda / \partial x^\mu$ from Equation (43) in Equation (44), we get,

$$- (\beta^\mu)^\beta_\lambda (\beta^\nu)^\lambda_\tau \frac{\partial^2 \chi_\tau}{\partial x^\mu \partial x^\nu} + m^2 \chi_\beta = 0,$$

which by Equation (42) becomes

$$(\square - m^2) \chi_\beta = 0. \quad (45)$$

We have explicitly written down Equations (41) to (44) since the corresponding equations are useful for obtaining that of χ'_5 . Proceeding in the same way as above for the set of equations in curved space (where β'^μ are now functions of x), the equation for χ'_5 becomes

$$-(\beta'^\mu)^\lambda (\beta'^\nu)^\tau \frac{\partial^2 \chi'_\tau}{\partial x^\mu \partial x^\nu} - (\beta'^\mu)^\lambda \frac{\partial}{\partial x^\mu} (\beta'^\nu)^\tau \frac{\partial \chi'_\tau}{\partial x^\nu} + m^2 \chi'_5 = 0, \quad (46)$$

which, in the linear approximation reduces to

$$(\partial^{\mu\nu} + \theta^{\mu\nu}) \frac{\partial^2 \chi'_5}{\partial x^\mu \partial x^\nu} + \frac{1}{2} \frac{\partial \theta^{\mu\nu}}{\partial x^\mu} \frac{\partial \chi'_5}{\partial x^\nu} - m^2 \chi'_5 = 0. \quad (47)$$

We could also obtain the above equation by using the postulate that in the local coordinate system the equations of the fields are to be taken in the usual form. Thus we have

$$\left(\partial^{\mu\nu} \frac{\partial}{\partial X^\mu} \frac{\partial}{\partial X^\nu} - m^2 \right) \chi'_5 = 0. \quad (48)$$

But, in the linear approximation, we can take by Equation (16)

$$\frac{\partial}{\partial X^\mu} = \frac{\partial}{\partial x^\mu} + \frac{1}{2} \theta^\lambda_\mu \frac{\partial}{\partial x^\lambda},$$

and hence the changed D'Alembertian is given as

$$\begin{aligned} \square' &\equiv \square \text{ (local)} = \delta^{\mu\nu} \left(\frac{\partial}{\partial x^\mu} + \frac{1}{2} \theta^\lambda_\mu \frac{\partial}{\partial x^\lambda} \right) \left(\frac{\partial}{\partial x^\nu} + \frac{1}{2} \theta^\kappa_\nu \frac{\partial}{\partial x^\kappa} \right) \\ &= (\delta^{\mu\nu} + \theta^{\mu\nu}) \frac{\partial^2}{\partial x^\mu \partial x^\nu} + \frac{1}{2} \frac{\partial \theta^{\mu\kappa}}{\partial x^\mu} \frac{\partial}{\partial x^\kappa}, \end{aligned} \quad (49)$$

which is the operator acting on χ'_5 in Equation (48) instead of \square alone and thus is identical with Equation (47).

The exact expression for the D'Alembertian obtained in the above manner is

$$\begin{aligned} \square' &= \square \text{ (local geodesic)} = \delta^{\mu\nu} \left(t^\lambda_\mu \frac{\partial}{\partial x^\lambda} \right) \left(t^\kappa_\nu \frac{\partial}{\partial x^\kappa} \right) \\ &= g^{\mu\nu} \left(\frac{\partial^2}{\partial x^\mu \partial x^\nu} - t^\lambda_\alpha \frac{\partial}{\partial x^\nu} (T^\alpha_\mu) \frac{\partial}{\partial x^\lambda} \right) \end{aligned} \quad (50)$$

where we have used the equation similar to (15') for g and that $\partial/\partial x^\lambda (t^\kappa_\nu T^\nu_\alpha) = 0$.

It may be seen that \square' of Equation (50) is the same as the invariant differential parameter of the second order given as

$$g^{\mu\nu} \left(\frac{\partial^2}{\partial x^\mu \partial x^\nu} - \left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\} \frac{\partial}{\partial x^\lambda} \right) \quad (51)$$

provided that

$$\frac{\partial}{\partial x^\mu}(T^i_\nu) = \frac{\partial}{\partial x^\nu}(T^i_\mu) \quad (52)$$

and

$$\frac{\partial}{\partial X^\mu}(t^i_\nu) = \frac{\partial}{\partial X^\nu}(t^i_\mu), \quad (53)$$

so that we have

$$T^i_{\lambda\lambda} \left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\} = \frac{\partial}{\partial x^\mu}(T^i_\nu) \quad (54)$$

and

$$t^k_\mu t^j_\nu \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} = -\frac{\partial}{\partial X^\mu}(t^i_\nu). \quad (55)$$

In deducing Equations (54) and (55) we have used that the Christoffel symbols vanish in the local geodesic coordinate system. But it must be noted that Equations (52) and (53) mean that the equations (15') are integrable for the X^i , hence that the space is actually flat.

To write down the equations of the electromagnetic field in the Duffin-Kemmer form, we take the representation⁷⁾

$$\beta^\mu = \frac{1}{2}(I \times \gamma^\mu + \gamma^\mu \times I) \quad (56)$$

and have to consider the flat space-time equation

$$\left(\beta^\mu \frac{\partial}{\partial x^\mu} + \gamma \right) \Psi = 0, \quad (57)$$

where Ψ is a sixteen component column vector and γ is the singular matrix given as

$$\gamma = \text{diag}[1, 1, 0, 0, 1, 1, 0, 0, 0, 0, 1, 1, 0, 0, 1, 1].$$

Here we note that in Equation (57) and subsequently, we are to remember the difference in notation because of the choice of the flat space metric, i.e. γ^μ (Corson) $= (-i\gamma^\mu)$ (ours); otherwise the notation is the same as that of Corson. For the curved space-time, however, we are to replace β^μ by β'^μ , which is equivalent to replacing γ^μ by γ'^μ in Equation (56). When we write Ψ as a square matrix $\Psi^{..}$, Equation (57) takes the form

$$\frac{1}{2} \left(\gamma'^\mu \frac{\partial \Psi^{..}}{\partial x^\mu} + \frac{\partial \Psi^{..}}{\partial x^\mu} \tilde{\gamma}^\mu \right) + (\gamma \Psi)^{..} = 0. \quad (58)$$

While converting Equation (58) to curved space-time, we must take, in a way similar to the case of flat space-time,

$$\Psi'^{..} = -\frac{1}{4} \left[(i/2) G'_{\mu\nu} \tilde{\gamma}_5 \tilde{\gamma}'^{\mu\nu} + i U'_\mu \tilde{\gamma}_5 \gamma'^\mu + \tilde{\chi}'_\mu \gamma'^\mu - i \phi' - \rho' \gamma_5 \right] \epsilon^0. \quad (59)$$

It is to be noted that in Equation (59)

$$\gamma_5 = (-\sqrt{g(x)}/4!) \epsilon_{\mu\nu\lambda\kappa} \tilde{\gamma}'^\mu \tilde{\gamma}'^\nu \tilde{\gamma}'^\lambda \tilde{\gamma}'^\kappa$$

remains the same as in flat space-time. Thus, Equation (58) becomes, with the similar interpretation of $\Psi'^{..}$,

$$\frac{1}{2} \left(\gamma'^\sigma \frac{\partial \Psi'^{..}}{\partial x^\sigma} + \frac{\partial \Psi'^{..}}{\partial x^\sigma} \tilde{\gamma}'^\sigma \right) + (\gamma' \Psi')^{..} = 0. \quad (60)$$

To obtain Equation (60) by the alternative procedure of using the form of our equations in the local geodesic coordinate system, we first note that, at any point,

$$\begin{aligned} \frac{\partial}{\partial X^\sigma} (G_{\mu\nu} \text{ (local geodesic)} \gamma_5 \gamma^\mu \gamma^\nu) \gamma^\sigma &= t^\alpha_\sigma \frac{\partial}{\partial x^\alpha} (t^\lambda_\mu t^\kappa_\nu G'_{\lambda\kappa} \gamma_5 \gamma^\mu \gamma^\nu) \gamma^\sigma \\ &= \frac{\partial}{\partial x^\alpha} (G'_{\lambda\kappa} \gamma_5 \tilde{\gamma}'^\lambda \tilde{\gamma}'^\kappa) \gamma'^\alpha, \end{aligned} \quad (61)$$

where we have made use of the relationship

$$\gamma'^\mu = t^\mu_\nu \gamma^\nu \quad (62)$$

which are exact equations instead of approximations (8) or (10), and agree with the fact that γ^μ should transform like a tensor. The other terms of Equation (58) will, on expansion (59), also transform in a similar manner from the local coordinate system to the general coordinate system. Hence, taking Equation (57) to be true in the local coordinate system, and using Equation (62) where necessary, the same equation (60) is obtained. With the equivalence of the two procedures in mind, we can write down the electromagnetic equations in curved space-time by the use of postulate A from the knowledge of the electromagnetic equations in flat space

$$\partial^{\nu\sigma} \partial G_{\mu\nu} / \partial x^\sigma = 0 \quad (63)$$

and

$$G_{\mu\nu} = \partial A_\nu / \partial x^\mu - \partial A_\mu / \partial x^\nu. \quad (63')$$

The equations thus obtained can be solved in the manner already mentioned with the help of the corresponding Green functions.

§ 8. Affine relationship

It has already been pointed out that the differential operators (50) and (51) are equivalent provided Equations (52) and (53) are satisfied, which need not be true when the transformations to local coordinates are not integrable. But a glance at Equations (50) and (51) suggests that we may take an alternative affine relationship

$$\mathcal{A}^\lambda_{\mu,\nu} = t^\lambda_\alpha T^\alpha_{\mu,\nu} \quad (64)$$

where the comma denotes differentiation with respect to the corresponding space-

time variable. When T^α_{μ} exists only as a space-time function and *not* as a differential coefficient $\partial X^\alpha / \partial x^\mu$ of an (integrable) flat space coordinate system X^α , generally $T^\alpha_{\mu,\nu} \neq T^\alpha_{\nu,\mu}$ and thus $\Delta^\lambda_{\mu\nu}$ in Equation (64) is not symmetric in μ and ν .

This affine relationship was chosen by Einstein¹⁴⁾ in his description of the unitary field theory. We see that this affine relationship is a 'natural' one in the sense that it has been derived with a fairly physical assumption and defines the 'natural' derivatives of contravariant or covariant tensors just as the ordinary covariant derivatives are defined with the usual affine relationship. This is easily seen from the fact that

$$F^\lambda_{;\nu} \equiv t^\lambda_{\alpha} T^\beta_{\nu} F^\alpha_{;\beta} \text{ (Local geodesic)} = F^\lambda_{,\nu} + \Delta^\lambda_{\mu\nu} F^\mu, \quad (65)$$

where the semicolon denotes the natural derivative.

Green¹⁵⁾ has started with a different affine relationship given as

$$\frac{1}{2} (\gamma'^\lambda \gamma'_{\mu,\nu} + \gamma'_{\mu,\nu} \gamma'^\lambda), \quad (66)$$

which, in our case, with application of Equation (62) becomes

$$\frac{1}{2} t^\lambda_{\alpha} T^\beta_{\mu,\nu} (\gamma^\alpha \gamma_\beta + \gamma_\beta \gamma^\alpha) = t^\lambda_{\alpha} T^\alpha_{\mu,\nu},$$

and this is identical with the affine relationship defined by Equation (64). Hence we obtain that when the affine relationship of Green is a numerical multiple of the unit matrix¹⁵⁾, as it must be when the curved space γ'^μ are given by Equation (62), this affine relationship is the 'natural' one, and also, the necessary and sufficient condition imposed by Green for his affine relationship to be a numerical multiple of the unit matrix,

$$\gamma'_{\mu;\nu} = 0, \quad (67)$$

is automatically satisfied.

The affine relationship (64) defined by the postulate A has the advantage that this defines a field of parallel vectors at a distance, since the condition

$$\Delta^\kappa_{\lambda\mu,\nu} - \Delta^\kappa_{\lambda\nu,\mu} + \Delta^\sigma_{\lambda\mu} \Delta^\kappa_{\sigma\nu} - \Delta^\sigma_{\lambda\nu} \Delta^\kappa_{\sigma\mu} = 0, \quad (68)$$

is seen to be automatically satisfied.

The consideration of distance parallelism or teleparallelism by this affine relationship has been discussed in detail by Einstein¹⁴⁾ and Green¹⁵⁾.

§ 9. Discussions

In the above we have considered in some detail the effect of taking postulate A, i.e. assuming that our equations are the same as obtained in the framework of special theory of relativity when we consider the local coordinates at any point. As has already been mentioned, this procedure is not covariant under the transformations of general theory of relativity, since the ordinary derivatives occur in our equations, and these are still not covariantly defined in the local coordinate system. For this reason, we have further restricted that we should choose such local co-

ordinates, that the Christoffel symbols at the point where the local coordinates are taken, should also vanish. But, with this restriction, it has been impossible to solve any case exactly when we already assume to have known metric of the space, even for the simple example of a space that is conformal to a flat space-time.

The most interesting result that has been obtained, however, is the definition of an unsymmetric affine relationship from the curved space D'Alembertian with the help of postulate A or A' which enables us, in its turn, to define distant parallelism of vectors and natural derivatives. The advantages of these concepts have been particularly emphasized by Green.^{15),16)}

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On a Possible Generalization of Quantum Mechanics

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An attempt is made to enlarge the number field underlying quantum mechanics from complex to quaternion. It is not impossible to construct quaternion quantum mechanics in quite a similar manner to the usual complex quantum mechanics, though some limitations are necessary for the position of factors.

Further, as a special case, the charge properties of the spin-0 particles are considered. This example may show some new way to generalize the concept of the charge of the elementary particles without adoption of the iso-space.

§ 1. Introduction

It seems necessary to introduce some new degrees of freedom into the present theory, in order to describe the charge properties of various elementary particles hitherto known to us, especially of baryons and mesons. Usually, such new degrees of freedom have often been looked for outside the Hilbert space and the Minkowski space. So far the iso-space, which is usually three-dimensional and sometimes four and seven-dimensional, has provided us with room for such new degrees of freedom.

However, the iso-space is quite independent of the Hilbert space and the Minkowski space, which are closely related to quantum mechanics and special relativity, respectively. In a sense, we may think that the introduction of the iso-space into the theory is a formal and rather easy-going way to increasing the degrees of freedom, because, in principle, the more the number of the dimensions of a space, the more degrees of freedom we have: namely, we have no definite theoretical criterion to characterize the iso-space. Hence a question naturally arises: Will it be possible to increase the degrees of freedom without introduction of any other spaces, such as the iso-space?

For this purpose, two ways can be considered, namely, the modification of the Hilbert space and that of the Minkowski space. In this paper we try to investigate the former possibility and, as for the Minkowski space, no modification will be made.

As is well known, in the conventional quantum mechanics the neutral particles are represented by real numbers and the charged particles by complex numbers; namely, the concept of the charge is closely related to the use of complex numbers which consist of two real parameters, and the use of complex numbers in the conventional quantum mechanics is axiomatically made from the very beginning.

Hence, we expect that the enlargement of the number field underlying quantum mechanics from complex to quaternion may be one of the most natural ways to introduce some new degrees of freedom into the charge properties of the particles, because we have *four* real parameters in the quaternion case, instead of *two* in the complex case. In this sense, the investigation of the general features of quaternion quantum mechanics itself will be very interesting. (Hereafter, for the sake of brevity, we use the words "quaternion quantum mechanics" for the quantum mechanics over the quaternion field and similarly for "complex quantum mechanics".)

However, concerning the adoption of quaternions in the theory, C. N. Yang pointed out the following two difficulties:¹⁾

1. It seems very difficult to give the measured charge states of the nucleon, namely, the fact that the proton and the neutron has the charge distribution of 1 and 0.

2. The commutative law does not hold in the multiplication of quaternions.

As no attempt to introduce quaternions into quantum mechanics has been made until now*, we will show, in the following, that the formulation of quantum mechanics over the quaternion field will not be impossible, though we are forced to use some axiomatically introduced rules to restrict the position of operators and pure numbers, when they are applied to a bra or a ket vectors. This part would be an answer to the second problem.

For the first difficulty, we must investigate the quantum theory of field which is constructed over the quaternion field. This is not a simple task. Moreover, the case of spinor fields is very complicated, because, in this case, the linearization of the relativistically invariant quadratic forms is necessary and other imaginary units will appear than those which originate from the quaternions in quantum mechanics.

On the other hand, the bosons can be treated, because in this case the linearization of the quadratic form is unnecessary. Moreover, since the kaon is similar to the nucleon concerning the charge states and can be treated without detailed knowledge of field theory, we shall show that we can assign the desired charge distribution for the kaon; so we claim that a part of the solution to the first problem is given, namely, the existence of the strangeness-like quantity may be inferred by the quaternion theory.

Since quaternions are scarcely used in the current theory, we summarize some properties of quaternion in section 2 and investigate the quaternion quantum mechanics in section 3. Some considerations on the charge states of the spin-0 particles are given in section 4, and section 5 contains the discussions.

* The author is indebted to Dr. T. Nakano for showing him, after completion of the present research, a preprint of the work due to D. Finkelstein, J. M. Jauch and D. Speiser ("Notes on Quaternion Quantum Mechanics, Part I", CERN (1959)), which treats the similar subject from the purely logical point of view.

§ 2. Some properties of quaternions

2.1 Definitions and basic formulas

A quaternion ϕ is expressed as

$$\phi = \phi_0 + \phi_1 i_1 + \phi_2 i_2 + \phi_3 i_3,^* \quad (2.1)$$

where ϕ_σ ($\sigma=0, 1, 2, 3$) are real numbers and i_1, i_2 and i_3 are three different imaginary roots of -1 . They satisfy the following relations:

$$\begin{aligned} i_\sigma^2 &= -1 \quad (\sigma=1, 2, 3), \\ i_1 i_2 &= -i_3 i_1 = i_3 \quad (\text{cycl.}). \end{aligned} \quad (2.2)$$

Given a quaternion ϕ , $\bar{\phi}$ is defined by

$$\bar{\phi} = \phi_0 - \phi_1 i_1 - \phi_2 i_2 - \phi_3 i_3. \quad (2.3)$$

$\bar{\phi}$ will be called the "quaternion conjugate" or, more briefly, the " q -conjugate" of ϕ . The q -conjugate corresponds to the usual complex conjugate in the case of complex numbers.

Further, for any two quaternions ϕ and ϕ' , the relation

$$(\overline{\phi\phi'}) = \bar{\phi}' \bar{\phi} \quad (2.4)$$

holds. This relation plays an important role in later developments.

For any quaternion ϕ , we can define $|\phi|$:

$$|\phi|^2 \equiv \bar{\phi}\phi = \phi\bar{\phi} = \phi_0^2 + \phi_1^2 + \phi_2^2 + \phi_3^2 \geq 0. \quad (2.5)$$

By the inequality in (2.5), we can regard $|\phi|$ as the generalization of the absolute value in the case of complex numbers. When it is necessary to distinguish the former from the latter, we call the former the "quaternion absolute value". The equality $|\phi|=0$ holds, if and only if each of ϕ vanishes (i.e., null quaternion).

The q -conjugate of $\bar{\phi}$ is ϕ , i.e.,

$$\overline{(\bar{\phi})} = \phi. \quad (2.6)$$

Thus, (2.5) also shows that the absolute values of ϕ and $\bar{\phi}$ are equal.

Sometimes it is convenient to regard (i_1, i_2, i_3) and (ϕ_1, ϕ_2, ϕ_3) as the components of vectors in a three-dimensional space. Denoting these vectors by \mathbf{i} and $\boldsymbol{\phi}$, respectively, (2.1) and (2.3) can be rewritten as

$$\phi = \phi_0 + (\boldsymbol{\phi} \cdot \mathbf{i}), \quad (2.1')$$

$$\bar{\phi} = \phi_0 - (\boldsymbol{\phi} \cdot \mathbf{i}). \quad (2.3')$$

The formula

* In order to avoid unnecessary confusions with the ordinary imaginary unit i in complex numbers, we do not use the conventionally adopted notations i, j and k for quaternion imaginary units.

$$(\phi \cdot i)(\phi' \cdot i) = -(\phi \cdot \phi') + ([\phi \times \phi'] \cdot i) \quad (2.7)$$

is easily verified and is very useful for actual calculations.

Especially, a quaternion which has the absolute value unity will be called the "normalized quaternion". Any quaternion ϕ can be rewritten as

$$\phi = |\phi|(\alpha + \beta \cdot i), \quad (2.8)$$

where

$$\begin{aligned} \alpha &= \phi_0/|\phi|, \quad \beta = \phi/|\phi|, \\ \alpha^2 + \beta^2 &= 1, \end{aligned} \quad (2.9)$$

that is, a quaternion can be expressed as the product of its absolute value and a normalized quaternion. It is seen that the form (2.8) is the generalization of the polar form in the case of complex numbers and the normalized quaternion corresponds to $e^{i\gamma}$ (γ : real)*.

2.2 Relations between quaternions and complex numbers

The addition of quaternions is defined by the addition of the corresponding components term by term. However, in general, the commutative law does not hold in the multiplication of quaternions. Instead, the relation

$$\phi\phi' - \phi'\phi = 2([\phi \times \phi'] \cdot i) \quad (2.10)$$

exists. Hence, two quaternions commute, either if $\phi=0$ and/or $\phi'=0$, that is, if, at least, one of ϕ and ϕ' is real, or if $[\phi \times \phi']=0$, that is, if two vectors ϕ and ϕ' are colinear. In terms of the components, the latter condition can be expressed as

$$\begin{aligned} \phi_1/\phi'_1 = \phi_2/\phi'_2 = \phi_3/\phi'_3 = m, \\ (m: \text{real, not zero}). \end{aligned} \quad (2.11)$$

Thus a collection of quaternions which commute each other can be expressed by a general form

$$\phi = a + b\mathcal{Q}, \quad (2.12)$$

where

$$\begin{aligned} a, b &: \text{real,} \\ \mathcal{Q} &= (\beta \cdot i) = \beta_1 i_1 + \beta_2 i_2 + \beta_3 i_3, \\ \beta_1, \beta_2, \beta_3 &: \text{real and fixed,} \\ \beta_1^2 + \beta_2^2 + \beta_3^2 &= 1. \end{aligned} \quad (2.13)$$

and

* In fact, the formula

$$e^{\phi \cdot i} = \cos|\phi| + \sin|\phi| \cdot \left(\frac{\phi \cdot i}{|\phi|} \right)$$

is easily verified.

It is seen that these quaternions are equivalent to the usual complex numbers and \mathcal{Q} , with *two* undetermined parameters, corresponds to the usual imaginary unit i .

If we call a quaternion ϕ , which has the form

$$\phi = (\phi \cdot \mathbf{i}) = \phi_1 i_1 + \phi_2 i_2 + \phi_3 i_3, \quad (2.14)$$

the "pure quaternion imaginary", we can say that the complex numbers consist of real numbers and a normalized *fixed* pure quaternion imaginary which serves as the imaginary unit.

As mentioned above, the general quaternions can be obtained if we do not fix three parameters β_1 , β_2 and β_3 in the complex numbers (2.12) and (2.13). Hence, if we consider a three-dimensional space with respect to the quaternion suffices 1, 2 and 3 and imagine a unit sphere with its center at the origin, there exists one kind of the complex numbers corresponding to one point on this surface. Two kinds of complex numbers corresponding to two different points can be connected, only if we regard them as the special cases of quaternions.

2.3 Connection with three-dimensional rotations

As is well known, we may use a 2×2 unitary unimodular matrix

$$\mathcal{V} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}, \quad |a|^2 + |b|^2 = 1, \quad (2.15)$$

for the two-dimensional representation of the rotation group in a three-dimensional space. Here a and b are complex numbers and $*$ means the usual complex conjugation.

Splitting up a and b into the real and the imaginary parts, \mathcal{V} can be rewritten as

$$\mathcal{V} = \begin{pmatrix} \phi_0 + \phi_3 i & \phi_2 + \phi_1 i \\ -\phi_2 + \phi_1 i & \phi_0 - \phi_3 i \end{pmatrix}, \quad \phi_0^2 + \phi_1^2 + \phi_2^2 + \phi_3^2 = 1. \quad (2.16)$$

Let \mathcal{V} and \mathcal{V}' be the matrices representing two rotations r and r' , respectively, and the product matrix $\mathcal{V}'' = \mathcal{V}'\mathcal{V}$, representing the successive rotation rr' , can be obtained by the calculations of ϕ_0'' , ϕ_1'' , ϕ_2'' and ϕ_3'' . The parameters ϕ_0 , ϕ_1 , ϕ_2 and ϕ_3 are called "Euler-Olinde-Rodrigues' parameters". The composition laws of these parameters can be expressed by the use of the normalized quaternions

$$\phi = \phi_0 + \phi_1 i_1 + \phi_2 i_2 + \phi_3 i_3, \quad (2.17)$$

instead of explicitly taking the product of the matrices \mathcal{V} and \mathcal{V}' .

Thus, inversely speaking, given a normalized quaternion (2.17), we can always construct a 2×2 matrix \mathcal{V} which is unitary and unimodular with Euler-Olinde-Rodrigues' parameter ϕ_σ ($\sigma=0, 1, 2$ and 3) and can represent a rotation in a three-dimensional space. Such a space has nothing to do with the coordinate transformations and is quite independent of the Minkowski space. This space will be called the "internal space".

Furthermore, we can write \mathcal{V} in the spinor notations :

$$\mathcal{V} = (\phi_I \phi_{II}) = \begin{pmatrix} \phi^1 & \phi^{\dot{1}} \\ \phi^2 & \phi^{\dot{2}} \end{pmatrix}, \quad (2.18)$$

where

$$\phi_I = \begin{pmatrix} \phi^1 \\ \phi^2 \end{pmatrix} = \begin{pmatrix} \phi_0 + \phi_3 i \\ -\phi_2 + \phi_1 i \end{pmatrix}, \quad (2.19)$$

and

$$\phi_{II} = \begin{pmatrix} \phi_2 + \phi_1 i \\ \phi_0 - \phi_3 i \end{pmatrix} = \begin{pmatrix} -\phi^{\dot{2}} \\ \phi^{\dot{1}} \end{pmatrix} = \begin{pmatrix} \phi^{\dot{1}} \\ \phi^{\dot{2}} \end{pmatrix}. \quad (2.20)$$

Here the dot means the usual complex conjugation and the use is made of the relation in the spinor analysis

$$\phi^{\dot{1}} = -\phi^2, \quad \phi^{\dot{2}} = \phi^1, \quad (2.21)$$

for raising and lowering the spinor suffices. Also we may write

$$\phi_{II} = \frac{1}{i} \sigma_y \phi_I^*, \quad (2.22)$$

where σ_y is the usual Pauli spin matrix.

At this point we should notice the following fact. Denoting the quaternion corresponding to \mathcal{V}^* by ψ^* , ψ^* is not always equal to $\bar{\psi}$. This fact infers the following idea. If we write ψ as

$$\psi = \phi_0 + i_3 \phi_3 + i_2 (\phi_2 + i_3 \phi_1), \quad (2.23)$$

ψ^* is expressed by

$$\psi^* = \phi_0 - i_3 \phi_3 + i_2 (\phi_2 - i_3 \phi_1). \quad (2.24)$$

Hence the $*$ operation means, in the languages of quaternions, the conjugation with respect to i_3 only. Since the usual complex conjugation is closely related to the concept of the charge and corresponds to a special case of the q -conjugations, we might expect that something will enter into the quaternion theory in addition to the usual charge. We will return to this point later.

2.4 "Quaternion spineur" and "quaternion vrais"

According to Ruelle,²⁾ there are two types of quaternions, i. e., "quaternion spineur," and "quaternion vrais". He used them in the reformulation of the usual iso-spin formalism in strong interactions by means of quaternions.

First, quaternion spineur is transformed according to

$$\psi \rightarrow \psi' = u\psi, \quad (2.25)$$

where u is a normalized quaternion. The transformation (2.25) does not change the absolute value of ψ and can be regarded as a three-dimensional generalization of the usual gauge transformation of the first kind. In this case, we have to use

all imaginary units i_1 , i_2 and i_3 ; namely, quaternion spineur is irreducible.

On the other hand, quaternion vrais is transformed according to

$$\psi \rightarrow \psi' = u\psi u. \quad (2.26)$$

This transformation also does not change the absolute value of ψ . However, in this case, ψ is reducible, namely, (2.26) is equivalently described by two independent transformations

$$\psi_0 \rightarrow \psi'_0 = \psi_0, \quad (2.27)$$

$$\psi \rightarrow \psi' = \psi + 2u_0[u \times \psi] + u(u \cdot \psi) - \psi(u \cdot u). \quad (2.28)$$

In other words, a reducible quaternion is split up into a scalar and a vector in the internal space. For the sake of brevity, we call them the internal scalar and the internal vector, respectively.

§ 3. Generalization of quantum mechanics

3.1 Rules for the position of operators and pure numbers

The considerations in the previous section can be generalized to the case of the vectors each component of which is described by a quaternion and we are led to the concepts of the bra and the ket vectors in quaternion quantum mechanics. Also, the operators and matrices which operate on them may be obtained.

However, since in this case even constant multipliers, as far as they are quaternions, cannot be placed in an arbitrary way, because of their non-commutativity, we must determine the position of factors from the beginning. In order that the theory is parallel to the usual quantum mechanics, the following rules will be used axiomatically:

For a ket vector $|A\rangle$, operators α should be placed on the left, i.e., we write $\alpha|A\rangle$, and constant multipliers c on the right, i.e., we write $|A\rangle c$. For a bra vector $\langle A|$, we write in the reversed order.

It is to be noted that only the case of $|A\rangle c^*$ is different from the conventional case, where it is written as $c|A\rangle$ ³⁾, though in the latter case, the position of c is not the matter of importance.

The above rules imply that $|A\rangle$ and $|A\rangle c$ represent the same state and similarly for $\langle A|$ and $c\langle A|$. In a sense, the position of c for $|A\rangle$ and $\langle A|$ are necessarily more symmetrical in the present case than in the usual case. Also, for example, the eigenvalue equations should be written in the following form:

$$\begin{aligned} \alpha|A\rangle &= |A\rangle c, \\ \langle A|\alpha &= c\langle A|. \end{aligned} \quad (3.1)$$

3.2 Bra and ket vectors

* D. Finkelstein, J. M. Jauch and D. Speiser (loco. cit.) also gives the same prescriptions.

Given two states corresponding to the ket vectors $|A_1\rangle$ and $|A_2\rangle$, the general state formed by superposing them corresponds to a ket vector $|R\rangle$,

$$|R\rangle = |A_1\rangle c_1 + |A_2\rangle c_2, \quad (3.2)$$

where c_1 and c_2 are quaternion multipliers (not zero). Multiplying a constant quaternion $\bar{c}_1/|c_1|^2$ from the right in the both sides of (3.2), we have

$$|R\rangle \bar{c}_1/|c_1|^2 = |A_1\rangle + |A_2\rangle c_2 \bar{c}_1/|c_1|^2.$$

Since $|R\rangle$ and $|R\rangle \bar{c}_1/|c_1|^2$ represent the same state as mentioned above ($\bar{c}_1/|c_1|^2$ is also a quaternion), this state is determined by one quaternion or by *four* real parameters. Thus, from the given two states a *fourhold* infinity of states may be obtained by superposition.

From a bra vector $\langle A|$ and a ket vector $|B\rangle$ we can construct a scalar product $\langle A|B\rangle$. The conditions that a scalar product $\langle A|B\rangle$ is a linear function of $|B\rangle$ may be expressed by

$$\langle A| \{ |B\rangle + |B'\rangle \} = \langle A|B\rangle + \langle A|B'\rangle, \quad (3.3)$$

$$\langle A| \{ |B\rangle c \} = \langle A|B\rangle c, \quad (3.4)$$

c being any quaternion.

The sum of two bra vectors is defined by

$$\{ \langle A| + \langle A'| \} |B\rangle = \langle A|B\rangle + \langle A'|B\rangle, \quad (3.5)$$

and the product of a bra vector and a quaternion c is defined by

$$\{ c \langle A| \} |B\rangle = c \langle A|B\rangle. \quad (3.6)$$

(3.3) and (3.5) show that the products of bra and ket vectors satisfy the distributive axiom of multiplication, as in the usual case. However, by (3.4) and (3.6) we cannot place c at an arbitrary position.

We now make the assumption that there is such a one-one correspondence between the bra and the ket vectors, that the bra vector corresponding to $|A\rangle + |A'\rangle$ is the sum of the bras corresponding to $|A\rangle$ and $|A'\rangle$ and the bra corresponding to $|A\rangle c$ is \bar{c} times the bra corresponding to $|A\rangle$, the bar being, in the present case, the quaternion conjugation.

By the relation (2.4), it will be very natural to assume that

$$\langle A|B\rangle = \overline{\langle B|A\rangle}, \quad (3.7)$$

as in complex quantum mechanics. Putting $|B\rangle = |A\rangle$ here, we find that the number $\langle A|A\rangle$ must be real. We make a further assumption

$$\langle A|A\rangle > 0, \quad (3.8)$$

except when $|A\rangle = 0$.

Thus, we can conclude that many statements on the bra and ket vectors can be generalized to quaternion quantum mechanics, only if we are careful enough of

the position of constant multipliers and regard the bar as the q -conjugate.

3.3 Linear operators

Similarly we can discuss linear operators. Let α be a linear operator and c a quaternion, the conditions of linearity may be expressed by the equations

$$\alpha\{|A\rangle + |A'\rangle\} = \alpha|A\rangle + \alpha|A'\rangle, \quad (3.9)$$

$$\alpha\{|A\rangle c\} = \{\alpha|A\rangle c\}. \quad (3.10)$$

For any two linear operators α and β we have the relations:

$$\{\alpha + \beta\}|A\rangle = \alpha|A\rangle + \beta|A\rangle, \quad (3.11)$$

$$\{\alpha\beta\}|A\rangle = \alpha\{\beta|A\rangle\}. \quad (3.12)$$

However, in general, $\alpha\beta$ differs from $\beta\alpha$, as in the usual case.

Further, we have

$$\langle B|\bar{\alpha}|A\rangle = \overline{\langle A|\alpha|B\rangle}, \quad (3.13)$$

$$\overline{(\bar{\alpha})} = \alpha, \quad (3.14)$$

$$\overline{\alpha\beta} = \bar{\beta}\bar{\alpha}. \quad (3.15)$$

A linear operator α for which $\bar{\alpha} = \alpha$ will be called the "real linear operator" and a linear operator β for which $\bar{\beta} = -\beta$, the "pure quaternion imaginary linear operator". Any linear operator may be split up into a real and a pure quaternion imaginary part.

If α and β are two real linear operators, $\alpha\beta$ is, in general, not real by (3.15). However, $\alpha\beta + \beta\alpha$ is a real and $\alpha\beta - \beta\alpha$ a pure quaternion imaginary linear operator.

We may regard a linear operator as a matrix. Thus a real operator will be called the "quaternion Hermitian matrix" and a pure quaternion imaginary operator the "quaternion anti-Hermitian matrix".

Taking the q -conjugate of each element of a quaternion matrix $A = (A_{ij})$ and transposing it, we have the matrix A^H :

$$A^H = (\bar{A})^T = (\bar{A}_{ji}). \quad (3.16)$$

If, in a special case, $A^H = A$, this matrix is a quaternion Hermitian matrix.

3.4 Examples

As examples of what we have given above, we state here two theorems, whose analogues are quite important in the conventional quantum mechanics.

Theorem. A quaternion Hermitian matrix has real eigenvalues.

Proof. Let λ be an eigenvalue of this matrix and x the eigenvector corresponding to λ , our eigenvalue problem may be expressed in the following form (see (3.1)):

$$A_{ij}x_j = x_i\lambda.$$

Taking the q -conjugate of the above equation, we have (see (2.4))

$$\bar{x}_j \bar{A}_{ij} = \bar{\lambda} \bar{x}_i.$$

We multiply \bar{x}_i from the left in the upper equation and x_i from the right in the lower. By comparison of both equations, we have

$$\bar{x}_i x_i \lambda = \bar{\lambda} \bar{x}_i x_i,$$

where use is made of the fact that $A^H = A$. Since $\bar{x}_i x_i$ is real and a real number commutes with all quaternions, the above equation implies that

$$(\lambda - \bar{\lambda}) \bar{x}_i x_i = 0.$$

If we consider such x_i as $\bar{x}_i x_i \neq 0$ for non-trivial solution, we can conclude that

$$\lambda = \bar{\lambda},$$

namely, λ is real.

Theorem. Two eigenvectors of a quaternion Hermitian matrix belonging to different eigenvalues are orthogonal.

Proof. Let $|\alpha'\rangle$ and $|\alpha''\rangle$ be two eigenkets of a quaternion Hermitian matrix α , belonging to the eigenvalues α' and α'' , respectively. Then we have the equations

$$\alpha |\alpha'\rangle = |\alpha'\rangle \alpha',$$

$$\alpha |\alpha''\rangle = |\alpha''\rangle \alpha''.$$

Taking the q -conjugate of the upper equation and multiplying $|\alpha''\rangle$ from the right gives

$$\langle \alpha' | \alpha | \alpha'' \rangle = \alpha' \langle \alpha' | \alpha'' \rangle,$$

where α' is real by the above theorem. Next, multiplying $\langle \alpha' |$ in the lower equation from the left gives

$$\langle \alpha' | \alpha | \alpha'' \rangle = \langle \alpha' | \alpha'' \rangle \alpha''.$$

Since α'' is also real and a real number always commutes with $\langle \alpha' | \alpha'' \rangle$, we have, by comparison of the both equations,

$$(\alpha' - \alpha'') \langle \alpha' | \alpha'' \rangle = 0.$$

Thus, if $\alpha' \neq \alpha''$, $\langle \alpha' | \alpha'' \rangle = 0$ and the two eigenvectors $|\alpha'\rangle$ and $|\alpha''\rangle$ are orthogonal.

It is seen from these examples that, only if we are careful enough of the position of constant multipliers which are not treated seriously in the conventional quantum mechanics and take account of the relation (2.4), which holds even for constant quaternions, quaternion quantum mechanics has the structure quite similar to the conventional quantum mechanics. However, since the quaternions have *four* real parameters, instead of two in complex numbers, we may expect the increase of the degrees of freedom in the theory.

3.5 Fundamental quantum conditions

At the end of this section we briefly mention about the fundamental quantum conditions. By comparison of the commutation relations and the classical Poisson brackets, as done by Dirac,³⁾ we have the equations

$$\begin{aligned} q_r q_s - q_s q_r &= 0, \quad p_r p_s - p_s p_r = 0, \\ q_s p_r - p_r q_s &= \hbar \delta_{rs} \mathcal{Q}, \end{aligned} \quad (3.17)$$

\mathcal{Q} being a normalized pure quaternion imaginary.

As mentioned in section 2, \mathcal{Q} contains two undetermined parameters, contrary to the i appearing in complex quantum mechanics. However, at the present stage of the theory, we can not definitely say whether this ambiguity brings difficulties to quaternion quantum mechanics or, rather, whether this point may be advantageous to the usual case. We will not discuss the problem of quantization any more in this paper.

§ 4. Charge states of the spin-0 particles

Now that we have established the possibility to construct quaternion quantum mechanics, we are in the position to explain the charge states of the particles with iso-spin 1/2. As mentioned in the Introduction, we will confine ourselves to the scalar fields and investigate the charge properties of the spin-0 particles.

Since we intend only to enlarge the complex numbers in quantum mechanics to quaternions, as emphasized in the Introduction, we should use only real quantities in relativistic considerations. Thus, for example, $x_0 = ct$ should be adopted as the time coordinate instead of $x_4 = ict$.

Contrary to the i in the quantum theory, where it is closely related to the physical quantity, i.e., the charge, we consider the i in relativity to be merely artificial, because its role is only to convert the pseudo-Euclidian fundamental form into the Euclidian form, although the latter is more convenient for us to handle than the former. Thus, we exclude the i 's appearing in relativity, in order not to disturb the essential roles played by the quaternion imaginary units.

Now, in the case of the complex scalar wave functions $\psi = \psi_0 + \psi_1 i$, each of two real functions ψ_0 and ψ_1 satisfies the same linear wave equation and, if we want to represent a charged field, we must combine them as the complex functions like $\psi_0 \pm \psi_1 i$, which satisfy the same equation. Thus, in this case, the way of combination is rather simple.

On the other hand, the situation is quite different in the quaternion theory from the case of the complex theory, because, in the former case, there are three non-commuting imaginary units and the way of combination becomes much more complicated.

As the wave equation for free quaternion scalar fields, we adopt the Klein-Gordon-type equation. Since we do not aim at the relativistic modification, as

stated in the Introduction, other choices could not be considered. Thus our equation is

$$(\square - \kappa^2)\psi = 0. \quad (4.1)$$

The meaning of this equation is as follows: Splitting up ψ into its real and imaginary parts, each one satisfies the same equation

$$(\square - \kappa^2)\psi_\sigma = 0, \quad (\sigma = 0, 1, 2, 3), \quad (4.2)$$

because the operator $(\square - \kappa^2)$ consists only of real quantities, as mentioned above. However, it is to be noted that, though (4.1) always implies (4.2), the inverse statement does not always hold, that is, from four ψ_σ in (4.2), (4.1) does not always follow. In other words, with four ψ_σ in (4.2), we can construct the irreducible and reducible quaternions. Further, the quaternion index σ should not be confused with the vector index in the Minkowski space, though both have quite similar appearance.

Our next step is to investigate the above mentioned combinations and give the measured charge states for the scalar particles, without the introduction of the isospace. Of course, in the present discussion, the word scalar may be replaced by pseudoscalar, because we do not take account of the space reflection property of wave functions in the Minkowski space. Indeed, pseudoscalar may be preferable, because pions and kaons which we will study in this section are known to be described by the pseudoscalar wave functions. However, we use the word scalar in the following for the sake of brevity.

Now we study the various combinations of ψ_σ case by case.

1. Internal scalar part of reducible quaternion

Among four ψ_σ , ψ_0 by itself is an irreducible representation, namely, the internal scalar part of a reducible quaternion is described by a single real number and cannot have the charge. Thus, the internal scalar wave function describes a neutral field. Actually, it will be very probable that π_0' , whose existence is proposed hypothetically by many physicists, is described by an internal scalar wave function.

2. Internal vector part of reducible quaternion

The remaining three ψ_σ ($\sigma = 1, 2, 3$) form a vector in the internal space. This vector can be decomposed by the following procedure. Since, as mentioned in section 2.3, the conjugation with respect to i_3 corresponds to the usual complex conjugation which is closely related to the charge of the particles, it seems natural to consider the special gauge transformation

$$u = \alpha + \beta i_3, \quad \alpha^2 + \beta^2 = 1. \quad (4.3)$$

Applying the above transformation to the internal vector gives

$$\begin{aligned} \psi_1 &\rightarrow \psi_1' = (\alpha^2 - \beta^2)\psi_1 - 2\alpha\beta\psi_2, \\ \psi_2 &\rightarrow \psi_2' = 2\alpha\beta\psi_1 + (\alpha^2 - \beta^2)\psi_2, \\ \psi_3 &\rightarrow \psi_3' = \psi_3, \end{aligned} \quad (4.4)$$

namely, the transformation of ϕ_1 and ϕ_2 is a real rotation and ϕ_3 remains invariant.

Using the imaginary unit i_3 (or, more generally, a normalized fixed quaternion imaginary), we can combine ϕ_σ ($\sigma=1, 2, 3$) into the well-known pion triplet:

$$\begin{aligned}\phi &= \pi, \\ \pi^\pm &= \frac{1}{\sqrt{2}}(\phi_1 \pm \phi_2 i_3), \quad \pi^0 = \phi_3.\end{aligned}\tag{4.5}$$

That ϕ_3 is neutral may be seen by the similar consideration as given above.

3. Irreducible quaternion

In the irreducible quaternion ϕ^1 and ϕ_2 are formed of the real and i_3 -part and they may be considered to describe the usual charged fields, if we take account of the fact that the i_3 -conjugation corresponds to the usual complex conjugation.

On the other hand, though ϕ^2 and ϕ_1 , which are formed of the i_1 - and i_2 -part, have a freedom with respect to the i_1 and i_2 , they remain unchanged under the i_3 -conjugation which is closely related to the charge. In other words, though ϕ^2 and ϕ_1 are *not real*, they seem as if they were neutral with respect to the usual complex conjugation, in a similar sense that the real functions suffer from no change by the complex conjugation.

This fact is very remarkable and implies that another charge-like quantity than the usual charge exists in the quaternion theory. Hence, it will be very natural to make the following assignments:

$$\begin{aligned}K^+ &= \phi^1, \quad \bar{K}^0 = \phi_1, \\ K^0 &= \phi^2, \quad K^- = \phi_2,\end{aligned}\tag{4.6}$$

where we have the correct property on charge conjugation by virtue of (2.22).

§ 5. Discussion

The peculiar situation in the case of kaon, which was mentioned in the previous section, is closely related to the irreducibility of quaternion spineur and cannot be considered at all by the usual complex theory, where only one imaginary unit appears.

Further, it is to be noted that, while our internal space is *three-dimensional* and can imply the concept of the strangeness-like quantity, the conventional *three-dimensional* iso-space can not exhibit the existence of the strangeness. In order to include the strangeness, the iso-space must have more number of dimensions than *four*. Hence, we may say that, though our three-dimensional internal space has the similar appearance to the usual three-dimensional iso-space, there exists a rather clear-cut distinction between them. This will be a remarkable fact.

As to the kaon, there is another remark. If we adopt the assignment (4.6), ϕ_1 and ϕ_2 in K^0 and \bar{K}^0 are seen to represent K_1^0 and K_2^0 , i. e., the so-called particle mixtures of the neutral kaons⁴⁾ are described only by the i_1 - and i_2 -part of

an irreducible quaternion, respectively.

Similar considerations may also be applied to the pions and the hypothetical neutral meson. If we transform the reducible quaternion (π_0', π) into the irreducible quaternion, we may expect that the similar particle-mixing phenomena will occur among the π_0' and the π^0 , namely, we may consider the existence of the π_1^0 and the π_2^0 similar to the K_1^0 and the K_2^0 . However, at present, we are not in a position to enter into these problems any more, because the physical meaning of the transformations from the irreducible to the reducible quaternion or from the latter to the former is not clear.

Only we will mention here that, for a charged field $\psi = \psi_0 + \psi_1 i$, no clear physical interpretation of ψ_0 and ψ_1 themselves exist, namely, they are considered to be the artificial fields, contrary to the case of the K_1^0 and K_2^0 which have some experimental supports.

As for the spinor particles which have not been treated in this paper, we might expect that similar situations will occur as in the case of the scalar particles, because the baryons have quite similar charge properties as those of the mesons which have been studied in section 4. This is a problem to be studied in the future.

Finally, as for the vector fields, the only example we know is the electromagnetic field. If we assume that a quaternion-valued vector field exists and that this is described by reducible quaternions, we might conjecture that its internal scalar part (i.e., the pure neutral part) should be the usual electromagnetic field. The remaining internal vector part might be considered to be something like Yang and Mills' b_μ -fields⁵⁾ (in strong interactions). Or, rather, they might be considered to govern the electromagnetic and weak interactions.

In conclusion we may say that the construction of quaternion quantum mechanics is not impossible and that some new features in the charge properties of the elementary particles could be interpreted from the view-point of the quaternion theory. At least, it will be permitted to claim that the introduction of quaternions into the theory will be one of the very promising ways to understand the charge properties of various elementary particles.

Among the problems which have not been given solutions in the present paper, two will be of vital importance: One is the problem of quantization and the other is the construction of the unified theory which comprises the fermions. These problems, together with their applications, are left for later consideration.

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However, they seem to be interested rather in the mathematical side of quaternion quantum mechanics.

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Measurement in Quantum Mechanics

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In quantum mechanics it is well known that, if any two states are superposed, they interfere with each other. It is true, we should not deny such interference in principle, but we may assert what follows. When two states different from each other in a great many degrees of freedom are superposed, the interference effect becomes obscure. If they are different in an infinitely many degrees of freedom, they do not interfere at all, and their superposition is nothing but a mere *probability function*. This assertion enables us to understand how the *probability amplitude* for a micro-system is converted into a *probability function* for a measuring apparatus in the course of measurement.

§ 1. Theory of measurement

Though quantum mechanics has achieved a brilliant success, yet there is lack of a completely unified interpretation of it, and there has been a lively discussion with regard to the subject. Especially the theory of measurement in quantum mechanics has aroused much discussion among the physicists, and it is still in dispute because there is no proper interpretation acceptable for all of them. In the present article this problem will be discussed in connection with the views of H. S. Green.¹⁾

At present the following may be considered to be the most "orthodox" interpretation on this subject, that is:

Let R be a Hermitian operator with a pure discrete and simple spectrum, and put

$$R\phi_i = \lambda_i \phi_i, \quad i=1, 2, \dots \quad (1)$$

Every wave function ϕ can, then, be represented by a series

$$\phi = c_1 \phi_1 + c_2 \phi_2 + \dots \quad (2)$$

When we measure R in the state ϕ and get the result λ_i , then the measurement changes ϕ into ϕ_i , and the probability of this transition is given by

$$P(\phi_i) = |(\phi_i, \phi)|^2 = |c_i|^2 \quad (3)$$

We cannot accept such an interpretation from a macroscopic point of view unless ϕ is only of a statistical nature, because the transition from ϕ to ϕ_i is abrupt and non-causal. On the other hand, it is well known that any two states interfere

with each other, and so ψ represents something more than a mere probability. In this sense ψ is usually called a *probability amplitude*. Now we are in a dilemma. From a macroscopic point of view ψ should be a mere *probability function*, that is, a mere statistical sum of ϕ 's, while from a microscopic point of view it should be a *probability amplitude*. The most typical illustration of this dilemma is afforded by Schrödinger's famous allegory about a cat.²⁾ It is believed in general that the abrupt and non-causal change of states by measurement is a characteristic feature of quantum mechanics in contrast to classical mechanics. There is, however, no conclusive evidence of this interpretation, and the acceptance of such an interpretation is not necessary to develop the most part of the theory of quantum mechanics. This is the reason why there is no unified theory of measurement as yet.

Some assert that we must treat measuring apparatus by quantum statistics, and that the above dilemma will be solved by such treatment.^{1), 3)} Let us cite, for instance, Durand III: "the initial state of the object system may be known, but that of the apparatus system is at best delimited only by the results of a few quasi-classical observations. Furthermore, complete knowledge of the quantum mechanical details of the apparatus state is probably impossible even in principle. . . . A statistical treatment of the measuring process relative to the apparatus state is required." Though there may be some truth in his view, we cannot wholly agree with him. If an apparatus can be treated by quantum statistics, it must be treated by quantum mechanics at least in principle, and so the theory will not be essentially improved by such a statistical treatment alone. Nevertheless, we agree substantially with H. S. Green in his following views:¹⁾

"The essential problem to be faced, therefore, concerns the detector. It has to be explained how the *probability amplitude* for the arrival of a micro-system in one of two or more channels gets converted into a *probability* for the transition between the metastable and stable states of a particular detector."[†]

Now, we interpret the process of "measurement" in the following way and explain it by a very simple example. Let [I] be a micro-system whose physical quantity R is measured by an apparatus [II]. Let, further, R have a pure discrete and simple spectrum and satisfy Eq. (1). At the beginning of "measurement" [II] is in a prepared metastable state ψ_0 , and in the course of "measurement" [I] interacts on [II] and converts ψ_0 into a state ψ . If [I] is in the state ϕ_i , then [II] is converted into a corresponding stable state ϕ_i , and we know that R has the value λ_i . As [II] is a measuring apparatus, these ϕ 's ought to differ from each other macroscopically. If the state ψ of [I] is given by Eq. (2), then

$$\psi = c_1 \phi_1 + c_2 \phi_2 + \cdots \quad (2')$$

In principle, this Eq. (2') is of the same nature as Eq. (2), that is, ψ is a super-

[†] Even if we accept this view of Green's completely, we cannot accept his explanation. At the end of § 2 we shall deal with this problem.

position of ϕ 's, and it is a *probability amplitude*. We shall show, however, in the following section that, *if a state is a superposition of two or more states different from each other macroscopically, it can be considered as a mere probability function*. This implies that Eq. (2') does not mean more than the following equation :

$$P(\phi_i) = |(\phi_i, \psi)|^2 = |c_i|^2. \quad (3')$$

Thus we can consider that, when we "observe" the state ψ , it changes into one of ϕ 's abruptly and non-causally. Even if the relation between a micro-system and a measuring apparatus be more complicated, the above interpretation will remain essentially unchanged.

In conclusion we assert what follows: *In the course of "measurement" the micro-system interacts on a measuring apparatus, and the prepared metastable state of the apparatus changes into a new state, which is a superposition of several stable states. As these stable states should differ macroscopically, we can treat the new state as a mere probability function, and so, when we "observe" it, it changes into one of these stable states abruptly and non-causally.* In this sense we can agree with H. S. Green, and the above statement enables us to accept the "orthodox" view on measurement in a slightly modified sense. Care must be taken of the fact that we have distinguished the word "observation" from the word "measurement" in the preceding assertion. "Observation" implies the final process of the usual measuring process, that is, the cognizance by our own organs of perception. On the contrary, "measurement" implies the usual measuring process except for the final process, "observation." In our terminology there may be an automatic "measuring" apparatus, but not an automatic "observing" apparatus.

§ 2. Systems with a great many degrees of freedom

Let S_N be a system with N degrees of freedom, where N is a very large number, and \mathfrak{h}_i , $i=1, 2, \dots, N$, be a Hilbert space corresponding to the i -th degree of freedom. Put

$$\mathfrak{H}_N = \bigotimes_{i=1}^N \mathfrak{h}_i, \quad (4)$$

and then it is a Hilbert space corresponding to S_N . For example, let S_N be made up of N particles with the same properties, and $\phi_i(x_i)$ be the wave-function of the i -th particle. Then every state $\psi \in \mathfrak{H}_N$ can be represented by one of direct products

$$\phi_1(x_1) \times \phi_2(x_2) \times \dots \times \phi_N(x_N), \quad \phi_i \in \mathfrak{h}_i,$$

or by their linear combination. In principle, S_N should obey quantum mechanics, but, if we apply the usual interpretation of quantum mechanics indiscriminately to S_N , we cannot but arrive at somewhat unphysical conclusions.

Let ψ' and ψ'' be given by

$$\left. \begin{aligned} \psi' &= \phi_1'(x_1) \times \phi_2'(x_2) \times \cdots \times \phi_N'(x_N), \\ \psi'' &= \phi_1''(x_1) \times \phi_2''(x_2) \times \cdots \times \phi_N''(x_N). \end{aligned} \right\} \quad (5)$$

At first, we assume that ψ' and ψ'' are the same except for the N -th degree of freedom, that is,

$$\left. \begin{aligned} \phi_i'(x_i) &= \phi_i''(x_i), \quad i=1, 2, \dots, N-1, \\ (\phi_N', \phi_N'') &= \int \overline{\phi_N'(x_N)} \phi_N''(x_N) dx_N = 0. \end{aligned} \right\} \quad (6)$$

(5) and (6) show that

$$(\psi', \psi'') = \prod_{i=1}^N (\phi_i', \phi_i'') = 0, \quad (7)$$

and so these two states are "completely" different from each other in the usual quantum mechanical sense. ψ' and ψ'' , however, should be very near to each other from a macroscopic point of view, because N is very large. In fact, it is not an operator q_i in \mathfrak{h}_i but a mean of q_1, q_2, \dots, q_N , that has a macroscopically important meaning, and ψ' and ψ'' give almost equal expectation values to the mean. This implies that, in order that two states ψ' and ψ'' in (5) can be said to be different macroscopically, they should differ in many degrees of freedom; namely, they should satisfy the following equations:

$$\prod_{i=1}^N (\phi_i', \phi_i'') = 0, \quad (7')$$

where each \prod' is a product of (ϕ_i', ϕ_i'') , $i=1, 2, \dots, N$, with the exception of a few arbitrary members. In other words, a great many of $|(\phi_i', \phi_i'')|$, $i=1, 2, \dots$, should be zero.

In quantum mechanics it is usually assumed that for any two states, ψ' and ψ'' , there is at least one such physically meaningful operator q that

$$(\psi', q\psi'') \neq 0, \quad (8)$$

and this assumption seems to have some grounds. The validity of the above assumption, however, becomes doubtful for the system S_N . In S_N every physically meaningful operator is given by

$$q_n = q_{i_1} \times q_{i_2} \times \cdots \times q_{i_n}, \quad (9)$$

or by their linear combination, where q_{i_j} is an operator in \mathfrak{h}_{i_j} . Let ψ' and ψ'' satisfy Eqs. (5) and (6), and put

$$q_N = q_1 \times q_2 \times \cdots \times q_N.$$

As

$$(\psi, q_N \psi) = (\phi_1, q_1 \phi_1) \times (\phi_2, q_2 \phi_2) \times \cdots \times (\phi_N, q_N \phi_N),$$

there may be such q_N that $|(\psi', q_N \psi') - (\psi'', q_N \psi'')| \gg 1$. This is unphysical at least from a macroscopic point of view, and so such q_N will be physically meaningless. Furthermore, it is obvious that any q_n corresponding to a proper physical quantity, such as energy and momentum, is a product of a few q 's. From these facts, we may conclude that, the larger n becomes, the less meaningful becomes q_n physically.[†] If ψ' and ψ'' are macroscopically different, every operator q satisfying

$$(\psi', q\psi'') \approx 0 \quad (8')$$

cannot be q_n with small n , and has little physical meaning. Therefore, we may say that

$$(\psi', q\psi'') = 0 \quad (10)$$

for almost every q which is physically meaningful.

Put

$$\psi^\theta = \psi' + e^{i\theta} \psi'' \quad (11)$$

for any macroscopically different states ψ' and ψ'' , where θ is any real number. From (10) it is easy to show that

$$(\psi^\theta, q\psi^\theta) = (\psi^{\theta'}, q\psi^{\theta'})$$

for any θ and θ' and for almost every physically meaningful operator q . Thus every ψ^θ can be considered as representing the same state, and this means that ψ' and ψ'' do not interfere at all. Accordingly,

$$\psi = \psi' + \psi'' \quad (11')$$

can be considered as a mere statistical sum of ψ' and ψ'' , that is, a mere *probability function*, though it is a *probability amplitude* as a matter of principle. This is what we desired to show in the preceding section.

We shall compare the foregoing assertion with that of H. S. Green.¹⁾ Let $\rho(x, y)$ be a statistical matrix

$$\rho(x, y) = \{\psi'(x) + \psi''(x)\} \cdot \{\psi'(y) + \psi''(y)\}^*. \quad (12)$$

The interference of ψ' and ψ'' can be represented by

$$\begin{aligned} \rho'(x, y) &= \psi'(x) \cdot \psi''(y)^* \\ &= [\phi_1'(x_1) \cdot \phi_1''(y_1)^*] \cdot [\phi_2'(x_2) \cdot \phi_2''(y_2)^*] \cdots [\phi_N'(x_N) \cdot \phi_N''(y_N)^*]. \end{aligned} \quad (13)$$

When ψ' and ψ'' are macroscopically different,

$$\begin{aligned} &\int \rho'(x, x) dx \\ &= \int \phi_1'(x_1) \overline{\phi_1''(x_1)} dx_1 \cdot \int \phi_2'(x_2) \overline{\phi_2''(x_2)} dx_2 \cdots \int \phi_N'(x_N) \overline{\phi_N''(x_N)} dx_N \\ &= (\psi', \psi'') = 0. \end{aligned} \quad (14)$$

[†] The validity of this assertion will be clear in the quantum theory of fields. (See also § 3.)

It seems to us that Green's assertion is, in essence, nothing but to assert what follows:

Eq. (14) shows that $\rho'(x, y)$ is a very small operator, accordingly it has no physical effect.

This seems to us incorrect, however, because the norm of the operator $\rho'(x, y)$ is 1. On the contrary, our assertion is as follows:

For any physically meaningful operator $q(x, y)$,

$$\int \rho'(x, y) q(x, y) dx dy = 0, \quad (15)$$

therefore $\rho'(x, y)$ has no physical effect.

Lastly, it will be necessary to add a few words about the relations between ϕ 's in (2'). At first sight, it seems paradoxical to apply (10) to these ϕ 's, because all these states result from the same state ψ_0 . For example, if

$$\phi_i = U_i \psi_0, \quad i=1, 2, \dots, N,$$

then

$$\phi_i = U_i U_i^{-1} \phi_i,$$

and

$$(\phi_i, U_i U_i^{-1} \phi_i) = (\phi_i, \phi_i) = 1$$

in contradiction to (10). In order to solve this dilemma, it is necessary to remember the fact that ψ_0 is the prepared *metastable* state; therefore, even if the operator U_i be physically meaningful, U_i^{-1} has no physical meaning. *Maxwell's demon* will be responsible for taking these operators into consideration. Only in this respect, it is necessary to take account of the statistical nature of the apparatus system. It is noteworthy that the statistical consideration is necessary not to show how ψ in (2') can be a mere probability function but to explain how various macroscopically different states can result from the same metastable state.

§ 3. Systems with an infinitely many degrees of freedom

In the preceding section we have shown that a system with a great many degrees of freedom is fairly different from a system with a few degrees of freedom in its physical import. The difference is, however, only quantitative and not qualitative. In order to clarify the distinction between these systems, we shall consider a system S_∞ with an infinitely many degrees of freedom, because it can be considered as an extreme case of S_N . Comparing S_∞ with S_N , we can show how they are qualitatively different, and this serves as an extreme illustration of the aforementioned difference.

In place of Eq. (4), we put

$$\mathfrak{H}_\infty = \bigotimes_{i=1}^{\infty} \mathfrak{h}_i, \quad (16)$$

where the right-hand side is the infinite direct product,⁴⁾ and it is a Hilbert space

corresponding to S_∞ .[†] The set of all physically meaningful operators of this system forms an algebra \mathfrak{A} , which is generated by operators in all \mathfrak{h} 's.⁵⁾ For example, in the quantum theory of fields every physical quantity can be represented by creation and annihilation operators. This means that they belong to \mathfrak{A} or to its closure \mathfrak{A}^c in an appropriate topology. Even a unitary operator $\exp(iHt)$ belongs to \mathfrak{A}^c where H is the Hamiltonian. On the other hand, every state of this system is nothing but a linear functional f on \mathfrak{A} or \mathfrak{A}^c ,⁵⁾ and can be represented by a vector $\Psi_f \in \mathfrak{S}_\infty$ as

$$f(q) = (\Psi_f, q\Psi_f), \quad q \in \mathfrak{A}^c. \quad (17)$$

Now we consider the superposition of two states Ψ' and Ψ'' , each of which belongs to an incomplete direct product.^{††} There are two cases to be distinguished. First, we consider the case in which Ψ' and Ψ'' belong to the same incomplete direct product. This case is similar to that of quantum mechanics, and there is such an operator $q \in \mathfrak{A}^c$ that

$$(\Psi', q\Psi'') \neq 0. \quad (18)$$

Put

$$\Psi^\theta = \Psi' + e^{i\theta} \Psi''; \quad (19)$$

then Ψ^θ represent different states for different θ , and we may say that Ψ' and Ψ'' interfere. It is obvious that there are two operators q' and q'' and a vector Ψ^0 such that

$$\Psi' = q' \Psi^0, \quad \Psi'' = q'' \Psi^0, \quad q', q'' \in \mathfrak{A}^c. \quad (20)$$

As

$$\Psi' + e^{i\theta} \Psi'' = (q' + e^{i\theta} q'') \Psi^0, \quad q' + e^{i\theta} q'' \in \mathfrak{A}^c, \quad (21)$$

we can represent the superposition as the sum of operators, but we cannot represent it by using f . In fact,

$$(f' + e^{i\theta} f'')(q) = (\Psi', q\Psi') + e^{i\theta} (\Psi'', q\Psi''),$$

and it is not equal to

$$f^\theta(q) = (\Psi^\theta, q\Psi^\theta).$$

The second is the case in which Ψ' and Ψ'' belong to different incomplete direct products. In this case there is no operator satisfying Eq. (18), and Ψ^θ

[†] \mathfrak{S}_∞ is not separable. On the contrary, an incomplete direct product \mathfrak{S}_σ is separable, but we cannot tell which of them is more appropriate for S_∞ from a physical point of view. \mathfrak{S}_σ , however, resembles \mathfrak{S}_N in its character, so we use not \mathfrak{S}_σ but \mathfrak{S}_∞ in the following.

^{††} Essentially this means that Ψ' and Ψ'' are direct products

$$\Psi' = \times_{i=1}^{\infty} \psi_i', \quad \Psi'' = \times_{i=1}^{\infty} \psi_i'', \quad \psi_i', \psi_i'' \in \mathfrak{h}_i.$$

In the first case of the following, $(\psi_i', \psi_i'') = 0$ only for a finite number of i 's, and in the second case, $(\psi_i', \psi_i'') = 0$ for an infinite number of i 's.

represent the same state for all θ ; namely, Ψ' and Ψ'' do not interfere. Ψ^θ can be given by

$$f^\theta(\mathbf{q}) = (f' + f'')(\mathbf{q}) = (\Psi', \mathbf{q}\Psi') + (\Psi'', \mathbf{q}\Psi''). \quad (22)$$

As there are no operators satisfying (20), Ψ^θ cannot be represented by a sum of operators. *To sum up, in the first case $(\Psi' + \Psi'')$ is a probability amplitude and can be represented by a sum of operators as in Eq. (21). On the contrary, in the second case $(\Psi' + \Psi'')$ is a mere probability function and can be represented by a sum of states as in Eq. (22).* In each case the representation by operators or by f 's will be more essential than that by Ψ 's.

As an illustration of the above argument, we shall consider the difference between the classical field theory and the quantum field theory. Divide the whole space into many small domains V_i , $i=1, 2, \dots$. In the quantum theory there is a Hilbert space \mathfrak{H}_{V_i} corresponding to each V_i , and the Hilbert space \mathfrak{H}_{V_∞} which corresponds to V_∞ is given by⁵⁾

$$\mathfrak{H}_{V_\infty} = \bigotimes_{i=1}^{\infty} \mathfrak{H}_{V_i}. \quad (23)$$

That is, any state $\Psi \in \mathfrak{H}_{V_\infty}$ is given by a direct product

$$\Psi = \times_{i=1}^{\infty} \Psi_i, \quad \Psi_i \in \mathfrak{H}_{V_i}, \quad (24)$$

or by their linear combination

$$\Psi = \sum_j \{ \times_{i=1}^{\infty} \Psi_i^j \}. \quad (25)$$

Every state Ψ can be represented as in (25), but it is not necessarily represented as in (24). All these states, however, are equivalent. In fact, whether a state Ψ can be written as in (24) or not is relative to the division $\{V_i\}$. If a state is written as in (24) for any division $\{V_i\}$, it is nothing but the vacuum. On the contrary, in the classical theory a state is completely determined when we know one in each V_i . Therefore, we may say that a state is always given by (24) in the classical theory.

In principle, the classical theory should be an approximation to quantum theory. Hence, it is necessary to explain how to settle the difference above mentioned. A domain V_i sufficiently small from the view-point of the classical theory may be large enough to contain a great many degrees of freedom from the view-point of quantum theory. In the following we take only such divisions. Two states different with each other from the view-point of the classical theory should be different macroscopically at least in one V_i , and the sum of two such states is a mere probability function. This implies that a state able to be written as in (24) plays a principal role, and that a state unable to be in the form of (24) plays only a subordinate role. Of course, there is no state which can be written as in (24) for any division $\{V_i\}$ except for the vacuum, but there are states which can be written as in (24) for any division $\{V_i\}$ in an approximate sense, and it is

only such states that are fundamental in the classical theory.[†] We may say, therefore, that in the classical theory a "state" means only such a state as in (24), and this explains the aforementioned difference.

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[†] It is obvious that every ϕ_i in Eq. (2') should be such a fundamental state. On the contrary, a state ϕ which is a superposition of these ϕ_i 's cannot be such a fundamental state.

A Formal Theory of Collective Behavior

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The method of auxiliary variables is generalized so as to be applicable to the systems having strong interparticle interactions or obeying Fermi-Dirac statistics. The ground state energy and the excitation energy spectrum are calculated to the zeroth approximation. It is shown that Feynman's relation for liquid helium and Tomonaga's expression for a one-dimensional Fermi gas can be derived as the special cases of the present theory. The theory is applied also to the classical statistical mechanics.

§ 1. Introduction

Many theories have been proposed to deal with systems showing collective behavior.¹⁾ Since in many systems there can occur longitudinal waves, we shall here pay special attention to the case of such longitudinal waves. In such a case the Fourier components of the density fluctuation of the system are used most appropriately as collective coordinates. Bogoliubov and Zubarev²⁾ called these components "auxiliary variables" and used them for dealing with a Bose gas. These variables were used also by other authors, for example, by Bohm and Pines³⁾ in the case of an electron gas in metals and by Tomonaga⁴⁾ in the case of a one-dimensional Fermi gas.

In general it is necessary for a complete description of the behavior of a system to use individual coordinates as well as collective coordinates. Both coordinates were used in the theory of Bohm and Pines.³⁾ Takano⁵⁾ also introduced individual coordinates into the theory of liquid helium in addition to the finite number of auxiliary variables. On the other hand, if one pays attention only to collective behavior of the system, it is possible to construct a theory without explicit use of individual coordinates. Such an example can be found in Tomonaga's excellent work.⁶⁾ In the present paper we shall also make no use of individual coordinates, because we shall consider only a collective behavior of the system.

In the previous paper⁷⁾ the method of auxiliary variables used by Bogoliubov and Zubarev²⁾ was formulated in a manner somewhat different from theirs. The formulation has the advantage that the Hamiltonian is given in an Hermitian form and the number of auxiliary variables can be limited. The theory cannot, however, be applied to systems having strong interparticle interactions,* just like the one of

* An interparticle interaction will be called *strong* if it cannot be expressed in a form of Fourier series such as Eq. (2.5), otherwise it will be called *weak*.

Bogoliubov and Zubarev cannot. Furthermore, the systems obeying Fermi-Dirac statistics cannot be treated by both of the theories.

In the present paper we shall construct a theory which is applicable also to the above-mentioned systems by means of the method of auxiliary variables. The development of the theory will be of a rather formal nature. We shall attempt neither full discussion of underlying assumptions nor careful comparisons with the earlier theories as cited in reference 1). Such discussion and comparisons will be postponed until the present theory is applied to real physical problems.

We consider a system of N identical particles with mass m , which are contained in a cube of edge length L , hence of volume $V=L^3$. The auxiliary variables are defined by

$$\rho_k = \frac{1}{\sqrt{N}} \sum_{j=1}^N e^{-i\mathbf{k}\mathbf{r}_j}, \quad (1.1)$$

where the components k_x , k_y and k_z of \mathbf{k} are positive or negative integers multiplied by $2\pi/L$ and \mathbf{r}_j is a position vector of particle j . Real auxiliary variables are defined only for $k_z > 0$ as follows:

$$\sigma_k = \frac{1}{\sqrt{N}} \sum_{j=1}^N \cos \mathbf{k}\mathbf{r}_j \quad \text{and} \quad \tau_k = \frac{1}{\sqrt{N}} \sum_{j=1}^N \sin \mathbf{k}\mathbf{r}_j. \quad (1.2)$$

Therefore we have the following relations for $k_z > 0$

$$\rho_k = \sigma_k - i\tau_k \quad \text{and} \quad \rho_{-k} = \sigma_k + i\tau_k. \quad (1.3)$$

Our fundamental assumption is that a wave function \mathcal{P} describing the system can be represented as a product of a function depending on some set of auxiliary variables alone and a function which may be dependent on spin coordinates as well as spatial coordinates, namely

$$\mathcal{P} = \varphi(\rho_k) \chi(x), \quad (1.4)$$

where x stands for both spatial and spin coordinates of particles. In general the wave function must possess a symmetry property and satisfy a boundary condition. We assume that $\chi(x)$ possesses the symmetry property and satisfies the boundary condition. Then it may be said that such a form as (1.4) is a very natural one of the wave function, though it is not a necessary one. The state described by $\chi(x)$ will hereafter be referred to as the "basic state".

In a formal development of the theory, any set of ρ_k 's may be used as the variables of $\varphi(\rho_k)$ if it is restricted only by the condition that it does not contain the one for $\mathbf{k}=0$ but contains the pair ρ_k, ρ_{-k} whenever ρ_k is a member. The set of k 's at our disposal in the present paper will hereafter be denoted explicitly by the symbol $\{\mathbf{k}\}$.

The course of our theory is as follows. We assume the basic state $\chi(x)$ to be known. We regard $\varphi(\rho_k)$ as a new wave function in place of \mathcal{P} . Such a representation of the wave function will hereafter be called the " ρ_k -representation".

In § 2 we shall derive the general expressions for a Hamiltonian and a total momentum in the ρ_k -representation by imposing some restrictions on $\chi(x)$. In § 3 we shall calculate the ground state energy and the excitation energy spectrum to the zeroth approximation. Possible applications of our theory will be discussed in § 4.

§ 2. General expressions in the ρ_k -representation

We write the Hamiltonian for the system in the form

$$H(\mathbf{r}) = T(\mathbf{r}) + V(\mathbf{r}), \quad (2.1)$$

where $T(\mathbf{r})$ is the kinetic energy part,

$$T(\mathbf{r}) = -\frac{\hbar^2}{2m} \sum_{j=1}^N \frac{\partial^2}{\partial \mathbf{r}_j^2}, \quad (2.2)$$

and $V(\mathbf{r})$ the potential energy part. The periodic boundary condition is assumed in the theory. We decompose $V(\mathbf{r})$ into

$$V(\mathbf{r}) = v(\mathbf{r}) + U(\mathbf{r}) \quad (2.3)$$

so that $v(\mathbf{r})$ contains all the strong interactions and $U(\mathbf{r})$ is a sum of two-body interactions which can be expanded in a Fourier series. Namely $U(\mathbf{r})$ is written in the form

$$U(\mathbf{r}) = \frac{1}{2} \sum_{i \neq j}^N \sum_{j=1}^N \phi(r_{ij}) \quad (2.4)$$

and $\phi(r)$, which is assumed to be a function of the distance between two particles, is expressed as the Fourier series,

$$\phi(r) = \frac{\nu(0)}{V} + \sum_{\mathbf{k} \neq 0} \frac{\nu(\mathbf{k})}{V} e^{i\mathbf{k}\mathbf{r}}, \quad \text{where } \nu(\mathbf{k}) = \int \phi(r) e^{-i\mathbf{k}\mathbf{r}} d\mathbf{r}. \quad (2.5)$$

The way of decomposing $V(\mathbf{r})$ such as Eq. (2.3) cannot be uniquely determined. In particular, when $V(\mathbf{r})$ contains hard sphere interactions, the way of choosing the form of $\phi(r)$ contains much arbitrariness. The present author has failed to obtain a solution of this difficulty.

Since we use the set of ρ_k 's so that \mathbf{k} belongs to $\{\mathbf{k}\}$, it is convenient to divide the sum over \mathbf{k} in Eq. (2.5) into two parts so that the one is the sum over \mathbf{k} belonging to $\{\mathbf{k}\}$ and the other the sum over \mathbf{k} not belonging to $\{\mathbf{k}\}$. Then $U(\mathbf{r})$ is written in terms of Fourier components of $\phi(r)$ as

$$U(\mathbf{r}) = \frac{1}{2} N(N-1) \frac{\nu(0)}{V} + \frac{1}{2} \sum_{\{\mathbf{k}\}} \sum_{i \neq j}^N \frac{\nu(\mathbf{k})}{V} e^{i\mathbf{k}(\mathbf{r}_i - \mathbf{r}_j)} + U'(\mathbf{r}), \quad (2.6)$$

where

$$U'(\mathbf{r}) = \frac{1}{2} \sum_{\substack{\mathbf{k} \neq 0 \\ \mathbf{k} \text{ does not} \\ \text{belong to } \{\mathbf{k}\}}} \sum_{i \neq j}^N \frac{\nu(\mathbf{k})}{V} e^{i\mathbf{k}(\mathbf{r}_i - \mathbf{r}_j)}. \quad (2.7)$$

The expectation value of the Hamiltonian $H(\mathbf{r})$ for a state Ψ is

$$E = \int (dx) \Psi^* H(\mathbf{r}) \Psi / \int (dx) \Psi^* \Psi, \quad (2.8)$$

where $\int (dx)$ stands for the sums over spin coordinates as well as the integrals over spatial coordinates.

In order to introduce the auxiliary variables, we define a function of ρ_k 's (and so of σ_k 's and τ_k 's)

$$D(\rho_k) = \int (dx) \prod_{\{k\}} f_k g_k \chi^*(x) \chi(x), \quad (2.9)$$

where

$$f_k = \delta \left(\sigma_k - \frac{1}{\sqrt{N}} \sum_{j=1}^N \cos \mathbf{k} \mathbf{r}_j \right) \quad \text{and} \quad g_k = \delta \left(\tau_k - \frac{1}{\sqrt{N}} \sum_{j=1}^N \sin \mathbf{k} \mathbf{r}_j \right) \quad (2.10)$$

and $\prod_{\{k\}}$ means the product over \mathbf{k} with $k_z > 0$. Assuming $\chi(x)$ to be normalized, we obtain, with the aid of a property of δ -function,

$$\int (d\rho) D(\rho_k) = 1, \quad (2.11)$$

where $\int (d\rho)$ stands for $\int_{-\infty}^{+\infty} \prod_{\{k\}} \Pi'(d\sigma_k d\tau_k)$. Furthermore the following identity is obtained for an arbitrary function $F(\mathbf{r})$

$$\int (dx) \mathbf{F}(\mathbf{r}) = \int (d\rho) \int (dx) \prod_{\{k\}} f_k g_k \mathbf{F}(\mathbf{r}). \quad (2.12)$$

As was seen in the previous paper⁷⁾, the function $D(\rho_k)$ may be supposed as the Jacobian of the transformation of variables from \mathbf{r}_j 's to ρ_k 's. It may also be regarded as the weight function for the inner product in the ρ_k -representation. In order to introduce the inner product with no weight function, we adopt the following form of Ψ , instead of Eq. (1.4),

$$\Psi = \frac{1}{\sqrt{D(\rho_k)}} \phi(\rho_k) \chi(x). \quad (2.13)$$

Making use of identity (2.12) and Eq. (2.13), we obtain, after straightforward calculations,

$$\int (dx) \Psi^* T(\mathbf{r}) \Psi = \int (d\rho) \phi^*(\rho_k) T(\rho_k) \phi(\rho_k),$$

where

$$\begin{aligned} T(\rho_k) = & \frac{1}{\sqrt{N}} \sum_{\{k\}} \sum_{\{k'\}} \frac{\hbar^2(\mathbf{k}\mathbf{k}')}{2m} \left[\frac{1}{2} \left\{ A(\mathbf{k}+\mathbf{k}') \cdot \frac{\partial^2}{\partial \rho_k \partial \rho_{k'}} + \frac{\partial^2}{\partial \rho_k \partial \rho_{k'}} \cdot A(\mathbf{k}+\mathbf{k}') \right\} \right. \\ & \left. - \frac{1}{2} \frac{\partial^2 A(\mathbf{k}+\mathbf{k}')}{\partial \rho_k \partial \rho_{k'}} + \frac{3}{4} A(\mathbf{k}+\mathbf{k}') \frac{1}{D^2} \frac{\partial D}{\partial \rho_k} \frac{\partial D}{\partial \rho_{k'}} - \frac{1}{2} A(\mathbf{k}+\mathbf{k}') \frac{1}{D} \frac{\partial^2 D}{\partial \rho_k \partial \rho_{k'}} \right] \end{aligned}$$

$$\begin{aligned}
 & -\frac{1}{2} \sum_{\langle k \rangle} \frac{\hbar^2 k^2}{2m} B(\mathbf{k}) \frac{1}{D} \frac{\partial D}{\partial \rho_k} + t(\rho_k) \\
 & + \sum_{\langle k \rangle} \frac{\hbar^2 k^2}{2m} \left[B(\mathbf{k}) - \frac{1}{\sqrt{N}} \sum_{\langle k' \rangle} \frac{(\mathbf{k}\mathbf{k}')}{k^2} \left\{ A(\mathbf{k}+\mathbf{k}') \frac{1}{D} \frac{\partial D}{\partial \rho_{k'}} + \frac{\partial A(\mathbf{k}+\mathbf{k}')}{\partial \rho_{k'}} \right\} \right] \frac{\partial}{\partial \rho_k} \\
 & - \sum_{\langle k \rangle} \frac{\hbar^2 k^2}{2m} Q(\mathbf{k}) \frac{\partial}{\partial \rho_k} + \frac{1}{2} \sum_{\langle k \rangle} \frac{\hbar^2 k^2}{2m} Q(\mathbf{k}) \frac{1}{D} \frac{\partial D}{\partial \rho_k} - \frac{\hbar^2}{2m} R(\rho_k) \quad (2.14)
 \end{aligned}$$

and

$$\int (dx) \Psi^* V(\mathbf{r}) \Psi = \int (d\rho) \Phi^*(\rho_k) V(\rho_k) \Phi(\rho_k),$$

where

$$\begin{aligned}
 V(\rho_k) = & \frac{1}{2} N(N-1) \frac{\nu(0)}{V} - \frac{1}{2} \sum_{\langle k \rangle} \frac{N}{V} \nu(\mathbf{k}) + \frac{1}{2} \sum_{\langle k \rangle} \frac{N}{V} \nu(\mathbf{k}) \rho_k \rho_{-\mathbf{k}} \\
 & + U'(\rho_k) + v(\rho_k). \quad (2.15)
 \end{aligned}$$

Quantities appearing in Eqs. (2.14) and (2.15) are defined as follows.

$$A(\mathbf{k}) = \frac{1}{D(\rho_k)} \int (dx) \left(\prod_{\langle k \rangle} f_k g_k \right) \frac{1}{\sqrt{N}} \sum_{j=1}^N e^{-i\mathbf{k}\mathbf{r}_j} \chi^* \chi. \quad (2.16)$$

$$B(\mathbf{k}) = \frac{1}{D(\rho_k)} \int (dx) \left(\prod_{\langle k \rangle} f_k g_k \right) \frac{1}{\sqrt{N}} \sum_{j=1}^N \frac{i\mathbf{k}}{k^2} \cdot \frac{\partial}{\partial \mathbf{r}_j} (e^{-i\mathbf{k}\mathbf{r}_j} \chi^* \chi). \quad (2.17)$$

$$t(\rho_k) = \frac{1}{D(\rho_k)} \int (dx) \left(\prod_{\langle k \rangle} f_k g_k \right) \sum_{j=1}^N \left\{ -\frac{\hbar^2}{4m} \left(\chi^* \frac{\partial^2 \chi}{\partial \mathbf{r}_j^2} + \frac{\partial^2 \chi^*}{\partial \mathbf{r}_j^2} \chi \right) \right\}. \quad (2.18)$$

$$Q(\mathbf{k}) = \frac{1}{D(\rho_k)} \int (dx) \left(\prod_{\langle k \rangle} f_k g_k \right) \frac{1}{\sqrt{N}} \sum_{j=1}^N e^{-i\mathbf{k}\mathbf{r}_j} \frac{(-i\mathbf{k})}{k^2} \left(\chi^* \frac{\partial \chi}{\partial \mathbf{r}_j} - \frac{\partial \chi^*}{\partial \mathbf{r}_j} \chi \right). \quad (2.19)$$

$$R(\rho_k) = \frac{1}{D(\rho_k)} \int (dx) \left(\prod_{\langle k \rangle} f_k g_k \right) \frac{1}{2} \sum_{j=1}^N \left(\chi^* \frac{\partial^2 \chi}{\partial \mathbf{r}_j^2} - \frac{\partial^2 \chi^*}{\partial \mathbf{r}_j^2} \chi \right). \quad (2.20)$$

$$U'(\rho_k) = \frac{1}{D(\rho_k)} \int (dx) \left(\prod_{\langle k \rangle} f_k g_k \right) U'(\mathbf{r}) \chi^* \chi. \quad (2.21)$$

$$v(\rho_k) = \frac{1}{D(\rho_k)} \int (dx) \left(\prod_{\langle k \rangle} f_k g_k \right) v(\mathbf{r}) \chi^* \chi. \quad (2.22)$$

From these definitions it can be easily seen that $t(\rho_k)$, $U'(\rho_k)$ and $v(\rho_k)$ are real quantities and $R(\rho_k)$ a pure imaginary one and also that $A^*(\mathbf{k}) = A(-\mathbf{k})$, $B^*(\mathbf{k}) = B(-\mathbf{k})$ and $Q^*(\mathbf{k}) = -Q(-\mathbf{k})$. It can also be shown that there exist the following relations,

$$B(\mathbf{k}) = \frac{1}{\sqrt{N}} \sum_{\langle k' \rangle} \frac{(\mathbf{k}\mathbf{k}')}{k^2} \left\{ A(\mathbf{k}+\mathbf{k}') \frac{1}{D} \frac{\partial D}{\partial \rho_{k'}} + \frac{\partial A(\mathbf{k}+\mathbf{k}')}{\partial \rho_{k'}} \right\} \quad (2.23)$$

and

$$\sum_{\langle k \rangle} k^2 \left\{ Q(k) \frac{1}{D} \frac{\partial D}{\partial \rho_k} + \frac{\partial Q(k)}{\partial \rho_k} \right\} = 2R(\rho_k). \quad (2.24)$$

The proof of relations (2.23) and (2.24) is omitted here, since it is similar to the one given in the Appendix of the previous paper⁷⁾.

$T(\rho_k)$ is written, with the aid of relations (2.23) and (2.24), as

$$\begin{aligned} T(\rho_k) = & \frac{1}{\sqrt{N}} \sum_{\langle k \rangle} \sum_{\langle k' \rangle} \frac{\hbar^2(kk')}{2m} \frac{1}{2} \left\{ A(k+k') \cdot \frac{\partial^2}{\partial \rho_k \partial \rho_{k'}} + \frac{\partial^2}{\partial \rho_k \partial \rho_{k'}} \cdot A(k+k') \right\} \\ & + \sum_{\langle k \rangle} \frac{\hbar^2 k^2}{2m} \left[-\frac{1}{2} \frac{\partial B(k)}{\partial \rho_k} - \frac{1}{4} \frac{1}{D} \frac{\partial D}{\partial \rho_k} \left\{ B(k) - \frac{1}{\sqrt{N}} \sum_{\langle k' \rangle} \frac{(kk')}{k^2} \frac{\partial A(k+k')}{\partial \rho_{k'}} \right\} \right] \\ & + t(\rho_k) - \sum_{\langle k \rangle} \frac{\hbar^2 k^2}{2m} \frac{1}{2} \left\{ Q(k) \cdot \frac{\partial}{\partial \rho_k} + \frac{\partial}{\partial \rho_k} \cdot Q(k) \right\}. \end{aligned} \quad (2.25)$$

The operators $T(\rho_k)$ and $V(\rho_k)$ are regarded as the kinetic energy operator and the potential energy operator in the ρ_k -representation respectively. It can be easily verified that these operators are Hermitian. It is to be noted that the Hermitian nature of an operator is secured in the inner product with no weight function.

The basic state $\chi(x)$ has so far been arbitrary if only $v(\rho_k)$ defined by Eq. (2.22) remains finite for strong interactions contained in $v(\mathbf{r})$. We shall choose $\chi(x)$ to be an eigenfunction of $T(\mathbf{r}) + v(\mathbf{r})$ with eigenvalue \mathfrak{E} , namely

$$[T(\mathbf{r}) + v(\mathbf{r})]\chi(x) = \mathfrak{E}\chi(x). \quad (2.26)$$

Then it turns out to be $t(\rho_k) + v(\rho_k) = \mathfrak{E}$. Finally we obtain as the expectation value of the Hamiltonian or the energy

$$E = \int (d\rho) \Phi^*(\rho_k) H(\rho_k) \Phi(\rho_k) / \int (d\rho) \Phi^*(\rho_k) \Phi(\rho_k), \quad (2.27)$$

where

$$\begin{aligned} H(\rho_k) = & T(\rho_k) + V(\rho_k) \\ = & \mathfrak{E} + \frac{1}{2} N(N-1) \frac{\nu(0)}{V} - \frac{1}{2} \sum_{\langle k \rangle} \frac{N}{V} \nu(k) + \frac{1}{2} \sum_{\langle k \rangle} \frac{N}{V} \nu(k) \rho_k \rho_{-k} + U'(\rho_k) \\ & + \frac{1}{\sqrt{N}} \sum_{\langle k \rangle} \sum_{\langle k' \rangle} \frac{\hbar^2(kk')}{2m} \frac{1}{2} \left\{ A(k+k') \cdot \frac{\partial^2}{\partial \rho_k \partial \rho_{k'}} + \frac{\partial^2}{\partial \rho_k \partial \rho_{k'}} \cdot A(k+k') \right\} \\ & + \sum_{\langle k \rangle} \frac{\hbar^2 k^2}{2m} \left[-\frac{1}{2} \frac{\partial B(k)}{\partial \rho_k} - \frac{1}{4} \frac{1}{D} \frac{\partial D}{\partial \rho_k} \left\{ B(k) - \frac{1}{\sqrt{N}} \sum_{\langle k' \rangle} \frac{(kk')}{k^2} \frac{\partial A(k+k')}{\partial \rho_{k'}} \right\} \right] \\ & - \sum_{\langle k \rangle} \frac{\hbar^2 k^2}{2m} \frac{1}{2} \left\{ Q(k) \cdot \frac{\partial}{\partial \rho_k} + \frac{\partial}{\partial \rho_k} \cdot Q(k) \right\}. \end{aligned} \quad (2.28)$$

Expression (2.28) is the Hamiltonian in the ρ_k -representation. The Schrodinger equation in the ρ_k -representation is obtained from Eq. (2.27) as

$$H(\rho_k) \Phi(\rho_k) = E \Phi(\rho_k). \quad (2.29)$$

An expression for the total momentum in the ρ_k -representation is derived in the same way as the Hamiltonian. The total momentum is written in the original representation as

$$\mathbf{P}(\mathbf{r}) = \frac{\hbar}{i} \sum_{j=1}^N \frac{\partial}{\partial \mathbf{r}_j}.$$

After simple calculation we obtain

$$\int (d\mathbf{x}) \Psi^* \mathbf{P}(\mathbf{r}) \Psi = \int (d\rho) \Phi^*(\rho_k) \mathbf{P}(\rho_k) \Phi(\rho_k),$$

where

$$\mathbf{P}(\rho_k) = \hat{\mathbf{P}} + \sum_{\{\mathbf{k}\}} (-\hbar \mathbf{k}) A(\mathbf{k}) \left\{ \frac{\partial}{\partial \rho_k} - \frac{1}{2} \frac{1}{D} \frac{\partial D}{\partial \rho_k} \right\}. \quad (2.30)$$

The operator $\mathbf{P}(\rho_k)$ given by Eq. (2.30) is regarded as the total momentum in the ρ_k -representation. In deriving Eq. (2.30) we have assumed the basic state $\chi(x)$ to be the eigenfunction of $\mathbf{P}(\mathbf{r})$ with eigenvalue $\hat{\mathbf{P}}$.

By the use of the relation

$$\sum_{\{\mathbf{k}\}} \mathbf{k} A(\mathbf{k}) \frac{\partial D}{\partial \rho_k} = 0, \quad (2.31)$$

Eq. (2.30) is transformed into

$$\mathbf{P}(\rho_k) = \hat{\mathbf{P}} + \sum_{\{\mathbf{k}\}} (-\hbar \mathbf{k}) \frac{1}{2} \left\{ \rho_k \cdot \frac{\partial}{\partial \rho_k} + \frac{\partial}{\partial \rho_k} \cdot \rho_k \right\}, \quad (2.32)$$

where the facts that $\sum_{\{\mathbf{k}\}} \mathbf{k} = 0$ and that $A(\mathbf{k}) = \rho_k$ for \mathbf{k} belonging to $\{\mathbf{k}\}$ have also been used (The latter fact is readily derived from definition (2.16)). It can be verified that expression (2.32) commutes with $T(\rho_k)$ and $V(\rho_k)$, as it should. Although the proof of this fact is omitted here, the following relations will serve to prove it:

$$\left. \begin{aligned} \sum_{\{\mathbf{k}\}} \mathbf{k} A(\mathbf{k}) \frac{\partial A(\mathbf{k}')}{\partial \rho_k} &= \mathbf{k}' A(\mathbf{k}') \\ \sum_{\{\mathbf{k}\}} \mathbf{k} A(\mathbf{k}) \frac{\partial B(\mathbf{k}')}{\partial \rho_k} &= \mathbf{k}' B(\mathbf{k}') \\ \sum_{\{\mathbf{k}\}} \mathbf{k} A(\mathbf{k}) \frac{\partial Q(\mathbf{k}')}{\partial \rho_k} &= \mathbf{k}' Q(\mathbf{k}') \end{aligned} \right\} \text{ for all } \mathbf{k}' \text{ 's.} \quad (2.33)$$

The proof of relations (2.31) and (2.33) is also omitted here, because it is similar to the one given in the Appendix of the previous paper⁷⁾.

§ 3. Expressions to the zeroth approximation

In order to obtain a concrete form of the Hamiltonian (2.28), it is necessary to determine the explicit dependence of $D(\rho_k)$, $A(\mathbf{k})$, $B(\mathbf{k})$, $U'(\rho_k)$ and $Q(\mathbf{k})$ upon the ρ_k 's. We assume that $Q(\mathbf{k})=0$. This is allowed when $\bar{\epsilon}$ is the lowest eigenvalue of $T(\mathbf{r})+v(\mathbf{r})$, because in such a case $\chi(x)$ can be taken to be a real function (cf. Eqs. (2.19) and (2.26)).

As was done in the previous paper⁷⁾, we calculate $D(\rho_k)$, $A(\mathbf{k})$, $B(\mathbf{k})$ and $U'(\rho_k)$ in a form of a power series in $N^{-1/2}$. To the zeroth approximation they are independent of $N^{-1/2}$. To the next approximation they contain the terms proportional to $N^{-1/2}$, to the third approximation the terms proportional to N^{-1} , and so on. Strictly speaking, such a way of specifying a degree of approximation may be of little sense. However, as was seen in the previous paper, it leads to a "formal" expansion of the energy per particle in terms of V/N . The word "formal" means that the coefficients in the expansion themselves depend on V/N in complicated manners.

Here we shall be content to remain to the zeroth approximation. Quantities to the zeroth approximation will hereafter be denoted by subscript letter 0. The details of calculating $D(\rho_k)$ are given in Appendix A. To the zeroth approximation $D(\rho_k)$ is written in the form

$$D_0(\rho_k) = \exp \left[-\frac{1}{2} \sum_{\{\mathbf{k}\}} \left\{ \frac{1}{S(\mathbf{k})} \rho_k \rho_{-\mathbf{k}} + \log \pi S(\mathbf{k}) \right\} \right]. \quad (3.1)$$

The function $S(\mathbf{k})$ is defined in terms of the radial distribution function $g(r)$ for the basic state $\chi(x)$ as

$$S(\mathbf{k}) = 1 + \frac{N}{V} \int [g(r) - 1] e^{i\mathbf{k}\cdot\mathbf{r}} d\mathbf{r}. \quad (3.2)$$

In deriving these expressions we have assumed that $\chi(x)$ represents a homogeneous system, namely the one-particle distribution function is independent of the position of the particle. Furthermore, we have assumed that the two-particle distribution function for $\chi(x)$ depends only on the distance between the particles. The function $S(\mathbf{k})$ is called the structure factor and closely related to the intensity of X-rays or neutron-beams scattered by a fluid⁸⁾.

The calculation of $A(\mathbf{k})$ and $U'(\rho_k)$ can be carried out in quite a similar way as in Appendix A. To the zeroth approximation they are

$$A_0(\mathbf{k}) = \begin{cases} \sqrt{N} & \text{for } \mathbf{k}=0 \\ \rho_k & \text{for } \mathbf{k} \text{ belonging to } \{\mathbf{k}\} \\ 0 & \text{otherwise} \end{cases} \quad (3.3)$$

and

$$U'_0(\rho_k) = 0. \quad (3.4)$$

In Eq. (3.3) the expressions for $\mathbf{k}=0$ and for \mathbf{k} belonging to $\{\mathbf{k}\}$ are the exact ones. This is immediately seen from definition (2.16). When the forms of $D(\rho_{\mathbf{k}})$ and $A(\mathbf{k})$ are known, $B(\mathbf{k})$ can be calculated with the aid of relation (2.23). To the zeroth approximation it is

$$B_0(\mathbf{k}) = \rho_{\mathbf{k}}/S(\mathbf{k}) \quad \text{for } \mathbf{k} \text{ belonging to } \{\mathbf{k}\}. \quad (3.5)$$

The expression for other \mathbf{k} 's is unnecessary for our purpose.

Substituting Eqs. (3.1), (3.3), (3.4) and (3.5) into Eq. (2.28), we obtain the Hamiltonian to the zeroth approximation as follows:

$$H_0(\rho_{\mathbf{k}}) = \mathcal{E} + \frac{1}{2} N(N-1) \frac{\nu(0)}{V} - \frac{1}{2} \sum_{\{\mathbf{k}\}} \frac{N}{V} \nu(\mathbf{k}) - \frac{1}{2} \sum_{\{\mathbf{k}\}} \frac{\hbar^2 k^2}{2m} \frac{1}{S(\mathbf{k})} \\ - \sum_{\{\mathbf{k}\}} \frac{\hbar^2 k^2}{2m} \frac{\partial^2}{\partial \rho_{\mathbf{k}} \partial \rho_{-\mathbf{k}}} + \frac{1}{2} \sum_{\{\mathbf{k}\}} \left[\frac{N}{V} \nu(\mathbf{k}) + \frac{\hbar^2 k^2}{4m} \frac{1}{S(\mathbf{k})^2} \right] \rho_{\mathbf{k}} \rho_{-\mathbf{k}}. \quad (3.6)$$

In order to diagonalize $H_0(\rho_{\mathbf{k}})$, we introduce the operators

$$b_{\mathbf{k}} = \lambda_{\mathbf{k}} \frac{\partial}{\partial \rho_{-\mathbf{k}}} + \frac{1}{2\lambda_{\mathbf{k}}} \rho_{\mathbf{k}} \quad \text{and} \quad b_{\mathbf{k}}^+ = -\lambda_{\mathbf{k}} \frac{\partial}{\partial \rho_{\mathbf{k}}} + \frac{1}{2\lambda_{\mathbf{k}}} \rho_{-\mathbf{k}}, \quad (3.7)$$

where $\lambda_{\mathbf{k}} = \lambda_{-\mathbf{k}} > 0$ is a function of \mathbf{k} to be determined. The operators $b_{\mathbf{k}}$'s and $b_{\mathbf{k}}^+$'s satisfy the commutation relations

$$[b_{\mathbf{k}}, b_{\mathbf{k}'}^+] = \delta_{\mathbf{k}, \mathbf{k}'}, \quad \text{and} \quad [b_{\mathbf{k}}, b_{\mathbf{k}'}] = [b_{\mathbf{k}}^+, b_{\mathbf{k}'}^+] = 0, \quad (3.8)$$

from which it is seen that the eigenvalues of each $b_{\mathbf{k}}^+ b_{\mathbf{k}}$ are positive integers including zero. It is readily seen that the total momentum (2.32) is expressed in terms of $b_{\mathbf{k}}$'s and $b_{\mathbf{k}}^+$'s as

$$\mathbf{P}(\rho_{\mathbf{k}}) = \mathcal{P} + \sum_{\{\mathbf{k}\}} \hbar \mathbf{k} b_{\mathbf{k}}^+ b_{\mathbf{k}}. \quad (3.9)$$

If we take

$$\lambda_{\mathbf{k}}^4 = \frac{\hbar^2 k^2}{2m} \left/ \left[2 \frac{N}{V} \nu(\mathbf{k}) + \frac{\hbar^2 k^2}{2m} \frac{1}{S(\mathbf{k})^2} \right] \right., \quad (3.10)$$

$H_0(\rho_{\mathbf{k}})$ is written in the form:

$$H_0(\rho_{\mathbf{k}}) = E_0 + \sum_{\{\mathbf{k}\}} E(\mathbf{k}) b_{\mathbf{k}}^+ b_{\mathbf{k}}, \quad (3.11)$$

where

$$E_0 = \mathcal{E} + \frac{1}{2} N(N-1) \frac{\nu(0)}{V} + \frac{1}{2} \sum_{\{\mathbf{k}\}} \left[E(\mathbf{k}) - \frac{\hbar^2 k^2}{2m} \frac{1}{S(\mathbf{k})} - \frac{N}{V} \nu(\mathbf{k}) \right] \quad (3.12)$$

and

$$E(\mathbf{k}) = \frac{\hbar^2 k^2}{2m} / \lambda_{\mathbf{k}}^2 = \sqrt{\frac{\hbar^2 k^2}{2m} \frac{1}{S(\mathbf{k})} \left[2 \frac{N}{V} \nu(\mathbf{k}) S(\mathbf{k}) + \frac{\hbar^2 k^2}{2m} \frac{1}{S(\mathbf{k})} \right]}. \quad (3.13)$$

We may say that $b_{\mathbf{k}}$ ($b_{\mathbf{k}}^+$) is an annihilation (creation) operator of a phonon with momentum $\hbar \mathbf{k}$ and energy $E(\mathbf{k})$. Then $H_0(\rho_{\mathbf{k}})$ can be considered to repre-

sent free phonons. If we do not remain to the zeroth approximation, we shall obtain the Hamiltonian with terms representing phonon-phonon interactions. As is seen from Appendix A, such terms will depend on the distribution functions in sets of three particles or more. From such a point of view it may be said that the zeroth approximation adopted in the present section is the one to neglect the correlations among three particles or more in the basic state $\chi(x)$.

E_0 and $E(\mathbf{k})$ are the ground state energy and the excitation energy spectrum of the system, respectively. If we put $\xi=0$ and $S(\mathbf{k})=1$ in Eqs. (3.12) and (3.13), these expressions coincide with the ones derived by Bogoliubov and Zubarev²⁾ and also by the present author⁷⁾.

It is very difficult to know exactly the degree of approximations included in the above derivation. However, the applications (i) and (ii) in the following section may serve as a suggestion of the nature of approximations.

In concluding the present section, we shall derive the wave function of the system to the zeroth approximation. As is well-known, the ground state wave function and the excited state wave function of $H_0(\rho_k)$ are written, apart from normalization constants, as follows:

$$\Phi_0(\rho_k) \sim \exp\left[-\frac{1}{4} \sum_{\langle k \rangle} \frac{1}{\lambda_k^2} \rho_k \rho_{-k}\right]$$

and

$$\Phi_k(\rho_k) = b_k^+ \Phi_0(\rho_k) \sim \rho_{-k} \exp\left[-\frac{1}{4} \sum_{\langle k \rangle} \frac{1}{\lambda_k^2} \rho_k \rho_{-k}\right].$$

Therefore we obtain the ground state wave function and the excited state wave function of the system to the zeroth approximation, with the aid of Eq. (2.13), as

$$\Psi_0 \sim \chi(x) \exp\left[-\frac{1}{4} \sum_{\langle k \rangle} \left(\frac{1}{\lambda_k^2} - \frac{1}{S(\mathbf{k})}\right) \rho_k \rho_{-k}\right] \quad (3.14)$$

and

$$\Psi_k \sim \chi(x) \rho_{-k} \exp\left[-\frac{1}{4} \sum_{\langle k \rangle} \left(\frac{1}{\lambda_k^2} - \frac{1}{S(\mathbf{k})}\right) \rho_k \rho_{-k}\right], \quad (3.15)$$

where

$$\rho_k = \frac{1}{\sqrt{N}} \sum_{j=1}^N e^{-ikr_j}.$$

§ 4. Possible applications

In the preceding section we have derived the expressions for the ground state energy and the excitation energy spectrum of the system. These expressions depend on the basic state $\chi(x)$ which is the lowest eigenfunction of $T(\mathbf{r}) + v(\mathbf{r})$. There is nothing for it but to choose $v(\mathbf{r})$ case by case. In the following we shall show

possible applications of our theory to three cases: (i) a Bose gas with hard sphere interactions and other weak interactions, (ii) a Fermi gas with weak interactions and (iii) a classical fluid. It will also be shown that our theory gives Feynman's relation for liquid helium and Tomonaga's expression for a one-dimensional Fermi gas.

(i) A Bose gas

We suppose that $v(\mathbf{r})$ contains hard sphere interactions alone. Then $\chi(x)$ is the ground state wave function of a Bose system of hard spheres. Such a system has been investigated and it is known that there exists a phonon-like excitation for small k .⁹⁾ When we use Feynman's relation (see Eq. (4.2)) for the hard sphere system, the structure factor $S(\mathbf{k})$ can be written as $S(\mathbf{k}) = \hbar k / 2mc$, where c is the sound velocity of the hard sphere system. Therefore we obtain, from Eq. (3.13),

$$E(\mathbf{k}) = \hbar k \sqrt{c^2 + \frac{1}{m} \frac{N}{V} \nu(0)} \quad \text{for small } k. \quad (4.1)$$

Namely the boson system under consideration also has a phonon-like excitation. Since all the attractive interactions are contained in $U(\mathbf{r})$, the sign of $\nu(0)$ will be negative (cf. Eqs. (2.4) and (2.5)). Nevertheless, it will be probable that the sign of $c^2 + (1/m)(N/V)\nu(0)$ is positive and, accordingly, that the excitation given by Eq. (4.1) is a real one. The excitation energy spectrum for larger k can also be derived with the aid of the one for the hard sphere system and Feynman's relation.

Feynman's relation¹⁰⁾ itself is derived from our theory if we take $v(\mathbf{r}) = V(\mathbf{r})$, that is, $v(\mathbf{r})$ contains all the interactions. Then we have $\lambda_k^2 = S(\mathbf{k})$ and Eq. (3.13) is reduced to

$$E(\mathbf{k}) = \frac{\hbar^2 k^2}{2m} / S(\mathbf{k}), \quad (4.2)$$

which is Feynman's relation. The correspondence of our theory to Feynman's one appears more clearly in the form of the wave function. In the case under consideration Eq. (3.15) gives, as the form of the wave function for the excited state,

$$\Psi_k \sim \left(\sum_{j=1}^N e^{ikr_j} \right) \chi(x),$$

which is the form that Feynman adopted in his theory of liquid helium. The theory of Feynman and Cohen¹¹⁾ must also be derived from our general formulas provided that we use more accurate expressions for $D(\rho_k)$, $A(\mathbf{k})$, and so forth, than those used in the preceding section.

(ii) A Fermi gas

We shall take $v(\mathbf{r}) = 0$. Then $\chi(x)$ represents the ground state of a non-interacting Fermi gas, that is, a Fermi sphere. As is shown in Appendix B, the structure factor for a Fermi sphere is written in the form

$$S(\mathbf{k}) = \frac{3}{4} \frac{k}{k_F} - \frac{1}{16} \left(\frac{k}{k_F} \right)^3 \quad \text{for } 0 < k \leq 2k_F$$

$$= 1 \quad \text{for } k > 2k_F,$$
(4.3)

where k_F denotes the wave number corresponding to the Fermi top. Substituting this expression into Eqs. (3.12) and (3.13), we obtain the expressions for the ground state energy and the excitation energy spectrum of the Fermi gas. In particular, the excitation energy spectrum for small k has the form

$$E(\mathbf{k}) = \hbar k \sqrt{\left(\frac{2}{3} \frac{\hbar k_F}{m} \right)^2 + \frac{1}{m} \frac{N}{V} \nu(0)} \quad \text{for small } k, \quad (4.4)$$

which shows that the system under consideration has a phonon-like excitation. Recently Edwards¹²⁾ has attempted to deal with the Fermi gas by means of auxiliary variables. However, he has not yet obtained such a definite result as ours.*

It is of much interest to treat a one-dimensional Fermi gas from our standpoint, because such a system has been investigated in detail by Tomonaga⁴⁾. In the one-dimensional case the structure factor for the Fermi sphere has the form

$$S(\mathbf{k}) = \frac{1}{2} \frac{k}{k_F} \quad \text{for } 0 < k \leq 2k_F$$

$$= 1 \quad \text{for } k > 2k_F$$
(one-dimensional case), (4.5)

which is derived in Appendix B. Therefore we obtain from Eqs. (3.12) and (3.13)

$$E_0 = \zeta + \frac{1}{2} N(N-1) \frac{\nu(0)}{L} + \frac{1}{2} \sum_{\{\mathbf{k}\}} \left[E(\mathbf{k}) - \frac{\hbar^2 k^2}{m} \frac{k_F}{k} - \frac{N}{L} \nu(\mathbf{k}) \right]$$

and

$$E(\mathbf{k}) = \sqrt{\frac{\hbar^2 k^2}{m} \frac{k_F}{k} \left[\frac{\hbar^2 k^2}{m} \frac{k_F}{k} + \frac{N}{L} \nu(\mathbf{k}) \frac{k}{k_F} \right]} \quad \text{(one-dimensional case),} \quad (4.6)$$

where we have explicitly used length L of the system instead of volume V and written the expression of $E(\mathbf{k})$ for $2k_F \geq k > 0$ alone. We shall assume that the set $\{\mathbf{k}\}$ consists of all the \mathbf{k} 's so that $k \leq k_F$. (It is to be noted that the set $\{\mathbf{k}\}$ has in general no connection with the Fermi sphere.) On the other hand, we have the relation $N = Lk_F/\pi$, where we have ignored a spin weight. Therefore we may put $n_{\max} = Lk_F/2\pi$ in Tomonaga's expressions. And also the quantities T_n , J_n , $2U_n$ and $J(0)$ appearing in his expressions can be written in our notations as $(\hbar^2 k^2/m)(k_F/k)$, $\nu(\mathbf{k})/L$, $(N\nu(\mathbf{k})/L)(k/k_F)$ and $(\nu(0)/L) + \sum_{\{\mathbf{k}\}} (\nu(\mathbf{k})/L)$ respectively. Then our expression (4.6) is in exact agreement with Eq. (5.14) in his paper.

* After the manuscript had been written, I noticed that Zubarev treated a Fermi gas by means of auxiliary variables. (D. N. Zubarev, J. Exp. Theor. Phys. U. S. S. R. 25 (1953), 548.) His treatment seems to be similar to the one of Bohm and Pines³⁾, that is, he used the individual coordinates as well as the auxiliary variables. I would like to express my sincere thanks to Professor S. Ono at Tokyo University for his kindness in informing me of this paper.

It is seen from Eq.(4.4) that even without interaction there exists phonon-like excitation and the sound velocity is given by $(2/3)(\hbar k_F/m)$. On the other hand the sound velocity can be calculated also by means of a hydrodynamical formula $(1/m)(\partial p/\partial \rho)$, where p and ρ denote the pressure and the particle number density of the system respectively. It gives the sound velocity to be $(1/\sqrt{3})(\hbar k_F/m)$, which is different by a factor $\sqrt{3}/2$ from the one determined above. In the one-dimensional case such a discrepancy does not occur. A similar situation has been found in Bloch's method of sound waves¹³⁾.

(iii) A classical fluid

Auxiliary variables have been used to deal with a classical fluid by several authors¹⁴⁾. We shall here apply the method explained in § 2 to such a system. When we decompose the potential energy into two parts such as Eq. (2.3), the Helmholtz free energy of the system is written in the form

$$A = \mathcal{O} - \kappa T \log Q, \quad (4.7)$$

where \mathcal{O} is the free energy in the presence of $v(\mathbf{r})$ alone, κ the Boltzmann constant, T the temperature and Q is given by

$$Q = \int d\mathbf{r}^{(3N)} e^{-\beta[v(\mathbf{r}) + U(\mathbf{r})]} / \int d\mathbf{r}^{(3N)} e^{-\beta v(\mathbf{r})}. \quad (4.8)$$

In expression (4.8) β denotes $1/\kappa T$.

If we suppose as

$$\chi^* \chi = e^{-\beta v(\mathbf{r})} / \int d\mathbf{r}^{(3N)} e^{-\beta v(\mathbf{r})}, \quad (4.9)$$

Eq. (4.8) can be written in the form

$$Q = \int d\mathbf{r}^{(3N)} \chi^* \chi e^{-\beta U(\mathbf{r})}. \quad (4.10)$$

Eq. (4.10) has the same form as the numerator of Eq. (2.8) if we put $\varphi(\rho_k) = 1$ in the latter. Therefore we obtain in the same way as in § 2

$$Q = \int (d\rho) D(\rho_k) \exp \left[-\frac{\beta}{2} \left\{ N(N-1) \frac{\nu(0)}{V} - \sum_k \frac{N}{V} \nu(\mathbf{k}) + \sum_k \frac{N}{V} \nu(\mathbf{k}) \rho_k \rho_{-\mathbf{k}} \right\} \right], \quad (4.11)$$

where we have used all the \mathbf{k} 's except for $\mathbf{k}=0$.

$D(\rho_k)$ has been defined by Eq. (2.9) and, to the zeroth approximation, it has the form given by Eq. (3.1). We shall be content to remain to the zeroth approximation. Q is written as

$$Q_0 = \int (d\rho) \exp \left[-\frac{1}{2} \left\{ \beta N(N-1) \frac{\nu(0)}{V} - \sum_k \beta \frac{N}{V} \nu(\mathbf{k}) + \left(\frac{1}{S(\mathbf{k})} + \beta \frac{N}{V} \nu(\mathbf{k}) \right) \rho_k \rho_{-\mathbf{k}} + \log \pi S(\mathbf{k}) \right\} \right].$$

The integration on the right-hand side of the above expression can be readily carried out. We obtain

$$Q_0 = \exp \left[-\frac{\beta}{2} \left\{ N(N-1) \frac{\nu(0)}{V} - \sum_k \frac{N}{V} \nu(\mathbf{k}) + \frac{1}{\beta} \sum_k \log \left(1 + \beta \frac{N}{V} \nu(\mathbf{k}) S(\mathbf{k}) \right) \right\} \right]. \quad (4.12)$$

Therefore we obtain as the zeroth approximation to the free energy

$$A_0 = \mathcal{Q} + \frac{1}{2} N(N-1) \frac{\nu(0)}{V} - \frac{\kappa T}{2} \sum_k \left[\beta \frac{N}{V} \nu(\mathbf{k}) - \log \left\{ 1 + \beta \frac{N}{V} \nu(\mathbf{k}) S(\mathbf{k}) \right\} \right]. \quad (4.13)$$

It should be remembered that $S(\mathbf{k})$ is the structure factor for the system possessing the potential energy $v(\mathbf{r})$. When we consider a system of particles interacting through strong repulsions as well as weak interactions, we may assume that $v(\mathbf{r})$ contains hard sphere interactions alone. A knowledge of properties of the hard sphere system can be utilized in Eq. (4.13).

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Appendix A. Calculation of $D(\rho_k)$

We shall here adopt Mayer's notations¹⁵⁾. The spatial coordinates of particle i are denoted by (i) and the volume element by $d(i)$. The spatial coordinates of n particles are indicated by $\{n\}$ and the volume element by $d\{n\} = d(1)d(2)\cdots d(n)$. The spatial coordinates of a subset of m particular particles of the set of particles $\{n\}$ are denoted by $\{m\}_n$.

In our case the distribution function of the spatial coordinates of N particles is defined by (cf. Eq. (2.26))

$$P_N\{N\} = \int (d\theta) \chi^*(x) \chi(x), \quad (A.1)$$

where $\int (d\theta)$ means the sum over spin coordinates. Since $\chi(x)$ is assumed to be normalized, $P_N\{N\}$ is also normalized such as

$$\int P_N\{N\} d\{N\} = 1. \quad (A.2)$$

We define the reduced distribution function of n particles by

$$P_n\{n\} = \int P_N\{N\} d\{N-n\}, \quad (A.3)$$

which is normalized as

$$\int P_n\{n\} d\{n\} = 1. \quad (A.4)$$

We assume that the system under consideration is homogeneous. Then $P_1(1)$

should be independent of the position of the particle, and hence we have

$$P_1(1) = 1/V. \quad (\text{A} \cdot 5)$$

Furthermore the distribution function should have the property that

$$P_{n+m}\{n+m\} = P_n\{n\} P_m\{m\} \quad \text{if } \{n\} \text{ and } \{m\} \text{ are distant from each other.} \quad (\text{A} \cdot 6)$$

It is convenient for our purpose to introduce the functions W_m 's defined in terms of P_n 's as follows:

$$\begin{aligned} P_1(1) &= W_1(1), \\ P_2(1, 2) &= W_2(1, 2) + W_1(1) W_1(2), \end{aligned} \quad (\text{A} \cdot 7)$$

$$P_N\{N\} = \sum_{\{k\{m_i\}_N\}_u} \prod_{i=1}^k W_{m_i}(\{m_i\}_N),$$

where in the last expression a complete set of k unconnected subsets $\{m_i\}_N (1 \leq i \leq k, \sum_{i=1}^k m_i = N)$ of the set $\{N\}$ is denoted by the symbol $\{k\{m_i\}_N\}_u$ and the sum is taken over all possible sets of such unconnected subsets. As is well known^{1b)}, the inverse of Eq. (A·7) is written in the form

$$\begin{aligned} W_1(1) &= P_1(1), \\ W_2(1, 2) &= P_2(1, 2) - P_1(1) P_1(2), \end{aligned} \quad (\text{A} \cdot 8)$$

$$W_m\{m\} = \sum_{\{l\{n_j\}_m\}_u} (-)^{l-1} (l-1)! \prod_{j=1}^l P_{n_j}(\{n_j\}_m).$$

The above definition of W_m 's leads to the relations

$$\int W_1(1) d(1) = 1 \quad \text{and} \quad \int W_m\{m\} d(1) = 0 \quad \text{for } m \geq 2, \quad (\text{A} \cdot 9)$$

the latter of which can be proved by means of a mathematical induction. With the aid of Eqs. (A·5) and (A·6) we obtain

$$W_1(1) = 1/V \quad (\text{A} \cdot 10)$$

and

$$W_{n+m}\{n+m\} = 0 \quad \text{if } \{n\} \text{ and } \{m\} \text{ are distant from each other.} \quad (\text{A} \cdot 11)$$

The property (A·11) leads to

$$\int W_m\{m\} \exp(i\mathbf{k}_1 \mathbf{r}_1 + i\mathbf{k}_2 \mathbf{r}_2 + \cdots + i\mathbf{k}_m \mathbf{r}_m) d\{m\} = 0 \quad \text{unless } \mathbf{k}_1 + \mathbf{k}_2 + \cdots + \mathbf{k}_m = 0. \quad (\text{A} \cdot 12)$$

We can write $D(\rho_k)$, which has been defined by Eq. (2·9), in the form

$$D(\rho_k) = \int P_N\{N\} \prod_{\langle k \rangle} \delta\left(\sigma_k - \frac{1}{\sqrt{N}} \sum_{j=1}^N \cos \mathbf{k} \mathbf{r}_j\right) \delta\left(\tau_k - \frac{1}{\sqrt{N}} \sum_{j=1}^N \sin \mathbf{k} \mathbf{r}_j\right) d\{N\}. \quad (\text{A} \cdot 13)$$

With the aid of the Fourier representation of δ -function

$$\delta(x) = \int_{-\infty}^{\infty} \exp(2\pi i \omega x) d\omega,$$

Eq. (A·13) is reduced to

$$D(\rho_k) = \int_{-\infty}^{+\infty} \left(\prod_{\langle k \rangle} d\alpha_k d\beta_k \right) \int P_N\{N\} \exp \left[2\pi i \sum_{\langle k \rangle} \left\{ \left(\sigma_k - \frac{1}{\sqrt{N}} \sum_{j=1}^N \cos \mathbf{k} \mathbf{r}_j \right) \alpha_k + \left(\tau_k - \frac{1}{\sqrt{N}} \sum_{j=1}^N \sin \mathbf{k} \mathbf{r}_j \right) \beta_k \right\} \right] d\{N\}.$$

By the use of the variables $\omega_k = \alpha_k + i\beta_k$ and $\omega_{-k} = \alpha_k - i\beta_k$, the above expression is written as

$$D(\rho_k) = \int_{-\infty}^{+\infty} \left(\prod_{\langle k \rangle} d\alpha_k d\beta_k \right) [\exp(i\pi \sum_{\langle k \rangle} \rho_k \omega_k)] I(\omega_k), \quad (\text{A} \cdot 14)$$

where

$$I(\omega_k) = \int P_N\{N\} \exp \left[-\frac{i\pi}{\sqrt{N}} \sum_{\langle k \rangle} \sum_{j=1}^N \omega_k e^{-i\mathbf{k} \mathbf{r}_j} \right] d\{N\}. \quad (\text{A} \cdot 15)$$

Substituting Eq. (A·7) into Eq. (A·15), we obtain

$$I(\omega_k) = N! \sum_{\sum_l n_l = N} \prod_l (b_l^{n_l} / n_l!), \quad (\text{A} \cdot 16)$$

where

$$b_l = \frac{1}{l!} \int W_l\{l\} \exp \left[-\frac{i\pi}{\sqrt{N}} \sum_{\langle k \rangle} \sum_{j=1}^l \omega_k e^{i\mathbf{k} \mathbf{r}_j} \right] d\{l\}. \quad (\text{A} \cdot 17)$$

To calculate the integral on the right-hand side of Eq. (A·17), we expand the first exponential function into a power series in its argument and perform the integration term by term. Taking notice of Eqs. (A·9), (A·10) and (A·12), we obtain

$$b_1 = 1 - \frac{1}{N} \frac{\pi^2}{2} \sum_{\substack{k_1 + k_2 = 0 \\ (k_1, k_2)}} \omega_{k_1} \omega_{k_2} + o(1/N) \quad (\text{A} \cdot 18)$$

and

$$b_l = \frac{1}{l!} \left[(-i\pi)^l N^{-(3l/2)+1} \sum_{\substack{k_1 + k_2 + \dots + k_l = 0 \\ (k_1, k_2, \dots, k_l)}} \omega_{k_1} \omega_{k_2} \dots \omega_{k_l} \int N^{l-1} W_l\{l\} \right. \\ \left. \times \exp[-i\mathbf{k}_1 \mathbf{r}_1 - i\mathbf{k}_2 \mathbf{r}_2 - \dots - i\mathbf{k}_l \mathbf{r}_l] d\{l\} + o(N^{-(3l/2)+1}) \right]. \quad (\text{A} \cdot 19)$$

As is seen from Eq. (A·19), b_l ($l \geq 2$) is at most of order $N^{-(3l/2)+1}$. (It should be noted that the integral on the right-hand side of Eq. (A·19) is independent of N in the limit $N \rightarrow \infty$ while keeping the value of N/V fixed.) To the zeroth approximation we shall neglect the contributions of b_l 's ($l \geq 3$) to $I(\omega_k)$. Furthermore we shall write b_1 and b_2 in the forms

$$b_1 = 1 + p/N \quad \text{and} \quad b_2 = q/N^2, \quad (\text{A} \cdot 20)$$

where p and q are of order unity and defined respectively by

$$p = -\frac{\pi^2}{2} \sum_{\substack{k_1+k_2=0 \\ (k_1, k_2)}} \omega_{k_1} \omega_{k_2}$$

and

$$q = -\frac{\pi^2}{2} \sum_{\substack{k_1+k_2=0 \\ (k_1, k_2)}} \omega_{k_1} \omega_{k_2} \int d(1) d(2) N W_2(1, 2) e^{-ik_1 r_1 - ik_2 r_2}. \quad (\text{A} \cdot 21)$$

Substitution of Eq. (A·16) leads to

$$I_0(\omega_k) = \sum_{n=0}^{N/2} \frac{N!}{(N-2n)! n!} \left(1 + \frac{p}{N}\right)^{N-2n} (q/N^2)^n. \quad (\text{A} \cdot 22)$$

Since we are looking for an asymptotic form in the limit $N \rightarrow \infty$, the above expression can be replaced by

$$I_0(\omega) = e^{p+q}, \quad (\text{A} \cdot 23)$$

which is the limiting form of Eq. (A·22) in the case of $N \rightarrow \infty$. Eq. (A·23) is verified by making use of the Stirling formula on Eq. (A·22). It can be proved by a more rigorous treatment of the sum appearing in Eq. (A·22).^{*} By the use of Eq. (A·21), $I_0(\omega_k)$ is written as

$$I_0(\omega_k) = \exp \left[-\frac{\pi^2}{2} \sum_{(k)} S(\mathbf{k}) \omega_k \omega_{-k} \right], \quad (\text{A} \cdot 24)$$

where

$$S(\mathbf{k}) = 1 + N \iint d\mathbf{r}_1 d\mathbf{r}_2 W_2(\mathbf{r}_1, \mathbf{r}_2) e^{ik(r_2-r_1)}. \quad (\text{A} \cdot 25)$$

Substituting Eq. (A·24) into Eq. (A·14) and performing the integration over α_k 's and β_k 's, we obtain, as the zeroth approximation to $D(\rho_k)$,

$$D_0(\rho_k) = \exp \left[-\frac{1}{2} \sum_{(k)} \left\{ \frac{1}{S(\mathbf{k})} \rho_k \rho_{-k} + \log \pi S(\mathbf{k}) \right\} \right]. \quad (\text{A} \cdot 26)$$

When we assume two-particle distribution function $P_2(\mathbf{r}_1, \mathbf{r}_2)$ to be a function of the distance between the particles, $P_2(\mathbf{r}_1, \mathbf{r}_2)$ can be written such as $P_2(\mathbf{r}_1, \mathbf{r}_2) = (1/V^2)g(r)$, where $g(r)$ is the radial distribution function and r the distance.

^{*} I am much indebted to Dr. T. Morita for his kindness to tell me a rigorous proof of Eq. (A·23).

Therefore Eq. (A·25) is reduced, with the aid of Eqs. (A·5) and (A·8), to Eq. (3·2).

If we take account of the contributions from b_l 's ($l \geq 3$), we can obtain a more accurate expression for $D(\rho_k)$ than Eq. (A·26).

Appendix B. Structure factor for a Fermi sphere

The structure factor $S(\mathbf{k})$ can be calculated by the use of Eq. (3·2), because the radial distribution function for a Fermi sphere has been calculated by Wigner and Seitz¹⁶⁾. However, the integral appearing in Eq. (3·2) is in need of somewhat complicated calculations. We shall here show a more elementary calculation of $S(\mathbf{k})$.

Let us consider a state represented by

$$\chi(x) = \frac{1}{\sqrt{N!}} \sum_P (-)^P P \phi_1(x_1) \phi_2(x_2) \cdots \phi_N(x_N), \quad (\text{B} \cdot 1)$$

where the set of ϕ_i 's is an orthonormal one and P denotes a permutation of x_i 's. The two-particle distribution function $P_2(\mathbf{r}_1, \mathbf{r}_2)$ which is defined by Eqs. (A·1) and (A·3) is written in our case as follows:

$$P_2(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{N(N-1)} \sum_i \sum_j \iint d\theta_1 d\theta_2 \phi_i^*(x_1) \phi_j^*(x_2) \{ \phi_i(x_1) \phi_j(x_2) - \phi_i(x_2) \phi_j(x_1) \}, \quad (\text{B} \cdot 2)$$

where $\int d\theta$ stands for a sum over spin coordinates. We may assume $\phi_i(x)$ to be a product of a spatial wave function and a spin function, that is

$$\phi_i(x) = \phi_k(\mathbf{r}) \theta_\sigma(\theta). \quad (\text{B} \cdot 3)$$

We substitute Eq. (B·3) into Eq. (B·2) and, after the integration over spin coordinates, obtain

$$P_2(\mathbf{r}_1, \mathbf{r}_2) = \frac{(2s+1)^2}{N(N-1)} \sum_{k_1} \sum_{k_2} n_{k_1} n_{k_2} \left[|\phi_{k_1}(\mathbf{r}_1)|^2 |\phi_{k_2}(\mathbf{r}_2)|^2 - \frac{1}{2s+1} \phi_{k_1}^*(\mathbf{r}_1) \phi_{k_2}(\mathbf{r}_1) \phi_{k_2}^*(\mathbf{r}_2) \phi_{k_1}(\mathbf{r}_2) \right], \quad (\text{B} \cdot 4)$$

where s denotes a spin of the particle (in units of \hbar) and n_k the number of ϕ_k appearing in Eq. (B·1). It is needless to say that there exists the relation

$$(2s+1) \sum_k n_k = N. \quad (\text{B} \cdot 5)$$

By the use of $\phi_k = (1/\sqrt{V}) e^{ikr}$ Eq. (B·4) can be written in the form

$$P_2(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{V^2} \left[\frac{N}{N-1} - \frac{2s+1}{N(N-1)} \sum_{k_1} \sum_{k_2} n_{k_1} n_{k_2} e^{-i(k_1 - k_2)(r_1 - r_2)} \right]. \quad (\text{B} \cdot 6)$$

Taking account of Eqs. (A·8) and (A·25), we obtain the structure factor as

$$S(\mathbf{k}) = 1 - \frac{2s+1}{N} \sum_{\mathbf{k}'} n_{\mathbf{k}'} n_{\mathbf{k}'+\mathbf{k}}. \quad (\text{B} \cdot 7)$$

The sum on the right-hand side of Eq. (B·7) may be replaced by an integral and then the above expression is reduced to

$$S(\mathbf{k}) = 1 - \frac{2s+1}{N} \frac{V}{(2\pi)^3} \int n_{\mathbf{k}'} n_{\mathbf{k}'+\mathbf{k}} d\mathbf{k}'. \quad (\text{B} \cdot 8)$$

For a fully degenerate case, that is a Fermi sphere,

$$\begin{aligned} n_{\mathbf{k}} &= 1 & \text{for } k \leq k_F \\ &= 0 & \text{for } k > k_F. \end{aligned} \quad (\text{B} \cdot 9)$$

Then the integral appearing in Eq. (B·8) has a simple geometrical meaning. It is equal to the overlapping volume of two spheres, each of which has a radius k_F and the centers are at the distance of k . An elementary calculation shows that the overlapping volume is

$$\frac{4\pi}{3} k_F^3 \left[1 - \frac{3}{4} \frac{k}{k_F} + \frac{1}{16} \left(\frac{k}{k_F} \right)^3 \right] \quad \text{for } 0 \leq k \leq 2k_F \quad (\text{B} \cdot 10)$$

and zero for $k > 2k_F$. Taking notice of that

$$\frac{2s+1}{N} \frac{V}{(2\pi)^3} \frac{4\pi}{3} k_F^3 = 1, \quad (\text{B} \cdot 11)$$

we obtain the structure factor for a Fermi sphere as

$$\begin{aligned} S(\mathbf{k}) &= \frac{3}{4} \frac{k}{k_F} - \frac{1}{16} \left(\frac{k}{k_F} \right)^3 & \text{for } 0 < k \leq 2k_F \\ &= 1 & \text{for } k > 2k_F. \end{aligned} \quad (\text{B} \cdot 12)$$

An expression for one-dimensional or two-dimensional case can be obtained if only the sphere is replaced by a line or a circle.

$$\begin{aligned} S(\mathbf{k}) &= \frac{1}{2} \frac{k}{k_F} & \text{for } 0 < k \leq 2k_F \\ &= 1 & \text{for } k > 2k_F \end{aligned} \quad (\text{one-dimensional case}). \quad (\text{B} \cdot 13)$$

$$\begin{aligned} S(\mathbf{k}) &= \frac{2}{\pi} \sin^{-1} \frac{k}{2k_F} + \frac{1}{\pi} \frac{k}{k_F} \sqrt{1 - \frac{k^2}{4k_F^2}} & \text{for } 0 < k \leq 2k_F \\ &= 1 & \text{for } k > 2k_F \end{aligned} \quad (\text{two-dimensional case}). \quad (\text{B} \cdot 14)$$

It is seen that in all cases $S(\mathbf{k})$ is proportional to k when k is small.

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Dependence of p - p Scattering Parameters on Phase Shifts*

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An attempt is made to discuss the dependence of the double and triple p - p scattering parameters on the 1S_0 , 1D_2 , $^3P_{0,1,2}$, 3F_2 phase shifts with special reference to the SM1 potential at 150 Mev. It is found that a positive depolarization requires a small or even negative 3P_0 phase shift thus favouring a strong spin-orbit potential. The parameters R and A depend chiefly on the 1S_0 and 1D_2 phase shifts.

§ 1. Introduction

The study of the proton-proton scattering has been carried out with a considerable amount of success by means of two main procedures viz., (i) Phenomenological potential and (ii) Phase shift analysis of the experimental data. In the first approach, one starts with a phenomenological potential and calculates a set of phase shifts which finally enable one to calculate the scattering parameters. The most recent work along this line is that of Signell-Marshak¹⁾, Gammel-Thaler²⁾, Otsuki³⁾, Watari⁴⁾, Tamagaki⁵⁾. In the second approach, say in the work of Stapp et al.⁶⁾ and Moravcsik et al.⁷⁾, one obtains several sets of phase shifts which fit the experimental differential cross-section, polarization and some of the triple scattering parameters and the problem is that of discriminating among the various solutions on the basis of other experimental evidence. It has been found⁷⁾ however that the various sets of phase shifts converge to similar sets, yet there are some differences, e.g. in the depolarization at 150 Mev.

The purpose of the present work is to investigate whether it is possible to identify the qualitative modifications which may be required in a phenomenological potential, so that disagreements with some of the parameters are eliminated. In order to do this we have expressed the scattering parameters P , D , R , A , R' , A' and the spin correlation parameters C_{kp} and C_{nn} , where the symbols have their usual meaning⁸⁾, in terms of the phase shifts, neglecting all the coulomb effects and partial waves F (except 3F_2) and higher. This is done in section 2. In section 3, we have worked in detail with the SM 1 potential⁹⁾ at 150 Mev as an example and discussed whether the individual scattering parameters show any special

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dependence on one or more phase shifts and thereby indicate the qualitative changes of the potential required to obtain better agreement with experiment. Assuming the fact that depolarization is mainly positive¹⁰⁾, it is concluded in section 3 (ii) that a short-range attractive spin-orbit interaction is needed for the p - p scattering and it is difficult to understand how the attempt of Otsuki³⁾, Watari⁴⁾, Tamagaki⁵⁾ to fit the nucleon-nucleon scattering with central plus tensor potentials can succeed in this respect.

§ 2. Expansion of scattering parameters in terms of phase shifts

The antisymmetrized M matrix elements for the proton-proton system can be written as follows⁹⁾

$$M_{pq}(\theta, \varphi) = f_c^{pq}(\theta) + \sum_{\substack{j=0 \\ \text{odd } l}} \{ a_{jj}^{pq}(\theta, \varphi) \alpha_{jj} + (a_{j-1j}^{pq}(\theta, \varphi) \alpha_{j-1j} + a_{j+1j}^{pq}(\theta, \varphi) \alpha_{j+1j}) \\ + a^{jpq}(\theta, \varphi) \alpha^j \} + \sum_{\text{even } l} a_l^{pq}(\theta) \alpha_l \quad (1)$$

where (θ, φ) is the scattering angle in the center of mass, the index pair pq can take the values 11, 00, 01, 10, 1-1, ss and the appropriate coefficients a^{pq} and the matrix elements of α are defined in Table III and Eqs. (3.12)–(3.14) of Stapp, et al.⁶⁾ The coulomb scattering amplitude $f_c^{pq}(\theta)$ is non-zero only for $pq=ss$, 11 and 00. Expressing α in terms of phase shifts (Eqs. (3.12)–(3.14) of reference 6) in Eq. (1), we obtain

$$M_{pq}(\theta, \varphi) = f_c^{pq}(\theta) + \sum_{j=0} \{ \sum_{mn} a_{mn}^{pq}(\theta, \varphi) \langle m'n, \Phi_m | \\ + f(\epsilon_j, a_{j-1j}^{pq}, a_{j+1j}^{pq}, a^{jpq}/2) \langle j-1j, j+1j | \rangle \}, \quad (2)$$

where mn take values jj , $j-1j$, $j+1j$ and j , and $m'=m$ except when $m=j-1$, $m'=j+1$.

$$\langle A, B | = 2ie^{i(\delta_A + \delta_B)} \sin(\delta_A - \delta_B) \\ f(\epsilon_j, a, b, c) = a \cos^2 \epsilon_j + b \sin^2 \epsilon_j + c \sin 2\epsilon_j. \quad (3)$$

Thus

$$M_{pq}(\theta, \varphi) M_{rs}^*(\theta, \varphi) = f_c^{pq}(\theta) f_c^{rs*}(\theta) \\ + f_c^{pq}(\theta) \left[\sum_{j'} \left\{ \sum_{xy} a_{xy}^{rs} \langle x'y, \Phi_x | + f(\epsilon_{j'}, a_{j'-1j'}^{rs}, a_{j'+1j'}^{rs}, a^{j'rs}/2) \langle j'-1j', j'+1j' | \rangle \right\} \right] \\ + f_c^{rs*}(\theta) \left[\sum_j \left\{ \sum_{mn} a_{mn}^{pq} \langle m'n, \Phi_m | + f(\epsilon_j, a_{j-1j}^{pq}, a_{j+1j}^{pq}, a^{jpq}/2) \langle j-1j, j+1j | \rangle \right\} \right] \\ + \sum_{j'j'} \left\{ \sum_{mn, xy} a_{mn}^{pq} a_{xy}^{rs*} \langle m'n, \Phi_m | x'y, \Phi_x \rangle \right. \\ \left. + f(\epsilon_j, a_{j-1j}^{pq}, a_{j+1j}^{pq}, a^{jpq}/2) f^*(\epsilon_{j'}, a_{j'-1j'}^{rs}, a_{j'+1j'}^{rs}, a^{j'rs}/2) \right. \\ \left. \langle j-1j, j+1j | j'-1j', j'+1j' \rangle \right\} \\ + \sum_{j'} \sum_{mn} a_{mn}^{pq}(\theta, \varphi) f^*(\epsilon_{j'}, a_{j'-1j'}^{rs}, a_{j'+1j'}^{rs}, a^{j'rs}/2) \langle m'n, \Phi_m | j'-1j', j'+1j' \rangle \\ + \sum_j \sum_{xy} a_{xy}^{rs*}(\theta, \varphi) f(\epsilon_j, a_{j-1j}^{pq}, a_{j+1j}^{pq}, a^{jpq}/2) \langle j-1j, j+1j | xy, \Phi_x \rangle \quad (4)$$

where

$$\langle A, B|C, D \rangle = (A, B|C, D) + i\{A, B|C, D\}$$

$$(A, B|C, D) = 4 \sin(\delta_A - \delta_B) \sin(\delta_C - \delta_D) \cos(\delta_A + \delta_B - \delta_C - \delta_D) \quad (5)$$

$$\{A, B|C, D\} = 4 \sin(\delta_A - \delta_B) \sin(\delta_C - \delta_D) \sin(\delta_A + \delta_B - \delta_C - \delta_D).$$

We now use Eq. (4) to express the polarization P , the triple scattering parameters D, R, A, R', A' and the spin correlation parameters C_{kp} and C_{nn} , where all the symbols have their usual meaning³⁾, in terms of phase shifts. Neglecting all the coulomb effects and partial waves F (except 3F_2) and higher, we obtain

$$I_0 P = \frac{\sqrt{2}}{4} \operatorname{Re} \{ i(M_{10} - M_{01})(M_{11} - M_{1-1} + M_{00})^* \} \quad (6)$$

$$= \frac{1}{k^2} \sin \theta \sum_{\text{odd } n} A_n(P) \cos n\theta \quad (6a)$$

$$I_0 D = \frac{1}{2} \operatorname{Re} \{ (M_{11} + M_{1-1})M_{ss}^* + (M_{11} - M_{1-1})M_{00}^* \} - \operatorname{Re} M_{10}M_{01}^* \quad (7)$$

$$= \frac{1}{k^2} \sum A_n(D) \cos n\theta \quad (7a)$$

$$\begin{aligned} \frac{I_0 R}{\cos \theta/2} &= \frac{1}{2} \operatorname{Re} \left\{ \left[M_{00} + (\cos \theta - 1) \frac{\sqrt{2} M_{10}}{\sin \theta} \right] (M_{11} + M_{1-1})^* \right. \\ &\quad \left. + \left[M_{00} + \cos \theta \frac{\sqrt{2} M_{10}}{\sin \theta} + \frac{\sqrt{2} M_{01}}{\sin \theta} \right] M_{ss}^* \right\} \end{aligned} \quad (8)$$

$$= \frac{1}{k^2} \sum A_n(R) \cos n\theta \quad (8a)$$

$$\begin{aligned} \frac{I_0 A}{\sin \theta/2} &= -\frac{1}{2} \operatorname{Re} \left\{ \left[M_{00} + \cos \theta \frac{\sqrt{2} M_{10}}{\sin \theta} - \frac{\sqrt{2} M_{01}}{\sin \theta} \right] (M_{11} + M_{1-1})^* \right. \\ &\quad \left. + \left[M_{00} + (\cos \theta + 1) \frac{\sqrt{2} M_{10}}{\sin \theta} \right] M_{ss}^* \right\} \end{aligned} \quad (9)$$

$$= -\frac{1}{k^2} \sum A_n(A) \cos n\theta \quad (9a)$$

$$\begin{aligned} \frac{I_0 R'}{\sin \theta/2} &= \frac{1}{2} \operatorname{Re} \left\{ \left[M_{00} + (\cos \theta + 1) \frac{\sqrt{2} M_{10}}{\sin \theta} \right] (M_{11} + M_{1-1})^* \right. \\ &\quad \left. + \left[M_{00} + \cos \theta \frac{\sqrt{2} M_{10}}{\sin \theta} - \frac{\sqrt{2} M_{01}}{\sin \theta} \right] M_{ss}^* \right\} \end{aligned} \quad (10)$$

$$= \frac{1}{k^2} \sum A_n(R') \cos n\theta \quad (10a)$$

$$\frac{I_0 A'}{\cos \theta/2} = \frac{1}{2} \operatorname{Re} \left\{ \left[M_{00} + \cos \theta \frac{\sqrt{2} M_{10}}{\sin \theta} + \frac{\sqrt{2} M_{01}}{\sin \theta} \right] (M_{11} + M_{1-1})^* \right. \\ \left. + \left[M_{00} + (\cos \theta - 1) \frac{\sqrt{2} M_{10}}{\sin \theta} \right] M_{ss}^* \right\} \quad (11)$$

$$= \frac{1}{k^2} \sum A_n(A') \cos n\theta \quad (11a)$$

$$I_0 C_{kp} = \{ |M_{01}|^2 - |M_{10}|^2 \} / 2 \sin \theta \quad (12)$$

$$= \frac{\sin \theta}{k^2} \sum_{\text{even } n} A_n(C_{kp}) \cos n\theta \quad (12a)$$

$$I_0 C_{nn} = \frac{1}{4} |M_{00}|^2 + \frac{1}{2} |M_{01}|^2 + \frac{1}{2} |M_{10}|^2 - \operatorname{Re} M_{11} M_{1-1} - \frac{1}{4} |M_{ss}|^2 \quad (13)$$

$$= \frac{1}{k^2} \sum_{\text{even } n} A_n(C_{nn}) \cos n\theta \quad (13a)$$

where

$$I_0 = \frac{1}{2} |M_{11}|^2 + \frac{1}{4} |M_{00}|^2 + \frac{1}{4} |M_{ss}|^2 + \frac{1}{2} |M_{10}|^2 + \frac{1}{2} |M_{01}|^2 + \frac{1}{2} |M_{1-1}|^2 \quad (14)$$

and the coefficients A_n are defined in the Appendix.

§ 3. Scattering parameters for Signell-Marshak potential 1

We mainly confine our attention to the SM1 potential⁹⁾ at 150 Mev in the laboratory and choose this potential as the basis for our discussion of the dependence of the scattering parameters on the phase shifts. The various phase shifts* at 150 Mev are

$$\left. \begin{aligned} \delta_0 = \delta(^1S_0) &= 19.2^\circ, \quad \delta_2 = \delta(^1D_2) = 5.7^\circ \\ \delta_{10} = \delta(^3P_0) &= 9.8^\circ, \quad \delta_{11} = \delta(^3P_1) = -19.5^\circ, \quad \delta_{12} = \delta(^3P_2) = 13.1^\circ, \quad \delta_{32} = \delta(^3F_2) = -2.0^\circ \end{aligned} \right\} \\ \text{and the mixing parameter } \epsilon_2 = -17.4^\circ. \quad (15)$$

Using Eqs. (A.1)–(A.13) of the Appendix, the calculated values of the coefficients A_n for the scattering parameters are listed in Table I. The curves for most of parameters are shown in Figs. 1–5. In the following we discuss the dependence of the parameters on the phase shifts and try to see if they exhibit any special dependence on one or more phase shifts. We will then investigate if an adjustment of these phase shifts will improve the fit with the experimental data and what

* I am thankful to Mr. R. Bryan for supplying me with the phase shifts for SM1 at 150 Mev.

Table I. Values of the coefficient A_n for SM1 at 150 Mev. The scattering parameters in column 1 are expanded into $\sum A_n \cos n\theta$. The contribution from the P and F phase shifts is listed in the first row while that arising from the S and D phase shifts or due their interference with the P and F waves is listed in the second row. D_M and R_M refer to D and R when the SM1 phase shifts are modified.

	A_0	A_1	A_2	A_3	A_4
$I_0 k^2 \frac{P}{\sin \theta}$.315		.027	
$I_0 k^2 D$	-.066	.067	.173	.007	-.072
$I_0 k^2 D_M$.059	.146	.082	.026	-.086
$I_0 k^2 \frac{R}{\cos \theta/2}$.025 -.543	-.015 .699	.025 -.360	-.004 .204	.074
$I_0 k^2 \frac{R_M}{\cos \theta/2}$.025 -.443	-.015 .744	.025 -.656	-.004 .312	.113
$I_0 k^2 \frac{A}{\sin \theta/2}$	-.025 -.192	.091 -.699	-.025 -.315	.015 -.204	-.100
$I_0 k^2 \frac{C_{ep}}{\sin \theta}$.537		-.329		-.016
$I_0 k^2 C_{nn}$.432 -.272		-.436 -.331		.038 -.069

this implies in terms of qualitative modifications of the SM1 potential.

i) Polarization: It is interesting to note here that from the fact that polarization in the p - p scattering at 135 Mev is positive, Wolfenstein¹¹⁾ has concluded that it is impossible to have a solution for which $\delta(^3P_0)$ is large and positive. However, Tamagaki⁹⁾, working with only central and tensor potentials in which case the $\delta(^3P_0)$ phase shift is large and positive, has obtained a good fit of the polarization at 150 Mev. Tamagaki's phase shifts are

$$\delta(^3P_0) = 28.6^\circ, \delta(^3P_1) = -17.2^\circ, \delta(^3P_2) = 7.4^\circ, \delta(^3F_2) = -4.6^\circ \text{ and } \epsilon_2 = -14.9^\circ. \quad (16)$$

For this set of phase shifts we have obtained $I_0 k^2 P = \sin \theta [.301 \cos \theta + .031 \cos 3\theta]$ which is in very good agreement with the corresponding expression for the SM1 potential which has a strong spin orbit interaction. This indicates that the p - p polarization at 150 Mev does not distinguish between a purely central plus tensor interaction and the addition of a spin-orbit interaction to it. One can understand this quite easily by an examination of the coefficients $A_1(P)$ and $A_3(P)$ in Eq.

(A.1). Denoting the phase shifts for a purely central plus tensor potential by a superscript zero, the effect of an addition of an attractive spin orbit potential is to make $\delta_{10} = \delta_{10}^{(0)} - 2J$, $\delta_{11} = \delta_{11}^{(0)} - J$, $\delta_{12} = \delta_{12}^{(0)} + J$, $\delta_{32} = \delta_{32}^{(0)} + J'$ where J and J' are positive and the changes in the phase shifts are indicated in Born approximation. If we observe that $\delta_{10}^{(0)}$, $\delta_{12}^{(0)}$ are positive and $\delta_{11}^{(0)}$, $\delta_{32}^{(0)}$ are negative, we find that $\{10|12, 32\}$ is decreased in value, $\{11|12, 32\}$ increases while $\{12, 32|32\}$ remains almost unaffected with the result that with suitable amounts of various interactions the coefficients $A_1(P)$ and $A_3(P)$ have the same values.

ii) Depolarization: As shown in Fig. 1 (Full curve), the depolarization for SM1 at 150 Mev goes negative. This result is in agreement with the Harwell¹²⁾

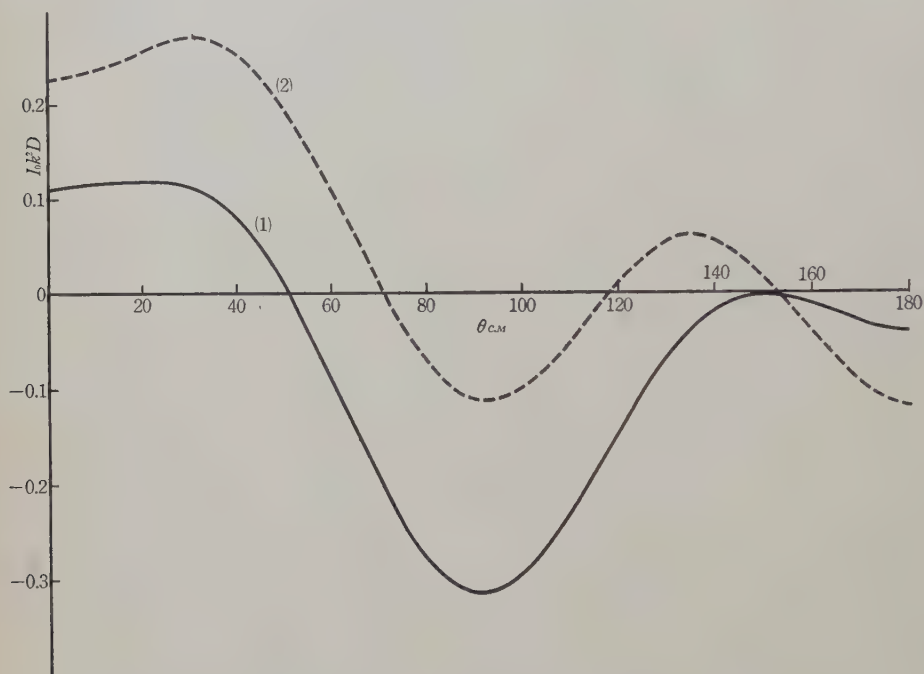


Fig. 1. Curve (1) shows the predictions for the triple scattering parameter $I_0 k^2 D(\theta)$ at 150 Mev for the SM1 potential. Curve (2) shows $I_0 k^2 D(\theta)$ when the ${}^3P_{012}$ phase shifts are modified to take into account an additional spin-orbit force.

data but does not agree with the Harvard¹⁰⁾ data in which $D(\theta)$ is mainly positive. Let us try to see what qualitative modifications are needed in the SM1 potential so that $D(\theta)$ would remain positive. It is obvious from the expression for $A_0(D)$ (Eq. (A.2)) that the easiest way to make a change in $I_0 k^2 D(\theta)$ by a constant amount will be to change the $\delta({}^3P_0)$ phase shift by a suitable amount. In order to predict a positive $D(\theta)$ we will need to reduce $\delta({}^3P_0)$ and even make it negative. This can be accomplished by increasing the attractive spin-orbit potential. Assuming Born approximation, the changes introduced by a

spin-orbit potential in the $\delta(^3P_0)$, $\delta(^3P_1)$, $\delta(^3P_2)$ phase shifts will be in the ratio $-2:-1:1$. The dashed curve in Fig. 1 is obtained by taking $\delta_{10}=3.8^\circ$, $\delta_{11}=-22.5^\circ$, $\delta_{12}=16.1^\circ$ and all the others as before. This indicates that a closer fit with the Harvard data is possible by the addition of an appropriate amount of spin-orbit potential.

It is inferred here that the experimental requirement that $D(\theta)$ is positive, favours the strong spin orbit potential for the nucleon-nucleon scattering. This follows from the need of a small $\delta(^3P_0)$ as discussed above. One can see this qualitatively in the following manner. In the Born approximation one can use the 3P phase shifts to predict the ratio

$$\frac{\langle V_{TS} \rangle}{\langle V_T \rangle} = \frac{\frac{1}{2}[-\frac{1}{3}\delta(^3P_0) - \frac{1}{2}\delta(^3P_1) + \frac{5}{6}\delta(^3P_2)]}{\frac{5}{12}[-\frac{1}{3}\delta(^3P_0) + \frac{1}{2}\delta(^3P_1) - \frac{1}{6}\delta(^3P_2)]}, \quad (17)$$

where

$$\langle V_{LS} \rangle = -\frac{mk}{\hbar^2} \int_0^\infty dr r^2 j_1^2(kr) V_{LS}(r),$$

$$\langle V_T \rangle = -\frac{mk}{\hbar^2} \int_0^\infty dr r^2 j_1^2(kr) V_T(r), \quad (18)$$

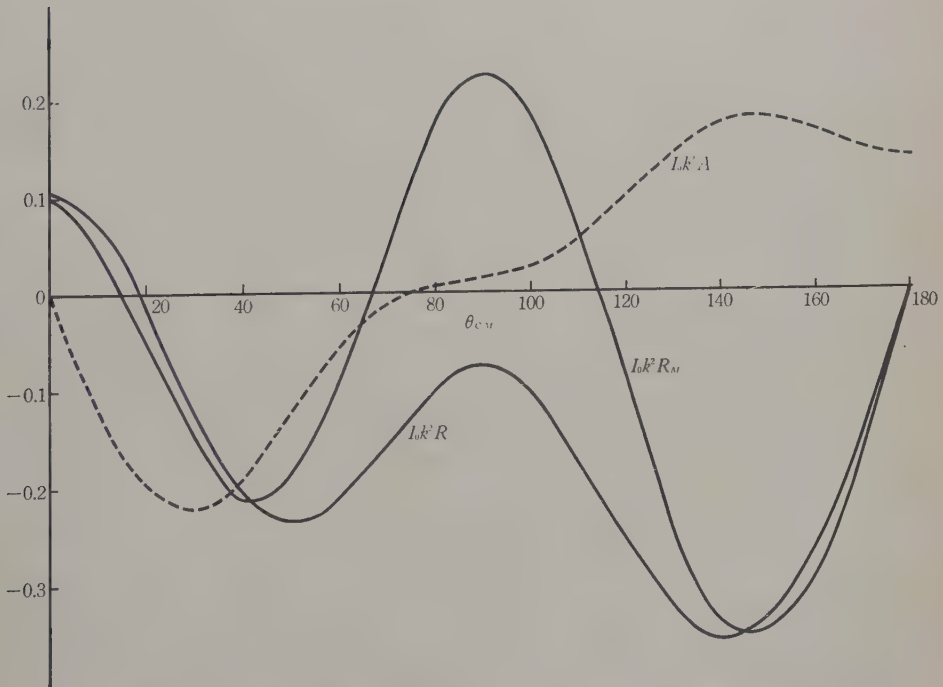


Fig. 2. Triple scattering parameters $I_0 k^2 R(\theta)$ and $I_0 k^2 A(\theta)$ for the SM1 potential at 150 Mev. The curve $I_0 k^2 R_M(\theta)$ is obtained instead of $I_0 k^2 R(\theta)$ when the 1S_0 and 1D_2 phase shifts used are those for the Gammel-Thaler potential at 150 Mev.

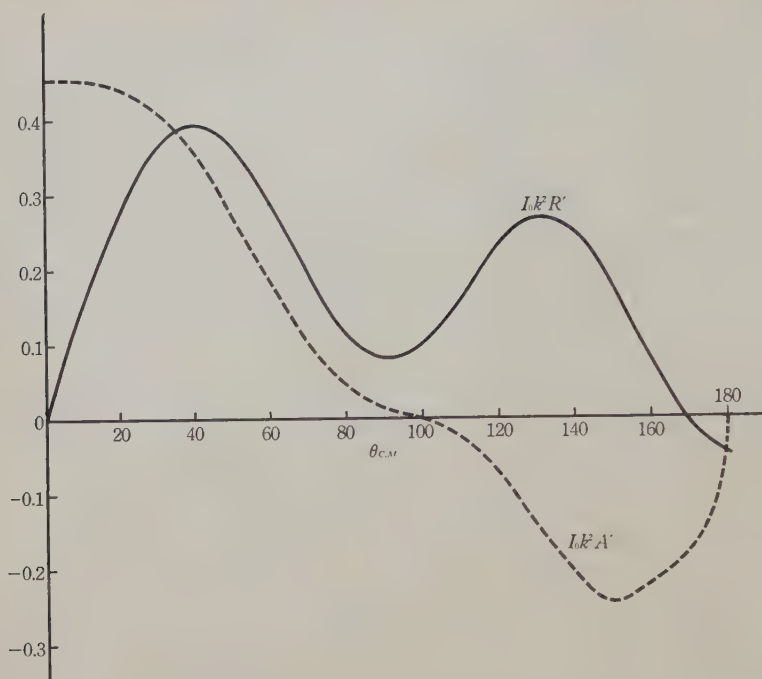


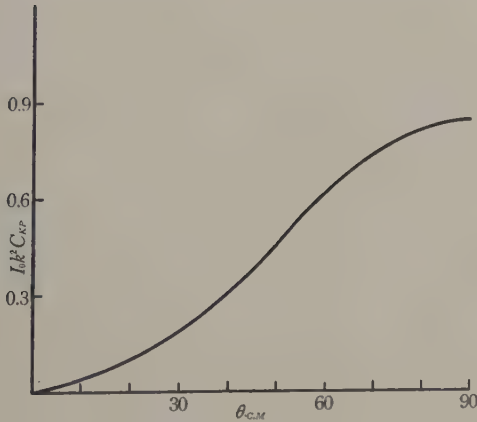
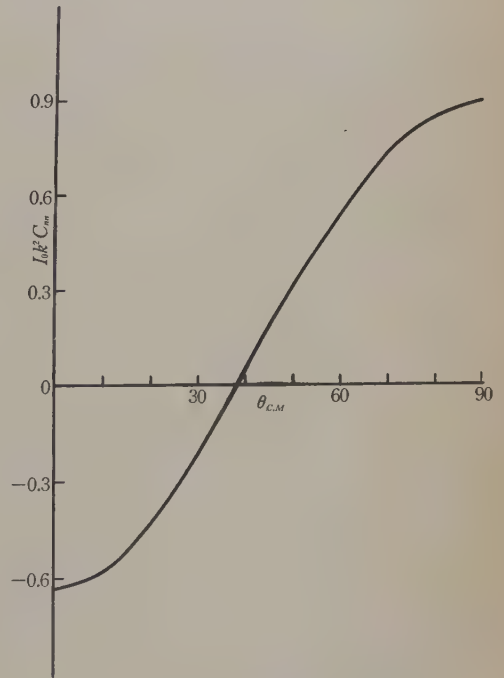
Fig. 3. Triple scattering parameters $I_0 k^2 R'(\theta)$ and $I_0 k^2 A'(\theta)$ for the SM1 potential at 150 Mev.

and a decrease in $\delta(^3P_0)$ increases this ratio. This point has a bearing on the attempts of Otsuki³⁾ and Watari⁴⁾ and Tamagaki⁵⁾ to fit the nucleon-nucleon scattering at 90 Mev and 150 Mev by 'purely central and tensor potentials. They have used the one pion exchange central and tensor potential to obtain a set of phase shifts and have then adjusted these phase shifts to some extent in order to obtain a fit of the cross-section and polarization. In the case of Tamagaki for the adjusted phase shifts, Eq. (16), the above ratio $\langle V_{LS} \rangle / \langle V_T \rangle$ is already $\sim 1/3$ (the value of this ratio for SM1 potential is ~ 1). Our analysis seems to suggest that such a large and positive $\delta(^3P_0)$ is very unlikely to give a positive depolarization and a fit of a positive depolarization with purely central and tensor potentials seems rather improbable.

iii) The parameters R, A, R' and A' : The curves for these parameters are given in Figs. 2 and 3. From Table I and Eqs. (A.3)–(A.7) one sees that the interference of the S and D waves with the P and F waves plays an important role in determining the values of the coefficients A_n . Thus the experimental values of these parameters will give information mainly about the 1S_0 and 1D_2 phase shifts, the former one will in turn tell us about the core region. In Table II we have computed the values of R and A at $\theta=0, \pi/2$ and π for the SM1 potential at 150 Mev and 310 Mev by an actual calculation of the matrices

Table II. Values of R and A at $\theta=0, \pi/2, \pi$ for SM1 at 150 Mev and 310 Mev.

SM1 at		$(k^2/4 \operatorname{Re}[\dots])$ $(M_{11}+M_{1-1})^*$	$(k^2/4 \operatorname{Re}[\dots])$ $(M_{33})^*$	Sum
150 Mev	$\left(I_0 k^2 \frac{R}{2 \cos \theta/2}\right)_0$	$[-.070+i.170]$ $(-.278-i.009)^* = .018$	$[-.278-i.009]$ $(-.158+i.805)^* = .037$.055
	$\left(I_0 k^2 \frac{R}{2 \cos \theta/2}\right)_{\pi/2}$	[] (0)*	$[-.056-i.985]$ $(-.084+i.064)^*$	-.055
310 Mev	$\left(I_0 k^2 \frac{R}{2 \cos \theta/2}\right)_0$	$[-.036-i.216]$ $(-.390-i.246)^* = .067$	$[-.390-i.246]$ $(-.199+i.100)^* = -.169$	-.102
	$\left(I_0 k^2 \frac{R}{2 \cos \theta/2}\right)_{\pi/2}$	[] (0)*	$[-.250-i.982]$ $(.099-i.466)^*$.433
150 Mev	$\left(I_0 k^2 \frac{A}{\sin \theta/2}\right)_{\pi/2}$	[] (0)*	$[-.099-i.167]$ $(-.084+i.064)^*$.011
	$\left(I_0 k^2 \frac{A}{\sin \theta/2}\right)_{\pi}$	$[-.278+i.009]$ $(.278+i.009)^* = -.078$	$[-.070-i.170]$ $(-.158+i.805)^* = .148$.070
310 Mev	$\left(I_0 k^2 \frac{A}{\sin \theta/2}\right)_{\pi/2}$	[] (0)*	$[-.017-i.076]$ $(.099-i.466)^*$	-.034
	$\left(I_0 k^2 \frac{A}{\sin \theta/2}\right)_{\pi}$	$[-.390+i.246]$ $(.390+i.246)^* = -.213$	$[-.036+i.216]$ $(-.199+i.100) = -.209$	-.422


 Fig. 4. Spin correlation parameter $I_0 k^2 C_{kp}(\theta)$ for the SM1 potential at 150 Mev.

 Fig. 5. Spin correlation parameter $I_0 k^2 C_{nn}(\theta)$ for the SM1 potential at 150 Mev.

M_{pq} , Eq. (1). The dependence of these parameters on M_{ss} , where $ik/2 \cdot M_{ss} = \frac{1}{2}\alpha_0 + \frac{5}{2}P_2(\cos\theta)\alpha_2$, is again obvious. The change in sign of $R(\pi/2)$ and $A(\pi/2)$ at 150 Mev and 310 Mev is mainly on account of $\text{Im}(ik/2 \cdot M_{ss})$ being $+0.064$ and -0.466 respectively. At $\theta = \pi/2$, $\delta(^1S_0)$ and $\delta(^1D_2)$ act destructively (at $\theta = 0$ and π , constructively) and a large positive $R(\pi/2)$ is an indication that $\delta(^1D_2) \sim$ or $> \delta(^1S_0)$ (at 310 Mev $\delta(^1D_2) = 11.5^\circ$, $\delta(^1S_0) = 1.3^\circ$), a large positive $R(\pi/2)$ (negative $A(\pi/2)$) requiring even a negative $\delta(^1S_0)$ thus suggesting a repulsive core in the singlet-even states. As an illustration, the curve $I_0 k^2 R_M$ in Fig. 2 shows the effect of modifying only the 1S_0 and 1D_2 phase shifts; the modified values of these phase shifts are those for the Gammel-Thaler potential at 150 Mev, viz. $\delta(^1S_0) = 11.7^\circ$, $\delta(^1D_2) = 8.7^\circ$.

iv) Spin correlation parameters C_{kp} and C_{nn} :—The curves for these parameters are shown in Figs. 4 and 5. It is interesting to note the dependence of $A_n(C_{kp})$, Eqs. (A.8) and (A.9), on the $\delta(^3P_0)$. For a very large (negative) $\delta(^3P_0)$, the coefficient $A_0(C_{kp})$ will become large and negative. This roughly explains why for solution 6 of Stapp et al, C_{kp} is negative¹³⁾ at $\theta = \pi/2$, which result is not in agreement with the measurement of C_{kp} at $E = 380$ Mev, $\theta = \pi/2$ at Liverpool¹⁴⁾.

Acknowledgements

I wish to express my thanks to Professor R. E. Marshak for leading me into this work and for several helpful discussions.

Appendix Expressions for the coefficients A_n

Using Eq. (4) and the values of the coefficients a^{pq} as obtained from Table III of Stapp et al,⁹⁾ we obtain from Eqs. (6)–(13) the following expression for the coefficients A_n occurring in Eqs. (6a)–(13a). We have made the approximation of neglecting all the coulomb effects and partial waves F (except 3F_2) and higher:

$$A_1(P) = g \left[-\frac{1}{2} \{10|12, 32\} + \frac{3}{4} \{11|12, 32\} - \frac{35}{16} \{12, 32|32\} \right] \quad (\text{A.1})$$

$$A_3(P) = -\frac{25}{16} g \{12, 32|32\}$$

$$\begin{aligned} A_0(D) = & \frac{3}{4} (11|10) + \frac{3}{8} \left(1 + \frac{1}{2} f_1 \right) (11|12, 32) + \frac{15}{16} (11|32) \\ & + \frac{3}{8} f_2 (12, 32|12, 32) + \frac{75}{32} (32|12, 32) + \frac{75}{32} (32|32) \end{aligned}$$

$$A_1(D) = \frac{3}{4} (11|0) + \frac{75}{32} (11|2) + \frac{1}{4} (3 - f_1) (12, 32|0)$$

$$\begin{aligned}
& + \frac{25}{32} (3-f_1) (12, 32|2) + \frac{5}{4} (32|0) + \frac{125}{32} (32|2) \\
A_2(D) = & \frac{9}{8} \left(1 + \frac{1}{2} f_1\right) (11|12, 32) + \frac{45}{16} (11|32) + \frac{1}{4} (3-f_1) (12, 32|10) \\
& + \frac{1}{8} f_2 (12, 32|12, 32) + \frac{5}{4} (32|10) + \frac{25}{16} (32|12, 32) + \frac{25}{16} (32|32)
\end{aligned} \tag{A.2}$$

$$A_3(D) = \frac{45}{32} (11|2) + \frac{15}{32} (3-f_1) (12, 32|2) + \frac{75}{32} (32|2)$$

$$A_4(D) = \frac{75}{32} (32|12, 32) + \frac{75}{32} (32|32),$$

$$A_n(R) = b_n^{11}(R) + b_n^{ss}(R) \tag{A.3}$$

where

$$\begin{aligned}
b_0^{11}(R) = & \frac{3}{8} g (11|12, 32) + \frac{1}{2} g h_4 (12, 32|12, 32) \\
& + \frac{5}{8} \left(3 - \frac{1}{4} h_2\right) (32|12, 32) = b_2^{11}(R) \\
b_0^{ss}(R) = & \frac{3}{4} (11|0) + \frac{15}{16} (11|2) - \frac{1}{4} g (12, 32|0) \\
& - \frac{5}{8} \left(\frac{1}{2} g + 3h_5\right) (12, 32|2) + \frac{75}{32} (32|2) \\
b_1^{11}(R) = & \frac{3}{4} (11|10) + \frac{3}{2} \left(h_1 + \frac{1}{2} h_3\right) (11|12, 32) + \frac{75}{32} (11|32) + h_4 (12, 32|10) \\
& + 2 \left(h_1 + \frac{1}{2} h_3\right) h_4 (12, 32|12, 32) + \frac{5}{4} (32|10) \\
& + \frac{5}{32} \left(7 + \frac{3}{2} h_2\right) (32|12, 32) + \frac{23}{8} (32|32) \\
b_1^{ss}(R) = & \frac{1}{2} g \left[(12, 32|0) + \frac{25}{8} (12, 32|2) \right] \\
b_2^{ss}(R) = & \frac{45}{16} (11|2) - h_5 (12, 32|0) - \frac{5}{4} \left(\frac{3}{4} g + h_5\right) (12, 32|2) \\
& + \frac{5}{4} (32|0) + \frac{25}{16} (32|2) \\
b_3^{11}(R) = & \frac{3}{4} h_3 (11|12, 32) + \frac{45}{32} (11|32) + h_3 h_4 (12, 32|12, 32)
\end{aligned} \tag{A.4}$$

$$+\frac{5}{32}\left(9+\frac{1}{2}h_2\right)(32|12, 32)+\frac{75}{32}(32|32)$$

$$b_3^{ss}(R)=\frac{15}{16}g(12, 32|2)$$

$$b_4^{11}(R)=0, \quad b_4^{ss}(R)=-\frac{15}{8}h_5(12, 32|2)+\frac{75}{32}(32|2),$$

$$A_n(A)=-\{b_n^{11}(A)+b_n^{ss}(A)\} \quad (\text{A} \cdot 5)$$

where

$$b_0^{11}(A)=\frac{1}{2}g\left[\frac{3}{4}(11|12, 32)+h_4(12, 32|12, 32)+\frac{5}{4}(32|12, 32)\right]=b_2^{11}(A)$$

$$b_0^{ss}(A)=-\frac{1}{2}(10|0)-\frac{5}{8}(10|2)-h_1(12, 32|0)-\frac{5}{4}\left(h_1+\frac{3}{2}h_3\right)(12, 32|2) \\ -\frac{5}{8}(32|0)-\frac{275}{64}(32|2)$$

$$b_1^{11}(A)=-\frac{9}{8}(11|11)+\frac{9}{4}h_5(11|12, 32)-\frac{45}{16}(11|32) \\ +\left(\frac{1}{2}g+h_5\right)h_4(12, 32|12, 32)+\frac{5}{4}\left(-\frac{1}{2}g+h_5-h_4\right)(32|12, 32) \\ -\frac{25}{16}(32|32) \quad (\text{A} \cdot 6)$$

$$b_1^{ss}(A)=b_1^{ss}(R), \quad b_3^{ss}(A)=b_3^{ss}(R)$$

$$b_2^{ss}(A)=-h_3(12, 32|0)-\frac{5}{4}(3h_1+h_5)(12, 32|2)-\frac{15}{8}(10|2) \\ -\frac{15}{8}(32|0)-\frac{75}{16}(32|2)$$

$$b_3^{11}(A)=\frac{3}{4}h_5(11|12, 32)-\frac{15}{16}(11|32)+h_4h_5(12, 32|12, 32) \\ +\frac{5}{4}(-h_4+h_5)(32|12, 32)-\frac{25}{16}(32|32)$$

$$b_4^{11}(A)=0, \quad b_4^{ss}(A)=-\frac{15}{8}h_3(12, 32|2)-\frac{225}{64}(32|2),$$

$$A_n(R')=(-1)^n\{b_n^{11}(R)-b_n^{ss}(R)\} \quad (\text{A} \cdot 7)$$

$$A_n(A')=(-1)^n\{b_n^{11}(A)-b_n^{ss}(A)\}$$

$$A_n(C_{kp})=\frac{1}{2}\{b_n^{01}(C_{kp})-b_n^{10}(C_{kp})\} \quad (\text{A} \cdot 8)$$

where

$$\begin{aligned}
 b_0^{01}(C_{kp}) &= \frac{9}{8}(11|11) - \frac{3}{4}g(11|12, 32) + \frac{1}{2}\left(\frac{1}{4}g^2 + 2h_5^2\right)(12, 32|12, 32) \\
 &\quad - \frac{5}{2}h_5(12, 32|32) + \frac{25}{16}(32|32) \\
 b_0^{10}(C_{kp}) &= \frac{1}{2}(10|10) + 2h_1(10|12, 32) + \frac{5}{4}(10|32) \\
 &\quad + \frac{1}{2}(4h_1^2 + 2h_3^2)(12, 32|12, 32) \\
 &\quad + \frac{5}{4}(2h_1 + 3h_3)(12, 32|32) + \frac{275}{64}(32|32) \\
 b_2^{01}(C_{kp}) &= -3h_5(11|12, 32) + \frac{15}{4}(11|32) + gh_5(12, 32|12, 32) \\
 &\quad - \frac{5}{4}g(12, 32|32)
 \end{aligned} \tag{A.9}$$

$$\begin{aligned}
 b_2^{10}(C_{kp}) &= 2h_3(10|12, 32) + \frac{15}{4}(10|32) + 4h_1h_3(12, 32|12, 32) \\
 &\quad + \frac{5}{2}(3h_1 + h_3)(12, 32|32) + \frac{75}{32}(32|32) \\
 b_4^{01}(C_{kp}) &= h_5^2(12, 32|12, 32) - \frac{5}{2}h_5(12, 32|32) + \frac{25}{16}(32|32) \\
 b_4^{10}(C_{kp}) &= h_3^2(12, 32|12, 32) + \frac{15}{4}h_3(12, 32|32) + \frac{225}{64}(32|32), \\
 A_n(C_{nn}) &= \frac{1}{4}b_n^{00}(C_{nn}) + \frac{1}{2}b_n^{01}(C_{nn}) + \frac{1}{2}b_n^{10}(C_{nn}) - b_n^{11,1-1}(C_{nn}) - \frac{1}{4}b_n^{ss}(C_{nn})
 \end{aligned} \tag{A.10}$$

$$\begin{aligned}
 b_0^{00}(C_{nn}) &= \frac{1}{2}(10|10) + k_1(10|12, 32) + \frac{25}{8}(10|32) \\
 &\quad + \frac{1}{2}(k_1^2 + h_3^2)(12, 32|12, 32) \\
 &\quad + \frac{5}{8}(5k_1 + 3h_3)(12, 32|32) + \frac{425}{64}(32|32), \\
 b_0^{01}(C_{nn}) &= \frac{1}{2}\left[b_0^{01}(C_{kp}) - \frac{1}{2}b_2^{01}(C_{kp})\right], \\
 b_0^{10}(C_{nn}) &= \frac{1}{2}\left[b_0^{10}(C_{kp}) - \frac{1}{2}b_2^{10}(C_{kp})\right],
 \end{aligned}$$

$$\begin{aligned}
b_0^{11,1-1}(C_{nn}) &= -\frac{3}{8}h_5(11|12, 32) + \frac{15}{32}(11|32) + \frac{1}{8}h_5(k_2-h_5)(12, 32|12, 32) \\
&\quad - \frac{5}{32}(k_2+h_5)(12, 32|32) + \frac{25}{64}(32|32) \\
b_0^{ss}(C_{nn}) &= (0|0) + \frac{5}{2}(0|2) + \frac{275}{32}(2|2) \\
b_2^{00}(C_{nn}) &= \frac{1}{2}(10|10) + (k_1+h_3)(10|12, 32) + 5(10|32) \\
&\quad + \frac{1}{2}k_1(k_1+2h_3)(12, 32|12, 32) \\
&\quad + 5\left(k_1+\frac{5}{8}h_3\right)(12, 32|32) + \frac{1375}{128}(32|32), \tag{A.11}
\end{aligned}$$

$$\begin{aligned}
b_2^{01}(C_{nn}) &= \frac{1}{2}\left[b_2^{01}(C_{kp}) - b_0^{01}(C_{kp}) - \frac{1}{2}b_4^{01}(C_{kp})\right] \\
b_2^{10}(C_{nn}) &= \frac{1}{2}\left[b_2^{10}(C_{kp}) - b_0^{10}(C_{kp}) - \frac{1}{2}b_4^{10}(C_{kp})\right] \\
b_0^{11,1-1}(C_{nn}) &= \frac{1}{8}h_5^2(12, 32|12, 32) - \frac{5}{16}h_5(12, 32|32) + \frac{25}{128}(32|32) \\
b_2^{ss}(C_{nn}) &= \frac{15}{2}(0|2) + \frac{75}{8}(2|2) \\
b_4^{00}(C_{nn}) &= h_3(10|12, 32) + \frac{15}{8}(10|32) + k_1h_3(12, 32|12, 32) \\
&\quad + \frac{5}{8}(3k_1+5h_3)(12, 32|32) + \frac{375}{64}(32|32), \\
b_4^{01}(C_{nn}) &= \frac{1}{2}\left[b_4^{01}(C_{kp}) - \frac{1}{2}b_2^{01}(C_{kp})\right] \\
b_4^{10}(C_{nn}) &= \frac{1}{2}\left[b_4^{10}(C_{kp}) - \frac{1}{2}b_2^{10}(C_{kp})\right] \\
b_4^{11,1-1}(C_{nn}) &= -b_0^{11,1-1}(C_{nn}), \quad b_4^{ss}(C_{nn}) = \frac{225}{32}(2|2)
\end{aligned}$$

where

$$\begin{aligned}
g(\epsilon_2) &= 3 \cos 2\epsilon_2 + \frac{\sqrt{6}}{4} \sin 2\epsilon_2 \\
f_1(\epsilon_2) &= \sin^2 \epsilon_2 + \sqrt{6} \sin 2\epsilon_2 \\
f_2(\epsilon_2) &= \cos 2\epsilon_2 \left(3 \cos 2\epsilon_2 + \frac{\sqrt{6}}{2} \sin 2\epsilon_2 \right) + \frac{1}{8} \sin^2 2\epsilon_2
\end{aligned}$$

$$h_1(\epsilon_2) = -\frac{1}{2}\cos^2\epsilon_2 + \frac{9}{8}\sin^2\epsilon_2 + \frac{\sqrt{6}}{16}\sin 2\epsilon_2 \quad (\text{A} \cdot 12)$$

$$h_2(\epsilon_2) = 24\sin^2\epsilon_2 - \sqrt{6}\sin 2\epsilon_2$$

$$h_3(\epsilon_2) = \frac{15}{8}\sin^2\epsilon_2 + \frac{5}{16}\sqrt{6}\sin 2\epsilon_2$$

$$h_4(\epsilon_2) = \frac{3}{4}\cos^2\epsilon_2 + \frac{1}{2}\sin^2\epsilon_2 - \frac{\sqrt{6}}{4}\sin 2\epsilon_2$$

$$h_5(\epsilon_2) = -\frac{5}{4}\sin^2\epsilon_2 + \frac{5}{16}\sqrt{6}\sin 2\epsilon_2$$

$$k_1(\epsilon_2) = 2\cos^2\epsilon_2 + \frac{9}{8}\sin^2\epsilon_2 + \frac{11}{16}\sqrt{6}\sin 2\epsilon_2$$

$$k_2(\epsilon_2) = -3\cos^2\epsilon_2 - \frac{3}{4}\sin^2\epsilon_2 + \frac{11}{16}\sqrt{6}\sin 2\epsilon_2$$

and

$$(A, B|C, D) = 4\sin(\delta_A - \delta_B)\sin(\delta_C - \delta_D)\cos(\delta_A + \delta_B - \delta_C - \delta_D),$$

$$\{A, B|C, D\} = 4\sin(\delta_A - \delta_B)\sin(\delta_C - \delta_D)\sin(\delta_A + \delta_B - \delta_C - \delta_D), \quad (\text{A} \cdot 13)$$

$\delta_0, \delta_2, \delta_{10}, \delta_{11}, \delta_{12}, \delta_{32}$ being the $^1S_0, ^1D_2, ^3P_0, ^3P_1, ^3P_2, ^3F_2$ phase shifts respectively.

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On the Derivation of the Optical Potential in Infinite Nuclear Matter

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The optical potential, for a nucleon in infinite nuclear matter, is derived, all nucleons and interactions being symmetrically treated. The exchange effects are exhibited; a previously neglected exchange graph, which has an important effect for the real part at low energy, is discussed and numerically computed.

§ 1. Introduction and general formalism

The present note describes a derivation of the optical potential for a nucleon propagating through infinite nuclear matter, all nucleons and interactions being symmetrically treated. Such a formalism exhibits more clearly the exclusion effects, some of which will be expressly studied in the following.

An S -matrix element for elastic scattering can be defined between states $|\mathbf{k}\rangle$, consisting of the target in the state $|0\rangle$ of a Fermi gas without interactions ("bare vacuum"), plus an incident particle in a plane wave state of momentum \mathbf{k} . All interactions V are adiabatically switched on and off, producing both the scattering process and the building up of the real target with interactions ("real vacuum"). The S -matrix element is given by a series expansion

$$\langle \mathbf{k}' | S | \mathbf{k} \rangle = \left\langle \mathbf{k}' \left| V \sum_{n=0}^{\infty} \left(\frac{1}{E - K + i\epsilon} V \right)^n \right| \mathbf{k} \right\rangle, \quad (1)$$

where K is the kinetic energy operator of the system, and E its initial value. (1) can be represented by a series of graphs,¹⁾ some of which have unlinked parts. It is however well known²⁾ that

$$\langle \mathbf{k}' | S | \mathbf{k} \rangle = \langle \mathbf{k}' | S | \mathbf{k} \rangle_L \langle 0 | S | 0 \rangle, \quad (2)$$

where L means that the sum is to be restricted to linked graphs in expansion (1); we also know that $\langle 0 | S | 0 \rangle$ represents the vacuum to vacuum amplitude (it has a modulus unity), so that $\langle \mathbf{k}' | S | \mathbf{k} \rangle_L$ is actually the observed particle to particle amplitude when we do not consider the unphysical change of phase of the unperturbed target.

The most general graph in $\langle \mathbf{k}' | S | \mathbf{k} \rangle_L$ is drawn in Fig. 1, and can be obtained by the iteration

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$$\langle \mathbf{k}' | S | \mathbf{k} \rangle_L = \left\langle \mathbf{k}' | \mathcal{V} \sum_{n=0}^{\infty} \left(\frac{1}{E - K + i\epsilon} \mathcal{V} \right)^n | \mathbf{k} \right\rangle \quad (3)$$

of the irreducible part

$$\langle \mathbf{k}' | \mathcal{V} | \mathbf{k} \rangle = \left\langle \mathbf{k}' | V \sum_{n=0}^{\infty} \left(\frac{Q}{E - K + i\epsilon} V \right)^n | \mathbf{k} \right\rangle, \quad (4)$$

where Q is the projector outside the one-particle states. Eq. (4) defines a one-body potential \mathcal{V} which produces the same elastic scattering matrix (3) as the complete interaction V . Therefore \mathcal{V} is the optical potential.

Because of momentum conservation in infinite nuclear matter, (3) is actually of the form $(2\pi)^3 \delta(\mathbf{k} - \mathbf{k}') \mathcal{V}(k)$. k is the wave number of the incident nucleon when it is already inside the nuclear matter, and it is not evident whether k can be unambiguously related to the kinetic energy \mathcal{E} of the incident nucleon outside the target. Here we just put

$$\mathcal{E} = k^2/2M + \text{Re } \mathcal{V}(k) \quad (5)$$

where M is the nucleon mass.

The first- and second-order graphs of (4) are drawn in Fig. 2. Graphs (1a), (1b), (2a), (2b) depict the interaction of the incident nucleon with one of the target nucleons.

These graphs have been previously studied,³⁾ and their iterations, which are the Brueckner t -matrix, have also been considered.⁴⁾ On the other hand, graphs (2c) and (2d) describe modifications in the mutual interaction of two target particles,

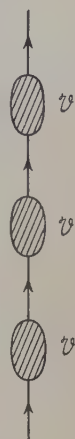


Fig. 1. The general term of $\langle \mathbf{k}' | S | \mathbf{k} \rangle_L$.

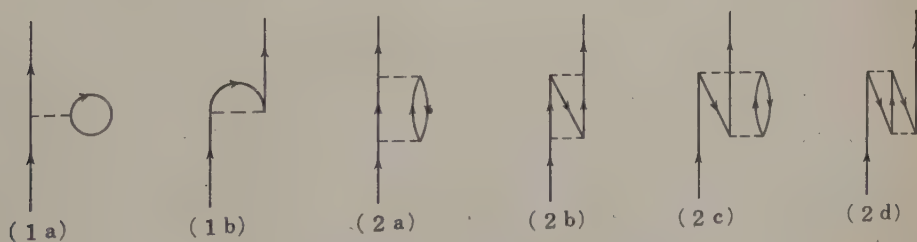


Fig. 2. The first and second order graphs.

for which the incident particle state is no longer available as an intermediate state because of the exclusion principle. These latter graphs did not appear in the treatments^{3,4)} which neglected the interactions between the target particles. The legitimacy of neglecting these graphs in the Brueckner theory has already been questioned.⁵⁾ It may however be noted that these graphs (2c) and (2d) contribute only to the real part of the optical potential, $\text{Re } \mathcal{V}$, because the energy denominators

cannot vanish: the energy of the excluded intermediate state under consideration is always higher than the initial state energy.

§ 2. Practical computation

We shall now estimate these graphs. The technique of the computation is the same as in Verlet.³⁾ The Fourier transform of the two-body interaction is $\mathcal{O}J(K)$, where \mathcal{O} is a linear combination of the operators 1, P_σ (spin exchange), P_τ (isotopic spin exchange), $P_\sigma P_\tau$. We now call k_N the momentum of the incident nucleon. The contributions to $\mathcal{V}(k_N)$ of graphs (2c) and (2d) are respectively

$$\mathcal{V}^{(2c)}(k_N) = \frac{\text{Tr } \mathcal{O}^2}{2} \frac{M}{(2\pi)^6} \int_{>} d^3 \mathbf{k} \int_{<} d^3 \mathbf{k}' \int_{<} d^3 \mathbf{k}_N' \frac{J^2(|\mathbf{k}_N' - \mathbf{k}_N|)}{k_N'^2 + k^2 - k_N'^2 - k'^2} \delta(\mathbf{k}_N' + \mathbf{k}' - \mathbf{k}_N - \mathbf{k}) \quad (6)$$

and

$$\mathcal{V}^{(2d)}(k_N) = \frac{\text{Tr } \mathcal{O}^2 P_\sigma P_\tau}{2} \frac{M}{(2\pi)^6} \int_{>} d^3 \mathbf{k} \int_{<} d^3 \mathbf{k}' \int_{<} d^3 \mathbf{k}_N' \frac{J(|\mathbf{k}_N - \mathbf{k}'|) J(|\mathbf{k}_N - \mathbf{k}_N'|)}{k_N'^2 + k'^2 - k_N'^2 - k^2} \times \delta(\mathbf{k}_N' + \mathbf{k}' - \mathbf{k}_N - \mathbf{k}), \quad (7)$$

where $>$ and $<$ mean integration on momenta above and below the Fermi momentum k_0 .

The exchange operator \mathcal{O} was chosen so that

$$\text{Tr } \mathcal{O}^2 P_\sigma P_\tau = 0, \quad (8)$$

and consequently $\mathcal{V}^{(2b)}$ and $\mathcal{V}^{(2e)}$ vanish. To compute $\mathcal{V}^{(2c)}$, we choose as integration variables $\mathbf{K} = \mathbf{k}_N' - \mathbf{k}_N$, \mathbf{k} , \mathbf{k}' ; after having performed the integrations on \mathbf{k}' , \mathbf{k} , and the angles of \mathbf{K} , we obtain

$$\mathcal{V}^{(2c)}(k_N) = \text{Tr } \mathcal{O}^2 (M/2^7 \pi^4) k_0 \int_{k_N - k_0}^{k_N + k_0} dK K^2 C(K, k_N) J^2(K), \quad (9)$$

where $C(K, k_N)$ is a function which is given in the Appendix. The integral (9) must be performed numerically.

We used two types of nuclear forces, a gaussian force and a Yukawa force, with the same parameters as in Ref. 3).^{*} The various contributions, including results from Ref. 3), are listed in Table I.

^{*} The parameters of the gaussian force as quoted in Ref. 3) (Nuovo Cimento) are inadequate because of some confusion in the units. The force which has actually been used both in Ref. 3) and in the present paper is $\mathcal{O}A \exp(-\alpha^2 r^2)$, where $A=72$ Mev and $\alpha=0.674 \times 10^{13} \text{cm}^{-1}$; the Fermi momentum was $k_0=1.2 \times 10^{13} \text{cm}^{-1}$. We are indebted to Dr. Verlet for kindly clarifying this point to us.

The exchange dependence, as stated in Ref. 3), involves an important Majorana part.

Table I. The contributions to $\text{Re } \mathcal{V}(k)$, in Mev.

k/k_0	\mathcal{V} (Mev)	Gaussian interaction			Yukawa interaction		
		(1a) + (1b)	(2a)	(2c)	(1a) + (1b)	(2a)	(2c)
1	-21	-44	-15	8	-45	-13	7
1.5	29 ± 1	-28	-12	1.5	-27	-11	0.9
2	90	-20	-9	0.2	-21	-8	0.2

§ 3. Discussion

It appears from Table I that, for a low energy incident particle, graph (2c) brings a correction to $\text{Re } \mathcal{V}$; although relatively small, this correction is comparable with the other second-order terms and should not be neglected in a consistent second-order calculation.* For an incident particle of higher energy, the exclusion effect (2c) becomes negligible, because, with the regular interactions which were used here the occurrence of self-excitations of the target to high momenta is unlikely. In real nuclear matter, there are however high momenta, due to the actual singular nuclear forces, and it is possible that graph (2c) and higher-order graphs of the same kind would be important at fairly high energies.

In problems in which the first-order graphs vanish, the second-order graph (2c) may happen to be of extreme qualitative importance. Such would be the case in the calculation of the spin-orbit part of the optical potential, from a tensor force interaction.^{6,7)}

Appendix

$C(K, k_N)$ is defined by

$$2\pi^2 k_0 C(K, k_N) = \int_{|\mathbf{K} + \mathbf{k}_N| < k_0} d\Omega_{\mathbf{K}} \int_{\substack{|\mathbf{k} - \mathbf{K}| < k_0 \\ k > k_0}} \frac{d^3 \mathbf{k}}{(\mathbf{k} - \mathbf{k}_N) \cdot \mathbf{K} - K^2}. \quad (\text{A} \cdot 1)$$

— If $k_N - k_0 < K < 2k_0$,

$$\begin{aligned} C(K, k_N) = & (1/12 k_0 k_N K^2) [K^4 - 4(k_0 + k_N) K^3 + (4k_N + 8k_0 k_N) K^2 \\ & + 4k_0(k_0^2 - k_N^2) K - (k_N^2 - k_0^2)^2] \\ & + (1/3 k_0 k_N K) [-K^3 + 3k_N K^2 + 3(k_0^2 - k_N^2) K + k_N^3 - 3k_0^2 k_N - 2k_0^3] \\ & \text{Log} | (k_N + k_0) K - K^2 | \\ & + (1/24 k_0 k_N K^4) [K^6 - (3k_N^2 + 9k_0^2) K^4 + 16k_0^3 K^3 + (3k_N^4 + 6k_0^2 k_N^2 \\ & - 9k_0^4) K^2 - (k_N^2 - k_0^2)^3] \text{Log} | (-K^2 + 2k_0 K + k_N^2 - k_0^2) / 2 | \end{aligned}$$

* A former estimate of graph (2c) in Ref. 5) gave 12 Mev, for a charge and spin independent Yukawa force. The result is very sensitive to the exchange dependence of the interaction.

$$\begin{aligned}
& + (1/4 k_0 k_N) (K^2 - 4k_N K + 4k_N^2) \text{Log} | (K^2/2) - k_N K | \\
& - (1/4 k_0 k_N K^2) (k_N^2 - k_0^2)^2 \text{Log} | (k_N^2 - k_0^2)/2 | \\
& + (1/3 k_0 k_N K) (-k_N^3 + 3k_0^2 k_N + 2k_0^3) \text{Log} | (k_N + k_0) K | \\
& + (1/24 k_0 k_N K^4) [K^6 + (3k_N^2 - 15k_0^2) K^4 - 16k_0^3 K^3 \\
& + (k_N^2 - k_0^2) (3k_N^2 - 15k_0^2) K^2 + (k_N^2 - k_0^2)^3] \text{Log} | (K^2 + 2k_0 K + k_N^2 - k_0^2)/2 |.
\end{aligned} \tag{A.2}$$

— If $2k_0 < K < k_N + k_0$,

$$\begin{aligned}
C(K, k_N) = & (1/6 k_N K^3) [-K^4 + 2(k_N^2 + k_0^2) K^2 - (k_N^2 - k_0^2)^2] \\
& + (1/3 k_0 K) (k_N^2 - 3k_0^2) \text{Log} | (k_N - k_0) / (k_N + k_0) | \\
& + (2k_0^2/3 k_N K) \text{Log} | (k_N^2 - k_0^2) K^2 / [(K^2 - 2k_0 K + k_N^2 - k_0^2) (K^2 + 2k_0 K + k_N^2 - k_0^2)] | \\
& + (1/24 k_0 k_N K^4) [-K^6 + (15k_0^2 - 3k_N^2) K^4 + (-3k_N^4 + 18k_0^2 k_N^2 - 15k_0^4) K^2 \\
& - (k_N^2 - k_0^2)^3] \text{Log} | (K^2 - 2k_0 K + k_N^2 - k_0^2) / (K^2 + 2k_0 K + k_N^2 - k_0^2) |.
\end{aligned} \tag{A.3}$$

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Bremsstrahlung in High Density Media at High Energies

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Qualitative discussions are presented concerning the effects of the collective motion of media for the bremsstrahlung process in high density at high energies. The derived formula of the bremsstrahlung cross section has similar structure as the Nozieres-Pines' cross section of Compton scattering obtained by taking into account effects of the collective motion of electrons.

§ 1. Introduction

In the past few years, much interest has been focused on the bremsstrahlung and pair production processes taking place in high density media at very high energies. As it is well known,¹⁾ the characteristic feature of the bremsstrahlung of high energy electron is determined by the magnitude of recoil momentum q , which defines an effective interaction range r_0 as follows,

$$r_0 = (2\lambda_0 E_0 E / \mu k) \quad (1)$$

where E_0 is incident energy of the electron, k is energy of an emitted quantum, λ_0 is the Compton wave length and μ is the rest energy of the electron. The energy of outgoing electron E is equal to $E_0 - k$. If $k \sim E$, the range r_0 becomes of the order of magnitude of the radius of the K-shell for $E_0 \sim 137\mu/Z$. Thus, the emission of the quantum is influenced by the outer electrons distributed in the region of the radius r_0 . In the usual treatment of the problem¹⁾, the effects of the outer electrons are taken into account as the screening of the Coulomb potential of atomic nucleus.

Landau and Pomeranchuk²⁾ have pointed out that at an extremely high energy the range r_0 becomes so large that the multiple scattering taking place over a distance of r_0 may disturb the emission of the bremsstrahlung; and lately Migdal³⁾ has developed a quantum theory of the Landau-Pomeranchuk effect. Experimental studies of the effect⁴⁾ appear to be in agreement with the theoretical prediction.

Dyson and Uberall,⁵⁾ on the other hand, have discussed a possible crystal interference effect which may be observed in the spectrum of bremsstrahlung emitted from a single crystal. The effect may be appreciable even at the energy of an incident electron of several hundreds Mev, since the r_0 covers a region involving several atoms of the crystal. Uberall,⁶⁾ using the Born approximation, has predicted

the amount of the enhancement of the interference effect. The preliminary report of Panofsky and Saxena⁷⁾ has revealed that there is complete suppression of the expected interference effect. Recently, on the other hand, Frisch and Olson⁸⁾ have detected an enhancement of the bremsstrahlung emitted from a single crystal of germanium, and have assured themselves that the observed enhancement is the Dyson-Uberall effect, though the enhancement seems to be greatly reduced from the predicted amount of the effect.

The purpose of the present paper is to discuss another kind of effect of the high density media for the bremsstrahlung process at high energies. The effective interaction range r_0 is determined by the recoil momentum q , and thence the process depends on details of the mechanism of momentum transfer between the incident electron and the target. In the usual theory of the bremsstrahlung process, the target is an isolated atom and it is treated as an external field to take care of the momentum q . In the high density media, the momentum q can be transferred either to a single atom of the media, or it can be shared by many atoms, namely by a collective mode of the system. If the momentum transfer q is less than a critical momentum q_c , the collective motion will be excited by the momentum transfer. In the following sections, we will discuss the inter-particle correlation effects for the bremsstrahlung process taking place in high density media.

§ 2. Interaction between collective mode of medium and electron

Theoretical investigations on dynamical behaviours of the many-particle system have established various kinds of the collective motion of the systems. These collective motions will be excited by a suitable external disturbance. For the present problem, in which collision between the incident electron and heavy particle of the medium plays the fundamental role, the collective mode of heavy particles is an essential one.

In a non-crystalline state of matter, whether it may be in a phase of gas, liquid or solid, the collective motion of atoms is determined through a short range interaction between atoms. In fully ionized plasma, the collective motion of heavy ion is determined through a screened Coulomb interaction. The general theory of the collective motion of many particle system can show that these collective motions of heavy particles interacting through a short range force can be described in terms of a collection field, of which normal coordinate is the density fluctuation, having an eigen frequency $\omega(q) = uq$, where u is the sound velocity of the medium. In the usual treatment of metals, the ionic motion is described by the lattice vibration around the equilibrium position. At the long wave-length limit, however, it seems to be justified to use the plasma model of metals, in which the ionic motion is described in terms of the density fluctuation. In fact, Zyrianov⁹⁾ has been able to develop a successful theory of the electric conductivity of metals on a basis of the plasma model of metals. (Appendix)

According to the studies of the dynamical behaviour of many particle system,¹⁰⁾

we can write down the interaction Hamiltonian between the incident electron and the target medium as follows,

$$H_I = \sum_{q < q_c} \left(\frac{n\hbar}{2M\omega} \right)^{1/2} q V(q) \{ a_q e^{-iqx} + a_q^* e^{iqx} \} + \sum_{q > q_c} V(q) \sum_i e^{iq \cdot (x - x_i)} \quad (2)$$

where $V(q)$ is the Fourier transform of the interaction potential of the incident electron and a single atom of the target. n is the number density of atoms of the medium. M is the mass of the atom. Here we are considering a unit volume of the medium for convenience. a_q and a_q^* are the destruction and creation operator of quantum of the collective motion, respectively. The eigen frequency of the collective motion is given as

$$\omega = (nU(q)/M)^{1/2} q \quad (3)$$

where $U(q)$ is the Fourier transform of the interaction potential between atoms of the medium. The critical momentum q_c must be determined by detailed investigations of the dynamical behaviour of the medium. For a solid, though relationship between the collective mode and the lattice-vibrational mode is not clear, it may be reasonable to take the q_c as an order of the Debye cut-off ($\kappa\theta_d/\hbar u$), where θ_d is the Debye temperature of the solid and κ is the Boltzmann constant. For a completely ionized classical plasma, q_c is determined to be the order of $(4\pi e^2 n / \kappa T_e)^{1/2}$ where T_e is the electron temperature, provided that the ion temperature is lower than the electron temperature.

§ 3. Differential cross section of bremsstrahlung in medium

Now, let us calculate the differential cross section of the bremsstrahlung in the medium. In the interaction Hamiltonian Eq. (2), the part of $q > q_c$ describes the individual collision between the electron and the atom. Thus, our main problem is to calculate the matrix element of the process using the part of $q < q_c$.

As discussed in the preceding sections, the bremsstrahlung process in the region of $q < q_c$ accompanies the emission and absorption of the collecton. A straightforward calculation yields the following formula of the differential cross section of the bremsstrahlung per one atom,

$$\begin{aligned} d\phi(k; q) = & \frac{R(q)}{137} \cdot \frac{PEE_0}{P_0} \cdot \frac{V(q)}{(4\pi^2)^2} d\Omega d\Omega_k k dk \\ & \times \left\{ \left| \sum \left(\frac{(U^* U') (U'^* \alpha U)}{E - E'} + \frac{(U^* \alpha U'') (U''^* U)}{E_0 - E'' + \hbar\omega} \right) \right|^2 n(q) \right. \\ & \left. + \left| \sum \left(\frac{(U^* U') (U'^* \alpha U)}{E - E'} + \frac{(U^* \alpha U'') (U''^* U)}{E_0 - E'' - \hbar\omega} \right) \right|^2 (n(q) + 1) \right\} \quad (4) \end{aligned}$$

where U, U^* are Dirac's spinors, $n(q)$ is an occupation number of the collecton

in the initial state and $R(q)$ is given by

$$R(q) = \left(\frac{\hbar^2 q^2}{2M} \middle| \hbar\omega \right). \quad (5)$$

At the temperature T , $n(q)$ may be given as

$$n(q) = \{\exp(\hbar\omega/\kappa T) - 1\}^{-1}. \quad (6)$$

In the denominators of Eq. (4), we can disregard the collecton energy $\hbar\omega$, considering the fact that $u/c \sim 10^{-5}$. Thus, we can finally write the effective differential cross section of the bremsstrahlung per one atom at $T=0$ as follows:

$$d\phi(k; q) = R(q) d\phi_{B-H}(k; q), \quad q < q_c \quad (7)$$

where $d\phi_{B-H}(k; q)$ is the Bethe-Heitler formula of the bremsstrahlung process. It is interesting to observe that the form of Eq. (7) has a similar structure as the cross section of Compton scattering in the electron gas obtained by Nozieres and Pines.¹¹⁾

§ 4. Concluding discussions

Obtaining the differential cross section in the region of $q < q_c$ as Eq. (7), we may derive the energy spectrum of the bremsstrahlung. At the extreme relativistic limit, according to Bethe's calculation, we get the energy spectrum of the bremsstrahlung as follows:

$$\begin{aligned} \phi_\kappa d\left(\frac{k}{E_0}\right) = & 4 \cdot \frac{Z^2}{137} \left(\frac{e^2}{\mu}\right)^2 \frac{dk}{k} \frac{E}{E_0} \left\{ \left(\frac{E_0^2 + E^2}{E_0 E} - \frac{2}{3} \right) \right. \\ & \times \left[\int_{\delta}^{q_c} (1 - F(q))^2 R(q) q^{-1} dq + \int_{q_c}^{mc} (1 - F(q))^2 q^{-1} dq + 1 \right] + 1/9 \Big\} \end{aligned} \quad (8)$$

where $F(q)$ is a form factor of the atom. The term with the range of integration between $\delta = (\mu^2 k / 2E_0 E c)$ to q_c represents the contribution of the process of bremsstrahlung accompanying the emission of the collecton. It turns out to be completely negligible as compared with other terms, since the factor $R(q)$ is less than the order of magnitude of 10^{-2} . The difference between our formula and the Bethe-Heitler formula for an isolated atom is apparent in the fact that the contribution arising from the range of the momentum transfer between δ and q_c has been cut down completely. This effect, however, is not appreciable as far as we are considering of a solid, since the screening of atomic potential due to the outer electrons suppresses the contribution of the momentum range between δ and q_c .

Finally, let us briefly discuss Uberall's calculation of the Dyson-Uberall effect. Uberall has used for a whole range of q an interaction Hamiltonian

$$H_I = \sum_q V(q) \sum_i e^{-iq(x-x_i)} \quad (9)$$

where \mathbf{x}_i is the position vector of the atom at the i -th lattice point. In the Born approximation, the interference effect appears through a multiplicative factor $|\sum \exp(i\mathbf{q} \cdot \mathbf{x}_i)|^2$. Using Eq. (2) instead of Eq. (9), we have recalculated the Dyson-Uberall effect and found that the effect of inter-particle correlation is not at all appreciable. Therefore, we have to seek for other effects to explain the possible suppression of the enhancement observed by Frisch and Olson.

In conclusion, it is our great pleasure to acknowledge Professor M. Kimura for calling our attention to this problem and we are much obliged to Professor K. Nakabayashi and Professor I. Sato for their constant encouragement and critical discussions through our investigations.

Appendix

Here, we shall comment on the plasma model of solids in comparison with the usual treatment of the lattice vibration of solids.

Let the inter-particle interaction be

$$U = \frac{1}{2} \sum_{i \neq j} U(\mathbf{x}_i - \mathbf{x}_j) = \frac{1}{2} \sum_{\mathbf{q}} \sum_{i \neq j} U(\mathbf{q}) e^{i\mathbf{q}(\mathbf{x}_i - \mathbf{x}_j)}. \quad (\text{A} \cdot 1)$$

In the plasma model of solids, according to the Tomonaga theory of the collective motion,¹⁰⁾ we can show that the normal coordinate $Q(\mathbf{q})$, which is defined by

$$Q(\mathbf{q}) = \left(\frac{M}{n} \right)^{1/2} \sum_i \frac{1}{q} e^{i\mathbf{q} \cdot \mathbf{x}_i} \quad (\text{A} \cdot 2)$$

executes a simple harmonic oscillation with the eigen frequency

$$\omega = (nU(\mathbf{q})/M)^{1/2} q. \quad (\text{A} \cdot 3)$$

If the interaction is a short range force, the density fluctuation $Q(\mathbf{q})$ describes the sound-wave with the propagating velocity given as

$$V_s = (nU(\mathbf{q}=0)/M)^{1/2} \quad (\text{A} \cdot 4)$$

at the long wave-length limit. The interaction Hamiltonian Eq. (2) is derived by using the expression (A.2).

Contrary to the above deduction of the sound velocity, in the usual treatment of the lattice vibration, the sound velocity is determined as

$$V_l = (a^2 \sum_{\mathbf{q}} U(\mathbf{q}) q^2 / (2M))^{1/2} \quad a: \text{distance between atoms} \quad (\text{A} \cdot 5)$$

for the one-dimensional case. The expression (A.5) is derived under the nearest neighbour approximation. For the longitudinal lattice vibration, the sound velocity depends on the value of second derivative of the potential at the origin. While in the plasma model of the solids, the sound velocity is determined by behaviour of the potential at the far distance, and thus it is not sensitive to detailed features of the potential. In fact, assuming a screened Coulomb potential of the atom,

$$V(r) = Ze^2 r^{-1} \exp(-r/a), \quad a = (\hbar^2/me^2 Z^{1/3}), \quad (\text{A} \cdot 6)$$

we get $V_c = (4\pi nZe^2a^2/M)^{1/2}$, which gives nearly correct values of the sound velocity for various materials.

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Spin-Orbit Splitting and Tensor Force. I*

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The effect of the tensor force on the spin-orbit splitting in He^5 and N^{15} is examined by using the meson-theoretic potential and the phenomenological Serber potential which are consistent with the experimental data of two nucleon systems. About a half of the experimental values of the spin-orbit splitting in the mentioned nuclei are obtained by the accurate computation of the second order effect in perturbation theory, whereas several previous calculations of this effect have yielded the splitting of wrong sign or of too small magnitude.

As is pointed out in the present computation, the deformation of the closed shell core induced by the tensor interaction between the nucleons in the core is restricted so as to satisfy the Pauli principle with the outside nucleon. This restriction is mainly responsible for the present result of splitting energy.

§ 1. Introduction

The origin of the spin-orbit coupling in the nuclear shell model and the nucleon-nucleus polarization experiments seems to be obscure at present. The Thomas term due to the relativistic effect¹⁾ and the spin-orbit force²⁾ induced from the many-body force have been found to be too small. Recently, a rather strong spin-orbit force of the two-body type has been introduced phenomenologically in the analysis of the high energy nucleon-nucleon scattering.³⁾ In a certain approximation, such spin-orbit force of two-body type has been shown to lead to that of one-body type available in the nuclear shell model.⁴⁾ The field-theoretic derivation of this force, however, is still unsuccessful, because the spin-orbit force predicted by the present-day meson theory seems to be very small.⁵⁾ On the other hand, it has been shown by Japanese group⁶⁾ that the meson-theoretic potential without the explicit term of spin-orbit force can successfully explain not only the low energy data of the two-nucleon system but also the high energy one. One of the typical properties of this potential is that its tensor part is strong and has a Serber-like exchange character. Then, it might be expected that the spin-orbit force of the one-body type could be obtained from the meson theoretic tensor force or the phenomenological strong tensor force of the Serber type. Indeed Wigner and Feingold^{7) 7 bis)} showed first that an appreciable part of the spin-orbit splitting in light nuclei could be

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explained by the tensor force, but their tensor force was not so strong as the meson theoretic one. In the same spirit as above, several other calculations⁸⁾ have been made, some of which have contained undesirable approximations, leading to the doublet splitting of wrong sign or of too small magnitude. Recently, the appreciable doublet splitting was also obtained in the calculations using the Fermi gas model,^{9,10)} and in the calculations of the p -wave phase shifts in the n -He⁴ scattering, by Sugie, Hodgson and Robertson,¹¹⁾ and by Nagata, Sasakawa, Sawada and Tamagaki¹²⁾ who used the meson theoretic potential. The comparison of the present calculation with those ones will be discussed.

The purpose of the present paper is to reexamine the second order perturbation effect of the tensor force on the spin-orbit splitting, using the procedure which enables us to compute with sufficient accuracy the second order perturbation energy, and adopting the meson theoretic potential and phenomenological one of the Serber type with the strong tensor part. It has been found that about a half of the experimental values of the doublet splittings in He⁵ and N¹⁵ can be derived by our computation. The main effect responsible for the present result of the doublet splitting is as follows. In the nuclei of the zeroth order configuration, closed shell + one nucleon (nl), the intermediate configurations of $[\text{closed shell}]^{-2} \cdot (nl)^2 \cdot (n'l')$ and $[\text{closed shell}]^{-2} \cdot (nl)^3$ play a decisive role in producing the doublet splitting from the tensor force. In these intermediate configurations, one or two of the two interacting nucleons in the closed shell can jump into the nl -orbit (see Fig. 1c, d). A certain sub-orbit of (nl), however, is already occupied by the outmost nucleon, i.e., the nucleon outside the core, and then the jumping nucleons cannot get into the sub-orbit of (nl) on account of the Pauli principle. That is, the corresponding intermediate substates belonging to the above configurations should be forbidden. The effect of this prohibition is stronger in the $J=l-1/2$ state than in the $J=l+1/2$ state, as illustrated in § 3, so the depression of the state due to the second order perturbation is smaller in the former state than in the latter, and this fact gives rise to the decisive influence upon the final result of the doublet splitting. In other words, the closed shell core is deformed by the tensor interaction between the nucleons in the core, and this deformation is restricted so as to satisfy the Pauli principle with the outmost nucleon. This restriction is stronger in the $J=l-1/2$ state than in the $J=l+1/2$ state. A disregard or underestimate of the mentioned effect is the reason why many of the previous calculations⁸⁾ of the effect of the tensor force on the spin-orbit splitting gave the wrong results. The exchange effect^{11),12)} in the n -He⁴ scattering corresponds to the above mentioned effect of the Pauli principle and has been responsible for getting the appreciable doublet splitting.

In § 2, the method of the calculation is described, and the doublet splittings in He⁵ and N¹⁵ are calculated in § 3, and 4. In § 5 the comparison of the present calculation with the previous ones, especially with Feingold's one, is discussed, and finally the conclusion and general discussion are described in § 6.

§ 2. The method of calculation

The total Hamiltonian of the nuclear system is given by

$$H = - \sum_i \frac{\hbar^2}{2m} \nabla_i^2 + V_c + V_T, \quad (2.1)$$

where m is the nucleon mass, and $V_c = \sum_{i>j} v_c(\mathbf{r}_{ij})$ and $V_T = \sum_{i>j} v_T(\mathbf{r}_{ij})$ are the central and tensor part of the two-body potential, respectively. As the zeroth order system in perturbation theory, we take the system of the independent harmonic oscillators:

$$H_0 = - \sum_i \frac{\hbar^2}{2m} \nabla_i^2 + \sum_i \frac{1}{2} m \omega^2 r_i^2$$

$$H_0 \phi_n = E_n \phi_n, \quad (2.2)$$

where ω is the angular frequency of the oscillator, and E_n and ϕ_n are the energy and wave function of the n -th state of H_0 . In H_0 the centre-of-mass coordinates are not separated, but this separation has an important effect, particularly in the light nuclei. Allowing for this situation, therefore, we have introduced Hamiltonian H' following the procedure adopted by Bolsterli and Feenberg;¹⁸⁾

$$\begin{aligned} H' &= H + \frac{1}{2} m A \omega^2 R^2 \\ &= H_r + \frac{\hbar^2}{2m A} \nabla_R^2 + \frac{1}{2} m A \omega^2 R^2 \\ &= H_0 + \sum_{i>j} \left\{ v_c(\mathbf{r}_{ij}) + v_T(\mathbf{r}_{ij}) - \frac{m}{2A} \omega^2 r_{ij}^2 \right\} \\ &\equiv H_0 + W, \end{aligned} \quad (2.3)$$

where A is the number of the nucleons in this system, \mathbf{R} the centre-of-mass coordinate and H_r the Hamiltonian of the internal motion. In H' , the perturbation W cannot change the centre-of-mass motion, because W depends only on the relative coordinate \mathbf{r}_{ij} . Therefore, the perturbation energy of H' , i.e.,

$$E - E_0 = W_{00} + \sum_n' \frac{W_{0n} W_{n0}}{E_0 - E_n} + \sum_{m,n}' \frac{W_{0m} W_{mn} W_{n0}}{(E_0 - E_m)(E_0 - E_n)} + \dots, \quad (2.4)$$

becomes equal to the perturbation energy of H , if the zeroth order wave function is taken to be an eigenstate of the centre-of-mass motion.

Thus, Eq. (2.3) can be adopted for the calculation of the second order energy. It would be very difficult to get a closed expression for the second order perturbation series on the right-hand side of (2.4), after calculating each matrix element. Bolsterli and Feenberg¹³⁾ proposed the interesting method which enables us to get a closed form for the second order energy if the harmonic oscillator wave function is used. However, this method has not successfully been applied to the heavier nuclei than He^4 . So we adopt another method for calculating the second order

energy with sufficient accuracy, which can be easily applied to the nuclei heavier than He⁴.

Now, first, we reduce the matrix elements for many particles W_{on} in Eq. (2.4) to the matrix elements for two particles, and then the second order perturbation energy is expressed as a linear combination of a few terms as follows:

$$\begin{aligned} \Delta \mathcal{E} &\equiv \sum_{n_a l_a n_b l_b} \frac{|\langle n_0 l_0 n_1 l_1 TSL | W | n_a l_a n_b l_b TS' L' \rangle|^2}{(E_0 + E_1) - (E_a + E_b)} \\ &= \sum_{N=1}^{\infty} \sum_{E_a + E_b - E_0 - E_1 = 2N\hbar\omega} \frac{|\langle n_0 l_0 n_1 l_1 TSL | W | n_a l_a n_b l_b TS' L' \rangle|^2}{2N\hbar\omega}, \quad (2.5) \end{aligned}$$

where $E_i = \{2(n_i - 1) + l_i\}\hbar\omega$ is the single particle energy of the (n_i, l_i) state, n_i the principal quantum number, $l_i\hbar$ the angular momentum of the i -state of the single particle and $2N\hbar\omega$ is the energy difference between the zeroth order state and the intermediate state. T, S and L represent the total isotopic spin, the total ordinary spin and the total angular momentum of the two particles, respectively.

There are many degenerate intermediate states whose excitation energies are $2N\hbar\omega$, and this situation makes the second order perturbation calculation very tedious. The summation over these many degenerate states, i.e.,

$$\Delta \mathcal{E}_{2N} \equiv \frac{1}{2N\hbar\omega} \sum_{n_a l_a n_b l_b} |\langle n_0 l_0 n_1 l_1 TSL | W | n_a l_a n_b l_b TS' L' \rangle|^2 \quad (2.6)$$

with the condition of $2n_a + l_a + 2n_b + l_b - 2n_0 - l_0 - 2n_1 - l_1 = 2N$, can be carried out through the following procedure. By transforming the two particle coordinates into the relative coordinate and the centre-of-mass one,¹⁴⁾ the Hamiltonian of the two harmonic oscillators, i.e.,

$$\mathcal{H} = \frac{1}{2m} p_1^2 + \frac{1}{2} m\omega^2 r_1^2 + \frac{1}{2m} p_2^2 + \frac{1}{2} m\omega^2 r_2^2 \dots (\mathbf{r}_1, \mathbf{r}_2)\text{-system},$$

can be rewritten as

$$\mathcal{H} = \frac{1}{2\mu} p^2 + \frac{1}{2} \mu\omega^2 r^2 + \frac{1}{2M} P^2 + \frac{1}{2} M\omega^2 R^2 \dots (\mathbf{r}, \mathbf{R})\text{-system},$$

where \mathbf{p} and \mathbf{P} are the momenta of the relative and the centre-of-mass motions, \mathbf{r} and \mathbf{R} the corresponding coordinates of these motions, μ and M express the reduced and total masses in these systems.

The wave function in the $(\mathbf{r}_1, \mathbf{r}_2)$ -system can be expressed as the linear combination of the wave functions in the (\mathbf{r}, \mathbf{R}) -system, i.e.,

$$\begin{aligned} &|n_a l_a n_b l_b TSL, M_T M_S M_L\rangle \\ &= \sum_{\tilde{n} \tilde{l} \tilde{N} \tilde{L}} |\tilde{n} \tilde{l} \tilde{N} \tilde{L}, TSL, M_T M_S M_L\rangle \langle \tilde{n} \tilde{l} \tilde{N} \tilde{L} TSL | n_a l_a n_b l_b TSL \rangle, \quad (2.7) \end{aligned}$$

where \tilde{n} and \tilde{N} are the principal quantum numbers of the relative and the centre-of-mass motions, $\tilde{l}\hbar$ and $\tilde{L}\hbar$ the angular momenta of these motions, and M_T, M_S

and M_z the magnetic quantum numbers of T , S and L , respectively. In the above transformation, the following quantities are conserved;¹⁴⁾

- 1) the total energy,
- 2) total isotopic spin T , total ordinary spin S and total orbital angular momentum L and their Z -components,
- 3) the symmetry of the wave function,
- 4) the parity of the wave function.

Furthermore, there are also the orthogonality relations of the transformation coefficients;

$$\sum_{\tilde{n} \tilde{l} \tilde{N} \tilde{L}} \langle n'_a l'_a n'_b l'_b T S L | \tilde{n} \tilde{l} \tilde{N} \tilde{L} T S L \rangle \langle \tilde{n} \tilde{l} \tilde{N} \tilde{L} T S L | n_a l_a n_b l_b T S L \rangle \\ = \delta_{n'_a n_a} \delta_{l'_a l_a} \delta_{n'_b n_b} \delta_{l'_b l_b}$$

and

$$\sum_{n'_a l'_a n'_b l'_b} \langle \tilde{n}' \tilde{l}' \tilde{N}' \tilde{L}' T S L | n_a l_a n_b l_b T S L \rangle \langle n_a l_a n_b l_b T S L | \tilde{n} \tilde{l} \tilde{N} \tilde{L} T S L \rangle \\ = \delta_{\tilde{n}' \tilde{n}} \delta_{\tilde{l}' \tilde{l}} \delta_{\tilde{N}' \tilde{N}} \delta_{\tilde{L}' \tilde{L}}. \quad (2.8)$$

Using the above properties, the summation (2.6) can be carried out as follows. $|n_0 l_0 n_1 l_1 T S L\rangle$ and $|n_a l_a n_b l_b T S L\rangle$ are transformed into the (\mathbf{r}, \mathbf{R}) -system:

$$|n_0 l_0 n_1 l_1 T S L\rangle = \sum_{\tilde{n}_0 \tilde{l}_0 \tilde{N}_0 \tilde{L}_0} |\tilde{n}_0 \tilde{l}_0 \tilde{N}_0 \tilde{L}_0 T S L\rangle \langle \tilde{n}_0 \tilde{l}_0 \tilde{N}_0 \tilde{L}_0 T S L | n_0 l_0 n_1 l_1 T S L \rangle$$

where the summation includes only one or two terms when $n_0 l_0$ and $n_1 l_1$ are 1s or 1p, and

$$|n_a l_a n_b l_b T S L\rangle = \sum_{\tilde{n} \tilde{l} \tilde{N} \tilde{L}} |\tilde{n} \tilde{l} \tilde{N} \tilde{L} T S L\rangle \langle \tilde{n} \tilde{l} \tilde{N} \tilde{L} T S L | n_a l_a n_b l_b T S L \rangle.$$

Then, we get

$$\mathcal{E}_{2N} = \frac{1}{2N\hbar\omega} \left[\sum \sum \sum \sum \sum \langle n_0 l_0 n_1 l_1 T S L | \tilde{n}'_0 \tilde{l}'_0 \tilde{N}'_0 \tilde{L}'_0 T S L \rangle \right. \\ \times \langle \tilde{n}'_0 \tilde{l}'_0 \tilde{N}'_0 \tilde{L}'_0 T S L | W | \tilde{n}' \tilde{l}' \tilde{N}' \tilde{L}' T S L \rangle \langle \tilde{n}' \tilde{l}' \tilde{N}' \tilde{L}' T S L | n_a l_a n_b l_b T S L \rangle \\ \times \langle n_a l_a n_b l_b T S L | \tilde{n} \tilde{l} \tilde{N} \tilde{L} T S L \rangle \langle \tilde{n} \tilde{l} \tilde{N} \tilde{L} T S L | W | \tilde{n}_0 \tilde{l}_0 \tilde{N}_0 \tilde{L}_0 T S L \rangle \\ \left. \times \langle \tilde{n}_0 \tilde{l}_0 \tilde{N}_0 \tilde{L}_0 T S L | n_0 l_0 n_1 l_1 T S L \rangle \right], \quad (2.9)$$

where the summations must be carried out under the conditions,

$$2(\tilde{n}_0 - 1) + \tilde{l}_0 + 2(\tilde{N}_0 - 1) + \tilde{L}_0 + 3 \\ = 2(\tilde{n}'_0 - 1) + \tilde{l}'_0 + 2(\tilde{N}'_0 - 1) + \tilde{L}'_0 + 3 = (E_0 + E_1)/\hbar\omega \\ 2(\tilde{n} - 1) + \tilde{l} + 2(\tilde{N} - 1) + \tilde{L} + 3 \\ = 2(\tilde{n}' - 1) + \tilde{l}' + 2(\tilde{N}' - 1) + \tilde{L}' + 3 = (E_a + E_b)/\hbar\omega.$$

After carrying out the summations over $n_a l_a$, $n_b l_b$, $\tilde{n}' \tilde{l}'$ and $\tilde{N}' \tilde{L}'$ by the aid of the orthogonality relations of the coefficients, we get

$$\begin{aligned} \Delta \mathcal{E}_{2N} = & \frac{1}{2N\hbar\omega} \left[\sum \sum \sum \langle n_0 l_0 n_1 l_1 TSL | \tilde{n}_0' \tilde{l}_0' \tilde{N}_0' \tilde{L}_0' TSL \rangle \right. \\ & \times \langle \tilde{n}_0' \tilde{l}_0' \tilde{N}_0' \tilde{L}_0' TSL | W | \tilde{n} \tilde{l} \tilde{N} \tilde{L} TSL \rangle \langle \tilde{n} \tilde{l} \tilde{N} \tilde{L} TSL | W | \tilde{n}_0 \tilde{l}_0 \tilde{N}_0 \tilde{L}_0 TSL \rangle \\ & \left. \times \langle \tilde{n}_0 \tilde{l}_0 \tilde{N}_0 \tilde{L}_0 TSL | n_0 l_0 n_1 l_1 TSL \rangle \right]. \end{aligned} \quad (2.10)$$

Since the interaction W depends only on the relative coordinates, the centre-of-mass motion cannot be disturbed by it. Hence

$$(\tilde{N}_0, \tilde{L}_0) = (\tilde{N}, \tilde{L}) = (\tilde{N}_0', \tilde{L}_0'). \quad (2.11)$$

Furthermore, in the case of τ_c , the angular momentum is not changed, i.e.,

$$\tilde{l}_0 = \tilde{l} = \tilde{l}_0'. \quad (2.12)$$

The energy conservation leads to

$$\tilde{n}_0 = \tilde{n}_0' \quad \text{and} \quad \tilde{n} = \frac{1}{2}(2\tilde{n}_0 + \tilde{l}_0 + 2N - \tilde{l}) = \tilde{n}_0 + N. \quad (2.13)$$

Finally, it follows, in view of the above relations,

$$\Delta \mathcal{E}_{2N} = \frac{1}{2N\hbar\omega} \sum_{n_0 l_0} \left[\sum_{\tilde{n}_0 \tilde{l}_0} |\langle n_0 l_0 n_1 l_1 TSL | \tilde{n}_0 \tilde{l}_0 \tilde{N}_0 \tilde{L}_0 TSL \rangle|^2 \right] \cdot |\langle \tilde{n}_0 \tilde{l}_0 TSL | \tau_c | \tilde{n} \tilde{l} TSL \rangle|^2, \quad (2.14)$$

where the summation actually includes only a few terms and the matrix element $\langle \tilde{n}_0 \tilde{l}_0 TSL | \tau_c | \tilde{n} \tilde{l} TSL \rangle$ becomes a simple function of N . Thus, the resulting second order energy is expressed as a simple series of N , which can be easily summed up for sufficiently large N so that the error owing to the neglect of the remaining higher N terms is sufficiently small. In the special case of the Gaussian form of nuclear potential, this energy can be expressed in an analytic form. In the case of τ_T , the number of the terms increases a little more.

The similar procedure to the above mentioned one has recently been used by Eden and Emery¹⁵⁾ in the calculation of the Green function. Now, the effect of the Pauli principle in many-body system is not included in the above results. However, this effect is important only in the intermediate states of the lower excitation energies, and the correction due to this effect can be rather easily performed in the actual examples as shown in the following sections.

§ 3. The P -state doublet splitting of He^5

Although the P -states of He^5 are virtual states, the harmonic oscillator wave function is assumed to be valid, following Feingold.^{7 (b)(8)} In this case the induced deformation of He^4 -core by the outside nucleon can be easily taken into account and the qualitative features of the doublet splitting in the nuclei with the zeroth order configuration of (closed shell + one nucleon) may be obtained by such procedure.

The zeroth order wave function of He^5 is given by

$$|(1s)^4(000)1p, T=1/2(T_z)S=1/2 L=l; JM\rangle, \quad J=1/2 \text{ or } 3/2, \quad (3.1)$$

where (TSL) expresses the isotopic spin T , the ordinary spin S and the angular momentum L , respectively, and J is the total spin. The harmonic oscillator wave function is

$$\phi_{nlm} = N_{nl} r^l \exp(-\frac{1}{2}\nu r^2) L_{n+l-1/2}^{l+1/2}(\nu r^2) Y_{lm}(\theta, \varphi) \quad (3.2)$$

where N_{nl} is a normalization constant, $L_{n+l-1/2}^{l+1/2}$ the associated Laguerre polynomial and $\nu = m\omega/\hbar$. The wave function (3.1) is the eigen function of the centre-of-mass motion, because there is only one unclosed shell.¹⁶ As was mentioned in § 2, therefore, we may forget the centre-of-mass motion for the time being.

Now, the intermediate states which appear in the second order perturbation are as follows:

$$(I) \quad [(1s)^2(010)(n_a l_a n_b l_b)(012)](022)1p, T=\frac{1}{2}(T_z)SL; JM, \quad (3.3)$$

where $n_a l_a$ and $n_b l_b$ are neither $1s$ nor $1p$. This configuration represents "the self-deformation" of the He^4 -core induced by the perturbing interaction between the two $1s$ -nucleons. (see Fig. 1a).

$$(II) \quad |(1s)^2(\frac{1}{2}\frac{1}{2}0)(n_a l_a n_b l_b)(T1L), T=\frac{1}{2}(T_z)S=\frac{3}{2} L; JM\rangle. \quad (3.4)$$

This represents "the induced deformation" of the He^4 -core induced by the perturbing interaction between the outside nucleon and one of the nucleons in the He^4 -core (see Fig. 1b).

$$(III) \quad (a) \quad |(1s)^2(010)\{nl \cdot (1p)^2(TSL)\}(\frac{1}{2}S'L'), T=\frac{1}{2}(T_z)S''L'; JM\rangle \\ (b) \quad |(1s)^2(010)(1p)^3(\frac{1}{2}SL), T=\frac{1}{2}(T_z)S'L; JM\rangle \quad (3.5)$$

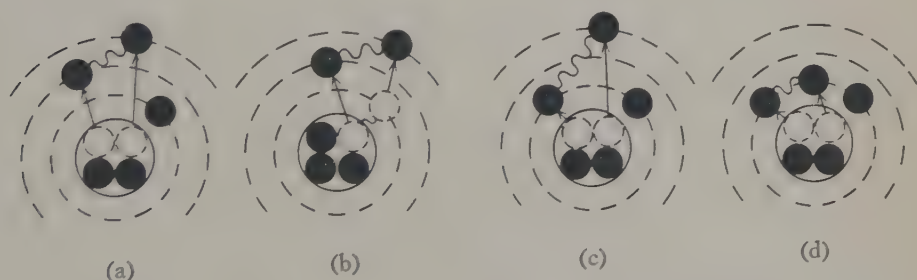


Fig. 1. Figures (a), (b), (c) and (d) show configurations (I), (II), (IIIa) and (IIIb) in He^5 , respectively.

These configurations are a part of the configurations of the type (I), but they must be carefully treated because of the following reason. In these configurations, one or two nucleons in the $1s$ -core jump into the $1p$ -orbit. Since one of the $1p$ -orbit is already occupied by the outmost nucleon, the excitation from the core, i.e.,

the self-deformation of the He^4 -core, is restricted so as to satisfy the Pauli principle with the nucleon. This restriction is important (see Fig. 1c, d).

As can be easily shown, the matrix elements with respect to the configuration (I) do not depend on the total spin J , and the matrix elements with respect to the configurations (II) and (III) become

$$\begin{aligned}
 \text{(II)}^\circ & \langle (1s)^4(000)1p, T=\frac{1}{2}(T_z)S=\frac{1}{2}L=1; JM|V_T|(1s)^3(\frac{1}{2}\frac{1}{2}0)(n_a l_a n_b l_b)(T1L), \\
 & T=\frac{1}{2}(T_z)S=\frac{3}{2}L; JM\rangle \\
 & = (-)^{3/2+L-J-T} \sqrt{\frac{2T+1}{2}} W(\frac{1}{2}1\frac{3}{2}L; J2) \langle 1s1p11||\mathbf{S}^{(2)}\mathbf{L}^{(2)}||n_a l_a n_b l_b 1L\rangle \Phi_\tau(T), \\
 \text{(III)}^\circ & \langle (1s)^4(000)1p, T=\frac{1}{2}(T_z)S=\frac{1}{2}L=1; JM|V_T|(1s)^2(010)\{nl \cdot (1p)^2(TSL)\} \\
 & (\frac{1}{2}S'L'), T=\frac{1}{2}(T_z)S''L'; JM\rangle \\
 & = (-)^{L'-S'-T-J+1} \sqrt{\frac{3}{2} \cdot (2T+1)(2S+1)(2L+1)(2S'+1)(2L'+1)(2S''+1)} \\
 & \times W(\frac{1}{2}1S''L'; J2) W(111L'; 2L) \\
 & \times W(\frac{1}{2}\frac{1}{2}\frac{1}{2}S'; 1S) W(12S'S''; 1\frac{1}{2}) \langle (1s)^2 10||\mathbf{S}^{(2)}\mathbf{L}^{(2)}||n l 1p 12\rangle \Phi_\tau(0),
 \end{aligned}$$

where $W(abcd; ef)$ and $\langle n_1 l_1 n_2 l_2 1L||\mathbf{S}^{(2)}\mathbf{L}^{(2)}||n_3 l_3 n_4 l_4 1L'\rangle$ are the Racah coefficient and the reduced matrix element, respectively. $\Phi_\tau(T) = \langle T|I|T\rangle$, $S_m^{(2)} = [\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2]_m^{(2)}$ and $L_m^{(2)} = \sqrt{24\pi/5} \cdot Y_{2m}(\theta, \varphi) V(r)$ are related with the tensor potential $v_T(\mathbf{r}_{ij}) = I \cdot S_{ij} \cdot V(r_{ij})$ by the equation

$$v_T(\mathbf{r}_{ij}) = I(\boldsymbol{\tau}_i, \boldsymbol{\tau}_j) \mathbf{S}^{(2)} \cdot \mathbf{L}^{(2)}, \quad (3.6)$$

where $I(\boldsymbol{\tau}_i, \boldsymbol{\tau}_j)$ is a scalar function of the isotopic spin operator $\boldsymbol{\tau}_i$ and $\boldsymbol{\tau}_j$, $\boldsymbol{\sigma}_i$ the Pauli spin operator, $Y_{2m}(\theta, \varphi)$ the spherical harmonics, $V(r)$ the radial part of $v_T(\mathbf{r}_{ij})$ and $S_{ij} = 3(\boldsymbol{\sigma}_i \cdot \mathbf{r}_{ij})(\boldsymbol{\sigma}_j \cdot \mathbf{r}_{ij})/r_{ij}^2 - (\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j)$.

Using the procedure mentioned in § 2, the corresponding doublet-splitting energies induced by the various intermediate states with the excitation energy $2N\hbar\omega$ are given by

$$\begin{aligned}
 \Delta E_{2N}(II) & = \frac{1}{2N\hbar\omega} \left[\sum'_{\substack{n_a l_a n_b l_b \\ T, L}} |\langle \text{Eq. (3.1)} | V_T | \text{Eq. (3.4)} \rangle|_{J=3/2}^2 \right. \\
 & \quad \left. - \sum'_{\substack{n_a l_a n_b l_b \\ T, L}} |\langle \text{Eq. (3.1)} | V_T | \text{Eq. (3.4)} \rangle|_{J=1/2}^2 \right] \\
 & = \left[-\frac{9}{2^{2N}} \langle 1\tilde{s} | V(r) | N\tilde{d} \rangle^2 \Phi_\tau^2(0) \right. \\
 & \quad \left. - \left(1 - \frac{1}{2^{2N}} \right) \cdot \frac{9^2}{5} \cdot \left\{ \langle 1\tilde{p} | V(r) | (N+1)\tilde{p} \rangle^2 - \langle 1\tilde{p} | V(r) | N\tilde{f} \rangle^2 \right\} \Phi_\tau^2(1) \right] / 2N\hbar\omega,
 \end{aligned} \quad (3.7)$$

and

$$\begin{aligned}
\Delta E_{2N}(III) &\equiv \frac{1}{2N\hbar\omega} \left[\sum'_{\substack{TS L, S' L' S'' \\ nl}} |\langle \text{Eq. (3.1)} | V_T | \text{Eq. (3.5)} \rangle|_{J=3/2}^2 \right. \\
&\quad \left. - \sum'_{\substack{TS L, S' L' S'' \\ nl}} |\langle \text{Eq. (3.1)} | V_T | \text{Eq. (3.5)} \rangle|_{J=1/2}^2 \right] \\
&= \left[2 \cdot \frac{9}{2^{2N}} \cdot \langle 1\tilde{s} | V(r) | N\tilde{d} \rangle^2 \phi_{\tau}^2(0) \right] / 2N\hbar\omega, \quad (3.8)
\end{aligned}$$

where the prime over the summation symbol means that $n_a l_a$ and $n_b l_b$ are not to be $1s$ or $1p$, and that nl is not to be $1s$. Then, the total doublet splitting energy becomes

$$\begin{aligned}
\Delta E &\equiv \sum_{N=1}^{\infty} \{ \Delta E_{2N}(II) + \Delta E_{2N}(III) \} = \sum_{N=1}^{\infty} \left[\frac{9}{2^{2N}} \langle 1\tilde{s} | V(r) | N\tilde{d} \rangle^2 \phi_{\tau}^2(0) \right. \\
&\quad \left. - \left(1 - \frac{1}{2^{2N}} \right) \cdot \frac{9^2}{5} \cdot \left\{ \langle 1\tilde{p} | V(r) | (N+1)\tilde{p} \rangle^2 - \langle 1\tilde{p} | V(r) | N\tilde{f} \rangle^2 \right\} \phi_{\tau}^2(1) \right] / 2N\hbar\omega, \quad (3.9)
\end{aligned}$$

where

$$\begin{aligned}
\langle 1\tilde{s} | V(r) | N\tilde{d} \rangle &= \sqrt{\frac{(2N+3)!!}{2^{N-1} \cdot (N-1)!}} \sum_{m=1}^N (-)^{m-1} \frac{N-1}{2m+3} C_{m-1} I_m \\
\langle 1\tilde{p} | V(r) | (N+1)\tilde{p} \rangle &= \sqrt{\frac{(2N+3)!!}{3 \cdot 2^N \cdot N!}} \sum_{m=1}^{N+1} (-)^{m-1} N C_{m-1} I_m \\
\langle 1\tilde{p} | V(r) | N\tilde{f} \rangle &= \sqrt{\frac{(2N+5)!!}{3 \cdot 2^{N-1} \cdot (N-1)!}} \sum_{m=2}^{N+1} (-)^m \frac{N-1}{2m+3} C_{m-2} I_m
\end{aligned} \quad (3.10)$$

and

$$I_m = \frac{\sqrt{2} \nu^{m+3/2}}{\sqrt{\pi} \cdot (2m+1)!!} \int_0^{\infty} V(r) \exp(-\frac{1}{2} \nu r^2) r^{2m+2} dr. \quad (3.11)$$

Some details of the calculations will be found in the Appendix. In these equations, the terms including the factor 2^{-2N} express the effect of the Pauli exclusion principle and decrease rapidly as N increases.

It will be interesting to note a few of important features of the above results.

(i) $\Delta E_{2N}(III) > 0$, i.e., the restriction of the self-deformation of the He^4 -core due to the Pauli principle always favours the inverted splitting ($P_{3/2} < P_{1/2}$).

(ii) The splitting due to the triplet even state ($T=0$) always has a good sign. On the other hand, the sign of the splitting due to the triplet odd state ($T=1$) depends on the shape of the two-body potential. The inverted splitting, therefore, is obtained, if the tensor force in the triplet even states is stronger than in the triplet odd states.

These qualitative features can be more clearly illustrated as follows. The tensor operator S_{12} can be written as

$$S_{12} = \sum_{m=-2}^2 \sqrt{\frac{24\pi}{5}} S_{-m}^{(2)} Y_{2m},$$

from Eq. (3.6). For the sake of simplicity, we shall take a two-dimensional picture, i.e., $m = -2$ or 2 . Here, for example, the operator $S_{-2}^{(2)} Y_{22}$ decreases the z -component of the ordinary spin by $2\hbar$ and increases that of the angular momentum by $2\hbar$.

(1) Illustration of the property (i)

We consider the case where the resultant spin of the two interacting $1s$ -nucleons in the He^4 -core, $(1s)^2 (010)^*$, and the orbital angular momentum of the outside nucleon $1p$ are directed upward (see Fig. 2). By the operator $S_{-2}^{(2)} Y_{22}$ in S_{12} , the zeroth order state $(1s)^2 (010)$ is transformed into the intermediate states $(n_1 l_1, n_2 l_2) (012)$, where the ordinary spin ($S=1$) is directed downward and the angular momentum ($L=2$) is upward as illustrated in Fig. 2. Now, in the $2\hbar\omega$ -excitation, $(n_1 l_1, n_2 l_2)$ can generally be $(1s, 1d)$ or $(1p)^2$. However, $(1s, 1d)$ is not allowed because the matrix element of the tensor force, $\langle (1s)^3 (\frac{1}{2} \frac{1}{2} 0) 1d, 012 || S^{(2)} L^{(2)} || (1s)^4 (000), 000 \rangle$, is zero. And $(1p)^2$ must be $(1p_{1/2})^2$, since the spin ($S=1$) and the angular momentum ($L=2$) are antiparallel as shown in Fig. 2. In the $P_{1/2}$ -

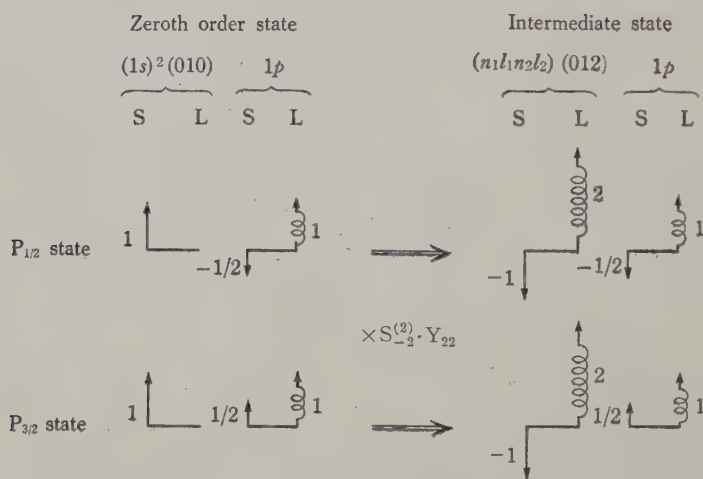


Fig. 2. S and L are ordinary spin and angular momentum, respectively.

state, this intermediate state $(1p_{1/2})^2 (012)$ is forbidden by the Pauli principle because the outside nucleon is already in the $1p_{1/2}$ -orbit. On the other hand, it is not forbidden in the $P_{3/2}$ -state because the outside nucleon is not in the $1p_{1/2}$ -orbit, but in the $1p_{3/2}$ -orbit. Therefore the $P_{3/2}$ -state is more depressed by the second order perturbation than the $P_{1/2}$ -state. In the higher intermediate states, this

* The resultant spin of $(1s)^2$ must be 1, because the tensor force has non-vanishing two-body matrix element only between the spin triplet states.

tendency will be weakened. The effect of the operator $S_2^{(2)} Y_{2-2}$ is nothing, because the resultant spin cannot become larger than 1. In the case where the resultant spin of the core nucleons $(1s)^2 (010)$ is directed upward and the orbital angular momentum of $1p$ is downward, there appears no difference between the $P_{1/2}$ - and $P_{3/2}$ -state.

(2) Illustration of the property (ii)

The transition of the zeroth order state to the intermediate states in the induced deformation ($T=1$) is illustrated in Fig. 3. The ordinary spins of $1s$ - and $1p$ -nucleon must be parallel to each other so that the resultant spin is 1. Accordingly, the resultant spin and the relative angular momentum are parallel in the $P_{3/2}$ -state and are antiparallel in the $P_{1/2}$ -state. Hence, by the operator $S_2^{(2)} Y_{22}$, the relative angular momentum in the intermediate state becomes larger in the $P_{3/2}$ -state, but becomes smaller in the $P_{1/2}$ -state as can be seen from Fig. 3. Therefore the $P_{3/2}$ -state is more depressed by

the effect of the intermediate states with the large relative angular momentum than the $P_{1/2}$ -state is depressed by it. On the other hand, the $P_{1/2}$ -state is more depressed by the effect of the intermediate states with the small relative angular momentum than the $P_{3/2}$ -state. This property can be seen in Eq. (3.7), where the coefficient of $\langle 1\tilde{p} | V(r) | N\tilde{f} \rangle^2$ is positive and that of $\langle 1\tilde{p} | V(r) | (N+1)\tilde{p} \rangle^2$ is negative. Furthermore, the long range potential will be favourable for the correct splitting, because the contribution from $\langle 1\tilde{p} | V(r) | N\tilde{f} \rangle^2$ becomes larger than that from $\langle 1\tilde{p} | V(r) | (N+1)\tilde{p} \rangle^2$ in this case.

After such a qualitative discussion, we shall compute numerically the doublet splitting by using the meson-theoretic potential and the phenomenological tensor potential of the Serber type both of which are consistent with the experimental data of the two-body system.

Meson potential⁽⁶⁾ for the triplet odd state :

$$V_T = \begin{cases} V_T^{(1\pi)}(\kappa r) & (\kappa r \geq 1.0) \\ 0 & (\kappa r < 1.0) \end{cases} \quad (3.12)$$

and that for the triplet even state :

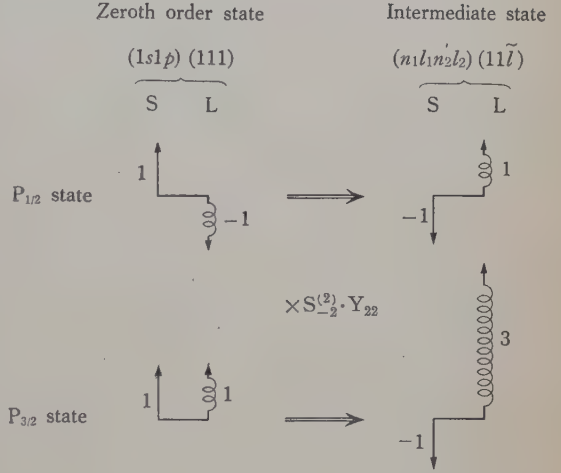


Fig. 3. S and L are ordinary spin and angular momentum, respectively.

$$(I) \quad V_T = \begin{cases} V_T^{(1\pi)}(\kappa r) & (\kappa r \geq 0.7) \\ 3 V_T^{(1\pi)}(0.7) & (\kappa r < 0.7) \end{cases} \quad (3.13)$$

$$\text{or (II)} \quad V_T = \begin{cases} V_T^{(1\pi)}(\kappa r) & (\kappa r \geq 0.7) \\ V_T^{(1\pi)}(0.7) & (\kappa r < 0.7) \end{cases} \quad (3.14)$$

in which

$$V_T^{(1\pi)}(\kappa r) = \frac{g_e^2}{4\pi} \mu c^2 \cdot \left(\frac{\tau_1 \cdot \tau_2}{3} \right) \cdot S_{12} \cdot \left(1 + \frac{3}{\kappa r} + \frac{3}{\kappa^2 r^2} \right) \cdot \exp(-\kappa r) \cdot \kappa r. \quad (3.15)$$

where $V_T^{(1\pi)}$ is the one-pion exchange potential, $\kappa^{-1} = \hbar/\mu c$ the Compton wave length of a pion, and 0.08 is used for the coupling constant $g_e^2/4\pi$.

Serber potential^[17]:

$$V_T = V_0 \{ (1 - \tau_1 \cdot \tau_2) / 4 \} \cdot S_{12} \cdot \exp(-r^2/r_i^2), \quad (3.16)$$

where

$$V_0 = -25.8 \text{ Mev}, \quad r_i = 2.41 \times 10^{-13} \text{ cm}.$$

The parameter $\rho (10^{-13} \text{ cm}) \equiv (\nu/2)^{-1/9}$ which measures the extension of the harmonic oscillator wave function can be determined by Coulomb energy difference of the mirror nuclei^[18] and the high energy electron scattering by nuclei,^[19] assuming the harmonic oscillator wave functions. For He^5 , the Coulomb energy difference of $\text{He}^5\text{-Li}^5$ gives $\rho \sim 2.34$ with some ambiguity, because of the rather large experimental error and it would be supposed that the above value corresponds to the radius of $1p$ -orbit, because the mentioned energy difference strongly depends on the wave function of the $1p$ -nucleon. On the other hand, the high energy electron scattering by He^4 gives $\rho = 1.80$ which corresponds to the radius of $1s$ -orbit. As mentioned above, the configuration (III) plays an important role in the calculation of the doublet splitting and the interaction of the two nucleons in $1s$ -core is responsible for this configuration. Then it would be reasonable to use 1.80 as an approximate value of ρ in this calculation.

The doublet splitting energies calculated by using the above nuclear potentials and the parameter ρ are listed in Table I.

Table I. The doublet splitting energy in He^5 (in Mev)

Configuration	(II)	(III)	Total
Meson I	-3.7	7.7	4.0
Meson II	-2.0	4.2	2.2
Serber	-2.1	4.2	2.1

[Experiment: Several Mev^{[20], [21]}]

We can see from this table that the effect of the Pauli principle on the self-deformation of the He^4 -core always gives the positive contribution and is responsible for getting about a half of the observed values and that the induced deformation

gives the negative contribution for our potential forms. Furthermore, we have examined the dependence of the doublet splitting on the parameter ρ (for Meson (I) and Serber) and on the force range r_t (for Serber). These results are shown in Fig. 4a, b. Finally, the results obtained in this section will be summarized as follows.

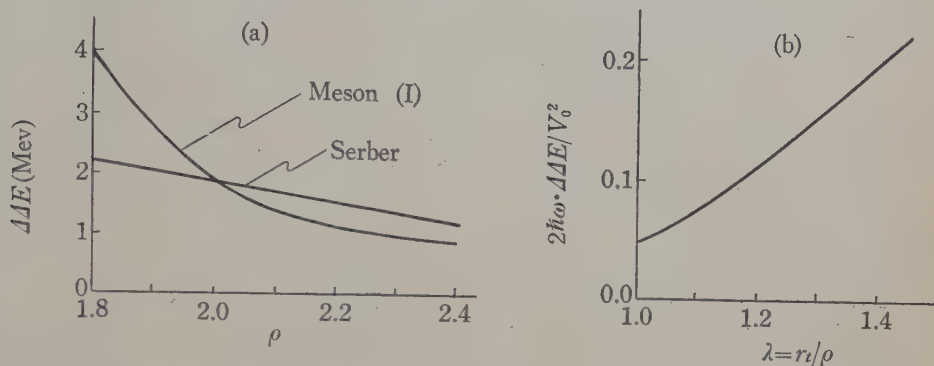


Fig. 4. Dependence of the doublet splitting energy $\Delta\Delta E$ in He^5 , (a) on the parameter ρ and (b) on the range r_t .

(A) The tensor forces, which are favourable for getting the inverted splitting, are of the following properties.

- (i) Its range is long,
- (ii) The tensor force in the triplet even states is stronger than in the triplet odd states.

This last property is important for the inverted splitting and, furthermore, it has already been shown to be consistent with the experimental data of the two-body system.

(B) The small nuclear radius is favourable.

(C) Self- and induced deformation of He^4 core:

(i) In general, the effect of the self-deformation of the He^4 -core, i.e., the configuration (I), does not contribute to the doublet splitting. However, when one or two nucleons in the $1s$ -core jump into the $1p$ -orbit which is already occupied by the outmost nucleon, that is to say, when the self-deformation of the He^4 -core overlaps with the $1p$ -orbit, the above deformation must be restricted so as to satisfy the Pauli principle with the outmost nucleon. The restriction is stronger in the $P_{1/2}$ state than in the $P_{3/2}$ state and always produces the favourable effect for the inverted splitting as has been illustrated above.

(ii) The sign of the splitting energy due to the induced deformation, i.e., the configuration (II), is not definite and may be negative according to the nature of the nuclear potential.

It will, therefore, be concluded that the restriction of the self-deformation of the He^4 -core due to the Pauli principle plays a decisive role for getting the correct splitting.

Now, we shall compare the present calculation with those of the n -He⁴ scattering.^{11),12)} The $1D$ -state mixing in He⁴ in the latter calculation corresponds to the configuration (III, b) in the present case and the higher D -states correspond to the configuration (III, a). Furthermore, the induced deformation of He⁴ by an incident neutron corresponds to the configuration (II). Such induced deformation effect has not been included in the previous calculations of the n -He⁴ scattering, but it seems to have negative effect and to be not so small as inferred from the present calculation. For example, in the case of the meson theoretic potential (I) with $\rho=1.80$ the contributions of the $1D$ -state, the higher D -states and the induced deformation are +5.7 Mev, +2.1 Mev and -3.8 Mev respectively. The sum of the latter two contributions is about 30% of the first one.

§ 4. The P -state doublet splitting of N¹⁵

The P -state doublet splitting of N¹⁵ has been calculated by the same method as in the case of He⁵, because the P -states of N¹⁵ are bound states and it provides an interesting example of the configuration, closed shell—one nucleon.

The zeroth order wave function is

$$|(1s)^4(000)(1p)^{11}(\frac{1}{2}\frac{1}{2}1), T=\frac{1}{2}(T_z)S=\frac{1}{2}L=1; JM\rangle$$

and the parameters ρ determined from the Coulomb energy difference¹⁸⁾ and the high energy electron scattering¹⁹⁾ are equal to each other, being 2.34.

The intermediate states, which are capable of coupling with the zeroth order configuration in the second order perturbation, are

$$(1) \quad | \{ (1s)^2(010)(n_a l_a n_b l_b)(012) \} (022)(1p)^{11}(\frac{1}{2}\frac{1}{2}1), T=\frac{1}{2}(T_z)S=\frac{3}{2}L=1; JM \rangle,$$

$$(2) \quad | (1s)^2(010) \{ nl \cdot (1p)^{12}(000) \} (\frac{1}{2}\frac{1}{2}l), T=\frac{1}{2}(T_z)S=\frac{3}{2}L=l; JM \rangle,$$

$$(3) \quad | \{ (1s)^3(\frac{1}{2}\frac{1}{2}0)(n_a l_a n_b l_b)(T1L) \} (\frac{1}{2}\frac{3}{2}L)(1p)^{10}(T'S'L'), T=\frac{1}{2}(T_z)S''L''; JM \rangle,$$

$$(4) \quad | (1s)^4(000) \{ (1p)^9(TSL)(n_a l_a n_b l_b)(T'1L') \} (\frac{1}{2}S''L''), T=\frac{1}{2}(T_z)S''L''; JM \rangle,$$

$$(5) \quad | (1s)^3(\frac{1}{2}\frac{1}{2}0) \{ nl \cdot (1p)^{11}(\frac{1}{2}\frac{1}{2}1) \} (T1L), T=\frac{1}{2}(T_z)S=\frac{3}{2}L; JM \rangle,$$

and

$$(6) \quad | (1s)^4(000) \{ nl \cdot (1p)^{10}(T1L) \} (\frac{1}{2}\frac{3}{2}L'), T=\frac{1}{2}(T_z)S=\frac{3}{2}L'; JM \rangle.$$

The important contributions for getting the correct doublet splitting are provided by the configurations (2), (5) and (6). In these configurations, two nucleons in $1s$ - or $1p$ -orbit interact with each other, on account of which one of them jumps into the $1p$ -orbit, where the $1p$ -nucleons already exist. Then the jumping nucleon must be correlated with these $1p$ -nucleons so as to satisfy the Pauli principle. Thus, the same effects as those in He⁵ also play an important role in N¹⁵. Using the same procedure as in the case of He⁵, the splitting energy ΔE is calculated, and the results are listed in Table II, where the splitting energies due to the configurations of various types are given.

Table II. The doublet splitting energy in N^{15} (in Mev)

Configuration	(2)	(3)	(4)	(5)	(6)	Total
Meson I	0.5	-0.2	-1.3	2.3	1.8	3.1
Meson II	0.3	-0.1	-0.6	1.3	1.0	1.9
Serber	0.5	-0.3	-0.5	3.1	3.3	6.1*

[Experiment: 6.3 Mev²⁰]* The value in the previous note²², 6.4 Mev, should be corrected to 6.1 Mev.

Very similar situations to those in Table I can be found in this table, and it is seen that at least about a half of the experimental value can also be obtained in this case.

§ 5. The comparison with the other calculations

We shall, at first, compare the present calculation with Feingold's one.^{7 bis)} The wave function used by Feingold is

$$\phi_a = \phi_0 + \lambda V_T' \phi_0 \quad (5.1)$$

where ϕ_0 is the zeroth order wave function of the harmonic oscillator, $V_T' = \sum_{i>j} (\sqrt{\nu/2} \cdot r_{ij})^m v_T(r_{ij})$ and λ , and m are the variational parameters. Varying the parameter λ , the minimum energy is given by

$$\Delta E = E_a - E_0 = \lambda \langle V_T' V_T \rangle_{00} \quad (5.2)$$

where λ is approximately taken as $-\langle V_T' V_T \rangle_{00} / \langle V_T' (H_0 + V_T - E_0) V_T \rangle_{00}$, if λ is small. To make the comparison easy, Eq. (5.1) is rewritten as

$$\phi_a = \phi_0 + \sum_n' \lambda \langle V_T' \rangle_{0n} \phi_n. \quad (5.3)$$

Using the approximate relation $\lambda \approx -\langle V_T' V_T \rangle_{00} / \langle V_T' (H_0 - E_0) V_T \rangle_{00}$, the corresponding energy is given by

$$E_a = \frac{\langle \phi_a, H \phi_a \rangle}{\langle \phi_a, \phi_a \rangle} \approx E_0 + \sum_n' \frac{1}{E_0 - E_n} \langle V_T \rangle_{0n}^2 x_n (2 - x_n), \quad (5.4)$$

where $x_n = \lambda (E_0 - E_n) \langle V_T' \rangle_{0n} / \langle V_T \rangle_{0n}$. The graph of $x_n (2 - x_n)$ vs. $(E_n - E_0)$ is shown in Fig. 5.

On the other hand, the present calculation yields

$$\phi_b = \phi_0 + \sum_n' \frac{1}{E_0 - E_n} \langle V_T \rangle_{0n} \phi_n, \quad (5.5)$$

and

$$E_b = \frac{\langle \phi_b, H \phi_b \rangle}{\langle \phi_b, \phi_b \rangle} \approx E_0 + \sum_n' \frac{1}{E_0 - E_n} \langle V_T \rangle_{0n}^2. \quad (5.6)$$

From Eqs. (5.4) and (5.6) together with Fig. 5, it can be easily seen that

Feingold's calculation underestimates the contributions from the intermediate states of the lower excitation energies. Therefore, the calculations of this type will underestimate the influence of the above mentioned Pauli principle on the doublet splitting. On the other hand, the contributions from the higher intermediate states are overestimated in his calculation. And the favourable terms for the correct doublet splitting in these states are the induced deformation terms as can be seen from Eq. (3.7) and Fig. 6, where the contribution from each intermediate configuration of the excitation energy $2N\hbar\omega$ is plotted as the function of N . Furthermore, in these terms, the triplet odd states are important. Therefore, it would also be expected that Feingold's calculation would give the larger splitting energy in the case of the Wigner type potential than in the case of the Serber one.

In the calculations based on the Fermi gas model, the previous papers⁸⁾ taking account of the induced deformation alone, have yielded the doublet splitting of wrong sign or of too small magnitude. The effect of the above mentioned Pauli principle, however, has recently been calculated by Takagi, Watari and Yasuno,⁹⁾ and Jancovici,¹⁰⁾ and the reasonable results have been obtained.

§ 6. Conclusion and discussions

It has been shown that about a half of the experimental values of the doublet splittings in He^5 and N^{15} can be derived, using the meson-theoretic potential or the Serber one, through the second order effect of perturbation theory. The splitting energies have been shown to be mainly ascribed to the facts that (1) the tensor force is strong and, especially, is stronger in the triplet even states than in the triplet odd states, and (2) the deformation of

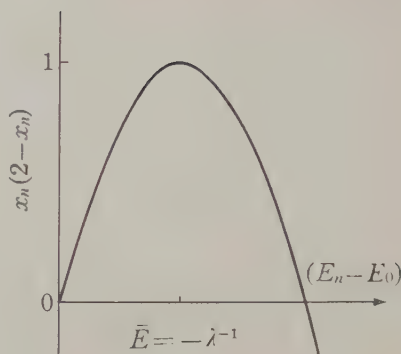


Fig. 5. The relation between $x_n(2-x_n)$ and $(E_n - E_0)$ in case of $m=0$.

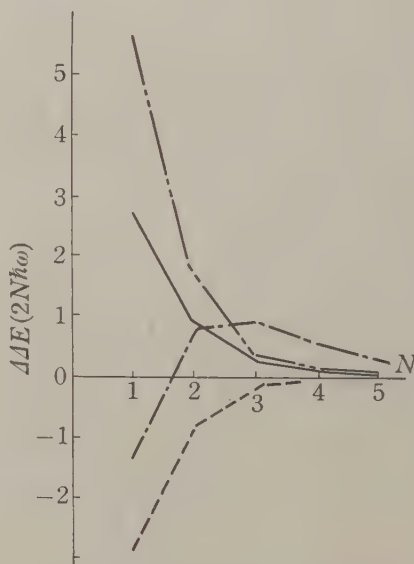


Fig. 6. Contributions to the doublet splitting energy $\Delta\Delta E$ in He^5 from various intermediate configurations, as a function of their excitation energy $2N\hbar\omega$, in case of Meson (I).
 ----- contribution from configuration (II) ($T=0$)
 - · - · - · - contribution from configuration (II) ($T=1$) $\times 10$
 · · · · · contribution from configuration (III)
 ——— total contribution

the closed shell core induced by the tensor force between the core-nucleons is restricted so as to satisfy the Pauli principle with the outside nucleon.

We shall discuss the approximation adopted in this calculation. Although only the second order perturbation has been used in the above calculation, we have actually solved the secular equation, allowing for all the configurations which include the states of the $2\hbar\omega$ excitation energy and the error in the second order perturbation, compared with the solutions of secular equation, is found to be about 10% for our potential forms. On the other hand, the rough estimation of the third order contribution from the higher excitation energies becomes about 30% of the second order one with a wrong sign. Therefore, the more accurate calculations will be desirable for getting the definite conclusion about the effect of the tensor force on the spin-orbit splitting.

Furthermore, the mixing ratio of the higher configurations coupled with the zeroth order one becomes 20~30%.

The detailed discussions about these problems will be given in the following paper where the *D*-state doublet splitting energy in O^{17} will be calculated in co-operation with A. Arima.

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Appendix. The outline of the calculation of the *P*-state doublet splitting in He^5

The splitting energies are expressed by the two-body matrix elements in the relative and centre-of-mass coordinate system, $(\tilde{n}\tilde{l}, \tilde{N}\tilde{L})$;

$$\begin{aligned}
 \Delta E_{2N}(II) = & [-3/80 \{1 - |\langle N\tilde{d}1\tilde{P}011|1s(N+1)p011\rangle|^2\} \\
 & \times |\langle 1\tilde{s}1\tilde{P}11||S^{(2)}L^{(2)}||N\tilde{d}1\tilde{P}11\rangle|^2 \Phi_{\tau}^2(0) \\
 & - 1/80 \{1 - 0\} \cdot |\langle 1\tilde{s}1\tilde{P}11||S^{(2)}L^{(2)}||N\tilde{d}1\tilde{P}12\rangle|^2 \Phi_{\tau}^2(0) \\
 & + 1/40 \{1 - |\langle N\tilde{d}1\tilde{P}013|1sNf013\rangle|^2\} \cdot |\langle 1\tilde{s}1\tilde{P}11||S^{(2)}L^{(2)}||N\tilde{d}1\tilde{P}13\rangle|^2 \Phi_{\tau}^2(0) \\
 & - 9/80 \{1 - |\langle (N+1)\tilde{p}1\tilde{S}111|1s(N+1)p111\rangle|^2\} \\
 & \times |\langle 1\tilde{p}1\tilde{S}11||S^{(2)}L^{(2)}||(N+1)\tilde{p}1\tilde{S}11\rangle|^2 \Phi_{\tau}^2(1) \\
 & + 3/40 \{1 - |\langle Nf1\tilde{S}113|1sNf113\rangle|^2\} \\
 & \times |\langle 1\tilde{p}1\tilde{S}11||S^{(2)}L^{(2)}||Nf1\tilde{S}13\rangle|^2 \Phi_{\tau}^2(1) / 2N\hbar\omega
 \end{aligned}$$

and

$$\begin{aligned}
 \Delta E_{2N}(III) = & [3/80 |\langle N\tilde{d}1\tilde{S}012|Np1p012\rangle|^2 - 1/40 |\langle N\tilde{d}1\tilde{S}012|(N-1)f1p012\rangle|^2] \\
 & \times (1 + \delta_{N,1}) \cdot |\langle 1\tilde{s}1\tilde{S}10||S^{(2)}L^{(2)}||N\tilde{d}1\tilde{S}12\rangle|^2 \Phi_{\tau}^2(0) / 2N\hbar\omega.
 \end{aligned}$$

The two-body matrix elements are calculated as follows:

$$\langle \tilde{n} \tilde{l} \tilde{N} \tilde{L} 1 L \| \mathbf{S}^{(2)} \mathbf{L}^{(2)} \| \tilde{n}' \tilde{l}' \tilde{N} \tilde{L} 1 L' \rangle = (-)^{\tilde{l}' - \tilde{L} + L - 2} \\ \times \sqrt{(2L+1)(2L'+1)} W(\tilde{l} \tilde{L} \tilde{l}' L'; \tilde{L} 2) \langle \tilde{n} \tilde{l}, S=1L=\tilde{L} \| \mathbf{S}^{(2)} \mathbf{L}^{(2)} \| \tilde{n}' \tilde{l}', S=1L=\tilde{L}' \rangle,$$

in which

$$\langle \tilde{n} \tilde{l}, S=1L=\tilde{L} \| \mathbf{S}^{(2)} \mathbf{L}^{(2)} \| \tilde{n}' \tilde{l}', S=1L=\tilde{L}' \rangle = \sqrt{120(2\tilde{l}'+1)} (\tilde{l}' 200 | \tilde{L} 0) \langle \tilde{n} \tilde{l} | V(r) | \tilde{n}' \tilde{l}' \rangle$$

where $(\tilde{l}' 200 | \tilde{L} 0)$ is Clebsch-Gordan coefficient,

$$\langle \tilde{n} \tilde{l} | V(r) | \tilde{n}' \tilde{l}' \rangle = \int_0^\infty R_{\tilde{n} \tilde{l}}(r) V(r) R_{\tilde{n}' \tilde{l}'}(r) r^2 dr$$

and $R_{nl}(r)$ is the radial part of the harmonic oscillator wave function.

The transformation coefficients $\langle \tilde{n} \tilde{l} \tilde{N} \tilde{L}; l | 1s nl; l \rangle$ and $\langle \tilde{n} \tilde{l} 1 \tilde{S}; \tilde{l} | n_1 l_1 n_2 l_2; \tilde{l} \rangle$ are given by

$$\langle \tilde{n} \tilde{l} \tilde{N} \tilde{L}; l | 1s nl; l \rangle = (-)^{\tilde{n} + \tilde{N} - n + 1} \cdot 2^{1/4(\tilde{L} + \tilde{L}' - 3l) - n + 1} \\ \times \sqrt{\frac{(n-1)! \cdot (2\tilde{l}+1) \cdot (2\tilde{L}+1) \cdot (2l+2n-1)!!}{(\tilde{n}-1)! \cdot (\tilde{N}-1)! \cdot (2l+1) \cdot (2\tilde{l}+2\tilde{n}-1)!! \cdot (2\tilde{L}+2\tilde{N}-1)!!}} (\tilde{l} \tilde{L} 00 | l 0),$$

and

$$\langle \tilde{n} \tilde{l} 1 \tilde{S}; \tilde{l} | n_1 l_1 n_2 l_2; \tilde{l} \rangle = (-)^{n_1 + n_2 + l_1 - \tilde{n} + 1} \cdot 2^{1/4(l_1 + l_2 - 3\tilde{l}) - \tilde{n} + 1} \\ \times \sqrt{\frac{(\tilde{n}-1)! (2\tilde{l}+2\tilde{n}-1)!! (2l_1+1) (2l_2+1)}{(n_1-1)! (n_2-1)! (2l_1+2n_1-1)!! (2l_2+2n_2-1)!! (2\tilde{l}+1)}} (l_1 l_2 00 | \tilde{l} 0).$$

The transformation coefficient $\langle \tilde{N} \tilde{L} \tilde{n} \tilde{l}; TSL | n_1 l_1 n_2 l_2; TSL \rangle$ of the antisymmetrized wave function $|n_1 l_1 n_2 l_2; TSL\rangle$ can be obtained by the following relation,

$$\langle \tilde{n} \tilde{l} \tilde{N} \tilde{L}; TSL | n_1 l_1 n_2 l_2; TSL \rangle \\ = \begin{cases} \frac{1 - (-)^{T+S+\tilde{l}}}{2} \cdot \sqrt{2} \langle \tilde{n} \tilde{l} \tilde{N} \tilde{L}; L | n_1 l_1 n_2 l_2; L \rangle & \text{for } n_1 l_1 \neq n_2 l_2 \\ \frac{1 - (-)^{T+S+\tilde{l}}}{2} \cdot \langle \tilde{n} \tilde{l} \tilde{N} \tilde{L}; L | n l n l; L \rangle & \text{for } n_1 l_1 = n_2 l_2 = n l. \end{cases}$$

The general derivation of the transformation coefficients will be discussed in the following paper.

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Electron Spin Resonance of Mn^{++} Ion in Ionic Crystals

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Fine structure constants of Mn^{++} ion are calculated by using overlap and covalent models. The results are compared with Watkins' and Tinkham's experiments. From the sign of D and E determined experimentally, it can be shown that in the case of Mn^{++} in alkaline chlorides covalency is dominant, while in the case of Mn^{++} in ZnF_2 the overlap effect predominates. It is also shown in each case that our model gives a correct order of magnitude of fine structure constants. Our model also accounts for the fact that the magnitudes of D increase from LiCl to KCl.

§ 1. Introduction

Recently, the electron spin resonance spectrum of Mn^{++} ion incorporated into ionic crystals has been extensively investigated by several workers.¹⁻¹²⁾ The results of these experiments give us information about the electronic structure around the Mn^{++} ion as well as dynamic processes involving the Mn^{++} ion and the vacancy of positive ion which is also incorporated into the crystal to restore the charge neutrality of the crystal. Especially the fine structure constants of D , E and a (in usual notations) are the subject of theoretical interest. In free state a Mn^{++} ion is in 6S state and there arises no anisotropy with respect to its spin direction. When it is incorporated into a crystal, there arises an anisotropy energy which depends on the spin orientation relative to the crystal axes. This may be accounted for by deformation of Mn^{++} ion, by overlap effects between Mn^{++} ion and the surroundings and by an admixture of excited configurations in which the Mn^{++} ion accepts an electron from the surrounding ions. Watanabe¹³⁾ has calculated D and a by taking account of deformation of Mn^{++} ion, but he replaced the effects of surroundings by a crystalline field. His results are, however, smaller than experimental values of D by a factor of about ten.³⁾ Although it is desirable to include overlap effects and covalency in the crystalline field, the deformation model cannot account for some experimental facts on Mn^{++} in alkaline chlorides (see § 3 (iii)). We shall here show calculations of fine structure constants D and E of Mn^{++} ion arising both from covalent model and from overlap model.

§ 2. Fine structure constants

We consider the system of one Mn^{++} ion and six negative ions surrounding

the Mn^{++} ion in nearly octahedral arrangement, as shown in Fig. 1. We assume that the 2-, 4- and 5- Cl^- ions occupy corners of regular octahedron at the distance a from the Mn^{++} ion, but that the positions of the 1-, 3- and 6- Cl^- ions deviate slightly from regular positions. The 1- Cl^- ion lies on the positive z axis at the distance a' from the Mn^{++} ion. The 3- and 5- ions lie on the x - y plane apart from x and y axes by the angle p at the distance a'' from the Mn^{++} ion (Fig. 1). We assume that the negative ions have a closed shell structure $(np)^6$ and that the Mn^{++} ion has its maximum spin which is directed to z' axis ($S_z = 5/2$).

We will calculate the energy of this system which depends on the direction cosines of the z' axis (l, m, n). We shall calculate terms only of the second power of l, m and n . As the atomic orbitals of these ions we take the Hartree-Fock solutions of free ions, and we take account of overlaps between the manganese $3d$ orbitals and the outermost np orbitals of the negative ions. We take account of only the largest overlaps, that is, those between those orbitals which have axial symmetry around the axis joining the

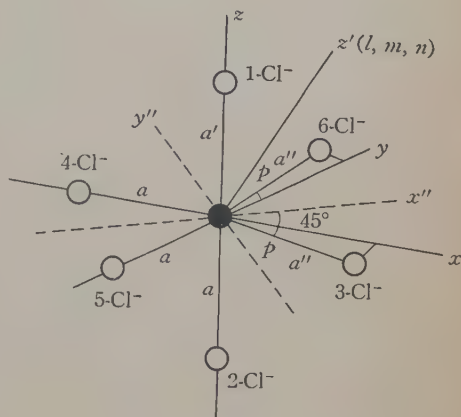


Fig. 1 ● Mn^{++} ○ Cl^-

Mn^{++} ion and the negative ion ($M_z = 0$). The maximum overlap integral between the $3d0$ orbital and $np0$ orbital of the i -th ion is denoted by S_i (here 0 means zero of the component of angular momentum with respect to the line joining the Mn^{++} ion and the i -th ion). We take account of covalency also. Suppose that the Mn^{++} ion has α spin direction. Then the np electrons of negative ions which have β spin can transfer to the empty $3d\beta$ orbitals. We take this effect into consideration by replacing the atomic orbital φ_{np0} of the $np\beta$ electron of the i -th ion by $N(\varphi_{np0} + \lambda_i \varphi_{3d0})$ where λ_i is the degree of covalency arising from the i -th ion and N is a normalization constant. We assume that the six φ_{np0} orbitals of six negative ions are simultaneously deformed in this way. Thus, we can write down our total wave function in the form of a single Slater determinant.

Now let $\mathbf{x} = (\mathbf{r}, \gamma)$ be the spatial and the spin coordinates of an electron. We take as perturbing Hamiltonian

$$H = H_1 + H_2, \quad (1)$$

$$H_1 = (1/2) \sum_{i \neq j} \sum V_{ij}, \quad (2)$$

$$H_2 = \sum_i \hat{\epsilon}(r_i) \mathbf{l}_i \cdot \mathbf{s}_i, \quad (3)$$

$$V_{12} = g^2 \beta^2 r_{12}^{-5} [(\mathbf{s}_1 \mathbf{s}_2) r_{12}^2 - 3(\mathbf{s}_1 \mathbf{r}_{12})(\mathbf{s}_2 \mathbf{r}_{12})]. \quad (4)$$

Here β is the Bohr magneton, $g=2$ for electron, \mathbf{s}_i and \mathbf{l}_i are the spin and orbital angular momenta of the i -th electron divided by \hbar and r_{12} is the distance between 1 and 2 electrons. H_1 is the dipole-dipole interaction between electron spins and H_2 is the spin-orbit interaction. The perturbed energy is

$$W = W_1 + W_2, \quad (5)$$

$$W_1 = (0|H_1|0), \quad (6)$$

$$W_2 = -\sum_n' |(0|H_2|n)|^2 / (E_n - E_0), \quad (7)$$

because the zero-th energy of H_2 vanishes. We first calculate W_1 .

$$W_1 = \int \Psi^* H_1 \Psi d\mathbf{x}_1 \cdots d\mathbf{x}_N = \frac{1}{2} \iint V_{12} \begin{vmatrix} \rho(\mathbf{x}_1', \mathbf{x}_1) & \rho(\mathbf{x}_1', \mathbf{x}_2) \\ \rho(\mathbf{x}_2', \mathbf{x}_1) & \rho(\mathbf{x}_2', \mathbf{x}_2) \end{vmatrix} d\mathbf{x}_1 d\mathbf{x}_2, \quad (8)$$

$$\rho(\mathbf{x}_1', \mathbf{x}_1) = N \int \Psi^*(\mathbf{x}_1', \mathbf{x}_2, \dots, \mathbf{x}_N) \Psi(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N) d\mathbf{x}_2 \cdots d\mathbf{x}_N. \quad (9)$$

In (8) V_{12} operates on the unprimed arguments of ρ , and then the primed arguments are set equal to the unprimed arguments ($\mathbf{x}_i' = \mathbf{x}_i$).¹⁴⁾ Since our total wave function is expressed by a single Slater determinant, ρ can be written in the form

$$\rho(\mathbf{x}_1', \mathbf{x}_1) = \rho_\alpha(\mathbf{r}_1', \mathbf{r}_1) \alpha^*(\gamma_1') \alpha(\gamma_1) + \rho_\beta(\mathbf{r}_1', \mathbf{r}_1) \beta^*(\gamma_1') \beta(\gamma_1). \quad (10)$$

Then substituting (10) and (4) into (8) and integrating over spin coordinates, we obtain

$$E = (1/8) g^2 \beta^2 \iint (1 - 3 \cos^2 \theta) r_{12}^{-3} [A\rho(\mathbf{r}_1, \mathbf{r}_1) A\rho(\mathbf{r}_2, \mathbf{r}_2) - A\rho(\mathbf{r}_1, \mathbf{r}_2) A\rho(\mathbf{r}_2, \mathbf{r}_1)] d\mathbf{r}_1 d\mathbf{r}_2, \quad (11)$$

$$A\rho = \rho_\alpha - \rho_\beta, \quad (12)$$

where θ is the angle between the spin direction (z' axis) and \mathbf{r}_{12} vector. A general scheme of obtaining ρ from a wave function is given by Löwdin.¹⁴⁾ When we assume that λ and S are small quantities and retain up to the second order quantities of them, we obtain from our wave function

$$\begin{aligned} A\rho = & \sum_{i=1}^5 \phi_i \phi_i + (3/2) (\mu + \mu'') (\phi_1 \phi_1 + \phi_2 \phi_2) + A\phi_1 \phi_1 - A'\phi_2 \phi_2 \\ & + A'\phi_3 \phi_3 + (A''/2) (\phi_1 \phi_3 + \phi_3 \phi_1), \end{aligned} \quad (13)$$

$$\left. \begin{aligned} A &= \mu' - \mu - (\mu'' - \mu), \\ A' &= (3/2) \mu'' \sin^2 2p, \\ A'' &= \sqrt{3} \mu'' \sin 2p, \end{aligned} \right\} \quad (14)$$

$$\left. \begin{aligned} \mu &= S^2 - \lambda^2, \quad \mu' = S'^2 - \lambda'^2, \quad \mu'' = S''^2 - \lambda''^2, \\ S &= S_2 = S_4 = S_6, \quad S' = S_1, \quad S'' = S_3 = S_5, \\ \lambda &= \lambda_2 = \lambda_4 = \lambda_6, \quad \lambda' = \lambda_1, \quad \lambda'' = \lambda_3 = \lambda_5, \end{aligned} \right\} \quad (15)$$

$$\phi_1 \sim (1/\sqrt{3})(3z^2 - r^2), \quad \phi_2 \sim x^2 - y^2, \quad \phi_3 \sim 2xy. \quad (16)$$

In (13) we retained only those terms containing $3d$ wave functions of the Mn^{++} ion. Substituting (13) into (11) and retaining only the second order terms of S and λ , we obtain

$$W_1 = (1/4)g^2\beta^2 \iint v_{12} [\rho_0(\mathbf{r}_1, \mathbf{r}_1)\rho'(\mathbf{r}_2, \mathbf{r}_2) - \rho_0(\mathbf{r}_1, \mathbf{r}_2)\rho'(\mathbf{r}_2, \mathbf{r}_1)] d\mathbf{r}_1 d\mathbf{r}_2, \quad (17)$$

$$v_{12} = (3 \cos^2 \theta - 1) r_{12}^{-3}, \quad (18)$$

$$\rho_0 = \sum_{i=1}^5 \phi_i \phi_i, \quad (19)$$

$$\rho' = \Delta \phi_1 \phi_1 - \Delta' \phi_2 \phi_2 + \Delta' \phi_3 \phi_3 + (\Delta''/2)(\phi_1 \phi_3 + \phi_3 \phi_1). \quad (20)$$

When integrations are carried out, we see

$$W_1 = -(3/14)g^2\beta^2(R_0 - \frac{8}{7}R_2)[\Delta(3n^2 - 1) - 2\sqrt{3}\Delta''lm], \quad (21)$$

$$R_l = \int_0^\infty dr_2 P_{3d}^2(r_2) r_2^{-l-3} \int_0^{r_2} P_{3d}^2(r_1) r_1^l dr_1, \quad (22)$$

where l, m and n are direction cosines of the spin direction z' . This anisotropy energy arises from non-spherical spin distribution of manganese ion. Non-sphericity comes both from overlap effect and from electron transfer. Overlap effect increases charge density near the nucleus of $3d$ orbital which has non-orthogonal overlap with negative ions, while transfer effect introduces electron density of the opposite spin direction. Thus two effects have opposite tendency to non-sphericity of the manganese ion. This can be seen from (15), where μ 's are given as the difference between S^2 and λ^2 . Next we calculate W_2 . If we replace $E_n - E_0$ in the denominator of (7) by some average $\langle \Delta E \rangle$, we have

$$W_2 = -(0|H_2^2|0)/\langle \Delta E \rangle, \quad (23)$$

$$H_2^2 = \sum_i \xi^2(r_i) (\mathbf{l}_i \mathbf{s}_i)^2 + \sum_{i \neq j} \xi(r_i) \xi(r_j) (\mathbf{l}_i \mathbf{s}_i) (\mathbf{l}_j \mathbf{s}_j). \quad (24)$$

Since H_2^2 is the sum of one-electron and two-electron operators, we can calculate $(0|H_2^2|0)$ in the same way as we calculate the kinetic and Coulomb energies of many-electron system.¹⁵⁾ The result is

$$W_2 = (1/2\langle \Delta E \rangle) \iint \xi(r_1) \xi(r_2) l_{1z'} l_{2z'} \rho_0(\mathbf{r}_1', \mathbf{r}_2) \rho'(\mathbf{r}_2', \mathbf{r}_1) d\mathbf{r}_1 d\mathbf{r}_2, \quad (25)$$

$$= -\{\langle \xi \rangle^2/2\langle \Delta E \rangle\} [\Delta(3n^2 - 1) - 2\sqrt{3}\Delta''lm], \quad (26)$$

$$\langle \xi \rangle = \int_0^\infty \xi(r) P_{3d}^2(r) dr. \quad (27)$$

Hence,

$$W = W_1 + W_2 = - \left\{ \frac{3}{14} g^2 \beta^2 \left(R_0 - \frac{8}{7} R_2 \right) + \frac{\langle \xi \rangle^2}{2 \langle \Delta E \rangle} \right\} [A(3n^2 - 1) - 2\sqrt{3} J'' lm]. \quad (28)$$

In the usual experiments the spin Hamiltonian is assumed to be of the form of

$$DS_z^2 + E(S_{x''}^2 - S_{y''}^2), \quad (29)$$

where x'' and y'' axes are shown in Fig. 1. Taking the expectation value of (29) and comparing it with (28), we get finally

$$D = -(9/70) \Delta P, \quad (30)$$

$$E = (9/70) \mu'' \sin 2p P, \quad (31)$$

$$P = g^2 \beta^2 (R_0 - 8R_2/7) + 7 \langle \xi \rangle^2 / 3 \langle \Delta E \rangle. \quad (32)$$

Watanabe¹³⁾ estimated R_0 and R_2 as $R_0 = 3.06 a_0^{-3}$ and $R_2 = 0.54 a_0^{-3}$. Then the first term of P becomes 28.6 cm^{-1} . As the value of $\langle \xi \rangle$ we take Kotani's value of 395 cm^{-1} determined from spectroscopic data.¹⁶⁾ The choice of the value of $\langle \Delta E \rangle$ introduces an ambiguity. But the term value of 29250 cm^{-1} , which is the energy of the excitation ${}^6S \rightarrow {}^4P$, may be pertinent. Then the second term of P becomes 10.6 cm^{-1} . Since this is about one-third of the first, the error introduced in the choice of the value of $\langle \Delta E \rangle$ may give rise to only a small ambiguity in the following discussion of each experimental data. Using above values of parameters, we have

$$D = -5.05 \Delta \text{ cm}^{-1}, \quad (33)$$

$$E = 5.05 \mu'' \sin 2p \text{ cm}^{-1}. \quad (34)$$

§ 3. Comparison with experiment

In this section we shall compare above results with experiment. In (i), (ii) and (iii) we refer to Watkins' spin resonance experiments on Mn^{++} in alkaline chlorides,³⁾ and in (iv) we refer to Shulman and Jaccarino's experiments¹⁷⁾ on F^{19} nuclear resonance in MnF_2 as well as Tinkham's Mn^{++} resonance⁹⁾ in ZnF_2 . In (v) we shall give a brief discussion about anisotropy energy of a Fe^{+++} ion in hexagonal oxides.

(i) Mn^{++} : NaCl , III_2 spectrum.

This is the resonance spectrum of the Mn^{++} ion in the NaCl crystal, which is accompanied by a cation vacancy in the next nearest cation site. We assume that the vacancy lies on the positive z axis. Then from symmetry consideration, we have $p=0$, and hence $E=0$. Since the 1-Cl^- ion is attracted by the Mn^{++} ion and repelled by the vacancy by virtue of electrostatic field, the 1-Cl^- ion approaches to the Mn^{++} ion, hence S' is larger than other five overlap integrals; but we assume approximately that the latter may be nearly equal to each other. Now λ 's should be determined so as to minimize the total energy of the system. If we

assume λ 's to be small quantities, the problem is similar to that we have treated in the theory of superexchange interaction.¹⁸⁾ Then $\lambda^2 = (b/B)^2$, where b is the transfer matrix of an np electron to the $3d$ orbital and B is the excitation energy necessary to transfer the np electron to the $3d$ orbital. Let us denote by B_i and b_i the excitation energy and the transfer matrix of the np electron of the i -th chlorine ion to the Mn^{++} ion. Then, following Mott's procedure, we have $B_1=3.3$ ev, $B_2=6.5$ ev, $B_3=\dots=B_6=6$ ev. We see that B_1 is very small and B_2 is nearly equal to B_3, \dots, B_6 . Then we may approximately assume that λ_2 to λ_6 are nearly equal to each other. Then from (14), we have $\Delta = \mu' - \mu$. From experimental results, $D = +130 \times 10^{-4} \text{ cm}^{-1}$, and from (33), we have $\mu' - \mu = -0.0026$. Since $S' > S$, the overlap model gives a wrong sign of D in this case. We must, therefore, invoke of the covalent model. If we assume tentatively that all the transfer matrices have the same value b ($\lambda' = b/3.3$, $\lambda = b/6.5$), we have $b = 0.2$ ev from $\lambda'^2 - \lambda^2 = 0.0026$. The value 0.2 ev of b is a reasonable one*. Since b_1 may be larger than the other b 's, we can get to $\lambda'^2 - \lambda^2 = 0.0026$ from a smaller value of b than 0.2 eV. If we take account of overlap effect simultaneously, the value of b must be larger, but the qualitative feature may not be altered. Then we conclude that the D term of the III_2 spectrum can be led from the covalent model in a reasonable way. We note that this is ascribed to the smallness of B_1 .

(ii) Mn^{++} : NaCl, III_1 spectrum.

This is the resonance spectrum of the Mn^{++} ion in the NaCl crystal which is accompanied by the cation vacancy in the nearest neighbour cation site. Let the vacancy lie on the first quadrant of the x - y plane. Then the 3- and 6- Cl^- ions should be distinguished from others. When we assume $\mu' = \mu$, we get $\Delta = -(\mu'' - \mu)$. From experiments, $D = -135 \times 10^{-4} \text{ cm}^{-1}$ and $|E| = 40 \times 10^{-4} \text{ cm}^{-1}$. Since the 3- and 6- Cl^- ions are expected to be repelled by the vacancy and attracted by the Mn^{++} ion, we may expect $S'' > S$. Then from the overlap model we have $D > 0$. We must, therefore, invoke of the covalent model in this case also. From Mott's procedure, $B_3 = B_6 = 4.3$ ev, $B_1 = B_2 = 6.5$ ev and $B_4 = B_5 = 7$ ev. Since B_1 and B_2 are nearly equal to B_4 and B_5 , we may set $\lambda'^2 \simeq \lambda^2$ as assumed. Then from the covalent model, $\Delta = \lambda''^2 - \lambda^2$. From $D = -135 \times 10^{-4} \text{ cm}^{-1}$ and $\lambda'' = b/B_3$, $\lambda = b/B_4$, we get $b = 0.3$ ev, which is also a value of a reasonable magnitude. We now determine the angle p from E term. From the above result, we have $\mu'' = -(b/B_4)^2 = -0.0049$. Then from $|E| = 40 \times 10^{-4} \text{ cm}^{-1}$, we have $|p| = 5.4^\circ$, which is also not unexpected. We notice that the sign of E cannot be determined from usual resonance experiments, that is, we cannot know whether the easy axis in the x - y plane is the line joining the vacancy and the Mn^{++} ion or it is at right angle to this line. Since from the consideration of the electrostatic energy we expect p is positive and our covalent model says that μ'' is negative, it is our theoretical prediction that the easy axis is the line joining the vacancy and the Mn^{++} ion.

* In the case of $MnO^{18)}$, our calculation gives 1 ev for b .

(iii) Mn^{++} : LiCl, KCl.

These cases are similar to the previous ones and we expect that the same qualitative conclusion holds in these cases. This can be most clearly seen from the fact that the absolute magnitude of D increases from LiCl to KCl. The deformation model, such as used by Watanabe, cannot account for this fact, because the crystalline field decreases from LiCl to KCl. In our model this fact may be explained by the following facts. First, since the Madelung energy and hence the transfer energy decreases from LiCl to KCl, the degree of covalency may increase, when the decrease of transfer matrix b , in going from LiCl to KCl, is slower than the decrease of B . Secondly, since the overlap effect becomes smaller from LiCl to KCl, the negative contribution to \mathcal{A} , that is $-S^2$, becomes smaller and hence \mathcal{A} becomes larger.

(iv) Mn^{++} : ZnF_2 .

Tinkham⁹⁾ experimentally determined the parameters of the spin Hamiltonian

$$H = DS_z^2 - E_1 S_x'^2 + E_2 S_y'^2,$$

as $D = 21 \times 10^{-4} \text{ cm}^{-1}$, $E_1 = 124 \times 10^{-4} \text{ cm}^{-1}$ and $E_2 = 103 \times 10^{-4} \text{ cm}^{-1}$. Since in this case the distances of six chlorine ions from the Mn^{++} ion are nearly equal, we may assume $\mu = \mu' = \mu''$. From crystallographic data we see $\sin 2p = -0.22$. Since in this case the 4- and 5- Cl^- ions are also rotated from octahedral positions in the x - y plane by the angle p (Fig. 2), the numerical factor of the right-hand side of (34) must be doubled. Then we have

$$D = 0, \quad (35)$$

$$E = 10.1 \mu \sin 2p \text{ cm}^{-1}. \quad (36)$$

Tinkham's data yield an approximate value of $D \simeq 0$ and $E_1 \simeq E_2 = 110 \times 10^{-4} \text{ cm}^{-1}$. Hence we have $-10.1 \mu (-0.22) = 110 \times 10^{-4}$ and this gives

$$\mu = S^2 - \lambda^2 = 0.005. \quad (37)$$

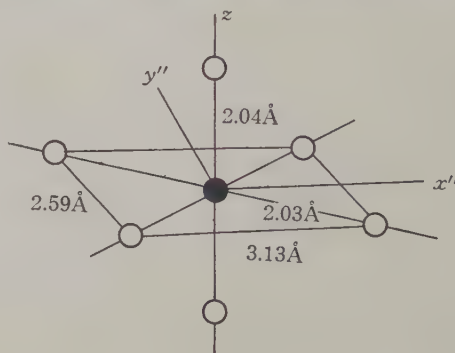


Fig. 2 ● Zn^{++} ○ F^-

Thus we see in this case that the overlap effect is larger than the covalency effect. This is as expected, because the transfer energy B amounts to 15 eV in this case. Now, Shulman and Jaccarino¹⁷⁾ observed hyperfine interactions between Mn^{++} spin and neighbouring F^{19} nuclei. From analysing the results, they have concluded that the effective fraction of $2p0$ orbital which is mixed into $3d0$ orbital is $0.6 \pm 0.3\%$. From (9) and using our wave function described early, we can see that the part of ρ which includes φ_{np0}^2 is $[1 + (S + \lambda)^2] \varphi_{np0} \alpha^* \alpha + \varphi_{np0}^2 \beta^* \beta$ up to the second order terms of λ and S . Then we can set $(S + \lambda)^2 = 0.006$. From this and (37) we have $S = 0.07$, $\lambda = 0.006$. From $\lambda = (b/15) = 0.006$ we have $b = 0.09 \text{ eV}$, which value is also not unexpected. Using the Hartree-Fock solutions of free ions,^{19), 20)} we have

calculated $S=0.07$. The agreement is satisfactory. Since the experimental error of Shulman and Jaccarino's result and the error introduced in analysing Tinkham's data are large, the agreement may be accidental, but this may not alter the qualitative conclusion that the overlap effect is larger than the covalency effect and that, the totally ionic model ($\lambda=0$) using the Hartree-Fock solutions of free ions gives correct order of magnitude of fine structure constant in this case.

(v) Casimir *et al.*⁽²¹⁾ have measured magnetic anisotropy constant K_1 of hexagonal oxide $BaFe_{12}O_{19}$. This anisotropy energy arises from dipole-dipole interaction between Fe^{+++} ions and from one-ion anisotropy such as discussed above. From measured K_1 and calculated contribution to K_1 from dipole interaction we are left to one-ion anisotropy of⁽²¹⁾

$$W = D'n^2 \text{ per molecule}$$

where $D' = -8 \text{ cm}^{-1}$ and n is the cosine of the angle between c -axis and spin direction. The value of -8 cm^{-1} of D' is very large compared with those found in usual resonance experiments. This may be ascribed to the presence of Fe^{+++} ions which are surrounded by five oxygen ions, as Casimir *et al.*⁽²¹⁾ pointed out. Three of the five oxygen ions lie in a c -plane which involves the Fe^{+++} ion and very close to the Fe^{+++} ion (at the distance of 1.70 \AA). The other two lie above and below the Fe^{+++} ion at the distance of 2.32 \AA . One-ion anisotropy of this Fe^{+++} ion can be calculated in the same way as in § 2. The result is $D' = 27/28 \cdot \mu P$, where we took account only of three oxygen ions in a c -plane, since they are much closer to Fe^{+++} ion than the other two. It will not introduce a serious error to ascribe the origin of the anisotropy energy of -8 cm^{-1} mainly to the Fe^{+++} ion surrounded by five oxygen ions. Since D' is negative, the covalent effect is larger than the overlap effect. If we neglect S and assume tentatively $\mu \simeq 0.1$, we have $P = 80 \text{ cm}^{-1}$, which is twice as that of Mn^{++} ion. Since a Fe^{+++} ion is smaller than a Mn^{++} ion, it is natural that P is larger in Fe^{+++} than in Mn^{++} . In any way $\mu P = 8 \text{ cm}^{-1}$ is not unexpected.

§ 4. Discussions

Since the fine structure constants of the Mn^{++} ion contain $S^2 - \lambda^2$, we can draw a definite conclusion about the dominant mechanism giving rise to fine structure constants of the Mn^{++} ion, when we can determine experimentally the sign of D or E . This is contrasted with the fact that the experiments of hyperfine interaction between the spin of the Mn^{++} ion and the nuclear spins of negative ions can give only information about $(S + \lambda)^2$ and that this cannot decide purely experimentally whether the covalent effect or the overlap effect predominates.

Usually the hyperfine interaction between the spin of Mn^{++} ion and the nuclear spins of negative ions is immediately related to the covalency and hence to the superexchange interaction when the Mn^{++} ions are not diluted by diamagnetic ions. As pointed out by Yamashita *et al.*⁽²²⁾ and by Mukherji and Das⁽²³⁾, not only the

covalency but also the overlap effects are responsible for this interaction. Both these effects are also responsible for the superexchange interaction. This is pointed out by Yamashita and the author,²⁴⁾ but it is a very difficult problem both in theoretical and in experimental aspects to know the dominant mechanism of superexchange interaction. The author has pointed out the importance of the Slater mechanism in the case of MnO .¹⁸⁾ Since the Slater mechanism may give only a small contribution both to D and E and to the hyperfine interaction between Mn^{++} spin and nuclear spin of negative ions, we cannot say much about superexchange interaction only from these resonance experiments.

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Spin-Orbit Splitting and Tensor Force. II

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General formulae of the second order perturbation energies due to the tensor force are given in the case of the closed shell+one nuclei, and useful formulae for calculating the two-body matrix elements are also derived. Using these formulae, the D -state doublet splitting in O^{17} is estimated and it is found that about a half of the observed value is explained in terms of the second order effect of the tensor force as in the case of He^6 and N^{16} .

§ 1. Introduction

In the preceding paper [I],¹⁾ one of the authors (T.T.) has estimated the second order effect of the tensor force on the spin-orbit splitting of He^6 and N^{16} using the meson theoretic potential and a phenomenological Serber one with a strong tensor part.

In this paper general formulae for the second order perturbation energy due to the tensor force are given in the case of the closed shell+one nuclei, and also useful formulae for calculating the two-body matrix elements are derived. These formulae are adopted for estimating the D -state doublet splitting of O^{17} due to the tensor force. It is found that the strong two-body tensor force can qualitatively explain the origin of the one-body spin-orbit force in the nuclear shell model, but quantitatively this force gives about a half of the observed value of the doublet splitting. It is mainly because the deformation of the closed shell core induced by the mutual tensor interaction among the core-nucleons is affected by the presence of the outmost nucleon so as to satisfy the Pauli principle. This situation is the same as in [I].

In § 2, the general formulae giving the second order effect of the tensor force are derived in the nuclei of the zeroth order configuration, closed shell (in LS -coupling sense)+one nucleon. In § 3, assuming the average field to be a harmonic oscillator well as in [I], the two-body wave functions are transformed into the wave functions in the system of the relative and the centre of mass coordinate.

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And formulae for the transformation coefficients are obtained when the state of the centre of mass coordinate is $(1s)$, $(1p)$, $(1d)$ and $(2s)$. Also, the recursion formula for the transformation coefficients is derived. The second order effect of the tensor force on the doublet splitting in O^{17} is calculated numerically in § 4, and the discussions are given in § 5.

§ 2. The second order effect of the tensor force

Throughout the present paper, the nuclei of the zeroth order configuration, closed shell + one nucleon, are treated. In these nuclei the first order perturbation energies due to the tensor force vanish. Therefore, we have computed the second order perturbation effect of the tensor force.

The zeroth order shell model wave functions for these nuclei are given in the LS -coupling scheme as follows,

$$\phi_0 = |(n_1 l_1)^{8l_1+4}(000)(n_2 l_2)^{8l_2+4}(000) \cdots (n_k l_k)^{8l_k+4}(000)nl, T=\frac{1}{2}(T_z) S=\frac{1}{2} L=l; JM\rangle \quad (2.1)$$

where (000) means all of the resultant isotopic spin T , ordinary spin S and angular momentum L are zero, and $n_1 l_1, \dots, n_k l_k$ are principal and azimuthal quantum numbers of the closed shells respectively and nl those of the outmost nucleon. The excited configurations ϕ_n , which can mix with the shell model wave function (2.1) in the first order perturbation, must not have more than two single particle orbitals different from the configuration (2.1). Furthermore, only the excited states which have the total spin $S=3/2$ or $5/2$ can have the non-vanishing matrix elements of the tensor interaction with the shell model wave function (2.1).

Now, the second order perturbation energy due to the tensor force is expressed as follows,

$$\Delta E = \sum_n' \frac{|\langle \phi_n | V_T | \phi_0 \rangle|^2}{E_0 - E_n}, \quad (2.2)$$

where E_0 and E_n are the energies of the zeroth order state and of the excited state n , respectively. The tensor potential, $V_T \equiv \sum_{i>j} v_T(\mathbf{r}_{ij})$, is written as

$$v_T(\mathbf{r}_{ij}) = [a + b(\boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j)] S_{ij} V(r_{ij}) \quad (2.3)$$

where a and b are constants, $\boldsymbol{\tau}_i$ the isotopic spin operator, $V(r_{ij})$ the radial part of the potential and $S_{ij} = 3(\boldsymbol{\sigma}_i \cdot \mathbf{r}_{ij})(\boldsymbol{\sigma}_j \cdot \mathbf{r}_{ij})/r_{ij}^2 - (\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j)$, with the Pauli spin operator $\boldsymbol{\sigma}_i$. In the following computation, it may be convenient to rewrite $S_{ij} V(r_{ij})$ as

$$S_{ij} V(r) = \mathbf{S}^{(2)} \cdot \mathbf{L}^{(2)}, \quad (2.4)$$

where $S_m^{(2)}$ is $[\boldsymbol{\sigma}_i \times \boldsymbol{\sigma}_j]_m^{(2)}$ and $L_m^{(2)}$ is proportional to the product of $V(r)$ and the spherical harmonics $Y_{2m}(\theta, \varphi)$, in which (r, θ, φ) are the relative coordinates between the particles i and j . Through the usual method, the matrix element of the tensor potential becomes

$$\langle TS'L'; J | V_T | TSL; J \rangle = (-)^{S'-L+J} W(S'L'SL; J2) \langle S'L' || \mathbf{S}^{(2)} \cdot \mathbf{L}^{(2)} || SL \rangle \phi_\tau(T), \quad (2.5)$$

where T , S and L are total isotopic spin, total ordinary spin and total angular momentum of the initial state, T , S' and L' those of the final state, J total spin and $\phi_\tau(T)$ is the expectation value of $a+b(\tau_1 \cdot \tau_2)$ in the state of the isotopic spin T . $W(S'L'SL; J2)$ and $\langle S'L' || \mathbf{S}^{(2)} \mathbf{L}^{(2)} || SL \rangle$ are the Racah coefficient and the reduced matrix element, respectively.

In the following part of this section, the excited configurations of various types will be presented and the contributions to the second order perturbation energy will be calculated.

(I) The first kind of the configurations are

$$\psi_{Ia,n} = [(n_1 l_1)^{8l_1+4} (000), \dots (n_i l_i)^{8l_i+2} (T_1 1 L_1) \dots, (n_k l_k)^{8l_k+4} (000) (n'_i l'_i n''_i l''_i) (T_1 1 L_2)] (022) nl, T = \frac{1}{2} (T_z) SL; JM \rangle \quad (2.6)$$

and

$$\psi_{Ib,n} = [(n_1 l_1)^{8l_1+4} (000), \dots \{ (n_i l_i)^{8l_i+3} (\frac{1}{2} \frac{1}{2} l_i) (n_j l_j)^{8l_j+3} (\frac{1}{2} \frac{1}{2} l_j) \} (T_1 1 L_1), \dots (n_k l_k)^{8l_k+4} (000) (n'_i l'_i n'_j l'_j) (T_1 1 L_2)] (022) nl, T = \frac{1}{2} (T_z) SL; JM \rangle \quad (2.7)$$

which are graphically shown in Figs. 1 and 2. In these configurations two nucleons are excited from the core into other unoccupied orbits than nl -orbit, i. e., $n'_i l'_i n''_i l''_i$ and $n'_j l'_j$ cannot coincide with nl . Both the total ordinary spin and total angular momentum of the excited core must be 2, because the tensor force is a scalar product of two second rank tensors $\mathbf{S}^{(2)}$ and $\mathbf{L}^{(2)}$ as can be seen from Eq. (2.4). These configurations interact with the shell model wave function (2.1) and give the same correction as that of the self energy of the closed shell nuclei in the second order perturbation. And these corrections cannot give the energy difference between two states $J=l+1/2$ and $J=l-1/2$. By the method of the tensor calculus, the contributions ΔE due to these configuration mixings may be easily estimated and result into

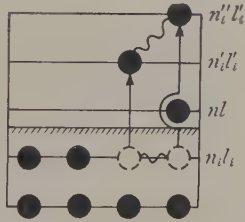


Fig. 1 Configuration Ia.

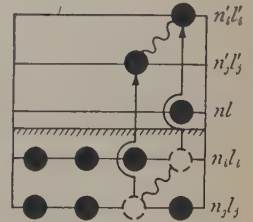


Fig. 2 Configuration Ib.

$\Delta E_{Ia} = -\frac{1}{5} \sum_{\substack{n'_i l'_i, n'_j l'_j, n''_i l''_i \\ n_i l_i}} \frac{1}{\Delta E_{on, T_1, L_1, L_2}} (2T_1+1) \times | \langle n'_i l'_i n'_j l'_j n''_i l''_i 1 L_2 || \mathbf{S}^{(2)} \mathbf{L}^{(2)} || (n_i l_i)^2 1 L_1 \rangle |^2 \phi_\tau(T_1) \quad (2.8)$

and

$$\Delta E_n = -\frac{1}{5} \sum_{\substack{n_i l_i, n_j l_j \\ n_i' l_i', n_j' l_j'}} \frac{1}{\Delta E_{on}} \sum_{T_1 T_2} (2T_1+1) \\ \times |\langle n_i' l_i' n_j' l_j' 1 L_2 \| \mathbf{S}^{(2)} \mathbf{L}^{(2)} \| n_i l_i n_j l_j 1 L_1 \rangle|^2 \Phi_\tau^2(T_1) \quad (2.9)$$

where ΔE_{on} is the zeroth order energy difference between the states (2.1) and (2.6) or (2.7), and all two-body wave functions $|n_\alpha l_\alpha n_\beta l_\beta T S L\rangle$ should be antisymmetrized.

Mixing percentage of these configurations is rather large as discussed in § 5, but it should be noted that these configuration mixings do not influence the expectation value of any single particle operator except for a scalar one. The deformation of the closed shell core due to these configuration mixings is caused by the mutual interaction between core-nucleons and then it will hereafter be briefly called "the self-deformation of the closed shell core".

(II) The second kind of the configurations are

$$\psi_{II,n} = |(n_1 l_1)^{8l+4} (000), \dots (n_i l_i)^{8l+3} (\frac{1}{2} \frac{1}{2} l_i), \dots (n_k l_k)^{8l+4} (000), (n_i' l_i' n' l') (T' 1 L'), \\ T = \frac{1}{2} (T_z) S = \frac{3}{2} L; JM\rangle \quad (2.10)$$

in which one core nucleon is excited by the interaction with the outmost nucleon (see Fig. 3). The resultant spin of $(n_i' l_i' n' l')$ is restricted to one, because the tensor force has non-vanishing two-body matrix element only in the spin triplet state. The second order perturbation energies due to the configuration mixings of this type result in

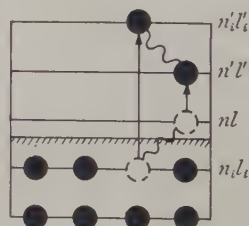


Fig. 3 Configuration II.

$$\Delta E_{J,II} = - \sum \frac{1}{\Delta E_{on}} \left[\frac{1}{20(2L+1)} \sum_{T' L' L''} (2T'+1) \right. \\ \times |\langle n_i' l_i' n' l' 1 L' \| \mathbf{S}^{(2)} \mathbf{L}^{(2)} \| n_i l_i n l 1 L'' \rangle|^2 \Phi_\tau^2(T') \\ - (-)^{1/2+l-J} W(\frac{1}{2} \frac{1}{2} l l; 1 J) \cdot \frac{3}{4\sqrt{5}} \sum_{T', L', L'', \tilde{L}''} (-)^{L''+\tilde{L}''} \\ \times (2T'+1) \sqrt{(2L''+1)(2\tilde{L}''+1)} W(l_i L'' l_1; l \tilde{L}'') W(L' L'' 21; 2\tilde{L}'') \\ \times \langle n_i' l_i' n' l' 1 L' \| \mathbf{S}^{(2)} \mathbf{L}^{(2)} \| n_i l_i n l 1 L'' \rangle \\ \left. \times \langle n_i' l_i' n' l' 1 L' \| \mathbf{S}^{(2)} \mathbf{L}^{(2)} \| n_i l_i n l 1 \tilde{L}'' \rangle \cdot \Phi_\tau^2(T') \right]. \quad (2.11)$$

In the brace of Eq. (2.11), the first term gives the common energy shift for both states of the $J=l+1/2$ and $J=l-1/2$. The second term gives the energy difference between these states. It is very interesting to note that this term has a factor $(-)^{1/2+l-J} W(\frac{1}{2} \frac{1}{2} l l; 1 J)$, which appears in the expectation value of $\langle \frac{1}{2} l; JM | \mathbf{s} \cdot \mathbf{l} | \frac{1}{2} l; JM \rangle$. $n_i' l_i'$ and $n' l'$ can be any orbit as far as the Pauli principle is not violated.

Only the second order effect of this type has been considered by Kisslinger²⁾ and Jancovici,^{3)*} whose calculations have shown this contribution to be small or of wrong sign. Brueckner et al.⁴⁾ have estimated more accurately the one-body spin-orbit force induced by the tensor force between the outside nucleon and the closed shell core, and have got the negative result. The present calculation and the preceding one¹⁾ have also led to the same conclusion. Therefore, it may be said that the configuration mixings of this type cannot explain the observed doublet splitting. The mixed configurations considered here will be called "the induced deformation of the closed shell core", because the mixings of this type are induced by the mutual interaction between the outside nucleon and the closed shell core.

(III) In the type I, it is not taken into consideration that the core-nucleons excited from the closed shells by the mutual interaction jump into the outmost orbit nl .

If the Pauli principle does not work, also in these cases the contributions are same for both spin states $J=l \pm \frac{1}{2}$. However, it has been shown in [I] that this exclusion effect is important for the doublet splitting. Therefore, these cases will be considered in this paragraph. There are several configurations in which the nucleons jump from the closed shells into the most outside orbit nl . The first of them is

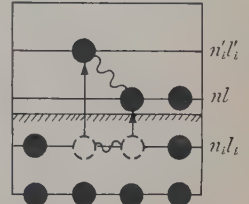


Fig. 4 Configuration IIIa.

$$\psi_{IIIa,n} = |(n_1 l_1)^{8l_1+4} (000), \dots (n_i l_i)^{8l_i+2} (T_1 S_1 L_1) \dots (n_k l_k)^{8l_k+4} (000) \\ \{n'_i l'_i \cdot (nl)^2 (T_2 S_2 L_2)\} (T_3 S_3 L_3), T = \frac{1}{2} (T_z) SL; JM\rangle, \quad (2 \cdot 12)$$

which corresponds to Fig. 4. The energy shift caused by these configuration interactions becomes

$$\Delta E_{J,IIIa} = - \sum \frac{1}{\Delta E_{on}} \left[\left\{ \frac{1}{5} - \frac{1}{20(2l+1)} \right\} \sum_{T_1, L_1, L_4} (2T_1+1) \right. \\ \times |\langle n'_i l'_i nl 1 L_4 \| S^{(2)} L^{(2)} \| (n_i l_i)^2 1 L_1 \rangle|^2 \phi_{\tau}^2 (T_1) \\ + (-)^{1/2+l-J} W(\frac{1}{2} \frac{1}{2} ll; 1J) \cdot \frac{3}{4\sqrt{5}} \sum_{T_1, L_1, L_4, \tilde{L}_4} (-)^{L_4+\tilde{L}_4} (2T_1+1) \\ \times \sqrt{(2L_4+1)(2\tilde{L}_4+1)} W(l'_i L_4 l 1; l \tilde{L}_4) W(L_1 L_4 2 1; 2\tilde{L}_4) \\ \times \langle (n_i l_i)^2 1 L_1 \| S^{(2)} L^{(2)} \| n'_i l'_i nl 1 L_4 \rangle \\ \left. \times \langle (n_i l_i)^2 1 L_1 \| S^{(2)} L^{(2)} \| n'_i l'_i nl 1 \tilde{L}_4 \rangle \phi_{\tau}^2 (T_1) \right]. \quad (2 \cdot 13)$$

The second is

* Recently, Takagi et al.¹²⁾ and Jancovici¹³⁾ have calculated the other effect which is discussed in the next paragraph, using the Fermi gas model.

$$\begin{aligned} \phi_{IIIb,n} = & |(n_1 l_1)^{8l_1+4} (000), \dots \{ (n_i l_i)^{8l_i+3} (\frac{1}{2} \frac{1}{2} l_i) (n_j l_j)^{8l_j+3} (\frac{1}{2} \frac{1}{2} l_j) \} (T_1 S_1 L_1), \dots \\ & (n_k l_k)^{8l_k+4} (000), \{ n'_i l'_i \cdot (nl)^2 (T_2 S_2 L_2) \} (T_3 S_3 L_3), T = \frac{1}{2} (T_z) SL; JM \rangle, \end{aligned} \quad (2.14)$$

which is shown graphically in Fig. 5. The contribution coming from this type is given by the equation

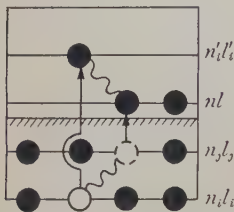


Fig. 5 Configuration IIIb.

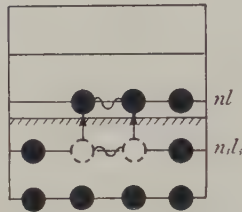


Fig. 6 Configuration IIIc.

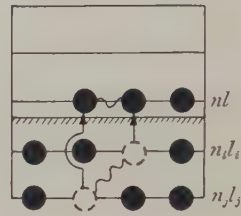


Fig. 7 Configuration IIId.

$$\begin{aligned} \Delta E_{J,IIIb} = & - \sum \frac{1}{\Delta E_{on}} \left[\left\{ \frac{1}{5} - \frac{1}{20(2l+1)} \right\} \sum_{T_1, L_1, L_4} (2T_1+1) \right. \\ & \times |\langle n_i l_i n_j l_j 1 L_1 \| S^{(2)} L^{(2)} \| n'_i l'_i nl 1 L_4 \rangle|^2 \phi_{\tau}^2(T_1) \\ & + (-)^{1/2+l-j} W(\frac{1}{2} \frac{1}{2} ll; 1J) \cdot \frac{3}{4\sqrt{5}} \sum_{T_1, L_1, L_4, \tilde{L}_4} (-)^{L_4 - \tilde{L}_4} (2T_1+1) \\ & \times \sqrt{(2L_4+1)(2\tilde{L}_4+1)} W(l'_i L_4 l 1; \tilde{L}_4) \\ & \times W(L_1 L_4 2 1; 2\tilde{L}_4) \cdot \langle n_i l_i n_j l_j 1 L_1 \| S^{(2)} L^{(2)} \| n'_i l'_i nl 1 L_4 \rangle \\ & \left. \times \langle n_i l_i n_j l_j 1 L_1 \| S^{(2)} L^{(2)} \| n'_i l'_i nl 1 \tilde{L}_4 \rangle \phi_{\tau}^2(T_1) \right]. \end{aligned} \quad (2.15)$$

In the brace of Eqs. (2.13) and (2.15), the first term gives no splitting between $J=l+\frac{1}{2}$ and $J=l-\frac{1}{2}$ states, and the correction $1/20(2l+1)$ in this term and the second term are due to the Pauli principle. It should be noted that the sign of the second term is opposite to that in Eq. (2.11). Now, the two particles excited from the closed shells may also jump into the same orbit nl . There are two possibilities whether both of the two particles come from a certain orbit or from two different orbits. Then, the third configuration is

$$\begin{aligned} \phi_{IIIc,n} = & |(n_1 l_1)^{8l_1+4} (000), \dots (n_i l_i)^{8l_i+2} (T_1 S_1 L_1), \dots (n_k l_k)^{8l_k+4} (000), (nl)^3 (T_2 S_2 L_2), \\ & T = \frac{1}{2} (T_z) SL; JM \rangle. \end{aligned} \quad (2.16)$$

Fig. 6. shows this configuration schematically. In this case the calculation of the second order perturbation is rather complicated but the result is

$$\begin{aligned}
\Delta E_{J,111c} = & - \sum \frac{1}{\Delta E_{on}} \left[\left\{ \frac{1}{5} - \frac{1}{10(2l+1)} \right\} \sum_{T_1, L_1, L_2} (2T_1+1) \right. \\
& \times |\langle (nl)^2 1L_2 \| \mathbf{S}^{(2)} \mathbf{L}^{(2)} \| (n_i l_i)^2 1L_1 \rangle|^2 \Phi_\tau^2(T_1) \\
& + (-)^{1/2+l-J} W(\tfrac{1}{2} \tfrac{1}{2} ll; 1J) \cdot \frac{3}{2\sqrt{5}} \sum (2T_1+1) (2L_2+1) \\
& \times W(lL_2 l1; lL_2) W(L_1 L_2 21; 2L_2) \\
& \left. \times |\langle (nl)^2 1L_2 \| \mathbf{S}^{(2)} \mathbf{L}^{(2)} \| (n_i l_i)^2 1L_1 \rangle|^2 \Phi_\tau^2(T_1) \right]. \quad (2.17)
\end{aligned}$$

Some details of this calculation are given in Appendix I. The last configuration (Fig. 7) which can be mixed with (2.1) and which can give the second order contribution to the doublet splitting is

$$\begin{aligned}
\phi_{111d,n} = & |(n_1 l_1)^{8l_1+4} (000), \dots \{ (n_i l_i)^{8l_i+3} (\tfrac{1}{2} \tfrac{1}{2} l_i) (n_j l_j)^{8l_j+3} (\tfrac{1}{2} \tfrac{1}{2} l_j) \} \\
& (T_1 S_1 L_1), \dots (n_k l_k)^{8l_k+4} (000), (nl)^3 (T_2 S_2 L_2), T = \tfrac{1}{2} (T_z) SL; JM \rangle. \quad (2.18)
\end{aligned}$$

By the same procedure as in Eq. (2.17), the second order correction due to these configurations may be calculated and becomes as follows,

$$\begin{aligned}
\Delta E_{J,111d} = & - \sum \frac{1}{\Delta E_{on}} \left[\left\{ \frac{1}{5} - \frac{1}{10(2l+1)} \right\} \sum_{T_1, L_1, L_3} (2T_1+1) \right. \\
& \times |\langle (nl)^2 1L_3 \| \mathbf{S}^{(2)} \mathbf{L}^{(2)} \| n_i l_i n_j l_j 1L_1 \rangle|^2 \Phi_\tau^2(T_1) \\
& + (-)^{1/2+l-J} W(\tfrac{1}{2} \tfrac{1}{2} ll; 1J) \cdot \frac{3}{2\sqrt{5}} \sum_{T_1, L_1, L_3} (2T_1+1) (2L_3+1) \\
& \times W(lL_3 l1; lL_3) W(L_1 L_3 21; 2L_3) \\
& \left. \times |\langle (nl)^2 1L_3 \| \mathbf{S}^{(2)} \mathbf{L}^{(2)} \| n_i l_i n_j l_j 1L_1 \rangle|^2 \Phi_\tau^2(T_1) \right]. \quad (2.19)
\end{aligned}$$

It is very interesting that, in Eqs. (2.17) and (2.19), the fractional parentage coefficients such as $\langle l^3 TSL \{ l^2 (T'S'L') l \rangle$ do not appear although they are inevitably used in the course of calculation.

In this section no special assumption about the average field has not been made, so that these formulae for the second order perturbation energy can be applied to any unperturbed system of independent particles. It is only necessary to estimate two-body matrix elements, for example, $\langle n_i l_i n_j l_j 1L \| \mathbf{S}^{(2)} \mathbf{L}^{(2)} \| n'_i l'_i n'_j l'_j 1L' \rangle$. However, these matrix elements are not easily calculated except by using the harmonic oscillator wave functions. If the harmonic oscillator wave functions are used and the wave functions of the two particle system are transformed into those of the relative and centre of mass coordinate system, the summations over the degenerate intermediate states of a same excitation energy can be carried out as

in [I]. However, the matrix elements for the states which are excluded by the Pauli principle should be subtracted. This can be done by using the transformation coefficients between the wave functions in the two coordinate systems mentioned above. Then, the general formulae for the transformation coefficients will be investigated in the next section.

§ 3. The transformation coefficients between the wave functions in the two-particle coordinate system and the relative and centre of mass coordinate system

Two-body matrix elements can be easily evaluated, if the shell model wave function of the two-particle system can be expressed in terms of the wave functions of the relative and centre of mass coordinate system. If the single particle wave function is a plane wave, the wave function can be transformed very easily into the new coordinate system. And also, when the average field is taken to be a harmonic oscillator well, this transformation coefficient may be calculated by an elementary method, although it is not so easy.^{5)*} In this section, the recurrence formula for the transformation coefficients between the wave functions in the two different coordinate systems is derived, and this formula is used to obtain the coefficients in the simple cases. At first, the spatial wave function of the two particles, $(n_1 l_1)$ and $(n_2 l_2)$, is expanded into the wave functions of the relative coordinate ($\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$) and the centre of mass coordinate ($\mathbf{R} = \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2)$), and vice versa, that is,

$$|n_1 l_1 n_2 l_2; LM\rangle = \sum_{\tilde{N} \tilde{L}, \tilde{n} \tilde{l}} |\tilde{N} \tilde{L} \tilde{n} \tilde{l}; LM\rangle \langle \tilde{N} \tilde{L} \tilde{n} \tilde{l}; L | n_1 l_1 n_2 l_2; L \rangle \quad (3.1a)$$

and

$$|\tilde{N} \tilde{L} \tilde{n} \tilde{l}; LM\rangle = \sum_{n_1 l_1, n_2 l_2} |n_1 l_1 n_2 l_2; LM\rangle \langle n_1 l_1 n_2 l_2; L | \tilde{N} \tilde{L} \tilde{n} \tilde{l}; L \rangle, \quad (3.1b)$$

where $\langle \tilde{N} \tilde{L} \tilde{n} \tilde{l}; L | n_1 l_1 n_2 l_2; L \rangle$ and $\langle n_1 l_1 n_2 l_2; L | \tilde{N} \tilde{L} \tilde{n} \tilde{l}; L \rangle$ are the transformation coefficients, (\tilde{n}, \tilde{l}) are the quantum numbers of the relative wave function and (\tilde{N}, \tilde{L}) are those of the centre of mass wave function. And it has to be borne in mind that the total energy in these two different systems must be equal, $2n_1 + l_1 + 2n_2 + l_2 = 2\tilde{n} + \tilde{l} + 2\tilde{N} + \tilde{L}$.

In this equation (3.1), the radial wave functions for the \mathbf{r}_1 and \mathbf{r}_2 coordinates are given by

$$R_{n_i l_i}(r_i) = N_{n_i l_i}(\nu) \cdot \exp(-\nu r_i^2/2) \cdot r_i^{l_i} \cdot v_{n_i l_i}(r_i), \quad (3.2)$$

where

$$N_{n_i l_i}(\nu) = \left[\frac{2^{l_i - n_i + 3} \cdot (2l_i + 2n_i - 1)!! \cdot \nu^{l_i + 3/2}}{\pi^{1/2} \cdot (n_i - 1)! \cdot \{(2l_i + 1)!!\}^2} \right]^{1/2}$$

* Lawson and Mayer have recently made a similar calculation to that in this section (private communication).

and

$$v_{n_i} u_i(r_i) = \sum_{k=0}^{n_i-1} (-)^k \cdot \frac{2^k \cdot (n_i-1)! \cdot (2l_i+1)!!}{k! \cdot (n_i-k-1)! \cdot (2l_i+2k+1)!!} \cdot (\nu r_i^2)^k,$$

The radial wave functions $R_{\tilde{n}\tilde{l}}(r)$ and $R_{\tilde{n}\tilde{l}}(R)$ for the relative and centre of mass motions are also written in the form of Eq. (3.2), but the ν in this equation must be replaced by $\nu/2$ and 2ν , respectively.

Now, the μ component of the operator $\mathbf{p} = (\nu\mathbf{r} - \nabla)/\sqrt{2\nu}$ brings the wave function $R_{n_l}(r) Y_{lm}(\theta, \varphi)$ into

$$\begin{aligned} p_\mu \cdot R_{n_l}(r) Y_{lm}(\theta, \varphi) &= \sqrt{\frac{(l+1)(2l+2n+1)}{(2l+3)}} \cdot (l1 m\mu | l+1 m+\mu) \\ &\quad \times R_{n, l+1}(r) Y_{l+1, m+\mu}(\theta, \varphi) \\ &+ \sqrt{\frac{l \cdot 2n}{(2l-1)}} \cdot (l1 m\mu | l-1 m+\mu) R_{n+1, l-1}(r) Y_{l-1, m+\mu}(\theta, \varphi). \end{aligned} \quad (3.3)$$

Therefore, by Eqs. (3.1b) and (3.3), the following equation is obtained,

$$\begin{aligned} &\sum_{\mu} (1L_1 \mu M_1 | LM) \cdot \frac{2\nu R_\mu - F_\mu}{2\sqrt{\nu}} \cdot |\tilde{N}\tilde{L}\tilde{n}\tilde{l}; L_1 M_1\rangle \\ &= \sqrt{(\tilde{L}+1)(2\tilde{L}+2\tilde{N}+1)(2L_1+1)} \cdot W(1\tilde{L}\tilde{L}\tilde{l}; \tilde{L}+1 L_1) \cdot |\tilde{N}\tilde{L}+1, \tilde{n}\tilde{l}; LM\rangle \\ &+ \sqrt{\tilde{L} \cdot 2\tilde{N} \cdot (2L_1+1)} \cdot W(1\tilde{L}\tilde{L}\tilde{l}; \tilde{L}-1 L_1) \cdot |\tilde{N}+1\tilde{L}-1, \tilde{n}\tilde{l}; LM\rangle \\ &= \sqrt{\frac{2L_1+1}{2}} \sum_{n_1 l_1, n_2 l_2} \left\{ \sqrt{l_1 \cdot (2l_1+2n_1-1)} \cdot W(1l_1-1 L l_2; l_1 L_1) \right. \\ &\quad \times \langle n_1 l_1-1, n_2 l_2; L_1 | \tilde{N}\tilde{L}, \tilde{n}\tilde{l}; L_1 \rangle \\ &\quad + \sqrt{(l_1+1)(2n_1-2)} \cdot W(1l_1+1 L l_2; l_1 L_1) \cdot \langle n_1-1 l_1+1, n_2 l_2; L_1 | \tilde{N}\tilde{L}, \tilde{n}\tilde{l}; L_1 \rangle \\ &\quad + (-)^{1+L_1-L} \cdot \sqrt{l_2 \cdot (2l_2+2n_2-1)} \cdot W(l_1 l_2-1 L 1; L_1 l_2) \\ &\quad \times \langle n_1 l_1, n_2 l_2-1; L_1 | \tilde{N}\tilde{L}, \tilde{n}\tilde{l}; L_1 \rangle \\ &\quad + (-)^{1+L_1-L} \cdot \sqrt{(l_2+1)(2n_2-2)} \cdot W(l_1 l_2+1 L 1; L_1 l_2) \\ &\quad \left. \times \langle n_1 l_1, n_2-1 l_2+1; L_1 | \tilde{N}\tilde{L}, \tilde{n}\tilde{l}; L_1 \rangle \right\} |n_1 l_1 n_2 l_2; LM\rangle. \end{aligned} \quad (3.4)$$

In this derivation, the relation

$$P_{R,\mu} = \frac{2\nu R_\mu - F_{R,\mu}}{2\sqrt{\nu}} = \frac{1}{\sqrt{2}} (p_{1,\mu} + p_{2,\mu}) \quad (3.5)$$

was used. By using the ortho-normality of the Racah coefficients, the following recurrence formulae are derived,

$$\begin{aligned}
& \langle n_1 l_1, n_2 l_2; L | \widetilde{N} \widetilde{L} + 1, \widetilde{n} \widetilde{l}; L \rangle \\
&= \frac{(2\widetilde{L}+3)}{\sqrt{2(\widetilde{L}+1)(2\widetilde{L}+2\widetilde{N}+1)}} \sum_{\widetilde{L}_1} (2L_1+1) W(1\widetilde{L} \widetilde{L} \widetilde{l}; \widetilde{L}+1 L_1) \\
&\times \left\{ \sqrt{l_1(2l_1+2n_1-1)} W(1l_1-1 L l_2; l_1 L_1) \langle n_1 l_1-1, n_2 l_2; L_1 | \widetilde{N} \widetilde{L}, \widetilde{n} \widetilde{l}; L_1 \rangle \right. \\
&+ \sqrt{(l_1+1)(2n_1-2)} W(1l_1+1 L l_2; l_1 L_1) \langle n_1-1 l_1+1, n_2 l_2; L_1 | \widetilde{N} \widetilde{L}, \widetilde{n} \widetilde{l}; L_1 \rangle \\
&+ (-)^{1+L_1-L} \sqrt{l_2(2l_2+2n_2-1)} W(1l_2-1 L l_1; l_2 L_1) \\
&\times \langle n_1 l_1, n_2 l_2-1; L_1 | \widetilde{N} \widetilde{L}, \widetilde{n} \widetilde{l}; L_1 \rangle \\
&+ (-)^{1+L_1-L} \sqrt{(l_2+1)(2n_2-2)} W(1l_2+1 L l_1; l_2 L_1) \\
&\times \langle n_1 l_1, n_2-1 l_2+1; L_1 | \widetilde{N} \widetilde{L}, \widetilde{n} \widetilde{l}; L_1 \rangle \left. \right\}. \quad (3.6)
\end{aligned}$$

Next the transformation coefficients for $\widetilde{N} \widetilde{L} = 1\widetilde{S}$ and $2\widetilde{S}$ are necessary to be computed for starting the calculation with the recurrence formula. This can be easily done as follows. The directions of \mathbf{r}_1 and \mathbf{r}_2 are assumed to be same, and this implies that

$$\theta_1 = \theta_2 = \theta = \theta \quad (3.7)$$

and

$$\phi_1 = \phi_2 = \phi = \phi \quad (3.8)$$

in the case of $r_2 \rangle r_1$. Then, on account of the identity,

$$\begin{aligned}
& \sum_{m_1, m_2} (l_1 l_2 m_1 m_2 | LM) Y_{l_1 m_1}(\theta, \phi) Y_{l_2 m_2}(\theta, \phi) \\
&= \sqrt{\frac{(2\widetilde{L}_1+1)(2\widetilde{L}_2+1)}{4\pi(2\widetilde{L}+1)}} (l_1 l_2 00 | L0) Y_{LM}(\theta, \phi), \quad (3.9)
\end{aligned}$$

Eq. (3.1a) becomes

$$\begin{aligned}
& N_{n_1 l_1}(\nu) N_{n_2 l_2}(\nu) \cdot r_1^{l_1} \cdot v_{n_1 l_1}(r_1) \cdot r_2^{l_2} \cdot v_{n_2 l_2}(r_2) \cdot \sqrt{(2\widetilde{L}_1+1)(2\widetilde{L}_2+1)} \cdot (l_1 l_2 00 | L0) \\
&= \sum_{\widetilde{N} \widetilde{L}, \widetilde{n} \widetilde{l}} N_{\widetilde{N} \widetilde{L}}(2\nu) N_{\widetilde{n} \widetilde{l}}(\nu/2) \cdot R^{\widetilde{L}} \cdot v_{\widetilde{N} \widetilde{L}}(R) \cdot r^{\widetilde{l}} \cdot v_{\widetilde{n} \widetilde{l}}(r) \cdot \sqrt{(2\widetilde{L}+1)(2\widetilde{l}+1)} \\
&\times (\widetilde{L} \widetilde{l} 00 | L0) \cdot \langle \widetilde{N} \widetilde{L}, \widetilde{n} \widetilde{l}; L | n_1 l_1 n_2 l_2; L \rangle, \quad (3.10)
\end{aligned}$$

where the common factors are omitted. From Eqs. (3.7) and (3.8),

$$r_1 = R - r/2 \quad (3.11)$$

and

$$r_2 = R + r/2. \quad (3.12)$$

Then, in the left-hand side of Eq. (3.10), r_1 and r_2 can be replaced by R and r . After expanding the left-hand side of Eq. (3.10) in terms of R and r , and com-

paring the terms of $R^0 \cdot r^{2\tilde{n}+\tilde{l}-2}$ in both sides, the transformation coefficient $\langle 1\tilde{S}, \tilde{n}\tilde{l}; \tilde{l} | n_1 l_1, n_2 l_2; \tilde{l} \rangle$ are obtained as follows,

$$\begin{aligned} \langle 1\tilde{S}, \tilde{n}\tilde{l}; \tilde{l} | n_1 l_1, n_2 l_2; \tilde{l} \rangle &= (-)^{n_1+n_2+l_1-\tilde{n}+1} \cdot 2^{(1/4)(l_1+l_2-3\tilde{l})-\tilde{n}+1} \\ &\times \sqrt{\frac{(\tilde{n}-1)! \cdot (2\tilde{l}+2\tilde{n}-1)!! (2l_1+1) (2l_2+1)}{(n_1-1)! (n_2-1)! (2l_1+2n_1-1)!! (2l_2+2n_2-1)!! (2\tilde{l}+1)}} (l_1 l_2 00 | \tilde{l} 0), \end{aligned} \quad (3.13)$$

where

$$2\tilde{n}+\tilde{l}=2n_1+l_1+2n_2+l_2-2.$$

In the right-hand side of Eq. (3.10), there are two terms that have factor $R^2 \cdot r^{2\tilde{n}+\tilde{l}-4}$, and one of them comes from $|1\tilde{S}, \tilde{n}\tilde{l}; \tilde{l}\rangle$ and the other from $|2\tilde{S}, \tilde{n}-1\tilde{l}; \tilde{l}\rangle$. By inserting Eq. (3.13) into Eq. (3.10) and comparing the terms of $R^2 \cdot r^{2\tilde{n}+\tilde{l}-4}$ in both sides, the transformation coefficients

$$\begin{aligned} \langle 2\tilde{S}, \tilde{n}\tilde{l}; \tilde{l} | n_1 l_1, n_2 l_2; \tilde{l} \rangle &= (-)^{(1/2)(\tilde{l}+l_1-l_2)} \cdot 2^{(1/4)(l_1+l_2-3\tilde{l})-\tilde{n}} \\ &\times \sqrt{\frac{(\tilde{n}-1)! \cdot (2\tilde{l}+2\tilde{n}-1)!! \cdot (2l_1+1) (2l_2+1)}{3 \cdot (n_1-1)! (n_2-1)! (2l_1+2n_1-1)!! \cdot (2l_2+2n_2-1)!! \cdot (2\tilde{l}+1)}} \\ &\times [2(n_1-n_2)^2 + 2(n_1-n_2)(l_1-l_2) + (n_2+n_2-2) - \frac{1}{2}(l_1+l_2+\tilde{l}+1)(l_1+l_2-\tilde{l})] \\ &\times (l_1 l_2 00 | \tilde{l} 0) \end{aligned} \quad (3.14)$$

are derived. These procedures may be applied to obtain the transformation coefficients $\langle \tilde{N}\tilde{L}, \tilde{n}\tilde{l}; \tilde{l} | n_1 l_1, n_2 l_2; \tilde{l} \rangle$.

For $(\tilde{N}\tilde{L}) = (1\tilde{P})$, Eq. (3.13) and the recurrence formula (3.6) give

$$\begin{aligned} \langle 1\tilde{P}, \tilde{n}\tilde{l}; \tilde{l}+1 | n_1 l_1, n_2 l_2; \tilde{l}+1 \rangle &= -\frac{1}{(2\tilde{l}+3)\sqrt{\tilde{l}+1}} \cdot [2(\tilde{l}+1)(n_1-n_2) + (l_1+l_2+\tilde{l}+2)(l_1-l_2)] \\ &\times (l_1 l_2 00 | \tilde{l}+10) \cdot C(n_1 l_1, n_2 l_2; \tilde{n}\tilde{l}), \\ \langle 1\tilde{P}, \tilde{n}\tilde{l}; \tilde{l} | n_1 l_1, n_2 l_2; \tilde{l} \rangle &= \sqrt{\frac{(l_1+l_2+\tilde{l}+2)(l_1+l_2-\tilde{l}+1)(\tilde{l}-l_2+\tilde{l}+1)(l_2-l_1+\tilde{l})}{\tilde{l} \cdot (\tilde{l}+1) \cdot (2\tilde{l}+1)}} \\ &\times (l_1+1 l_2 00 | \tilde{l} 0) \cdot C(n_1 l_1, n_2 l_2; \tilde{n}\tilde{l}), \end{aligned}$$

and

$$\begin{aligned} \langle 1\tilde{P}, \tilde{n}\tilde{l}; \tilde{l}-1 | n_1 l_1, n_2 l_2; \tilde{l}-1 \rangle &= \frac{1}{(2\tilde{l}-1)\sqrt{\tilde{l}}} \cdot [2\tilde{l} \cdot (n_1-n_2) + (l_1+l_2-\tilde{l}+1)(l_1-l_2)] \\ &\times (l_1 l_2 00 | \tilde{l}-10) \cdot C(n_1 l_1, n_2 l_2; \tilde{n}\tilde{l}), \end{aligned}$$

where

$$C(n_1 l_1, n_2 l_2; \tilde{n} \tilde{l}) = (-)^{(1/2)(\tilde{l}+l_1-l_2+1)} \cdot 2^{(1/4)(l_1+l_2+1-3\tilde{l})-\tilde{n}} \\ \times \sqrt{\frac{(\tilde{n}-1)! \cdot (2\tilde{l}+2\tilde{n}-1)!! \cdot (2l_1+1) (2l_2+1)}{(n_1-1)! \cdot (n_2-1)! \cdot (2l_1+2n_1-1)!! \cdot (2l_2+2n_2-1)!!}}.$$

Further the transformation coefficients for $(\tilde{N}\tilde{L}) = (1\tilde{D})$ are derived as

$$\langle 1\tilde{D}, \tilde{n} \tilde{l}; \tilde{l}+2 | n_1 l_1, n_2 l_2; \tilde{l}+2 \rangle = -\frac{1}{2(2\tilde{l}+5)\sqrt{(2\tilde{l}+2)(2\tilde{l}+3)(2\tilde{l}+4)}} \\ \times [\{2(\tilde{l}+1)(n_1-n_2) + (l_1+l_2+\tilde{l}+1)(l_1-l_2)\} \\ \times \{2(\tilde{l}+2)(n_1-n_2) + (l_1+l_2+\tilde{l}+3)(l_1-l_2)\} \\ - (\tilde{l}+2)(l_1+l_2+\tilde{l}+1)(l_1+l_2+\tilde{l}+3) - 4(\tilde{l}+1)(\tilde{l}+2)(n_1+n_2-2) \\ - 2(l_1-l_2)(n_1-n_2)(l_1+l_2-\tilde{l}-1)] \times (l_1 l_2 00 | \tilde{l}+20) \cdot K(n_1 l_1, n_2 l_2; \tilde{n} \tilde{l}), \\ \langle 1\tilde{D}, \tilde{n} \tilde{l}; \tilde{l}+1 | n_1 l_1, n_2 l_2; \tilde{l}+1 \rangle \\ = \sqrt{\frac{(l_1+l_2+\tilde{l}+2)(l_1-l_2+\tilde{l}+1)(l_2-l_1+\tilde{l}+1)(l_1+l_2-\tilde{l})}{2\tilde{l} \cdot (2\tilde{l}+1)(2\tilde{l}+2)(2\tilde{l}+3)(2\tilde{l}+4)}} \\ \times \{2\tilde{l} \cdot (n_1-n_2) + (l_1-l_2)(l_1+l_2+\tilde{l}+1)\} \cdot (l_1 l_2 00 | \tilde{l}0) K(n_1 l_1, n_2 l_2; \tilde{n} \tilde{l}), \\ \langle 1\tilde{D}, \tilde{n} \tilde{l}; \tilde{l} | n_1 l_1, n_2 l_2; \tilde{l} \rangle = \frac{1}{\sqrt{2 \cdot 3 \cdot (2\tilde{l}-1)2\tilde{l} \cdot (2\tilde{l}+1)(2\tilde{l}+2)(2\tilde{l}+3)}} \\ \times [2(n_1-n_2)(l_1-l_2) \{3(l_1+l_2+1) + 2\tilde{l}(\tilde{l}+1)\} + (l_1+l_2-\tilde{l})(l_1+l_2+\tilde{l}+1) \\ \times \{-3(l_1-l_2)^2 + 2\tilde{l}(\tilde{l}+1)\} + 4(n_1-n_2)^2 \tilde{l}(\tilde{l}+1) - 4(n_1+n_2-2)\tilde{l}(\tilde{l}+1)] \\ \times (l_1 l_2 00 | \tilde{l}0) \cdot K(n_1 l_1, n_2 l_2; \tilde{n} \tilde{l}), \\ \langle 1\tilde{D}, \tilde{n} \tilde{l}; \tilde{l}-1 | n_1 l_1, n_2 l_2; \tilde{l}-1 \rangle \\ = \sqrt{\frac{(l_1+l_2+\tilde{l}+1)(l_1-l_2+\tilde{l})(l_2-l_1+\tilde{l})(l_1+l_2-\tilde{l}+1)}{(2\tilde{l}-2)(2\tilde{l}-1)2\tilde{l} \cdot (2\tilde{l}+1)(2\tilde{l}+2)}} \\ \times \{2(\tilde{l}+1)(n_1-n_2) - (l_1-l_2)(l_1+l_2-\tilde{l})\} \cdot (l_1 l_2 00 | \tilde{l}0) \cdot K(n_1 l_1, n_2 l_2; \tilde{n} \tilde{l}),$$

and

$$\langle 1\tilde{D}, \tilde{n} \tilde{l}; \tilde{l}-2 | n_1 l_1, n_2 l_2; \tilde{l}-2 \rangle = -\frac{1}{2 \cdot (2\tilde{l}-3)\sqrt{(2\tilde{l}-2)(2\tilde{l}-1) \cdot 2\tilde{l}}} \\ \times [\{-2\tilde{l}(n_1-n_2) + (l_1+l_2-\tilde{l})(l_1-l_2)\} \\ \times \{-2(\tilde{l}-1)(n_1-n_2) + (l_1+l_2-\tilde{l}+2)(l_1-l_2)\} \\ + (\tilde{l}-1)(l_1+l_2-\tilde{l})(l_1+l_2-\tilde{l}+2) - 4\tilde{l} \cdot (\tilde{l}-1)(n_1+n_2-2) \\ - 2(l_1-l_2)(n_1-n_2)(l_1+l_2+\tilde{l})] (l_1 l_2 00 | \tilde{l}-20) \cdot K(n_1 l_1, n_2 l_2; \tilde{n} \tilde{l}),$$

where

$$K(n_1 l_1, n_2 l_2; \tilde{n} \tilde{l}) = (-)^{(1/2)(\tilde{l}+l_1-l_2)} \cdot 2^{(1/4)(l_1+l_2-8\tilde{l})-\tilde{n}+1} \\ \times \sqrt{\frac{(\tilde{n}-1)! \cdot (2\tilde{l}+2\tilde{n}-1)!! \cdot (2l_1+1)(2l_2+1)}{(n_1-1)! \cdot (n_2-1)! \cdot (2l_1+2n_1-1)!! \cdot (2l_2+2n_2-1)!!}}.$$

By these methods, other transformation coefficients may be calculated. However, in the present paper, the above coefficients are enough for estimating the second order effect of the tensor force.

The transformation coefficients used in the following section are tabulated in Appendix II.

Finally, some remarks are given on the relations between the transformation coefficients. By the transformation of the coordinates \mathbf{r}_1 and \mathbf{r}_2 into the coordinates $\mathbf{x}_1 = \sqrt{2}\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2)/\sqrt{2}$ and $\mathbf{x}_2 = \mathbf{r}/\sqrt{2} = (\mathbf{r}_2 - \mathbf{r}_1)/\sqrt{2}$ we obtain the relation,

$$\langle \tilde{N} \tilde{L}(\mathbf{R}) \tilde{n} \tilde{l}(\mathbf{r}); L | n_1 l_1(\mathbf{r}_1) n_2 l_2(\mathbf{r}_2); L \rangle \\ = (-)^{\tilde{l}-l_2} \langle n_1 l_1(\mathbf{X}) n_2 l_2(\mathbf{x}); L | \tilde{N} \tilde{L}(\mathbf{x}_1) \tilde{n} \tilde{l}(\mathbf{x}_2); L \rangle$$

where $\mathbf{X} = (\mathbf{x}_1 + \mathbf{x}_2)/2$ and $\mathbf{x} = \mathbf{x}_2 - \mathbf{x}_1$.

The transformation coefficient $\langle \tilde{N} \tilde{L} \tilde{n} \tilde{l}; TSL | n_1 l_1 n_2 l_2; TSL \rangle$ of the antisymmetrized wave function $|n_1 l_1 n_2 l_2; TSL\rangle$ can be obtained by the following relation:

$$\langle \tilde{N} \tilde{L} \tilde{n} \tilde{l}; TSL | n_1 l_1 n_2 l_2; TSL \rangle \\ = \begin{cases} \frac{1 - (-)^{T+S+\tilde{l}}}{2} \sqrt{2} \langle \tilde{N} \tilde{L} \tilde{n} \tilde{l}; L | n_1 l_1 n_2 l_2; L \rangle & \text{for } n_1 l_1 \neq n_2 l_2 \\ \frac{1 - (-)^{T+S+\tilde{l}}}{2} \langle \tilde{N} \tilde{L} \tilde{n} \tilde{l}; L | n l n l; L \rangle & \text{for } n_1 l_1 = n_2 l_2 = n l. \end{cases}$$

§ 4. Numerical calculation

In this section the general formulae are applied to the case of the D -state doublet splitting in O^{17} . The numerical results have been obtained using the two kinds of the potentials as in [I]. One of them is the phenomenological tensor potential of the Serber type,⁶⁾

$$v_T = \frac{1}{4} (1 - \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2) \cdot S_{12} \cdot V_0 \cdot \exp(-r^2/r_i^2), \quad (4.1)$$

where $V_0 = -25.8$ Mev, $r_i = 2.41 \times 10^{-13}$ cm and S_{12} is the tensor force operator. The others are the meson theoretic tensor potential⁷⁾ for the triplet odd state;

$$v_T = \begin{cases} V_T^{(1\pi)}(\kappa r) & (\kappa r \geq 1.0) \\ 0 & (\kappa r < 1.0) \end{cases} \quad (4.2a)$$

and that for the triplet even state:

$$v_T = \begin{cases} V_T^{(1\pi)}(\kappa r) & (\kappa r \geq 0.7) \\ 3V_T^{(1\pi)}(0.7) & (\kappa r < 0.7) \end{cases} \quad (4.2b)$$

In the above equation, the one pion exchange potential is given by

$$V_T^{(1\pi)}(\kappa r) = \frac{g_\pi^2}{4\pi} \cdot \mu c^2 \cdot \frac{(\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2)}{3} \cdot S_{12} \cdot \left(1 + \frac{3}{\kappa r} + \frac{3}{\kappa^2 r^2}\right) \cdot \exp(-\kappa r)/\kappa r,$$

where $\boldsymbol{\tau}$ is the isotopic spin operator, $\kappa^{-1} = \hbar/c\mu$ the Compton wave length of a pion, and 0.08 is used for the coupling constant $g_\pi^2/4\pi$.

The calculated splitting energies, ΔE , are graphically shown in Fig. 8. as the function of the parameter ρ (10^{-13}cm) $\equiv (\nu/2)^{-1/2}$ which measures the extension of the harmonic oscillator wave function. Now, ρ can be estimated from the Coulomb energy difference⁸⁾ between O^{17} and F^{17} , and also from the high energy electron scattering experiment,⁹⁾ if the wave function of O^{17} is assumed to be a shell model one. From these experiments the following numerical value is obtained, $\rho = 2.37$. In the case of the Gaussian potential (4.1), the doublet splitting energy corre-

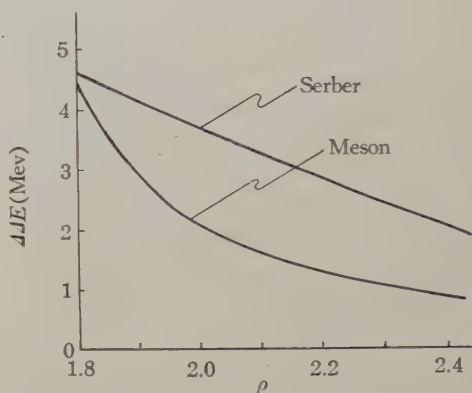


Fig. 8 Dependence of the doublet splitting energy ΔE in O^{17} on the parameter ρ .

sponding to this value of ρ is about a half of the observed value (5.08 Mev), while the splitting energy is too small in the case of the meson theoretic one. However, the results are rather sensitive to ρ , as can be seen from Fig. 8, and the splitting of the correct magnitude can be obtained, if ρ is some 20% smaller than the above value. There appears to be some reasons for using a smaller ρ value than the above value, $\rho = 2.37$. At first, the numerical calculation shows the large mixing probability of higher configurations into the zeroth order shell model configuration, and the wave functions of these higher configurations spread out more than that of the zeroth order configuration. Therefore, if the effect of the mixed higher configurations is taken into account, ρ should become smaller than the above value. Next, the effects of the strong correlation in the closed shell core are very important. The main effect of the correlation on the doublet splitting seems to come from the change of ρ , for the effect of the short range correlation function taking care of the singular repulsive core potentials is much reduced. It is because of r^2 or higher power of r in the integrand of the tensor matrix element which comes from the radial wave function, as the matrix element vanishes between S -states. Dabrowski¹⁰⁾ calculated the binding energy of O^{16} using the variational trial function in the form

$$\Phi(1, \dots, A) = N \cdot \prod_{i>j}^A f(r_{ij}) \Phi_0(1, \dots, A) \quad (4.3)$$

where N is a normalization constant and

$$f(r_{ij}) = \begin{cases} 0 & \text{for } r_{ij} < a \\ 1 - \exp[-\beta \{(r_{ij}/a)^2 - 1\}] & \text{for } r_{ij} \geq a \end{cases} \quad (4.4)$$

are the short range correlation function and

$$\Phi_0(1, \dots, A) = (A!)^{-1/2} \cdot \det \{\phi_k(i)\}, \quad (4.5)$$

with single particle orbitals $\phi_k(i)$'s. By this method he determined $\rho=1.57$ for $a=0.2 \times 10^{-13}$ cm which is rather small. Sawicki and Folk¹¹⁾ used this result in their calculation of the effect of the two-body spin-orbit force on the doublet splitting in O^{17} . If we also take this value, the calculated doublet splitting becomes larger than 5 Mev for both potentials, as can be expected from Fig. 8. Furthermore, for example, if $\rho=2.37$ and 2.00 are used in cases of the induced deformation (II in § 2) and the self-deformation (III in § 2), respectively, we can obtain $\Delta E \sim 4$ Mev for both potentials because the positive contribution from the self-deformation becomes larger. From these considerations, it may be said that at least a considerable amount of the observed doublet splitting can be explained in terms of the tensor force.

§ 5. Discussions

Through the present calculation it has been found that qualitatively same situations as those in He^5 and N^{15} also hold in the case of O^{17} , i.e., the important effects on getting the splitting are that (1) the tensor force is strong and (2) the deformation of the closed shell core induced by the tensor interaction between the core-nucleons are restricted so as to satisfy the Pauli principle with the outside nucleon (see Table I). Therefore it may generally be concluded that at least a considerable part of the experimental spin-orbit splitting is explained in terms of the second order effect of the tensor force on account of the above mentioned effects.

Table 1. The doublet splitting energies due to the configurations (II) and (III), in case of $\rho=2.00$. (in Mev)

Configuration	(II)	(III)
Serber	-1.6	5.2
Meson	-5.5	7.5

Some problems should be solved for obtaining more definite conclusion. Numerical calculations show that the mixing percentage of the configurations of the higher excitation energies into the zeroth order configuration is very large

and becomes, for example, about 50%. However, the major part of the mixed configurations comes from the self-deformation of the closed shell core and does not contribute to the moments, i.e., magnetic moment, quadrupole moment, etc., because the total spin of the closed shell part in these mixed configurations is zero. On the other hand, the induced deformation of the closed shell core and the effect of the Pauli principle on the self-deformation induce only small mixing of the higher configurations as can be seen from Fig. 9, where the mixing percentage (P) of the configurations of the excitation energy $2N\hbar\omega$ is plotted as a function of N . From these results, if the wave function of the closed shell core could be obtained in good accuracy, it might be expected that the effects of the outside nucleon added to the closed shell core could reasonably be treated by the perturbation method. And such calculation would be useful also for settling other problems, i.e., (1) there are some ambiguities in the determination of the parameter ρ , as discussed in § 4, and (2) the effect of the higher order perturbation seems to be not so small.

In the present paper and [I], only the doublet splitting in the bound states has been calculated. Then finally, we shall briefly discuss the spin-orbit coupling in the high energy nucleon-nucleus scattering. In this case, the effect of the Pauli principle mentioned above is much reduced. The reason for this is two-fold; first because the overlap of the self-deformed closed shell core with the incident nucleon becomes much smaller, and second because the corresponding energy denominator in the second order perturbation becomes larger. On the other hand, as has been shown by several authors,^{3,14)} the effect of the induced deformation can reasonably explain the spin-orbit coupling in the high-energy nucleon-nucleus scattering, although its contribution is very small or negative in the case of the bound states. Therefore, the spin-orbit couplings in these two cases seem to be caused by the different effects.

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Appendix I.

According to the usual method, the matrix element can be expressed as follows,

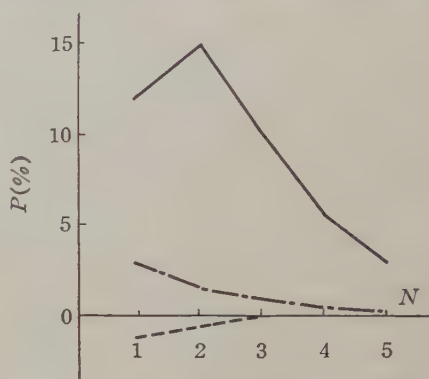


Fig. 9 Mixing percentages of various configurations.

— Configuration I and III without the Pauli principle.

- · - · - Configuration II

----- Effect of the Pauli principle in configuration III

$$\begin{aligned}
& \langle (n_1 l_1)^{8l_1+4} (000), \dots (n_i l_i)^{8l_i+2} (T_1 S_1 L_1), \dots (n_k l_k)^{8l_k+4} (000), (nl)^8 (T_2 S_2 L_2), \\
& T=\frac{1}{2} (T_z) SL; JM | V_T | (n_1 l_1)^{8l_1+4} (000), \dots (n_i l_i)^{8l_i+4} (000), \dots (n_k l_k)^{8l_k+4} (000) nl, \\
& T=\frac{1}{2} (T_z) S=\frac{1}{2} L=l; JM \rangle \\
& = \sum_{L_3} (-)^{T_2-T_1+L+L_1-L_3-J} \cdot \sqrt{\frac{3}{2}} \\
& \times \sqrt{(2T_2+1)(2S_2+1)(2L_2+1)(2S+1)(2L+1)} \langle l^3 T_2 S_2 L_2 \{ | l^2 (T_1 1 L_3) l \rangle \\
& \times W(SL \frac{1}{2} l; J_2) W^2(11S \frac{1}{2}; 2S_2) W(L_1 L_3 L l; 2L_2) \\
& \times \langle (nl)^2 1 L_3 \| \mathbf{S}^{(2)} \mathbf{L}^{(2)} \| (n_i l_i)^2 1 L_1 \rangle \Phi_\tau(T_1). \tag{A.1}
\end{aligned}$$

The corresponding second order energy is

$$\begin{aligned}
\Delta E_J = & -\frac{1}{\Delta E_{on}} \sum \frac{3}{2} \cdot (2T_2+1)(2S_2+1)(2L_2+1)(2S+1)(2L+1) \\
& \times W^2(SL \frac{1}{2} l; J_2) W^2(11S \frac{1}{2}; 2S_2) W(L_1 L_3 L l; 2L_2) W(L_1 \tilde{L}_3 L l; 2L_2) \\
& \times \langle l^3 T_2 S_2 L_2 \{ | l^2 (T_1 1 L_3) l \rangle \langle l^3 T_2 S_2 L_2 \{ | l^2 (T_1 1 \tilde{L}_3) l \rangle \\
& \times \langle (nl)^2 1 L_3 \| \mathbf{S}^{(2)} \mathbf{L}^{(2)} \| (n_i l_i)^2 1 L_1 \rangle \langle (nl)^2 1 \tilde{L}_3 \| \mathbf{S}^{(2)} \mathbf{L}^{(2)} \| (n_i l_i)^2 1 L_1 \rangle \Phi_\tau^2(T_1). \tag{A.2}
\end{aligned}$$

Using the relations

$$\begin{aligned}
W^2(SL \frac{1}{2} l; J_2) & = \sum_x (-)^{1/2+l-J-x} \cdot (2x+1) W(\frac{1}{2} l \frac{1}{2} l; Jx) W(L 2lx; l_2) \\
& \times W(S 2 \frac{1}{2} x; \frac{1}{2} 2),
\end{aligned}$$

$$\sum_S (2S+1) W^2(11S \frac{1}{2}; 2S_2) W(S 2 \frac{1}{2} x; \frac{1}{2} 2) = W(112x; 21) W(S_2 1 \frac{1}{2} x; \frac{1}{2} 1)$$

and

$$\begin{aligned}
\sum_L (2L+1) W(L 2lx; l_2) W(L_1 L_3 L l; 2L_2) W(L_1 \tilde{L}_3 L l; 2L_2) & = W(L_2 L_3 lx; l \tilde{L}_3) \\
& \times W(L_1 L_3 2x; 2 \tilde{L}_3),
\end{aligned}$$

and summing over S and L , we obtain

$$\begin{aligned}
\Delta E_J = & -\frac{1}{\Delta E_{on}} \cdot (-)^{1/2+l-J} \cdot \sum \frac{3}{2} \cdot (2T_2+1)(2S_2+1)(2L_2+1) \cdot (-)^x \cdot (2x+1) \\
& \times W(\frac{1}{2} l \frac{1}{2} l; Jx) W(112x; 21) W(S_2 1 \frac{1}{2} x; \frac{1}{2} 1) W(L_2 L_3 lx; l \tilde{L}_3) \\
& \times W(L_1 L_3 2x; 2 \tilde{L}_3) \langle l^3 T_2 S_2 L_2 \{ | l^2 (T_1 1 L_3) l \rangle \langle l^3 T_2 S_2 L_2 \{ | l^2 (T_1 1 \tilde{L}_3) l \rangle \\
& \times \langle (nl)^2 1 L_3 \| \mathbf{S}^{(2)} \mathbf{L}^{(2)} \| (n_i l_i)^2 1 L_1 \rangle \langle (nl)^2 1 \tilde{L}_3 \| \mathbf{S}^{(2)} \mathbf{L}^{(2)} \| (n_i l_i)^2 1 L_1 \rangle \Phi_\tau^2(T_1) \\
& = -\frac{1}{\Delta E_{on}} \left[\frac{1}{20(2l+1)} \sum \frac{(2T_2+1)(2S_2+1)(2L_2+1)}{(2L_3+1)} \right. \\
& \left. \times \langle l^3 T_2 S_2 L_2 \{ | l^2 (T_1 1 L_3) l \rangle \cdot | \langle (nl)^2 1 L_3 \| \mathbf{S}^{(2)} \mathbf{L}^{(2)} \| (n_i l_i)^2 1 L_1 \rangle |^2 \Phi_\tau^2(T_1) \right.
\end{aligned}$$

$$\begin{aligned}
& - (-)^{1/2+l-j} W(\tfrac{1}{2} l \tfrac{1}{2} l; J1) \cdot \frac{9}{4\sqrt{5}} \cdot \sum (2T_2+1) (2S_2+1) (2L_2+1) \\
& \times \langle l^3 T_2 S_2 L_2 \{ |l^2 (T_1 1 L_3) l\rangle^2 W(S_2 1 \tfrac{1}{2} 1; \tfrac{1}{2} 1) W(L_2 L_3 l1; l L_3) W(L_1 L_3 21; 2 L_3) \\
& \times | \langle (nl)^2 1 L_3 \| S^{(2)} L^{(2)} \| (n_i l_i)^2 1 L_1 \rangle|^2 \Phi_\tau^2(T_1) \rangle. \quad (A \cdot 3)
\end{aligned}$$

Combining the relations,

$$\begin{aligned}
\langle l^3 (T_2 S_2 L_2) \{ |l^2 (T_1 1 L_3) l\rangle^2 \rangle &= \frac{(8l+2) (2T_1+1) (2L_3+1)}{(2T_2+1) (2S_2+1) (2L_2+1)} \\
&\times \langle l^{8l+2} (T_1 1 L_3) \{ |l^{8l+1} (T_2 S_2 L_2) l\rangle^2 \rangle
\end{aligned}$$

and

$$\begin{aligned}
\langle l^{8l+2} T_1 1 L_3 \| \sum_i \mathbf{s}_i \cdot \mathbf{l}_i \| l^{8l+2} T_1 1 L_3 \rangle &= 3 \cdot (8l+2) \sum (2L_3+1) W(S_2 1 \tfrac{1}{2} 1; \tfrac{1}{2} 1) \\
&\times W(L_2 L_3 l1; l L_3) \langle \tfrac{1}{2} \| \mathbf{s} \| \tfrac{1}{2} \rangle \langle l \| \mathbf{l} \| l \rangle \langle l^{8l+2} T_1 1 L_3 \{ |l^{8l+1} (T_2 S_2 L_2) l\rangle^2 \rangle,
\end{aligned}$$

Eq. (A·3) is rewritten as

$$\begin{aligned}
\Delta E_J &= - \frac{1}{\Delta E_{on}} \left[\frac{(8l+2)}{20(2l+1)} \cdot \sum (2T_1+1) \cdot \langle (nl)^2 1 L_3 \| S^{(2)} L^{(2)} \| (n_i l_i)^2 1 L_1 \rangle^2 \Phi_\tau^2(T_1) \right. \\
&- (-)^{1/2+l-j} W(\tfrac{1}{2} l \tfrac{1}{2} l; J1) \cdot \frac{3}{4\sqrt{5}} \cdot \sum (2T_1+1) \cdot \frac{\langle l^{8l+2} T_1 1 L_3 \| \sum_i \mathbf{s}_i \cdot \mathbf{l}_i \| l^{8l+2} T_1 1 L_3 \rangle}{\langle \tfrac{1}{2} \| \mathbf{s} \| \tfrac{1}{2} \rangle \cdot \langle l \| \mathbf{l} \| l \rangle} \\
&\left. \times W(L_1 L_3 21; 2 L_3) | \langle (nl)^2 1 L_3 \| S^{(2)} L^{(2)} \| (n_i l_i)^2 1 L_1 \rangle|^2 \Phi_\tau^2(T_1) \right]. \quad (A \cdot 4)
\end{aligned}$$

Since

$$\begin{aligned}
\langle l^{8l+2} T_1 1 L_3 \| \sum_i \mathbf{s}_i \cdot \mathbf{l}_i \| l^{8l+2} T_1 1 L_3 \rangle &= - \langle l^2 T_1 1 L_3 \| \sum_i \mathbf{s}_i \cdot \mathbf{l}_i \| l^2 T_1 1 L_3 \rangle \\
&= -2(2L_3+1) W(l L_3 l1; l L_3) \langle \tfrac{1}{2} \| \mathbf{s} \| \tfrac{1}{2} \rangle \langle l \| \mathbf{l} \| l \rangle,
\end{aligned}$$

Eq. (A·4) reduces to Eq. (2·17) in § 2.

Appendix II.

Table of $2^{2N} \langle \tilde{N} \tilde{L} \tilde{n} \tilde{l}; L | n_1 l_1 n_2 l_2; L \rangle^2$

The transformation coefficient multiplied by 2^N , $2^N \langle \tilde{N} \tilde{L} \tilde{n} \tilde{l}; L | n_1 l_1 n_2 l_2; L \rangle$, is minus or plus the square root of the entry in the table according to whether this entry is, or is not, preceded by an asterisk. Here, it should be noted that, for example, the square root of $(\alpha N + \beta)^2$ means not $|\alpha N + \beta|$ but $(\alpha N + \beta)$.

$\tilde{N} \tilde{L} \tilde{n} \tilde{l}; n_1 l_1 \quad n_2 l_2 \quad ; L$	$2^{2N} \langle \tilde{N} \tilde{L} \tilde{n} \tilde{l}; L n_1 l_1 n_2 l_2; L \rangle^2$
$1S \ Nl \ ; \ 1s \quad Nl \quad ; \ l$	2^{2-l}
$1P \ Ns \ ; \ 1s \quad Np \quad ; \ 1$	$2(2N+1)/3$
$1P \ Np \ ; \ 1s \ (N+1) \ s \ ; \ 0$	$2N$

$1P \ Np ; 1s \ Nd \quad ; 2$	$2(2N+3)/5$
$1P \ Nd ; 1s \ (N+1) \ p ; 1$	$2N/3$
$1P \ Nd ; 1s \ Nf \quad ; 3$	$3(2N+5)/7 \cdot 2$
$1P \ Nf ; 1s \ (N+1) \ d ; 2$	$3N/5 \cdot 2$
$1P \ Nf ; 1s \ Ng \quad ; 4$	$(2N+7)/9$
$1P \ Ng ; 1s \ (N+1) \ f ; 3$	$N/7$
$1P \ Ng ; 1s \ Nh \quad ; 5$	$5(2N+9)/11 \cdot 8$
$1D \ Np ; 1s \ (N+1) \ p ; 1$	$2(2N+3)N/5 \cdot 3$
$1D \ Np ; 1s \ Nf \quad ; 3$	$3(2N+5) (2N+3)/7 \cdot 5 \cdot 2$
$1D \ Nd ; 1s \ (N+2) \ s ; 0$	$(N+1)N/3$
$1D \ Nd ; 1s \ (N+1) \ d ; 2$	$(2N+5)N/7 \cdot 3$
$1D \ Nd ; 1s \ Ng \quad ; 4$	$(2N+7) (2N+5)/7 \cdot 3 \cdot 2$
$1D \ Nf ; 1s \ (N+2) \ p ; 1$	$(N+1)N/5 \cdot 2$
$1D \ Nf ; 1s \ (N+1) \ f ; 3$	$(2N+7)N/9 \cdot 5$
$1D \ Nf ; 1s \ Nh \quad ; 5$	$5(2N+9) (2N+7)/11 \cdot 9 \cdot 4$
$2S \ Np ; 1s \ (N+1) \ p ; 1$	$(2N+3)N/3 \cdot 2$
$2S \ Nf ; 1s \ (N+1) \ f ; 3$	$(2N+7)N/4 \cdot 3 \cdot 2$
$1F \ Nd ; 1s \ (N+2) \ p ; 1$	$(2N+5) (N+1)N/7 \cdot 5 \cdot 2$
$1F \ Nd ; 1s \ (N+1) \ f ; 3$	$(2N+7) (2N+5)N/9 \cdot 7 \cdot 5$
$1F \ Nd ; 1s \ Nh \quad ; 5$	$5(2N+9) (2N+7) (2N+5)/11 \cdot 9 \cdot 7 \cdot 4$
$1S \ Nd ; 1p \ Np \quad ; 2$	$*2(2N+3)/5$
$1S \ Nd ; 1p \ (N-1) \ f ; 2$	$*6(N-1)/5$
$1S \ Ng ; 1p \ (N-1) \ h ; 4$	$*5(N-1)/9 \cdot 2$
$1S \ Nh ; 1p \ (N-1) \ i ; 5$	$*3(N-1)/11 \cdot 2$
$1P \ Np ; 1p \ Np \quad ; 1$	4
$1P \ Np ; 1p \ Np \quad ; 2$	$*16(N-1)^2/5^2$
$1P \ Np ; 1p \ (N-1) \ f ; 2$	$*12(2N+3) (N-1)/5^2$
$1P \ Nd ; 1p \ (N+1) \ s ; 1$	$*2(2N+3)N/9$
$1P \ Nd ; 1p \ Nd \quad ; 1$	$*2(2N-3)^2/9$
$1P \ Nd ; 1p \ Nd \quad ; 2$	2

$1P Nd; 1p Nd$; 3	$*(6N+1)^2/7^2 \cdot 2$
$1P Nd; 1p (N-1) g$; 3	$*12(2N+5)(N-1)/7^2$
$1P Nf; 1p Nf$; 2	$*(3N-5)^2/5^2$
$1P Nf; 1p Nf$; 3	1
$1P Nf; 1p Nf$; 4	$*(4N+5)^2/9^2$
$1P Nf; 1p (N-1) h$; 4	$*10(2N+7)(N-1)/9^2$
$1P Ng; 1p (N-1) i$; 5	$*15(2N+9)(N-1)/11^2 \cdot 2$
$1D Ns; 1p Np$; 2	$*2(2N+1)(2N-7)^2/5^2 \cdot 3$
$1D Ns; 1p (N-1) f$; 2	$*2(2N+3)(2N+1)(N-1)/5^2$
$1D Np; 1p (N+1) s$; 1	$*2(2N-7)^2 N/5 \cdot 3^2$
$1D Np; 1p (N-1) g$; 3	$*12(2N+5)(2N+3)(N-1)/7^2 \cdot 5$
$1D Nd; 1p (N+1) p$; 0	$*2N(N-1)^2/3$
$1D Nd; 1p (N+1) p$; 1	$2N$
$1D Nd; 1p (N+1) p$; 2	$*8N(N-1)^2/7 \cdot 5 \cdot 3$
$1D Nd; 1p Nf$; 2	$*2(2N+5)(N-1)^2/7 \cdot 5$
$1D Nd; 1p Nf$; 3	$2(2N+5)/7$
$1D Nd; 1p Nf$; 4	$*8(2N+5)(N-1)^2/9 \cdot 7 \cdot 3$
$1D Nf; 1p (N+2) s$; 1	$*(2N+5)(N+1)N/5 \cdot 3 \cdot 2$
$1D Nf; 1p Ng$; 3	$*(2N+7)(4N-5)^2/9 \cdot 7 \cdot 5 \cdot 2$
$1D Nf; 1p Ng$; 4	$(2N+7)/3 \cdot 2$
$1D Nf; 1p Ng$; 5	$*(2N+7)(10N+1)^2/11^2 \cdot 4 \cdot 3^2$
$1D Nf; 1p (N-1) i$; 5	$*5(2N+9)(2N+7)(N-1)/11^2 \cdot 3$
$1D Ng; 1p (N+2) p$; 2	$*3(2N+7)(N+1)N/7 \cdot 5^2$
$1D Ng; 1p (N+1) f$; 2	$*(3N-7)^2 N/7 \cdot 5^2$
$1D Ng; 1p (N+1) f$; 3	$5N/7 \cdot 2$
$1D Ng; 1p (N+1) f$; 4	$*5(4N+7)^2 N/11 \cdot 9 \cdot 7 \cdot 3 \cdot 2$
$2S Np; 1p Nd$; 1	$*2(2N+3)(N-2)^2/9$
$2S Nd; 1p (N+1) p$; 2	$*(2N+1)^2 N/5 \cdot 3 \cdot 2$
$2S Nd; 1p Nf$; 2	$*(2N+5)(N-2)^2/5 \cdot 2$
$2S Nf; 1p (N+1) d$; 3	$*(2N+3)^2 N/7 \cdot 4 \cdot 2$

$2S\ Nf ; 1p\ Ng ; 3$	$*(2N+7)(N-2)^2/7 \cdot 3$
$1F\ Nd ; 1p\ (N+2)\ s ; 1$	$*(2N-9)^2(N+1)N/7 \cdot 5 \cdot 3 \cdot 2$
$1F\ Nd ; 1p\ (N+1)\ d ; 1$	$*2(2N+5)N^3/7 \cdot 5 \cdot 3$
$1F\ Nd ; 1p\ (N+1)\ d ; 2$	$4(2N+5)N/7 \cdot 5$
$1F\ Nd ; 1p\ (N+1)\ d ; 3$	$*(2N+5)(2N-5)^2N/7^2 \cdot 5 \cdot 3$
$1F\ Nd ; 1p\ Ng ; 3$	$*(2N+7)(2N+5)(4N-3)^2/7^2 \cdot 5 \cdot 3^2 \cdot 2$
$1F\ Nd ; 1p\ Ng ; 4$	$(2N+7)(2N+5)/7 \cdot 3 \cdot 2$
$1F\ Nd ; 1p\ Ng ; 5$	$*(2N+7)(2N+5) \cdot (10N-21)^2/11^2 \cdot 7 \cdot 4 \cdot 3^2$
$1F\ Nd ; 1p\ (N-1)\ i ; 5$	$*5(2N+9)(2N+7)(2N+5)(N-1)/11^2 \cdot 7 \cdot 3$
$2P\ Nd ; 1p\ (N+2)\ s ; 1$	$*(2N+1)^2(N+1)N/9 \cdot 5 \cdot 2$
$2P\ Nd ; 1p\ (N+1)\ d ; 1$	$*(2N+5)(2N-5)^2N/9 \cdot 5 \cdot 2$
$2P\ Nd ; 1p\ (N+1)\ d ; 2$	$(2N+5) \cdot N/5 \cdot 2$
$2P\ Nd ; 1p\ (N+1)\ d ; 3$	$*(2N+5)(6N-5)^2N/7^2 \cdot 5 \cdot 4 \cdot 2$
$2P\ Nd ; 1p\ Ng ; 3$	$*3(2N+7)(2N+5)(N-2)^2/7^2 \cdot 5$
$1S\ Ns ; 1d\ (N-2)\ d ; 0$	$16(N-1)(N-2)/3$
$1S\ Nd ; 1d\ Ns ; 2$	$(2N+3)(2N+1)/5 \cdot 3$
$1S\ Nd ; 1d\ (N-1)\ d ; 2$	$4(2N+3)(N-1)/7 \cdot 3$
$1S\ Nd ; 1d\ (N-2)\ g ; 2$	$24(N-1)(N-2)/7 \cdot 5$
$1S\ Ng ; 1d\ Nd ; 4$	$(2N+7)(2N+5)/7 \cdot 3 \cdot 2$
$1S\ Ng ; 1d\ (N-1)\ g ; 4$	$10 \cdot (2N+7)(N-1)/11 \cdot 7 \cdot 3$
$1S\ Ng ; 1d\ (N-2)\ i ; 4$	$5(N-1)(N-2)/11 \cdot 3$
$1P\ Nd ; 1d\ Np ; 1$	$2(2N+3)(2N-1)^2/9 \cdot 5$
$1P\ Nd ; 1d\ (N-1)\ f ; 1$	$8(N-1)(N-3)^2/5 \cdot 3$
$1P\ Nd ; 1d\ Np ; 2$	$*2(2N+3)/5$
$1P\ Nd ; 1d\ (N-1)\ f ; 2$	$*16(N-1)/5$
$1P\ Nd ; 1d\ Np ; 3$	$(2N+3)(6N-13)^2/7^2 \cdot 5 \cdot 2$
$1P\ Nd ; 1d\ (N-1)\ f ; 3$	$4(N-1)(2N-1)^2/7 \cdot 5 \cdot 3$
$1P\ Nd ; 1d\ (N-2)\ h ; 3$	$20(2N+5)(N-1)(N-2)/7^3 \cdot 3$
$1D\ Nd ; 1d\ Nd ; 0$	$4\{(N-1)^2 - (N-1) + 1\}^2/9$
$1D\ Nd ; 1d\ Nd ; 1$	$*4(N-1)^3$

$1D Nd; 1d Nd ; 2$	$16 \{ (N-1)^2 - (N-1) + 7 \}^2 / 7^2 \cdot 3^2$
$1D Nd; 1d (N+1) s ; 2$	$(2N+3) (2N-9)^2 N / 9 \cdot 7 \cdot 5$
$1D Nd; 1d (N-1) g ; 2$	$8 (2N+5) (N-1) (N-2)^2 / 7^2 \cdot 5$
$1D Nd; 1d Nd ; 3$	$*16 (N-1)^2 / 7^2$
$1D Nd; 1d (N-1) g ; 3$	$*20 (2N+5) (N-1) / 7^2$
$1D Nd; 1d Nd ; 4$	$\{ 4 (N-1)^2 - 4 (N-1) - 21 \}^2 / 7^2 \cdot 3^2$
$1D Nd; 1d (N-1) g ; 4$	$80 (2N+5) (N-1) (N-2)^2 / 11 \cdot 7^2 \cdot 3^2$
$1D Nd; 1d (N-2) i ; 4$	$10 (2N+7) (2N+5) (N-1) (N-2) / 11 \cdot 9 \cdot 7$
$2S Nd; 1d (N+1) s ; 2$	$(2N+3) (2N-3)^2 N / 9 \cdot 5 \cdot 4$
$2S Nd; 1d Nd ; 2$	$\{ 2 (N-1)^2 + (N-1) - 7 \}^2 / 9 \cdot 7$
$2S Nd; 1d (N-1) g ; 2$	$2 (2N+5) (N-1) (N-4)^2 / 7 \cdot 5$

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A Field Theoretical Investigation of Multiple Meson Production. I*— Pion-Nucleon Collisions —*

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The multiple meson production in pion-nucleon collisions is investigated by the field theoretical method. By introduction of the interaction time and assuming that the free Hamiltonian can be approximately put as a C-number during the calculation of the effect due to the interaction Hamiltonian for the interaction time-interval, it is shown that the multiple meson production is caused by the pseudoscalar type interaction. The numerical results are obtained for the relative cross sections of n pion production, the angular distribution, the momentum distribution of final nucleons, etc. These results are compared with the experimental data in Bev and cosmic ray energy regions. The covariant phase space integration is carried out with the saddle point method and its final expression is comparatively simple. The probability distribution of charge states in the statistical theory is derived by an approximate formula instead of the tedious combination of Clebsch-Gordan coefficients.

§ 1. Introduction

As regards the multiple production of mesons a great deal of works has been carried out both on theoretical and experimental aspects.¹⁾ Recent experiments with accelerators have detected the multiple production of mesons not only in nucleon-nucleon and pion-nucleon collisions but also in antinucleon-nucleon annihilations. The mechanism of the multiple production in a high energy region is one of the most interesting problems because one may be able to explore the behaviour of the field in the very vicinity of a nucleon or to check the validity of field theory in this small region.

About twenty years ago, Heisenberg²⁾ pointed out the possibility of multiple production of particles based on the non-linear interaction which is different from the quantum theoretical interaction at low energies. In subsequent papers³⁾ he investigated this process as the collision of wave packets, turbulence or shock waves. Contrary to this somewhat classical approach, Lewis, Oppenheimer and

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Wouthuysen⁴⁾⁵⁾ (LOW.) have made a more straightforward attack on this problem from the field theoretical standpoint. They employed the Bloch-Nordsieck approximation⁶⁾ because of the fact that the low frequency part of photons interacts with an electron through an effectively strong coupling and that the multiple production of mesons can also be considered as due to the similar strong coupling between mesons and a nucleon. Subsequently Fukuda and Takeda⁷⁾ refined the calculation of the above authors by using the covariant formalism and separated nucleon-nucleon reactions into various charge states.

Fermi⁸⁾ has developed an ingenious theory based on the statistical equilibrium which is similar to some extent to Heisenberg's. The phase space calculation in his theory has been improved by some authors, and widely utilized in the analysis of the multiple production of particles.⁹⁾¹⁰⁾¹¹⁾¹²⁾¹³⁾ But this phase space integration has not a covariant form under Lorentz transformations and the basis of his theory is not always clear. Pomerancuk¹⁴⁾ pointed out the possibility that a strong final state interaction can considerably affect Fermi's results. Furthermore Landau¹⁵⁾ modified Fermi's theory by emphasizing the role of secondary interactions among produced particles and treated these highly excited particles as a kind of hydrodynamical flows.

On the other hand, one of the most fascinating formalism for the theoretical physicists is quantum field theory. It has given remarkable agreement with experiment in low energy regions, especially in quantum electrodynamics, although concerning the divergences quantum field theory is suffering from the difficulties. The validity of field theory in high energy regions is more or less doubtful and must be investigated. In this respect, the multiple production of mesons seems to be quite of use for checking the above mentioned validity. Some authors¹⁶⁾ have tried to check the hydrodynamical description and to derive the phenomenological quantities introduced by Heisenberg and Landau from the conventional quantum field theory. It is, however, not clear whether such phenomenological concepts as temperature and coefficient of friction are still well defined in extremely high energy regions with which we are concerned.

According to our opinion, it may be rather good to apply directly the conventional field theory to the present problem with some suitable approximations and modifications of field theory, no matter what the correspondence may be between the phenomenological and quantum field theoretical quantities. One of our aim is to obtain the numerical results which are capable of being compared with the experimental data in high energy phenomena and consequently we must be submitted to use some assumptions, as we cannot derive the cross section of multiple production of mesons exactly by making use of the present field theory.

The differences of the present treatment from those of LOW. or Fukuda and Takeda's theory are that in the present work the interaction between nucleons is not replaced by a potential, the nucleon recoil is taken into account and its self energy part is also considered. Furthermore, it will be seen that, contrary to

Fermi's statistical theory, we evaluate the matrix element from the view-point of meson theory and the integration over the phase space is covariantly performed. In connection with the latter point Srivastava and Sudarshan¹⁷⁾ independently gave a fully covariant formulation.

In the following sections we shall treat pion-nucleon collisions which are simpler than other ones. In the case of nucleon-nucleon collisions or pair annihilations the calculation is tedious but we can also obtain quantitative results. Detail of these two cases will be reported in near future. In the next section we shall give the cross section of multiple meson production under a few adequate assumptions. In § 3, the covariant phase space integration is performed with the saddle point method. The numerical results are compared in § 4 with experimental data obtained from accelerators and cosmic rays.

§ 2. Formulation of the cross section

Consider two incident particles whose mutual interaction can be neglected because of their large distance in the initial stage. At a certain moment t' these two particles begin to suffer a disturbance due to the interaction with each other which lasts until a time t'' . One may call $t=t''-t'$ the interaction time. The concept of the interaction time was already considered by Koba¹⁸⁾. If two incident particles are described as plane waves the interaction between them continues during an infinite time interval. In order to introduce the interaction time and to keep the energy conservation law, the collision of plane waves has to be reconsidered as a sum of collisions between a couple of various wave packets with average momenta \mathbf{p} and \mathbf{P} , and average positions \mathbf{r} and \mathbf{R} respectively at some early instant.

The first assumption is that the interaction between a nucleon and a pion effectively takes place only for a definite time interval $t=t''-t'$, namely no interaction exists for $-\infty$ to t' and t'' to ∞ . The instant t' when the interaction between a couple of packets begins depends on \mathbf{p} , \mathbf{P} , \mathbf{r} , and \mathbf{R} and in general can range from $-\infty$ to $+\infty$ while the interaction time t depends on the shape of the packets and is assumed to be constant in the present case. The matrix element corresponding to the collision of the above stated plane waves is

$$M = \langle \chi_f^*, U(\infty, -\infty) \chi_i \rangle / \langle \chi_0^*, U(\infty, -\infty) \chi_0 \rangle, \quad (2.1)$$

where χ_i , χ_f and χ_0 are an initial, final and free vacuum states, respectively, and $U(\infty, -\infty)$ is the transition matrix.* $|\chi_i\rangle$ consists of two plane waves with their momenta \mathbf{k} and \mathbf{K} , and is written down as follows,¹⁹⁾

* In the present paper the interaction Hamiltonian gives a non-vanishing contribution only for a finite interval t . Therefore, before and after this time interval the Schrödinger and interaction representations have no difference from each other. All the expression in what follows will be given in the interaction representation at t' which agrees with the Schrödinger picture.

$$|\chi_i\rangle = \int d\mathbf{r} d\mathbf{R} d\mathbf{p} d\mathbf{P} |\mathbf{p}, \mathbf{r}; \mathbf{P}, \mathbf{R}\rangle \langle \mathbf{p}, \mathbf{r}; \mathbf{P}, \mathbf{R} | \chi_i \rangle, \quad (2.2)$$

where $|\mathbf{p}, \mathbf{r}; \mathbf{P}, \mathbf{R}\rangle$ represents the state corresponding to incident packets. Eq. (2.2) is valid under the assumption of the completeness of wave packets. In Eq. (2.2) the two average momenta, \mathbf{p} and \mathbf{P} , can be chosen to be nearly equal to \mathbf{k} and \mathbf{K} respectively. As the incident state involves a pair of independent waves with no mutual interactions, the coefficient $\langle \mathbf{p}, \mathbf{r}; \mathbf{P}, \mathbf{R} | \chi_i \rangle$ can be factorized as

$$\langle \mathbf{p}, \mathbf{r}; \mathbf{P}, \mathbf{R} | \chi_i \rangle = F_1^*(\mathbf{k}, \mathbf{p}, \mathbf{r}) F_2^*(\mathbf{K}, \mathbf{P}, \mathbf{R}), \quad (2.3)$$

where $F_1^*(\mathbf{k}, \mathbf{p}, \mathbf{r})$ and $F_2^*(\mathbf{K}, \mathbf{P}, \mathbf{R})$ are zero except for $\mathbf{p} \approx \mathbf{k}$ and $\mathbf{P} \approx \mathbf{K}$ respectively owing to the above mentioned remark. As was already stated, t' is a function of $\mathbf{p}, \mathbf{P}, \mathbf{r}$ and \mathbf{R} , then we can choose t' as an integration variable in Eq. (2.2) in place of one of the variables $\mathbf{p}, \mathbf{P}, \mathbf{r}$ and \mathbf{R} , say x . Accordingly, if we perform the integration of Eq. (2.2) with respect to coordinate variables y, z and \mathbf{R} , we may get

$$|\chi_i\rangle = (1/T) \int dt' \int d\mathbf{p} d\mathbf{P} G(\mathbf{p}, \mathbf{k}; \mathbf{P}, \mathbf{K}) |\mathbf{p}, \mathbf{P}, t'\rangle, \quad (2.4)$$

where the factor $1/T$ has been introduced in order to make the expression under the t' integration have the same dimension as that of $|\chi_i\rangle$. Of course, T may be dependent on \mathbf{k} and \mathbf{K} but not on the number of mesons which will be produced after the collision. The state vector $|\mathbf{p}, \mathbf{P}, t'\rangle$ appearing in the integrand of Eq. (2.4) means an assembly of states describing the packets having average momenta \mathbf{p} and \mathbf{P} , and colliding with each other at t' without regard to the location of collisions.

Now the particular character of F_1^* and F_2^* allows us to replace Eq. (2.4) with

$$|\chi_i\rangle \cong (1/T) \int dt' |\mathbf{k}, \mathbf{K}, t'\rangle. \quad (2.5)$$

The matrix element (2.1) can be written as

$$\begin{aligned} M &= (1/T) \langle \chi_f^*, U(\infty, -\infty) \int dt' |\mathbf{k}, \mathbf{K}, t'\rangle / \langle \chi_0^*, U(\infty, -\infty) \chi_0 \rangle \\ &= (1/T) \int \langle \chi_f^*, U(\infty, -\infty) | \mathbf{k}, \mathbf{K}, t' \rangle dt' / \langle \chi_0^*, U(\infty, -\infty) \chi_0 \rangle. \end{aligned}$$

Owing to the character of wave packets $|\mathbf{k}, \mathbf{K}, t'\rangle$ and to our assumption on the interaction time t , the U -matrix under the t' -integration differs from unity only for the interval $(t', t' + t = t'')$. Accordingly, we can rewrite as

$$\begin{aligned} M &= (1/T) \int_{-\infty}^{\infty} dt' \langle \chi_f^*, U(t'', t') | \mathbf{k}, \mathbf{K}, t' \rangle / \langle \chi_0^*, U(\infty, -\infty) \chi_0 \rangle \\ &= (1/T) \int_{-\infty}^{\infty} dt' \langle \chi_f^*, U(t'', t') | \mathbf{k}, \mathbf{K}, t' \rangle \langle \chi_0^* U^{-1}(\infty, -\infty) \chi_0 \rangle, \end{aligned} \quad (2.6)$$

where the familiar relations (stability of the vacuum state)

$$\langle \chi_0^*, U(\infty, -\infty) \chi_0 \rangle = e^{i\alpha} \quad (\alpha: \text{real}) \quad (2.7a)$$

$$\text{and} \quad \langle \chi_0^*, U^{-1}(\infty, -\infty) \chi_0 \rangle = e^{-i\alpha}, \quad (2.7b)$$

have been used. In this stage of argument it may be plausible to replace the complicated expression of the state vector $|\mathbf{k}, \mathbf{K}, t'\rangle$ with the plane waves with momenta \mathbf{k} and \mathbf{K} as $|\mathbf{k}, \mathbf{K}, t'\rangle \rightarrow |\mathbf{k}, \mathbf{K}\rangle$ (=plane wave) $= \varphi_+^*(\mathbf{k}) u_+^*(\mathbf{K}) |\chi_0\rangle$ because the state $|\mathbf{k}, \mathbf{K}, t'\rangle$ has quite a similar appearance as that of the plane wave state $|\mathbf{k}, \mathbf{K}\rangle$ and the characteristic property of the wave packets, that is, the fact that the interaction between the couple of packets takes place only for a finite interval (t', t'') , has already been exclusively utilized in reducing $U(\infty, -\infty)$ to $U(t'', t')$. In the above equation $\varphi_+^*(\mathbf{k})$ and $u_+^*(\mathbf{K})$ represent the creation operator of the positive meson and the nucleon with momenta \mathbf{k} and \mathbf{K} respectively. Following our fundamental postulate that the interaction takes place during a finite time interval t , $U^{-1}(\infty, -\infty)$ appearing in the second factor of Eq. (2.6) may be substituted with $\langle \chi_0^*, U^{-1}(t, 0) \chi_0 \rangle$, since the instant when the interaction is switched on gives no difference.

Here it must be remarked that the assumption of the finite interaction time interval t does not necessarily mean the sudden switching on and off of interactions but this assumption should rather be regarded as follows; the interaction is to some extent gradually switched on or off and the mean interval is the finite t . The law of conservation of energy is, of course, more or less violated. Accordingly, it may be possible to expect that the interaction is so gradually inserted or removed that the magnitude of the violation of the energy conservation, $|E_i - E_f|$, is smaller than the error ΔE of the measurement of the energy, where E_i and E_f are the energy of the initial and final states respectively. This expectation may be satisfied if the interaction time is regarded as the time interval during which the packets of the meson and the nucleon overlap each other. Under this expectation, the matrix element $\langle \chi_n^*, U^{-1}(t, 0) \chi_0 \rangle$ for $\chi_n \neq \chi_0$ can be neglected by virtue of the approximate conservation of energy. Consequently our matrix element becomes

$$M = (1/T) \int_{-\infty}^{\infty} dt' \langle \chi_f^*, U(t'', t') \varphi_+^*(\mathbf{k}) u_+^*(\mathbf{K}) U^{-1}(t, 0) \chi_0 \rangle. \quad (2.8)$$

By inserting the definition of $U(t'', t')$

$$\begin{aligned} U(t'', t') &= e^{iH_0 t''} e^{-i(H_0 + H_1)(t'' - t')} e^{-iH_0 t'} \\ &= e^{iH_0 t'} U(t, 0) e^{-iH_0 t'} \end{aligned}$$

into Eq. (2.6) and taking into account the replacement $|\mathbf{k}, \mathbf{K}, t'\rangle \rightarrow |\mathbf{k}, \mathbf{K}\rangle$ (=plane wave), Eq. (2.8) becomes

$$M = (1/T) \int dt' e^{i(E_f - E_i)t'} \langle \chi_f^*, U(t, 0) \varphi_+^*(\mathbf{k}) u_+^*(\mathbf{K}) U^{-1}(t, 0) \chi_0 \rangle, \quad (2.9)$$

or equivalently it reads

$$M = (2\pi/T) \langle \chi_f^*, U(t, 0) \varphi_+^*(\mathbf{k}) u_+^*(\mathbf{K}) U^{-1}(t, 0) \chi_0 \rangle \delta(E_f - E_i), \quad (2.10)$$

with

$$U(t, 0) = e^{iH_0 t} e^{-i(H_0 + H_I)t}, \quad (2.11)$$

where H_0 and H_I are the free and interaction Hamiltonians respectively.

The production of many mesons in a high energy region seems to show that the interaction Hamiltonian H_I between a nucleon and a meson may be very strong. As we shall see in Appendix I, if we expand $U(t, 0)$ into a power series of t , we have

$$U(t, 0) = e^{-iH_I t} (1 + (t^2/2)[H_0, H_I] + \dots).$$

Accordingly, the contribution from the term $\exp(-iH_I t) \cdot t^2 \cdot [H_0, H_I]$ can be neglected in comparison with that from the first term $\exp(-iH_I t)$ provided that the interaction time t is moderately small. In other words, during the interaction time the contribution from H_I is so predominant that the quantum fluctuation due to H_0 can be neglected. Thus we may put

$$U(t, 0) \approx \exp(-iH_I t). \quad (2.12)$$

Some discussions will be made in Appendix I on the validity of (2.12). The indeterminate factor T need not be given explicitly as we are interested only in the relative cross section of n pion production, accordingly the factor $1/T$ does not appear in the final result. The creation operators φ_+^* and u_+^* are given as*

$$\varphi_+^*(\mathbf{k}) u_+^*(\mathbf{K}) = \int f_\rho(\mathbf{K}, \mathbf{k}, \mathbf{x}, \mathbf{x}') \phi^*(\mathbf{x}', 0) \bar{\psi}_\rho(\mathbf{x}, 0) \chi_0 d\mathbf{x} d\mathbf{x}', \quad (2.13)$$

where

$$f_\rho(\mathbf{K}, \mathbf{k}, \mathbf{x}, \mathbf{x}') = \sqrt{2} (2\pi)^{-3} \sqrt{k_0} e^{ik_0 x + ik_0 x'} \beta_{\rho\sigma} a_{\sigma}^r(\mathbf{K}) \quad (2.14)$$

and a is a Dirac spinor. Here $\phi^*(\mathbf{x}', 0)$ and $\bar{\psi}_\rho(\mathbf{x}, 0)$ are operators at $t=0$ in the interaction representation and will later be written as $\phi^*(\mathbf{x}')$ and $\bar{\psi}_\rho(\mathbf{x})$ respectively.

The symmetrical pseudoscalar interaction Hamiltonian is taken for H_I , that is,

$$H_I = ig \int \bar{\psi}(\mathbf{x}) \gamma_5 \tau_a \psi(\mathbf{x}) \phi_a(\mathbf{x}) d\mathbf{x}. \quad (2.15)$$

This interaction Hamiltonian seems to be well established in a low energy region and perhaps will also be applicable to high energy phenomena. Recently it has been shown by a perturbation method that the angular distribution of final mesons in one pion production at extremely high energies is better interpreted by means

* The same reduction can be made for a negative or neutral pion only by replacing φ_+^* with φ_+ or φ_3 respectively.

of the pseudoscalar interaction than the pseudovector coupling.²⁰⁾

By virtue of the symmetrical pseudoscalar interaction Hamiltonian, we get the following simple expressions,*

$$e^{-iH_I t} \phi^*(\mathbf{x}) e^{iH_I t} = \phi^*(\mathbf{x}), \quad (2.16)$$

$$e^{-iH_I t} \bar{\phi}_\rho(\mathbf{x}) e^{iH_I t} = \bar{\phi}_\rho(\mathbf{x}) \{e^{g\gamma_5 \gamma_4 \tau_\alpha \phi_\alpha(\mathbf{x})}\}_{\rho' \rho} \\ \equiv \bar{\phi}_\rho(\mathbf{x}, t). \quad (2.17)$$

The latter equation is derived as the solution of the following differential equation,

$$d_\rho \bar{\phi}(\mathbf{x}, t)/dt = -ie^{-iH_I t} [H_I, \bar{\phi}_\rho(\mathbf{x})] e^{iH_I t} \\ = \bar{\phi}_{\rho'}(\mathbf{x}, t) g(\gamma_5 \gamma_4)_{\rho' \rho} \tau_\alpha \phi_\alpha(\mathbf{x}). \quad (2.18)$$

By using Eqs. (2.13), (2.16) and (2.17), the matrix element becomes

$$M = (2\pi/T) \int \langle \chi_f^*, \phi^*(\mathbf{x}') (\bar{\phi}(\mathbf{x}) e^{g\gamma_5 \gamma_4 \tau_\alpha \phi_\alpha(\mathbf{x})})_\rho \chi_\sigma \rangle \\ \times f_\rho(\mathbf{K}, \mathbf{k}, \mathbf{x}, \mathbf{x}') d\mathbf{x} d\mathbf{x}' \delta(E_f - E_i). \quad (2.19)$$

The similar matrix element as the above expression was drawn out by Sawada, Takagi²¹⁾ and Glauber.²²⁾ It must be noticed that in deriving Eq. (2.19), the curious restriction on the magnitude of masses of pion and nucleon pointed out by Glauber is unnecessary although the expression (2.19) has a similar appearance as that of Glauber.

After a troublesome calculation shown in Appendix II, the exponential factor in Eq. (2.19) can be written down as follows,

$$e^{g\gamma_5 \gamma_4 \tau_\alpha \phi_\alpha(\mathbf{x})} = Z \sum_n (gt\gamma_5 \gamma_4)^n : \overbrace{\phi^*(\mathbf{x}) \cdots}^{n_+}, \overbrace{\phi(\mathbf{x}) \cdots}^{n_-}, \overbrace{\phi_3(\mathbf{x}) \cdots}^{n_0} : \\ \times F(n_+, n_-, n_0) G(n, n_0), \quad (2.20)$$

where

$$F(n_+, n_-, n_0) = \frac{\Gamma((n_0 + \delta_{n_0} + 1)/2) ((n - n_0 + \delta_n - \delta_{n_0})/2)!}{\Gamma((n + \delta_n + 3)/2) n_+! n_-! n_0!} \sqrt{2}^{n_0 - n} \quad (2.21)$$

and

$$G(n, n_0) = \{1 - \delta_{n_0}(1 - \delta_n)\} \{1 - 2\delta_n \delta_{n_0} P\} \quad (2.22)$$

with

$$\delta_n = \begin{cases} 0 & n = \text{even} \\ 1 & n = \text{odd} \end{cases} \quad \text{for}$$

* The authors got the hint of the use of these relations from the lecture given by Prof. R. Utiyama concerning the anti-interaction representation.

Here Z is a renormalization constant which comes from the meson propagator $\mathcal{A}^+(0)$, the symbol $::$ represents Wick's S -product, P is a projection operator to the neutron state and the final meson number is $n = n_+ + n_- + n_0$ where n_+ , n_- and n_0 are the numbers of the positive, negative and neutral mesons respectively.

The matrix element for the final state which consists of a nucleon with its momentum \mathbf{K}' and its spin state r' and n mesons with their momenta $\mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_n$ is as follows,

$$M_n = \left(\frac{1}{T}\right) \frac{(gt)^{n+1}}{2^{n/2} (2\pi)^{3n/2}} \frac{1}{\sqrt{k_0}} \left(\prod_{i=1}^n \frac{1}{\sqrt{k_{i0}}} \right) \delta(\mathbf{K} + \mathbf{k} - \mathbf{K}' - \sum_{i=1}^n \mathbf{k}_i) \\ \times \delta(K_0 + k_0 - K_0' - \sum_{i=1}^n k_{i0}) \bar{a}^{r'}(\mathbf{K}') (\gamma_5 \gamma_4)^{n+1} \gamma_4 a^r(\mathbf{K}) \\ \times \{n_+! n_-! n_0!\}^{1/2} G(n+1, n_0) \mathbf{F}(n_+, n_-+1, n_0) (n_-+1), \quad (2 \cdot 23)$$

where the renormalization constant and some numerical factors are omitted. Here the last two factors come from the fact that an initial meson is positive in charge. If it is a negative or neutral meson these factors are $\mathbf{F}(n_++1, n_-, n_0) (n_++1)$ or $\mathbf{F}(n_+, n_-, n_0+1) (n_0+1)$ respectively. The differential cross section for the definite final nucleon momentum and the definite number n of the final mesons ($n = n_+ + n_- + n_0$) is reduced from Eq. (2.23) as follows,

$$\frac{d\sigma_n}{d\mathbf{K}'} = \frac{(gt)^{2n+2}}{2^n (2\pi)^{3n}} \frac{1}{T^2} \frac{\{K_0 K_0' + (-)^{n+1} (\mathbf{K} \cdot \mathbf{K}' + m^2)\}}{K_0 k_0 \cdot K_0'} \{\mathbf{F}(n_+, n_-+1, n_0) (n_-+1)\}^2 \\ \times \{1 - \delta_{n_0} (1 - \delta_{n+1})\} \int \prod_{i=1}^n \frac{d\mathbf{k}_i}{k_{i0}} \delta^4(\mathbf{K} + \mathbf{k} - \mathbf{K}' - \sum \mathbf{k}_i). \quad (2 \cdot 24)$$

Finally one finds the relative cross section of n meson production as follows,

$$\sigma_n = \left(\frac{g^2 t^2}{16\pi^3}\right)^n \frac{1}{K_0 k_0} f(n) \sum_{n_0} S(n, n_0) \\ \times \int \frac{d\mathbf{K}'}{K_0'} \{K_0 K_0' + (-)^{n+1} (\mathbf{K} \cdot \mathbf{K}' + m^2)\} \\ \times \int \prod_{i=1}^n \frac{d\mathbf{k}_i}{k_{i0}} \delta(K_0 + k_0 - K_0' - \sum_{i=1}^n k_{i0}) \delta(\mathbf{K} + \mathbf{k} - \mathbf{K}' - \sum_{i=1}^n \mathbf{k}_i), \quad (2 \cdot 25)$$

where

$$f(n) = \left[\frac{2^{\delta_{n+1}+1} ((n+1+\delta_{n+1})/2)!}{(n+2+\delta_{n+1})! ((n+1-\delta_{n+1})/2)!} \right]^2, \quad (2 \cdot 26)$$

and

$$S(n, n_0) = \binom{(n+1-\delta_{n+1})/2}{(n_0-\delta_{n_0})/2} 2^{n+1-n_0} (n_-+1)^2 \{1 - \delta_{n_0} (1 - \delta_{n+1})\}. \quad (2 \cdot 27)$$

The last two equations are obtained by simplifying Eqs. (2.21), (2.22) and (2.2) in order to evaluate easily the different charge states.

§ 3. Covariant phase space integration

In Eq. (2.25) the covariant integral with respect to the phase space appears, which in the center of mass system reads

$$I_n(K') = \int \prod_{i=1}^n \frac{d\mathbf{k}_i}{k_{i0}} \delta(\mathbf{K}' + \sum_{i=1}^n \mathbf{k}_i) \delta(W - \sum_{i=1}^n k_{i0}), \quad (3.1)$$

where $W = K_0 + k_0 + K'_0$. This integral has recently been derived from some recurrence relation.¹⁷⁾ Since the cross section given by Eq. (2.25) contains the integration with respect to a nucleon momentum, in order to obtain a simpler form of the above multiple integral we shall here evaluate the integral $I_n(K')$ by using the same saddle point method as that used by Fialho.¹³⁾ We start with the following integral derived from Eq. (3.1),

$$I_n(K') = \frac{1}{(2\pi)^4} \int_{-\infty-i\epsilon}^{\infty-i\epsilon} d\alpha e^{i\alpha W} \int_{-\infty}^{\infty} d\lambda e^{i\lambda \mathbf{K}'} \prod_{i=1}^n J_i, \quad (3.2)$$

where

$$\begin{aligned} J_i &= \int \frac{d\mathbf{k}_i}{k_{i0}} \exp[i\lambda \mathbf{k}_i - i\alpha k_{i0}] \\ &= \frac{2\pi}{i\lambda} \int_{-\infty}^{\infty} \frac{k_i}{k_{i0}} dk_i \exp[i(\lambda k_i - \alpha \sqrt{k_i^2 + \mu^2})]. \end{aligned} \quad (3.3)$$

The J_i can be derived with a similar method to that of Lepore and Stuart.¹⁰⁾ Putting $k_i = \mu \sinh \theta$ and $k_{i0} = \mu \cosh \theta$, we obtain

$$J_i = -2\pi\mu^2 \frac{d}{d\lambda} \int_{-\infty}^{\infty} d\theta \exp[i\mu(\lambda \sinh \theta - \alpha \cosh \theta)].$$

By the transformation of variables the above equation is changed as follows,*

$$\begin{aligned} J_i &= 2\pi \frac{1}{z} \frac{d}{dz} \left\{ \int_{-\infty}^{\infty} d\theta' \exp[-iz \cosh \theta'] \right\} \\ &= 2\pi^2 i \frac{H_1^{(2)}(z)}{z} \quad \text{with } z = (\alpha^2 - \lambda^2)^{1/2}, \end{aligned} \quad (3.4)$$

where the definition and the relation between Hankel functions have been used. The Hankel function in Eq. (3.4) has the first rank owing to the integrand $1/k_{i0}$, whereas those appeared in Lepore and Stuart's formula is of the second rank.

Substituting Eq. (3.4) into Eq. (3.2) and integrating the angular part of (3.2), we have

* Throughout this paper the pion mass μ is put to unity.

$$I_n(K') = \frac{(2\pi^2 i)^n}{(2\pi)^4} \frac{2\pi}{iK'} \int_{-\infty - i\epsilon}^{\infty - i\epsilon} d\alpha e^{i\alpha W} \int_{-\infty}^{\infty} d\lambda \cdot \lambda e^{i\lambda K'} \\ \times \exp \left[n \log \frac{H_1^{(2)}(z)}{z} \right]. \quad (3.5)$$

If n is a large positive number, the integral with respect to λ may be evaluated asymptotically with the method of steepest descents. The asymptotic expansion of $\exp[n \cdot \log(H_1^{(2)}(z)/z)]$ are performed in the vicinity of $\lambda=0$ which is the root of the following equation,

$$\frac{\partial [\log(H_1^{(2)}(z)/z)]}{\partial \lambda} = 0.$$

By this method it is led to

$$(2\pi/iK') \int_{-\infty}^{\infty} d\lambda \cdot \lambda e^{i\lambda K'} \exp \left[n \log \frac{H_1^{(2)}(z)}{z} \right] \\ = (2\pi/iK') \left[\frac{H_1^{(2)}(\alpha)}{\alpha} \right]^n \int_{-\infty}^{\infty} d\lambda \lambda e^{i\lambda K'} e^{-ns(\alpha)/2} \\ = (2\pi)^{3/2} \left[\frac{H_1^{(2)}(\alpha)}{\alpha} \right]^n n^{-3/2} s(\alpha)^{-3/2} \exp \left[\frac{-K'^2}{2ns(\alpha)} \right], \quad (3.6)$$

$$\text{where} \quad s(\alpha) = -H_2^{(3)}(\alpha)/\alpha H_1^{(2)}(\alpha). \quad (3.7)$$

Thus Eq. (3.5) runs to

$$I_n(K') = \frac{(2\pi^2 i)^n}{(2\pi)^4} (2\pi)^{3/2} n^{-3/2} \int_{-\infty - i\epsilon}^{\infty - i\epsilon} d\alpha s(\alpha)^{-3/2} \exp \left[\frac{-K'^2}{2ns(\alpha)} \right] \\ \times \exp \left\{ n \log \frac{H_1^{(2)}(\alpha)}{\alpha} + i\alpha\gamma \right\}, \quad (3.8)$$

where W/n has been replaced by γ which is the average energy of mesons.

The above integral can also be performed in the same way at the saddle point defined by

$$\frac{\partial}{\partial \alpha} \left\{ \log \frac{H_1^{(2)}(\alpha)}{\alpha} + i\alpha\gamma \right\} = 0, \quad (3.9)$$

or equivalently

$$\frac{2}{x} + \frac{-iH_0^{(2)}(-ix)}{-H_1^{(2)}(-ix)} = \gamma, \quad (3.10)$$

where $-ix = \alpha$. It is seen that Eq. (3.10) has a real positive root a since the value of $-iH_0^{(2)}(-ix)$ and $-H_1^{(2)}(-ix)$ are positive when x is positive. The covariant integration with respect to the phase space of Eq. (3.1) finally becomes

$$I_n(K') = \frac{(2\pi^2)^n}{(2\pi)^2} n^{-2} \frac{(a/\gamma)^{3/2}}{J(a)^{1/2}}$$

$$\times \exp \left[n \left\{ \log \frac{-H_1^{(2)}(-ia)}{a} + \gamma a \right\} - \frac{K'^2 a}{2n\gamma} \right], \quad (3.11)$$

where a is the solution of Eq. (3.10) and

$$J(a) = 1 + \frac{2}{a^2} - \frac{3}{a} \left\{ \frac{-iH_0^{(2)}(-ia)}{-H_1^{(2)}(-ia)} \right\} - \left\{ \frac{-iH_0^{(2)}(-ia)}{-H_1^{(2)}(-ia)} \right\}^2. \quad (3.12)$$

Eq. (3.11) is simpler than the corresponding one given by Fialho, for we are concerned with the system of pions and the factor $1/k_{i0}$ exists in the definition of $I_n(K')$. If the mass of π -meson and the conservation law of momentum are neglected in an extremely relativistic case, the following equation is easily obtained

$$\int \prod_{i=1}^n \frac{dk_i}{k_{i0}} \delta(W - \sum_{i=1}^n k_{i0}) = (4\pi)^n (W - n\mu)^{2n-1} / (2n-1)!. \quad (3.13)$$

From Eqs. (2.25) and (3.11) we can obtain quantitative results which are the relative cross section of the n pion production, the angular distribution, the momentum distribution of final nucleons and so on. These quantitative results of this theory will be compared with experimental ones in the next section.

§ 4. Comparison with experiment

A. The weights of various charge states

We consider in the first place the weights of charge states. If the number n of final mesons is fixed, the value of $S(n, n_0) / \sum_{n_0} S(n, n_0)$ defined by Eq. (2.27) means the probability of the charge state given by n and n_0 . This probability is represented by

$$C(n, n_0) = S(n, n_0) / \sum_{n_0} S(n, n_0). \quad (4.1)$$

In general n and n_0 determine the numbers of charged mesons and the charge state of final nucleons in pion-nucleon collisions, for example, the charge state with $n=5$ and $n_0=2$ means $(p\pi^+\pi^0\pi^0\pi^-\pi^-)$ in a π^-p or a π^0n collision. The factor $\{1 - \delta_{n_0}(1 - \delta_{n+1})\}$ contained in Eq. (2.27) shows that the transitions to the states with $n=\text{odd}$ and $n_0=\text{odd}$ are exactly forbidden. This circumstance is very different from the ordinary one led by the combination of Clebsh-Gordan coefficients. In our case this forbidden rule is due to the fact that different Feynman graphs* corresponding to a final state have the same expression in the matrix element apart from the factor depending on τ -spins, accordingly, when all the terms are summed up, the contributions from the factors involving τ -matrices cancel out owing to the

* Strictly speaking, owing to the fact that the present representation of field quantities is different from the conventional interaction representation, the process under consideration cannot be schematically shown by Feynman's diagram. However, if remotely compared, our process corresponds to a Feynman's diagram in which every meson line is emerged from a single world point $(x, t=0)$, contrary to our expectation, since every field variable is represented by those defined at $t=0$.

commutation character of these matrices if both n and n_0 are odd. In Fermi's theory, on the other hand, we neglect in the cross section the interferences between different terms of the matrix element which in some sense correspond to our various Feynman's graphs.

In our opinion, the Fermi theory allows Feynman's diagrams in which the isobar states of nucleons can occur, whereas, owing to the charge independent interaction, the present theory involves only those intermediate states which have the same τ -spin as that of the initial state and the isobar states are excluded. This difference is a part of the reason for which gives rise to the above mentioned forbidden rule. When n is large, the probabilities of charge states, however, are derived after very troublesome calculations in his theory. One of us and Kakudo²³⁾ have calculated these probabilities (we represent them as $C'(n, n_0)$ hereafter) until $n=6$ in π - N collisions.

Let us consider, now, to make the approximate formula for C' , although a more roughly approximate one has been given by Kwon and Kakudo.²⁴⁾ When a final state is designated by $n=n_++n_-+n_0$, the number of Feynman's diagrams corresponding to the final state is $\binom{n+1}{n_0} \cdot (n_++1)$ in a π^- - p collision. One gets this product by the following considerations that the factor $\binom{n+1}{n_0}$ shows the location of τ -spins and the factor (n_++1) means the number of ways by which the initial π^- is absorbed in the diagram. Since we have the value $(\sqrt{2})^{n+1-n_0}$ for each diagram and in Fermi's theory we neglect the interference terms, we get approximately

$$C'(n, n_0) = S'(n, n_0) / \sum_{n_0} S'(n, n_0), \quad (4.2)$$

where

$$S'(n, n_0) = \binom{n+1}{n_0} 2^{n+1-n_0} (n_++1). \quad (4.3)$$

In Table I, we tabulate the approximate values $C'(n, n_0)$ given by the above equation in comparison with the correct values derived from the combination of Clebsch-Gordan coefficients for $n=5$ and $n=6$ in a π^- - p collision. One will find in this table a fairly good agreement between the approximate values and the correct ones, especially find a good correspondence of relative size of each charge state.

From the experimental view-point the probability having some fixed number of charged prong is more interesting. The values of $\sum_{n_0} C(n, n_0)$ for a fixed n and $p \pm (=1+n-n_0$ for the final proton and $=n-n_0$ for the final neutron) is shown for the case of a π^- - p collision in Table II. The corresponding probabilities $\sum_{n_0} C'(n, n_0)$ having fixed $p \pm$ given by Fermi's theory are tabulated in Table III. The latter for $n \leq 6$ are derived by combinations of Clebsch-Gordan coefficients and ones for $n > 6$ are obtained by Eq. (4.2). In order to compare the quantitative results predicted by the present theory with the experimental data

Table I. Comparison of the approximate values of $C'(n, n_0)$ derived from Eq. (4.2) with the correct ones derived from the combinations of Clebsch-Gordan coefficients for $n=5$ and $n=6$ in a π^-p collision.

$n=5$	p_{\pm}	$C'(n, n_0)$		$n=6$	p_{\pm}	$C'(n, n_0)$	
		C-G	approx.			C-G	approx.
$n \pi^0 \pi^0 \pi^0 \pi^0 \pi^0$	0	0.008	0.007	$n \pi^0 \pi^0 \pi^0 \pi^0 \pi^0 \pi^0$	0	0.003	0.002
$p \pi^0 \pi^0 \pi^0 \pi^0 \pi^-$	2	0.042	0.037	$p \pi^0 \pi^0 \pi^0 \pi^0 \pi^0 \pi^-$	2	0.018	0.015
$n \pi^+ \pi^0 \pi^0 \pi^0 \pi^-$	2	0.183	0.195	$n \pi^+ \pi^0 \pi^0 \pi^0 \pi^0 \pi^-$	2	0.095	0.099
$p \pi^+ \pi^0 \pi^0 \pi^0 \pi^-$	4	0.302	0.293	$p \pi^+ \pi^0 \pi^0 \pi^0 \pi^0 \pi^-$	4	0.205	0.198
$n \pi^+ \pi^+ \pi^0 \pi^0 \pi^-$	4	0.337	0.351	$n \pi^+ \pi^+ \pi^0 \pi^0 \pi^0 \pi^-$	4	0.337	0.357
$p \pi^+ \pi^+ \pi^0 \pi^0 \pi^-$	6	0.129	0.117	$p \pi^+ \pi^+ \pi^0 \pi^0 \pi^0 \pi^-$	6	0.249	0.238
				$n \pi^+ \pi^+ \pi^+ \pi^0 \pi^0 \pi^-$	6	0.094	0.091

Table II. Probabilities for charged prongs (p_{\pm}) emerged from a π^-p collision in the case of n final mesons according to the present theory; $\sum_{n_0} C(n, n_0)$ with the fixed p_{\pm} .

$n \backslash p_{\pm}$	0	2	4	6	8
1	0.000	1.000			
2	0.053	0.947			
3	0.000	0.200	0.800		
4	0.004	0.289	0.707		
5	0.000	0.030	0.485	0.485	
6	0.000	0.053	0.518	0.429	
7	0.000	0.004	0.147	0.588	0.261
8	0.000	0.008	0.176	0.582	0.234

Table III. Probabilities for charged prongs (p_{\pm}) emerged from a π^-p collision in the cases of n final mesons given by combinations of Clebsch-Gordan coefficients for $n \leq 6$ and by Eq. (4.2) for $n > 6$.

$n \backslash p_{\pm}$	0	2	4	6	8
1	0.444	0.556			
2	0.156	0.844			
3	0.061	0.662	0.277		
4	0.021	0.411	0.568		
5	0.008	0.224	0.639	0.129	
6	0.003	0.112	0.543	0.342	
7	0.000	0.077	0.658	0.263	0.002
8	0.000	0.028	0.491	0.460	0.021

in cosmic energy regions we must calculate the probabilities having a given p_{\pm} and a large n . In Table IV these probabilities for $n \geq 7$ are shown when the initial

Table IV. Relative probabilities with normalization 2 of that the charged prongs ($p\pm$) emerge from $\pi^\pm p$ and $\pi^\pm n$ collisions according to the present theory.

$n \backslash p\pm$	1	2	3	4	5	6	7	8	9
6	0.007	0.046	0.156	0.506	0.506	0.449	0.331		
7	0.004	0.004	0.147	0.147	0.588	0.588	0.261	0.261	
8	0.001	0.006	0.044	0.165	0.281	0.584	0.498	0.245	0.176
9	0.001	0.001	0.033	0.033	0.300	0.300	0.533	0.533	0.133
10		0.001	0.010	0.040	0.112	0.312	0.376	0.522	0.412
11			0.006	0.006	0.103	0.103	0.412	0.412	0.412
12			0.002	0.008	0.036	0.110	0.198	0.417	0.417
13			0.001	0.001	0.028	0.028	0.200	0.200	0.450
14				0.002	0.009	0.031	0.081	0.207	0.279
15					0.007	0.007	0.073	0.073	0.294
16					0.002	0.008	0.027	0.077	0.142
17					0.002	0.002	0.022	0.022	0.139

π -meson is incident with the equal abundance with respect to π^+ and π^- and at the same time the targets consist of the same number of protons and neutrons.

B. Multiplicity

In § 2, we have formulated the relative cross section of n final mesons. Eq. (2.25) contains an unknown factor t determined by the character of two incident wave packets. Since the determination of the magnitude of t is, however, difficult, we consider t as a parameter. If we apply Eq. (3.13) with $t=\text{constant}$ to collisions in extremely high energy regions, the most probable value of n is given by

$$\langle n \rangle \propto E^{1/2}, \quad (4.4)$$

where E is the total energy in c. m. system. This relation seems to be supported by experimental facts especially in pion-nucleon collisions. It might be curious that the value of t is independent of an initial energy. From our view-point stated in § 2 the incident particles have been represented by two mutually non-interacting wave packets with some average momenta. The width of the packet depends not on the magnitude of the momentum or energy but on their uncertainty owing to the Heisenberg principle. Therefore the interaction time t can be regarded as independent of the initial energy contrary to the customary picture of the meson cloud of the incident nucleon. We determine the value of t so as to agree approximately the multiplicity predicted by Eq. (2.25) with the experimental one, that is

$$t=0.68. \quad (4.5)$$

We can obtain the relative cross sections σ_n after a numerical integration with respect to \mathbf{K}' by using Eqs. (2.25), (3.11) and $g^2/4\pi=15$ which seems to be well established in low energy pion physics.

Table IV. (continued)

n	p_{\pm}	10	11	12	13	14	15	16	17	18
9		0.133								
10		0.126	0.090							
11		0.412	0.066	0.066						
12		0.402	0.304	0.063	0.044					
13		0.450	0.288	0.288	0.032	0.032				
14		0.449	0.402	0.281	0.207	0.031	0.021			
15		0.294	0.423	0.423	0.188	0.188	0.015	0.015		
16		0.298	0.336	0.419	0.350	0.184	0.132	0.014	0.010	
17		0.139	0.357	0.357	0.357	0.357	0.117	0.117	0.007	0.007

Table V. Relative probabilities of producing $(n-1)$ secondary pions and of producing p_{\pm} charged prongs, the angular distribution of protons predicted by Theory I and Theory II in π^-p collisions with 5 Bev and the experimental distribution of charged prongs.

n	1	2	3	4	5	6	7	\bar{n}
σ_n	0.004	0.059	0.453	0.251	0.207	0.020	0.006	3.68

$\sigma_{p_{\pm}}$	elastic	inelastic p_{\pm}				
		0	2	4	6	8
Exp.	0.017	0.077	0.547	0.333	0.026	
Th. I	0.004	0.004	0.227 $1-0.185 \cos\theta$	0.651 $1-0.263 \cos\theta$	0.112	0.002
Th. II	0.002	0.046	0.504 $1+0.051 \cos\theta$	0.414 $1+0.036 \cos\theta$	0.034	0.000

Employing Fermi's statistical theory some authors²³⁾²⁵⁾ have calculated the relative probabilities of n final mesons and of p_{\pm} charged prongs in a π^-p collision with 5 Bev. In Table V, the present theory with the above t value is compared with the experiments²⁵⁾²⁶⁾ in π^-p collisions with 5 Bev. Because the distribution of multiplicities cannot be directly compared with experimental results, the charged prongs distribution $\sigma_{p_{\pm}}$ is also tabulated and compared with experimental one. The notation Th. I represents the results derived from the theory in which $\sum_{n_0} C(n, n_0)$ tabulated in Table II has been used as the probabilities of charged prongs, and in Th. II $\sum_{n_0} C(n, n_0)$ has been replaced by $\sum_{n_0} C'(n, n_0)$ shown in Table III. In the case of Th. I the theoretical values of $\sigma_{p_{\pm}}$ are rather larger for large p_{\pm} than experimental ones. This fact is caused mainly by the factor $(n_+ + 1)^2$ in $C(n, n_0)$. If final state interactions exist and have only an effect on final charge states without giving rise to any change for σ_n , Th. II seems to be

justified. The similar numerical results in π^-p collisions with 8 Bev are shown in Table VI. In Table VII, comparison is given of the numerical results predicted by Th. I with the experimental ones²⁷⁾ in cosmic energy region: its average energy is about 200 Bev. We treat all events of secondary particles as pion-nucleon collisions and the fourteen events cited there have been selected by the following three criterions, a) the number of gray and black tracks ≤ 4 , b) the inelasticity is known, and total energy in c.m. system $< 50 mc^2$.

Table VI. Relative probabilities of producing $(n-1)$ secondary pion and of producing $p\pm$ charged prongs and the angular distribution of protons predicted by Theory I and Theory II in π^-p collisions with 8 Bev.

n	1	2	3	4	5	6	7	8	\bar{n}
σ_n	0.001	0.026	0.262	0.290	0.339	0.059	0.022	0.001	4.21

$\sigma_{p\pm}$	elastic	inelastic $p\pm$				
		0	2	4	6	8
Th. I	0.001	0.002	0.174 $1-0.218 \cos\theta$	0.613 $1-0.283 \cos\theta$	0.204 $1-0.330 \cos\theta$	0.006
Th. II	0.001	0.029	0.399 $1+0.048 \cos\theta$	0.501 $1+0.093 \cos\theta$	0.071 $1-0.092 \cos\theta$	0.000

Table VII. Relative probabilities of producing $(n-1)$ secondary pions and of producing $p\pm$ charged prongs, the inelasticity predicted by Theory I in $\pi^\pm p$ and $\pi^\pm n$ collisions with 200 Bev and the corresponding values in cosmic ray events.

n	7	8	9	10	11	12
σ_n	0.002	0.011	0.049	0.151	0.200	0.209

n	13	14	15	16	17	\bar{n}
σ_n	0.179	0.107	0.061	0.024	0.008	12.0

$p\pm$	3	4	5	6	7	8	9	10	11
$\sigma_{p\pm}$	0.003	0.007	0.035	0.061	0.131	0.171	0.186	0.173	0.111

$p\pm$	12	13	14	15	16	$\bar{p}\pm$	$\bar{p}\pm_{exp}$	\bar{K}	\bar{K}_{exp}
$\sigma_{p\pm}$	0.073	0.030	0.014	0.004	0.001	9.0	9.3	0.88	0.85

C. The momentum distribution of a nucleon

With respect to the momentum distributions of nucleons and π -mesons we can see its rough tendency from Eq. (2.24). In Fermi's statistical theory these distribu-

tions are symbolized by $d\mathbf{K}'$ and $d\mathbf{k}_i$ for a nucleon and a meson respectively. The corresponding ones in Eq. (2.24) involve, on the other hand, $d\mathbf{K}'$ and $d\mathbf{k}_i/k_{i0}$ as factors. The latter case shows that nucleons are inclined to have larger momentum but mesons have lower momentum compared with ones predicted by the statistical theory. This is a good tendency to fit the experimental distributions. The comparison of the experiment and the theory has been shown in Fig. 1 and Fig. 2 with respect to the momentum distribution of protons in π^-p collisions with 5 Bev in the case of 2 prong events and of 4 & 6 prongs respectively. In these figures the curves are predicted by Th. I but they are scarcely affected by replacing Th. I with Th. II. Fig. 5 shows the same distributions for the cases of 2, 4 and 6 & 8

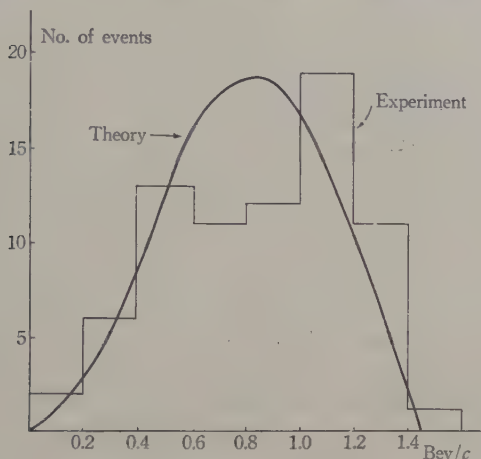


Fig. 1 Momentum distribution of protons from 2 prong events of π^-p collisions with 5 Bev.

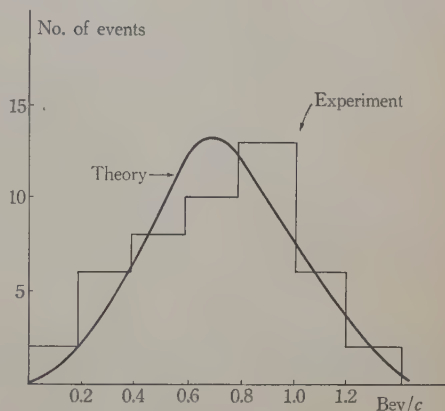


Fig. 2 Momentum distribution of protons from 4 and 6 prong events of π^-p collisions with 5 Bev.

prongs with 8 Bev. In Table VII, the inelasticity is compared with experiments with 200 Bev and its agreement is fairly good. Although some of the experimental inelasticities exceed unity, we took the mean value of these with no corrections.

D. Angular distribution

The angular distribution of mesons in the present theory is isotropic just as LOW's if the momentum conservation is neglected or n is large. By considering Eq. (2.25) we find that nucleons essentially have its angular distribution represented as follows,

$$\frac{d\sigma_n}{d\Omega} \propto 1 + (-)^n a_n \cos \theta \quad \text{with } 0 \leq a_n < 1, \quad (4.6)$$

where θ is the angle between the direction of an initial meson and of a final nucleon, and the factor $(-)^n$ shows the effect of the recoil of a nucleon. In order to compare the theoretical angular distribution with the experimental one we must recombine some states of the same prong as has been done in the above

parts. Using Th. I and II we tabulate the angular distributions of final protons with some type of prongs for the cases of the initial energy 5 and 8 Bev in Tables V and VI respectively. The comparison of the former result with experimental data is shown in Figs. 3 and 4. Because the final states having the same

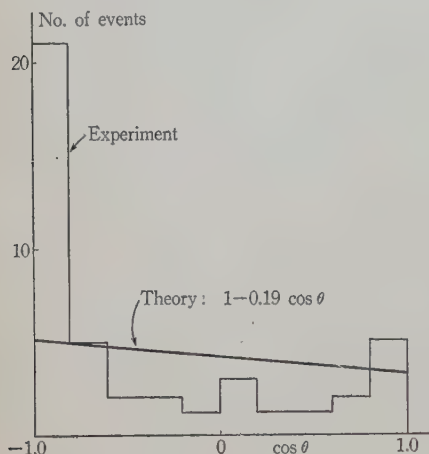


Fig. 3. Angular distribution of protons from 2 prong events of π^-p collisions with 5 Bev.

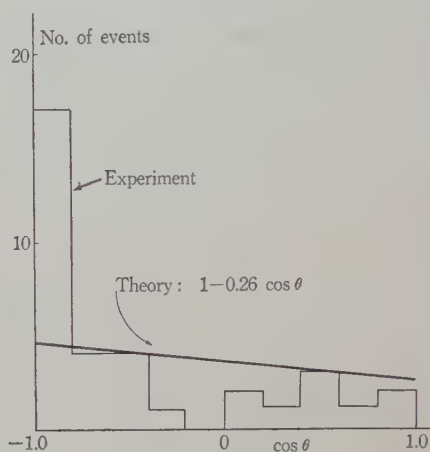


Fig. 4. Angular distribution of protons from 4 and 6 prong events of π^-p collisions with 5 Bev.

prong consists of states with various n 's and the factors $(-)^n a_n$ almost cancel out with each other, the angular distribution with a fixed prong is nearly isotropic contrary to the experimental distribution. The above fact seems natural from a consideration that under our assumption we have neglected the contributions from the internal nucleon line which in the conventional field theory is expected to give rise to sharp angular dependence. Namely, in an ordinary perturbation method,

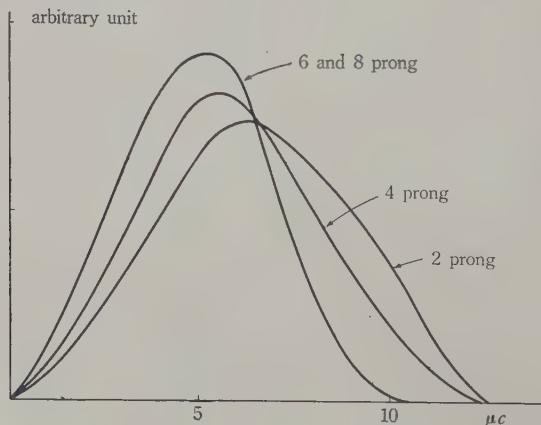


Fig. 5. Momentum distribution of protons from 2, 4 and 6 & 8 prong events of π^-p collisions with 8 Bev.

this nucleon line is replaced by S_F function having the character of $1/(1+a \cos \theta)$, thanks to which it may be possible to show the backward peak of the angular distribution of nucleons. Taking the ps-coupling in the latter case we must, however, meet a following difficulty that the direction of an internal nucleon is very changeable owing to its recoil due to the creation of mesons and $\Pi/(1+a_i \cos \theta_i)$ may be almost isotropic just as in the case of the present theory. In the high energy region the angular distribution of nucleons is one of the most interesting problems and remains open for theoretical physics.

§ 5. Summary and discussion

In preceding sections we have consistently studied the pion-nucleon collision problem in high energy regions by introducing the interaction time and assuming that the quantum effect of H_0 can be almost neglected during the interaction time interval. In § 2 we have formulated the relative cross section with n final mesons. By observing this cross section we can roughly investigate the momentum distribution and the angular distributions of both nucleons and mesons. The covariant phase integration has been done in § 3 and obtained as a simpler form than Srivastava and Sudarshan's. This phase space integration derived by the saddle point method has been estimated to have an error of about 10% even when the energy is 5 Bev and $n=2$. This error is, of course, more strongly depressed at higher energies and larger n . By using this form and an approximate Eq. (4.2) of the probability of the charge state we can easily establish a modified statistical theory apart from the present theory, to calculate easily the relative cross sections, however large n may be, is

$$\sigma(n, n_0) \approx F'(n) C'(n, n_0) \int d\mathbf{K}' \Pi \frac{d\mathbf{k}_i}{k_{i0}} \delta^4(K + k - K' - \sum \mathbf{k}_i), \quad (5.1)$$

where the factor $F'(n)$ corresponds to the weight of the states with different resultant isotopic spins in the original statistical theory. If $F'(n)$ is given the same weight, $\sigma(n, n_0)$ will give the large multiplicity. Comparing Fermi's theory with Eqs. (2.24) and (5.1) one finds that the interaction volume \mathcal{Q} in the former theory is replaced by $1/k_{i0}$. This point of view is more or less similar to Pomerancuk's.

A more resemblance of the present theory will be found in LOW's theory. The pseudoscalar type interaction gives too low multiplicity in their treatment. We cannot make the same formulation of cross sections by our procedure stated so far in the case of the pseudovector type interaction because of the occurrence of the $\partial\phi/\partial x$ in H_I which makes it difficult to compute Eq. (2.16). Contrary to LOW's theory, however, we have seen that the strong ps-interaction can give a high multiplicity. In the present theory the magnitude of cross section depends not on the separate g and t but on the product of these parameters. If t is restricted by inequality (A.1.6) in order to justify assumption (2.12), namely

if we take

$$t \ll 2/\langle k_{i0} \rangle, \quad (5.2)$$

where $\langle k_{i0} \rangle$ is the mean energy of final mesons, then $g^2/4\pi$ becomes too much larger than 15 when the multiplicity is fixed to agree with the experimental data. Concretely stating, since $2/\langle k_{i0} \rangle$ is about 0.2 in π -N collisions with 200 Bev we have $g^2/4\pi \gg 172$ in order to get a suitable magnitude of multiplicity which gives $(gt)^2/4\pi = 7.0$. Inequality (5.2) as shown in Appendix I comes from the fact that we have taken no account of the effect of pair creations in the $[H_0, H_I]$. The matrix element with a final state in which pairs of nucleons are included must be calculated by this term. The pair effect might also be neglected when $g^2/4\pi$ is very large, or equivalently t is sufficiently small. The above mentioned circumstance in the present theory is different from LOW's. Because the nucleon recoil is neglected from the latter theory and this recoil is quite effective in the ps-interaction, the cross sections of many meson production seem to be suppressed in their theory. In our case, on the other hand, the nucleon recoil is taken into account and the assumption (2.12) is essentially valid in the case of the strong interaction. Which is more appropriate ps- or pv-coupling seems to be an open question in high energy physics. We would rather elect ps-type interaction on account of its renormalizability.

The predicted numerical results with $(gt)^2/4\pi = 7.0$ are compared with experimental ones in § 4. The momentum distribution is in better agreement with experimental one than that of the ordinary statistical theory. The theoretical curves as shown in Figs. 1 and 2 seem still to be slightly depressed at high energies. With respect to the angular distribution of π -mesons the experiments even with 200 Bev show it almost isotropic and this is also predicted by the present theory as well as by LOW. and Fermi's theory. Concerning the nucleon angular distribution one finds the remarkable difference between the experimental and predicted one in π^-p collisions with 5 Bev as shown in Figs. 3 and 4. As already mentioned this comes mainly from the neglect of contributions of internal nucleon lines. If this line is replaced by S_F function we have no compact formulation for the symmetrical theory and it would be difficult to derive the backward peak of the nucleon angular distribution even though the S_F functions had been taken into account.

The difficulties in the present theory so far mentioned might be caused partly by final state interactions. Many authors²⁸⁾ have pointed out the possibility that final state interactions have effects fairly on the multiple production. Tables V and VI in § 4 give the comparison between the experimental data and the results of Th. II which is derived from the assumption that final state interactions change only the probability of charge states. It is seen that Th. II is more favourable with respect to prong distributions than Th. I. Final state interactions have, of course, the possibilities of changing other quantities. For example, it may happen

that the interaction time interval t is lengthened by its effects, hence the multiplicities become large. Some authors²⁹⁾³⁰⁾ have proposed that a collision of two energetic nucleons gives rise to two fire balls in which no nucleon is contained. If a π - N collision also creates such fire balls, although in this case there appears only a single fire ball and its velocity is small, it seems to be possible to show the backward peak of a final nucleon and the small momentum of final mesons. We could not, however, establish such a model only by means of the interaction between a nucleon and mesons. It may be necessary to consider other kinds of interactions, for example π - π or more complicated interactions. It may be said that the necessity of unfamiliar strong couplings, such as π - π , is reflected in the fact that in the present paper we had to take a very large coupling constant.

The authors are interested in the same problem for the cases of a nucleon-nucleon collision and an annihilation of a nucleon pair. In spite of the complexity due to the τ -matrix for the former case, a further approximation besides the assumptions of this paper leads to the expression for the differential cross section. A sharper distribution and a high elasticity of a nucleon will be expected compared with the case of a meson-nucleon collision.

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Appendix I.

A rough investigation with respect to the validity of assumption (2.12).

The validity of Eq. (2.12) will be roughly investigated below. As well known³¹⁾ the following relation is satisfied for the operator $H=H_0+H_I$,

$$e^{i(H_0+H_I)t} = e^{iH_0t} e^{iH_I t} e^{c_2/2} e^{c_3} \dots, \quad (\text{A} \cdot 1 \cdot 1)$$

where $c_2 = [H_0, H_I] t^2$

and $c_3 = (i/3) [c_2, H_I] t + (i/6) [c_2, H_0] t.$

If the following inequality,

$$\langle a | H_I | b \rangle t \gg \langle a | c_2 | b \rangle / 2, \quad (\text{A} \cdot 1 \cdot 2)$$

is well established between an arbitrary initial state $|b\rangle$ and final state $\langle a|$, Eq.

(2.12) is easily derived from Eq. (2.11). Here we roughly estimate the right-hand side of (A.1.2). Let us assume that $|a\rangle$ and $|b\rangle$ are eigenstates of H_0 with eigenvalues E_a and E_b , then we have

$$\langle a|[H_0, H_I]|b\rangle = (E_a - E_b)\langle a|H_I|b\rangle.$$

For the sake of simplicity it is also assumed that $|b\rangle$ shows a state in which one nucleon with the energy $E_b = \varepsilon_b$ exists, while $|a\rangle$ represents a state in which one meson with the energy ω exists together with one nucleon of its energy ε_a , namely, we put $E_a = \omega + \varepsilon_a$. In this simple case, (A.1.2) gives

$$\langle a|H_I|b\rangle t \gg (t^2/2)(\varepsilon_a + \omega - \varepsilon_b)\langle a|H_I|b\rangle. \quad (\text{A.1.3})$$

From this relation we have

$$t \ll 2/(\varepsilon_a + \omega - \varepsilon_b), \quad (\text{A.1.4})$$

$$\text{if} \quad \langle a|H_I|b\rangle \neq 0. \quad (\text{A.1.5})$$

The latter condition shows that at least one of the states $|a\rangle$ and $|b\rangle$ should be virtual, accordingly we cannot get any information about the magnitude of t from the relation (A.1.3). However, if the change of energy $|\varepsilon_a - \varepsilon_b|$ of the nucleon is assumed to be not so large as compared with the meson mass and, further, if ω may be put equal to the average energy of the emitted mesons, we have a relation on t

$$t \ll 2/\langle \omega \rangle. \quad (\text{A.1.6})$$

Appendix II.

The derivation of Eq. (2.20)

The exponential factor can be written as

$$e^{gt \tau_3 \tau_4 \tau_5 \tau_6 \tau_7 \tau_8 \tau_9 \tau_{10}} = \sum_n \sum_{s: \text{even}} \frac{(gt \gamma_5 \gamma_4)^n (gt)^s}{(n+s)!} \cdot h(n_+, n_-, n_0, s) \\ \times (\mathcal{A}^+(0))^{s/2} : \overbrace{\phi^*(\mathbf{x}) \cdots}^{n_+}, \overbrace{\phi(\mathbf{x}) \cdots}^{n_-}, \overbrace{\phi_3(\mathbf{x}) \cdots}^{n_0} :. \quad (\text{A.2.1})$$

$$\text{Here } h(n_+, n_-, n_0, s) = \sum_{s_0: \text{even}} G(n, n_0) \begin{pmatrix} (s_0 + n_0 - \delta_n)/2 \\ (s_0 - n_0 - \delta_{n_0})/2 \end{pmatrix} \begin{pmatrix} n_+ + (s - s_0)/2 \\ n_+ \end{pmatrix} \\ \times \begin{pmatrix} n_- + (s - s_0)/2 \\ n_- \end{pmatrix} \begin{pmatrix} s + n_0 \\ n_0 \end{pmatrix} \left(\frac{s - s_0}{2} \right)! (s_0 - 1)!! \sqrt{2}^{s - s_0 + n - n_0}, \quad (\text{A.2.2})$$

where s is the number of τ -matrices at the ends of internal lines, among which there are s_0 τ_3 -matrices. n_+ , n_- and n_0 respectively mean the numbers of τ_- , τ_+ and τ_3 -matrices at the ends of external lines. In order to get this expression, it is helpful to consider the diagram in Fig. 6. (cf. § 4, A. footnote) The first two factors come from the variety of the location of τ -matrix for fixed numbers of τ_- , τ_+ and τ_3 vertices. The last factor comes from the transformation of meson

fields from real to complex representation. The other factor is the number of diagrams for a fixed location of τ -matrices. Each internal nucleon and meson lines are represented as 1 and $\Delta^+(0)$ respectively. In (A.2.1) the singular quantity $\Delta^+(0)$ is in practice infinite but we may regard this as a finite constant however large it may be, since this quantity will be eliminated from the last expression by considering a relative cross section. It may be seen that among the terms of the right-hand side of (A.2.1) the main contributions come from those with

$$s \gg n \text{ and } s_0 \gg n_0. \quad (\text{A.2.3})$$

This situation greatly facilitates the following calculations. Eq. (A.2.2) becomes

$$h(n_+, n_-, n_0, s)$$

$$\begin{aligned} &= G(n, n_0) \sum_{s_0, \text{even}} \frac{(s/2)! (s/2)^{(n-\delta_n)/2} ((s-s_0)/2)^{n_++n_-}}{(s_0/2)! (s_0/2)^{(n_0-\delta_{n_0})/2} ((s-s_0)/2)!} \\ &\times \frac{2^{n_0} (s_0/2)^{n_0} (s-s_0/2)! s_0! \sqrt{2^{s-s_0+n-n_0}}}{((s-s_0)/2)^{(n-n_0-\delta_n+\delta_{n_0})/2} n_+! n_-! n_0! (s_0/2)! 2^{s_0/2}} \\ &= \frac{G(n, n_0) \sqrt{2^{s+n+n_0}} (s/2)!}{n_+! n_-! n_0! \sqrt{\pi}} \sum_{s_0, \text{even}} (s_0/2)^{(n_0+\delta_{n_0}-1)/2} ((s-s_0)/2)^{(n-n_0+\delta_n-\delta_{n_0})/2} (s/2)^{(n-\delta_n)/2} \end{aligned}$$

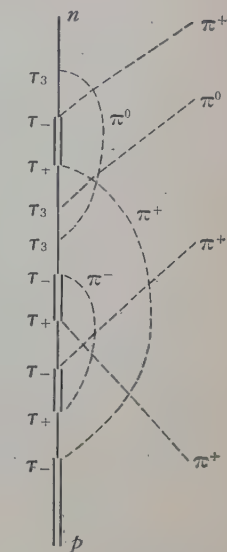


Fig. 6 A diagram of the process $\pi^+ + p \rightarrow n + \pi^+ + \pi^+ + \pi^0$ having $s=6$, $s_0=2$, $n_+=2$, $n_-=1$ and $n_0=1$.

Here the following two approximations have been used.

$$\Gamma(a+1) \cong a^a e^{-a} \sqrt{2\pi a} \quad \text{for } a \gg 1, \quad (\text{A.2.4})$$

$$\Gamma(a+b) \cong \Gamma(a) a^b \quad \text{for } a \gg b. \quad (\text{A.2.5})$$

The summation as to s_0 can be replaced by an integration and its result is

$$\begin{aligned} h(n_+, n_-, n_0, s) &= \frac{G(n, n_0) \Gamma((n_0+\delta_{n_0}+1)/2) ((n-n_0+\delta_n-\delta_{n_0})/2)! \sqrt{2^{n_0-n}}}{n_+! n_-! n_0! \Gamma((n+\delta_n+3)/2)} \\ &\times \frac{s! s^{n+1}}{(s/2)! \sqrt{2^s} 2} = F(n_+, n_-, n_0) G(n, n_0) \frac{s!}{(s/2)! 2^{s/2}} \cdot \frac{s}{2}. \end{aligned} \quad (\text{A.2.6})$$

Thus we can get the following expression,

$$\begin{aligned} \frac{e^{g\tau_5\tau_4\tau_3\tau_2\tau_1} \phi_\alpha(x)}{F(n_+, n_-, n_0) G(n, n_0)} &= \sum_n \sum_{s: \text{even}} \frac{-2^{-1/2} g^2 t^2 \Delta^+(0)}{(s/2-1)!} \{-2^{+1} g^2 t^2 \Delta^+(0)\}^{s/2-1} \\ &\times (\gamma_5 \gamma_4 g t)^n : \phi^*(x) \cdots, \phi(x) \cdots, \phi_3(x) \cdots : \\ &= Z \sum_n (\gamma_5 \gamma_4 g t)^n : \phi^*(x) \cdots, \phi(x) \cdots, \phi_3(x) \cdots : , \end{aligned} \quad (\text{A.2.7})$$

where

$$Z = -2^{-1/2} (g^2 t^2) \Delta^+(0) e^{-2^{-1} g^2 t^2 \Delta^+(0)}. \quad (\text{A} \cdot 2 \cdot 8)$$

The factor Z can be considered as a sort of renormalization constants for a state vector and Z has a similar form as Glauber's. Eq. (A·2·8) leads easily to Eq. (2·20) which have we intended to get.

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Direct Capture of Slow Neutrons by the Nuclear p States

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Cross sections for the capture of a slow neutron by unfilled bound p states are calculated with the p -state wave functions and slow neutron wave functions for a square well and a Woods-Saxon type potential in the case of Ca^{40} . The results are in good agreement with experiment. This allows us to suggest the possibility of obtaining minimum possible neutron cross sections for nuclei with given A . Also, neutron cross sections may give a sensitive test of nuclear potentials.

§ 1. Introduction

The slow neutron capture process is usually understood as a succession of two processes, namely, the formation of a compound state and its subsequent decay by the emission of gamma rays. Such compound state formations are pictorially seen in the energy dependence of the capture cross sections as one or many resonance peaks. When such a peak comes close to zero energy, the capture cross section for thermal neutron takes a large value.

When there is not a resonance very close to zero neutron energy, neutron cross section usually becomes very small but still takes various finite values. Some of them are explained as due to the tail of nearby resonances.

If the concerning *compound* state is of complicated nature, as is usually the case, or if the statistical theory may be applied to the concerning state, the number of the first capture gamma rays and also the multiplicity of the gamma rays must be rather large.¹⁾ Experimental results, however, are not always so. Sometimes, especially when the cross sections are not large, only a few strong gamma-rays contribute to the capture process. In such cases multiplicities are usually low. One may well suspect, therefore, that a *direct process* rather than a compound state formation may be governing the neutron capture in those cases when the neutron capture cross section is very small or the number of the gamma rays and their multiplicity is small.

In order to examine the situation in detail one had better use a rather clear-cut example. Fortunately, we have some.

One of the best examples is the case of slow neutron capture by Ca^{40} . The cross section is rather small, —220 mb. It is the basic condition that we can suspect that the capture may be of direct process type. The neutron capture gamma rays have been measured by a few authors. It is found that there are only three

strong first gamma rays accompanying the capture, among which two are far stronger than the third.²⁾ They are the gamma rays leading to the excited states of Ca^{41} at 1.95 Mev and 3.95 Mev. These states are also characterized by their outstandingly large width in the (d, p) reactions and l_n of the states is 1.³⁾ By now, there seems to be no doubt that they are fairly good $2p$ states split by the spin-orbit interaction by two million volts, the lower one corresponding to the $2p_{3/2}$ state and the upper one to the $2p_{1/2}$ state. Gamma-rays selection rules and systematics in other nuclei⁴⁾ near $A=40$ support this assignment. The neutron capture gamma-rays are therefore of E1 type.

Assuming that the above assignment is correct and also that the states are pure, one may obtain the radial transition matrix elements from the experimental intensities of the gamma-rays. The radial matrix element for low energy transition is higher than that of high energy one by a factor of ~ 3 . Since we do not see many other gamma-rays in this region of comparable intensity, it will be difficult to attribute this large difference in the magnitude to the configuration mixing which affects on the transition probability in small extent. It is more tempting to attribute this large difference of radial matrix element to the small difference of the $2p$ wave functions. In the following section we shall see if the magnitude and the ratio of the radial matrix element could be explained by the direct capture of slow neutrons by slightly different $2p$ state at experimentally observed energies.

§ 2. Calculation of the capture cross section

The radial Schrödinger equation for the wave function of a neutron under the influence of a nuclear potential with the spin-orbit interaction of the Thomas type is

$$-\frac{\hbar^2}{2m} \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\psi}{dr} \right) + \left[-V(r) + \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2} - \lambda \frac{\hbar^2}{4m^2 c^2} \frac{1}{r} \frac{dV(r)}{dr} \left(\frac{l}{-l-1} \right) \right] \psi = E\psi, \quad (1)$$

for $j = l \pm \frac{1}{2}$,

where l is the orbital angular momentum of the neutron, $V(r)$ is the nuclear potential and λ is the spin-orbit coupling constant. The neutron capture cross section due to E1 transitions is given as

$$\sigma = \sum_f \sigma_f = \frac{2}{\sqrt{\pi} v_0} \frac{4}{9} \frac{Z^2}{A^2} \left(\frac{e^2}{\hbar c} \right) \sum_f \frac{E_f^3}{c^2 \hbar^3} S(j_i, 1, j_f) \left| \int \psi_f r \psi_i r^2 dr \right|^2, \quad (2a)$$

where, ψ_i is the normalized wave function for the incoming neutron and ψ_f is the wave function of the state to which the gamma transitions take place, and E_f is the energy of the latter state. The summation should extend over all the states to which the E1 transitions can take place, but because of the reason stated in the former section, we take only two states, the $2p_{3/2}$ and the $2p_{1/2}$ state. $S(j_i, 1, j_f)$ is the statistical factor and

$$S(j_i, 1, j_f) = \begin{cases} 1, & \text{for } s \rightarrow p_{1/2} \text{ transition,} \\ 2, & \text{for } s \rightarrow p_{3/2} \text{ transition.} \end{cases} \quad (2b)$$

The factor Z^2/A^2 comes in since this is a neutron transition. $2/\sqrt{\pi}v_0$ takes care of the averaging of the neutron cross section over the thermal energies under the influence of the $1/v$ law.

A. Square well potential

In order to see what kind of results we would get with a simple potential, we tried a square well potential with $R_1 = 1.3A^{1/3} \times 10^{-13}$ cm. The depth of the potential was adjusted to find the $2p$ state without spin-orbit splitting at -5.74 Mev, which is the center of gravity of the experimental pair of the p states. Then, a surface spin-orbit interaction of δ -function type was added to find the experimental splitting correctly. The corresponding parameters are

$$V_0 = 49.6 \text{ Mev, } \lambda = 24.1. \quad (3)$$

The capture cross section and the ratio of the radial matrix elements thus obtained are

$$\rho = \frac{|\int \phi_{2p1/2} r \phi_s r^2 dr|^2}{|\int \phi_{2p3/2} r \phi_s r^2 dr|^2} = 1.7, \quad \sigma = 23.5 \text{ mb.} \quad (4)$$

A rather large difference in the radial matrix elements is already seen but the cross section is only one tenth of the experimental value.

B. Woods-Saxon type potential

As a more realistic nuclear potential, a potential of the following type⁵⁾ was assumed:

$$V(r) = \frac{U_0}{1 + \exp[1/a \cdot (r - R_2)]}. \quad (5)$$

Here, a and R_2 were fixed as

$$a = 0.65 \times 10^{-13} \text{ cm; } R_2 = 4.55 \times 10^{-13} \text{ cm, } (r_0 = 1.32 \times 10^{-13} \text{ cm}), \quad (6)$$

and U_0 and λ were taken to be the adjustable parameters. The PACE analogue computer at Japan Atomic Energy Research Institute was used for obtaining the necessary wave functions for this potential. Function generators were used for generating the potential U_0 and the Thomas type spin-orbit potential. Initial conditions (the binding energy, the value and its slope of the wave function) were set at $r = 0.5 \times 10^{-13}$ cm. It was assumed that the wave functions are spherical Bessel functions at $r = 0.5 \times 10^{-13}$ cm.

For the bound $2p$ states, the following parameters were obtained:

$$U_0 = 47.1 \text{ Mev}, \quad \lambda = 25. \quad (7)$$

This value of λ is about one half of the value by Ross, Mark, and Lawson.⁶⁾ The same parameters were used for finding the solutions of the zero energy s state. The results are shown in Fig. 1.

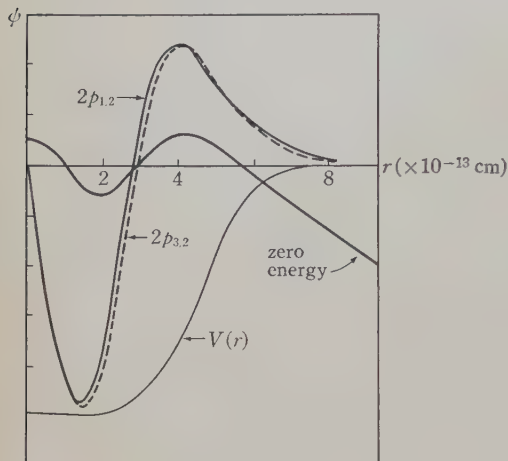


Fig. 1. Wave functions of a diffuse potential for the p states in Ca^{41} and of zero energy neutron state in same potential. $V(r)$ is the diffuse potential of Eq. (6).

It was noticed that the $3s$ resonance appeared at 12 Mev up in case of the square well potential, but in the diffused potential used here it was bound by about 4 Mev. Experimentally, such an s state is not established for Ca^{41} , but by extrapolating from the larger A nuclides,⁴⁾ this value of the binding energy seems to be too large. One of the reasons may be too large a diffuseness. Actually the analysis of the low energy neutron scattering by Feshbach, Porter and Weisskopf shows that $a = 0.57 \times 10^{-13} \text{ cm}$.⁷⁾ The cross section and the ratio of the radial matrix element using these wave functions are

$$\rho = 3.1, \quad \sigma = 550 \text{ mb}. \quad (8)$$

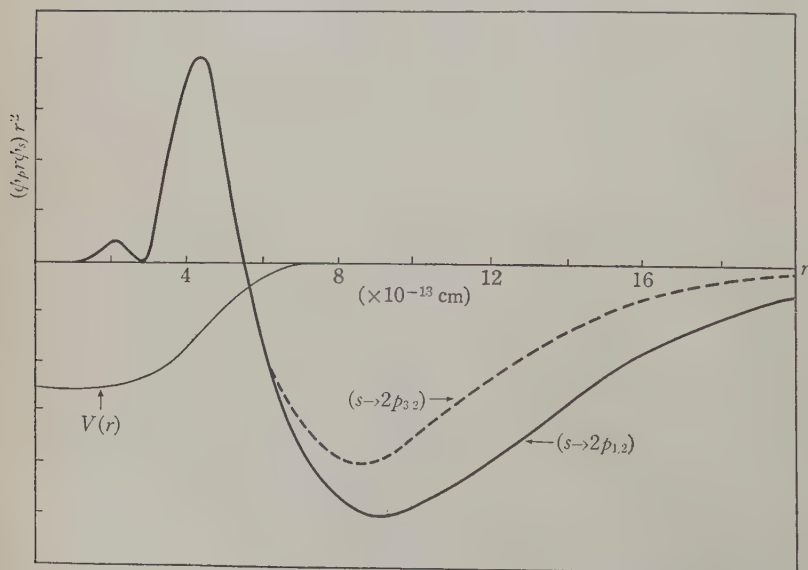


Fig. 2. The integrands of the radial matrix elements.

Here, the cross section is rather overestimated but the ratio is given correctly.

The integrand of the matrix element is shown in Fig. 2. It is interesting to see that the main contribution to the matrix element comes from outside the nucleus. This is the reason why the result was much improved compared with the case of the square well potential. However, the error introduced by the computing machine is increased at large distances. So, it was assumed that the wave function becomes the spherical Hankel function of the first kind at r larger than 7.0×10^{-13} cm.

The result seems to be very sensitive to the s wave function. It was found that the value of the cross section is reduced to 110 mb if one reduces the nuclear radius to $R = 4.23 \times 10^{-13}$ cm ($r_0 = 1.23 \times 10^{-13}$ cm). The ratio becomes 8.5 instead of 3.1.

The calculated transition probability and the ratio of the radial matrix element must be said to fit very well to the observed values. Here, again, the value will be improved if a slightly smaller diffuseness is employed.

§ 3. Minimum neutron cross sections

The good fit of the calculated cross sections to the experimental results in the case of Ca^{40} suggests that the neutron capture with small cross sections may be largely due to such direct capture of thermal neutrons by bound p states. Of course, when the cross section is small, other types of radiation than electric dipole radiation, such as magnetic dipole or electric quadrupole radiation may contribute to the cross section in appreciable extent. But they will become important only when the cross section is very small.

In other cases than Ca^{40} , situations are not so good for theoretical analyses. In most cases, the experimental energies of both members of pure p states are not known. In a few case, like Si^{28} , S^{32} , and around, $A = 40$ to $60^{4)}$ both members of the p states are known with good certainty. But if one wants to discuss more in detail, the fact that they are not closed core nuclei may be disturbing.

The more interesting thing seems to be a qualitative feature of the present result. If such a mechanism as suggested determines the minimum possible neutron cross section, there must be a rather smooth envelope or a lower limit in the plotted neutron cross section against the nuclear mass or the neutron number. It is because the height of the p states and the wave functions of a slow neutron changes smoothly with A or N . These p and s states are fairly safely obtained from the realistic nuclear potential.⁸⁾

The experimentally measured neutron cross sections plotted against the neutron number is shown in Fig. 3. The points were taken from W. H. Sullivan's Tri-linear Nuclear Chart.⁹⁾ Neutron cross sections are not always measured, but by now there are not very many unknown ones. Also, when it is not measured it is usually not because of their smallness. Therefore, in the following discussions the fact that this plot is incomplete is unimportant.

A point with a circle on the figure is the case of Ca^{40} . It is actually on the lower envelope of the points. There is a point below. But this corresponds to S^{36} , which has a smaller nuclear radius. Because of the sensitivity of cross sections around here (near the s resonance), it is not strange at all that the cross section becomes quite smaller.

There are three places where the neutron cross sections take extremely small values. They are around $N=8$, 50 and 126. They are magic numbers but also the place where $1p$, $2p$ and $3p$ states fill up. At other magic numbers, 20, 28 and 82, like the case of Ca^{40} , neutron cross sections do not take extremely small values. This is considered to be due to the unoccupied bound p states. (At $N=2$, the p state is unfilled but there is no bound state).

According to the calculations using a diffuse well,⁸⁾ the next p state does not become bound when a p shell fills up at $N=8$, 50 or 126. The p states start to become bound at around $N=15$ and 70. Therefore, between 8 and 15 and between 50 and 70 there is no definite single particle p state. Indeed, the minimum cross sections are small in this range but are not extremely small. It is probably due to the fact that being not at the closed core, the single particle p state may easily mix into the bound state when it is possible. At around $N=82$ relatively small cross sections are seen. Here, probably, the $3p$ states are almost bound to prevent the very small cross sections to appear. After 82, no small cross sections are observed. It may be partly due to the deformation of the nuclei, but certainly it is due to the $3p$ unfilled bound states.

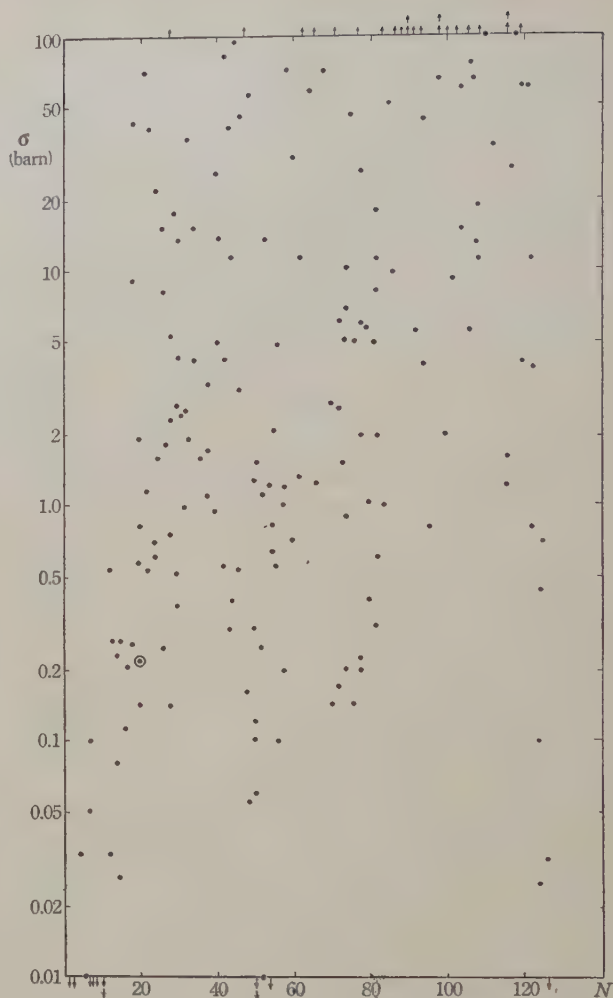


Fig. 3. Thermal neutron capture cross section as a function of N .

The s state, or the state of the incoming slow neutrons, has also influence on the cross sections. Especially, when the s state is bound, the cancellation of the integrand of the matrix element may occur. The cancellation, however, can not take place for both the $p_{1/2}$ and $p_{3/2}$ state at the same time. Therefore, an extremely small cross section may take place only for nuclei close to the filling of the $p_{1/2}$ state, that is, 8, 50, and 126. But at these places the s state, will not cause such an anomaly.

It is interesting to mention that the minimum possible neutron cross sections can be predicted. Very small cross sections can appear only in certain regions of A . Also, from theoretical aspects these small cross sections give, sometimes, a very sensitive check of nuclear potentials, especially of its diffuseness since the capture takes place almost entirely from outside the nucleus.

We should like to express our gratitude to Dr. M. Hara of the Japan Atomic Energy Research Institute for his help during the course of the machine calculations and to Prof. M. Kawai for discussions concerning this calculation.

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Letters to the Editor

The opinions expressed in these columns do not necessarily reflect those of the Board of Editors. Communications should be submitted in duplicate and should be held to within 100 lines (pica type) on standard size letter paper (approx. 21×30 cm.), so that each letter may be arranged into two pages when printed. Do not forget to count in enough space for formulas, figures or tables.

Meson Theoretical Potential and Nuclear Properties

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August 26, 1959

Brueckner et al.¹⁾ gave a comprehensive understanding of the nuclear saturation and pointed out that the effect of the repulsive core is particularly important. Iwamoto and Yamada²⁾ showed also that the hard core is essential to the saturation. In the present note, we will see explicitly and simply how the meson-theoretical potential³⁾ is related to the nuclear properties.

We assume that the nucleus contains an equal number of neutrons and protons and that both groups have an equal number of nucleons with spin up and down. Then the tensor force does not contribute to the binding energy in first order. The properties of the central part of the meson-theoretical potential are as follows. (i) In the region I ($r \gtrsim 2.0$), the one-pion-exchange potential is effective. (ii) In the region II ($2.0 \gtrsim r \gtrsim 1.0$) the singlet even state and the triplet odd state contribute mainly. The potentials of these states are strongly attractive. The contribu-

tions in this region come mainly from the two-pion-exchange potential. (iii) If we have such potential in the outer parts, it is quite probable that there is a hard core in the inmost region ($r \lesssim 0.5$). (iv) A suitable phenomenological potential is assumed between $1.0 \gtrsim r \gtrsim 0.5$, which is forced by many ambiguities in the potential derivation. Then we can express the two-body potential as

$$V = (\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2) (\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2) V^{(1)} - \frac{3 + (\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2)}{4} \left\{ \frac{3 + (\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)}{4} V^{(2)}(^3\text{O}) + \frac{1 - (\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)}{4} V^{(2)}(^1E) \right\}. \quad (1)$$

$V^{(1)}$ represents the central part of the one-pion-exchange potential in the region I and

$$V^{(1)} = (g^2/4\pi) (\mu c^2/3) \cdot e^{-\kappa r} / (\kappa r),$$

where $g^2/4\pi \sim 0.08$ and $\kappa = \hbar/\mu c = 1.4 \cdot 10^{-13} \text{ cm}$. μ is the pion mass. In $V^{(2)}$ in the expression (1), we understand that the phenomenological hard core in the short distance ($r \lesssim 0.5$) and the potential in the region $1.0 \gtrsim r \gtrsim 0.5$ are included together with the potential in the region II.

Now we evaluate the energy matrix in first order in the Born approximation. The method of calculation is given

in the text book by Rosenfeld.⁴⁾ We get the following result for the potential given by (1) :

$$\begin{aligned}\tilde{v} = & -(9/8) D^{(2)}(^3\text{O}) - (3/8) D^{(2)}(^1E) \\ & - (9/4) X^{(2)}(^3\text{O}) + (3/4) X^{(2)}(^1E) \\ & - (9/2) X^{(1)}. \quad (2)\end{aligned}$$

Here D means the direct term and X means the exchange term :

$$\begin{aligned}D^{(i)} = & \int d\mathbf{r}_1 \int d\mathbf{r}_2 \sum_k^{\infty} \phi_k^*(r_1) \phi_k(r_1) V^{(i)} \\ & \times \sum_{k'}^{\infty} \phi_{k'}^*(r_2) \phi_{k'}(r_2)\end{aligned}$$

and

$$\begin{aligned}X^{(i)} = & \int d\mathbf{r}_1 \int d\mathbf{r}_2 \sum_k^{\infty} \phi_k^*(r_1) \phi_k(r_2) V^{(i)} \\ & \times \sum_{k'}^{\infty} \phi_{k'}^*(r_2) \phi_{k'}(r_1). \quad (3)\end{aligned}$$

Now it is important to note that we can take $V^{(2)}(^1E) \simeq 3 V^{(2)}(^3\text{O})$, without introducing a serious error. See Fig. 1 of our previous paper.⁵⁾ Then the main contribution to the exchange term comes from the one-pion-exchange potential :

$$\tilde{v} \simeq -(3/4) D^{(2)} - (9/2) X^{(1)}. \quad (4)$$

As is well known, if the potential does not change sign as function of distance, the direct term cannot give the saturation. However, $V^{(2)}$ is not monotonic. Inside the main attractive part, there is a hard core. Also it can be easily seen that the exchange term in general does not give any large contribution at high density. Then this hard core included in $V^{(2)}$ is responsible and essential for the nuclear saturation. Thereby, the outer part of the two-body potential resulting from the one-pion-exchange is not significant to the saturation.

On the other hand, the exchange term in general gives rise to the velocity dependent potential seen by a nucleon in the nuclear matter. The expression (4) shows that the velocity dependent potential results from the one-pion-exchange part, the outer part of the two-body potential. In what follows, we shall estimate its effect.

The velocity dependent potential modifies the mass of nucleon in the nuclear matter. If we express the effective mass M^* in the nuclear matter as

$$M^* = M(1 + \beta)^{-1}$$

β is given by

$$\begin{aligned}\beta = & (2M/\hbar^2) (3/4\pi r_0^3) (2\pi/\kappa^3) \\ & \times \{8(\kappa a + 1)\kappa^{-2} + (4/3)\kappa a^3 + 4a^2\} e^{-\kappa a} \\ & \times (9/16) (g^2/4\pi) (\mu c^2/3), \quad (5)\end{aligned}$$

when the one-pion-exchange potential is effective for $r > a$. Putting $a = 2 \cdot 10^{-13}$ cm, $\kappa^{-1} = r_0$, we get $\beta \simeq 2.2$. This is very close to the estimated value by Weisskopf,⁶⁾ $\beta = 1.42$.

Finally, it can be shown that the surface energy is directly related to the exchange term.⁷⁾ Then at the surface, the contribution to the energy comes mainly from the one-pion-exchange potential.

The author wishes to thank Professor M. Kobayasi for his interest in this work.

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Possibility for or against the Existence of a Neutral Scalar Meson

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September 23, 1959

Recently there have been various attempts¹⁾ to determine such physical quantities as parities, coupling constants, by making use of the conjecture that dispersion relations exist in momentum transfer variables and that in the immediate vicinity of a pole the scattering amplitude is dominated by this pole term.

If an unknown pole term exists in nature, predictions can be made about it in principle by its extrapolation procedure. Now, the possible existence of a neutral heavy meson was speculated by several authors.²⁾ In this note we wish to discuss the possibility for or against the presence of such a pole term

with use of extrapolation analysis.

If there exists a neutral scalar meson, which we tentatively call π_0' , a pole contribution comes from the π_0' meson intermediate state in the process $\pi^- - p$ scattering :

$$\pi^- + p \longrightarrow \pi^- + p. \quad (1)$$

As to the π_0' meson mass μ' , we assume that

$$\mu < \mu' < 2\mu. \quad (2)$$

Because if $\mu' > 2\mu$, the π_0' meson will decay into two π mesons very rapidly, while if $\mu' < \mu$, one can expect that the incoming p-waves of π^- on the proton will lead mostly to an outgoing s-wave of π_0' in the low energy region.

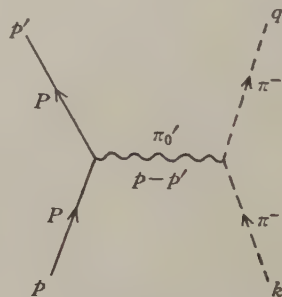


Fig. 1 A possible pole contribution for the process $\pi^- + p \rightarrow \pi^- + p$. $p, p'; k, q$ are respectively protons, and pions momenta.

From the analyticity conjecture³⁾ we can write the scattering amplitude for the process (1) of Fig. 1 as

$$M(\cos \theta) = \frac{i}{(2\pi)^2} \delta^{(4)}(p + k - p' - q) \\ \times \frac{M}{2p_0 k_0} \frac{1}{2k^2} \frac{f \bar{u}(p') u(p)}{\alpha_0 - \cos \theta} \\ + T(\cos \theta) \quad (3)$$

where θ is the angle between the incoming π meson and the outgoing π meson, and k is the π -meson momentum in the center-of-mass system. f is the renormalized and unrationalized coupling constant arising from the possible $(PP\pi_0')$ interaction, and λ is defined as the coupling constant of the $(\pi\pi\pi_0')$ interaction. In this case the matrix element as a function of $\cos\theta$ has a pole at

$$\alpha_0 = \frac{2k_0'^2 + \mu'^2 - 2\mu^2}{2k^2} \quad (4)$$

where μ' is the π_0' meson mass.

At 270 Mev lab. energy, for example,

$$\alpha_0 = \begin{cases} 1.125 & (\text{if } \mu' = \mu)^* \\ 1.50 & (\mu' = 2\mu). \end{cases} \quad (5)$$

Let us define

$$F(\cos\theta) = (\alpha_0 - \cos\theta)^2 \frac{d\sigma}{d\Omega}. \quad (6)$$

Then in the neighbourhood of a pole, we get the result

$$F(\cos\theta) \sim \frac{f^2 \lambda^2}{(2\pi)^2} \frac{1}{32k^4} \frac{p_0^2 - k^2 \cos\theta + M^2}{(p_0 + k_0)^2}. \quad (7)$$

Using the experimental values⁴⁾ of $d\sigma/d\Omega$, we have plotted in Fig. 2 the curve of $F(\cos\theta)$ in the region $1 \geq$

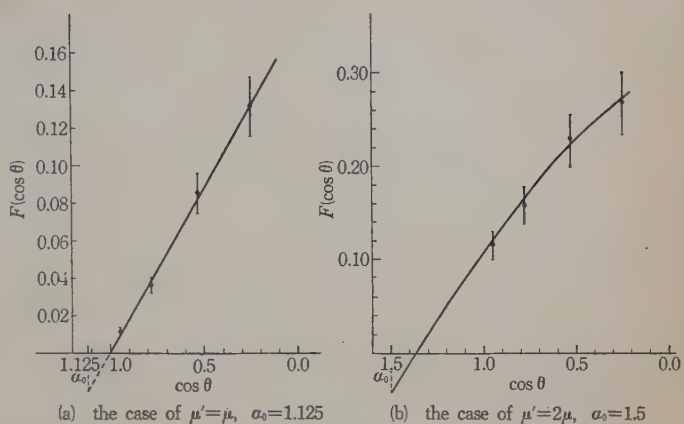


Fig. 2 Plot of $F(\cos\theta) = (\alpha_0 - \cos\theta)^2 d\sigma/d\Omega$ for $\pi^- - p$ elastic scattering at 270 Mev. The straight line is obtained on the assumption that the pole term is dominant. The unit is meson Compton wave length.

$\cos\theta \geq 0$ for the two extreme cases, $\mu' = \mu$ and $\mu' = 2\mu$. We see that the curve is very close to a straight line, but it does not cross the transversal axis at $\cos\theta = \alpha_0$ for both two cases.

For other energies the situation was similar to this case. This seems to show that the pole contribution is not zero, although the experimental data are not accurate enough to conclude anything about the existence of π_0' .

However, it is possible to estimate the upper limit of the coupling constant f of the $NN\pi_0'$ interaction. To this end we calculated the effective coupling constant λ as shown in Fig. 3.

$$\lambda = -\frac{g^2 f M \Lambda^2}{\pi^2} \delta_{ij} \left[\frac{1}{M^2 - \Lambda^2} + \frac{2}{(M^2 - \Lambda^2)^2} \log \frac{\Lambda}{M} \right] \quad (8)$$

where g is the renormalized, unrationalized $ps(ps)$ coupling constant and Λ is the cut-off parameter. From (5), (7), (8) and Fig. 2, we have

* We consider the two extreme cases in the interval (2).

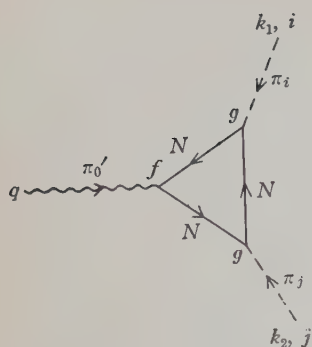


Fig. 3 Effective $(\pi\pi\pi')$ interaction.

$$f^2/4\pi < 0.03 \quad (\text{scalar coupling}) \quad (9)$$

where we used the nucleon mass as the cut-off parameter Λ . We can carry out similar procedures at other energies, with the result

$$f^2/4\pi < 0.03, \quad \text{at 240 Mev}, \quad (10)$$

$$f^2/4\pi < 0.07, \quad \text{at 915 Mev}, \quad (11)$$

respectively.

These values are very small compared with the π - N interaction. For instance, the scalar coupling constant as large as $f^2/4\pi \sim 0.4$ could only bind the two nucleons into a bound state. The neutral scalar meson, even if it existed at all, would play only a minor role in the world of strong interactions.

The author wishes to express his thanks to Professor H. Miyazawa for suggesting this problem.

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Polarization of Proton Scattered from Li^6 , Be^9 and B^{11}

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October 3, 1959

Bethe and others¹⁾ pointed out the validity of the direct interaction model in the high energy nuclear reactions by analysing the data of nucleon scattered from C^{12} with use of the nucleon-nucleon scattering phase shifts. Moreover they tried to discriminate the suitable sets of nucleon-nucleon scattering phase shifts from the available ones.²⁾ Also one of the present authors (Y.S.) tried to distinguish the sets of nucleon-nucleon scattering phase shifts by comparison of the experimental data with calculated results in the impulse approximation for the nucleon scattered from D^2 and He^4 .³⁾

If one uses the even-even nuclei like He^4 or C^{12} with spin zero and isotopic spin zero as the target, the expression for the polarization and the differential cross section will be very simple. And the amplitudes of nucleon scattered from such target nuclei are the simple sum of p - n and p - p scattering ones, so that p - n and p - p scattering amplitudes interfere in equal weight.

However, if one uses the even-odd nuclei such as Be^9 or B^{11} , or odd-odd nuclei, such as Li^6 for the target one, the expression for the differential cross section and the polarization of the nucleon scattered from such nuclei are somewhat complicated, and the ampli-

tudes of scattered nucleon are not merely the sum of p - n and p - p scattering amplitudes. They are composed of the terms of some weighted sum of p - n and p - p scattering amplitudes and the terms of p - n or p - p scattering amplitudes alone, or the terms of the sum of p - p and p - n scattering amplitudes. In the present note, we discuss these effects.

We write the scattering spin matrix as

$$M = A + C(\sigma_1 + \sigma_2)n + B(\sigma_1 n)(\sigma_2 n) + E(\sigma_1 k)(\sigma_2 k) + F(\sigma_1 P)(\sigma_2 P) \quad (1)$$

where A , C , etc. are the functions of the scattering angle and of the energy of incident particle, and n , p and k are the three mutually perpendicular unit vectors, chosen so that n is normal to the scattering plane.

Using the spin wave function constructed according to the coupling rule of the shell model for Be^9 at the ground state, one finds the following expression for the polarization of elastically scattered nucleon,

$$P(\theta) = \frac{2\text{Re}(A^*C + B_{pn}^*C_{pp})}{|A|^2 + |C|^2 + |B_{pn}|^2 + |E_{pn}|^2 + |F_{pn}|^2 + |C_{pn}|^2}, \quad (2)$$

where $A = 5A_{pn} + 4A_{pp}$ and $C = 5C_{pn} + 4C_{pp}$, A_{pn} and C_{pn} , etc. are given by (1) for p - n scattering and A_{pp} , C_{pp} , etc. for p - p scattering. In Eq. (2), A and C are the contributions of non spin flip of target nucleons, while B_{pn} , C_{pn} , E_{pn} and F_{pn} are the contributions of spin flip of target nucleons.

Using the spin wave function similar

to the case of Be^9 , one finds the expression for the polarization of the nucleon elastically scattered from Li^6 ,

$$P(\theta) = \frac{2\text{Re}(A^*C + \frac{2}{3}C'^*B')}{|A|^2 + |C|^2 + \frac{2}{3}(|C'|^2 + |E'|^2 + |F'|^2 + |B'|^2)}, \quad (3)$$

where $A = 3(A_{np} + A_{pp})$, $C = 3(C_{np} + C_{pp})$, $C' = C_{pp} + C_{pn}$, etc., A_{np} , C_{np} , etc., and A_{pp} , C_{pp} , etc. are given by (1) for p - n and p - p scattering, respectively. In Eq. (3) the non-spin flip of target nucleons contribute to A and C , while the spin flip of target nucleons contribute to C' , B' , E' and F' .

One also finds the expression for the polarization of the nucleon elastically scattered from B^{11} in a similar manner,

$$P(\theta) = \frac{2\text{Re}(A^*C + B_{pp}^*C_{pp})}{|A|^2 + |C|^2 + |C_{pp}|^2 + |E_{pp}|^2 + |F_{pp}|^2 + |B_{pp}|^2}, \quad (4)$$

where $A = 6A_{pn} + 5A_{pp}$, $C = 6C_{pn} + 5C_{pp}$.

Eqs. (2), (3) and (4) are different from the equation for the polarization of scattered nucleon from He^4 or C^{12} ,* i.e.,

$$P(\theta) = \frac{2\text{Re}(A^*C)}{|A|^2 + |C|^2} \quad (5)$$

where $A = 2(A_{pn} + A_{pp})$ and $C = 2(C_{pn} + C_{pp})$ for the He^4 -target, and $A = 6(A_{pn} + A_{pp})$ and $C = 6(C_{pn} + C_{pp})$ for the C^{12} -target.

However, Eqs. (2), (3) and (4) are composed of the spin flip terms and the non-spin flip terms of target nucleons, while Eq. (5) is constituted only of the non-spin flip terms. The contribu-

tions of the non-spin flip terms are, however, dominant for the polarization of the nucleon scattered from the light nuclei. Then the calculated results for Be^9 , Li^6 and B^{11} are very similar to each other.⁴⁾ In calculations of the present note, we use both the nucleon-nucleon scattering phase shifts of Gammel-Thaler and Signell-Marshak²⁾ at 180 Mev.

The next interesting thing is the comparison of the polarization of elastically scattered proton with inelastically scattered proton. In the present note, we consider the polarization for the B^{11} -target. The expression for polarization of nucleon scattered from 4.6 Mev level of B^{11} is given according to Egardt,⁵⁾ by

$$P(\theta) = \frac{2\text{Re}[(7A + \frac{1}{2}A_{pp})^*(7C + \frac{1}{2}C_{pp}) + B_{pp}^*C_{pp}]}{|7A + \frac{1}{2}A_{pp}|^2 + |7C + \frac{1}{2}C_{pp}|^2 + |C_{pp}|^2 + |\frac{1}{2}E_{pp}|^2 + |F_{pp}|^2 + |B_{pp}|^2} \quad (6)$$

where $A = \frac{1}{2}(A_{pp} + A_{pn})$ and $C = \frac{1}{2}(C_{pp} + C_{pn})$, and C_{pp} , E_{pp} , etc. are the same as those in (4).

The polarization of the nucleon scat-

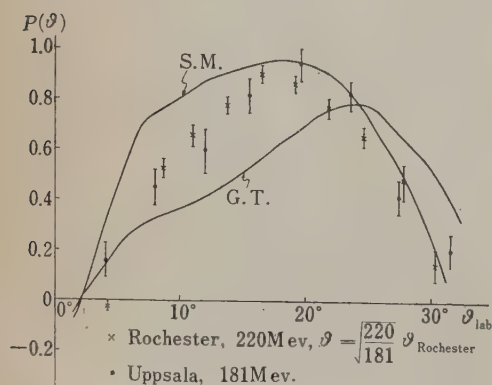


Fig. 1 Polarizations of 181 Mev protons elastically scattered from beryllium. Elastic polarizations (Uppsala)⁶⁾ of beryllium compared with asymmetries of Hafner (Rochester)⁷⁾, and calculated values.

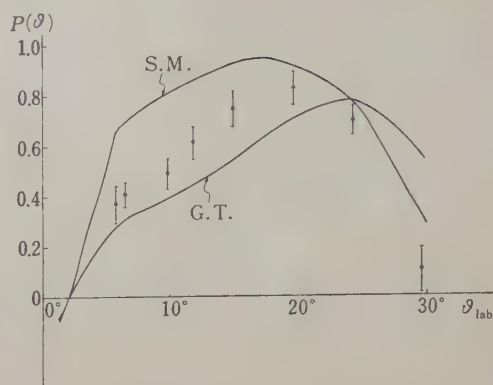


Fig. 2 Polarizations of 181 Mev protons elastically scattered from lithium. Experimental points are those of Uppsala group.⁶⁾

tered from light nuclei such as Be^9 , Li^6 and B^{11} depends on the character of nucleon-nucleon scattering rather than on the character of the nucleus, that is,

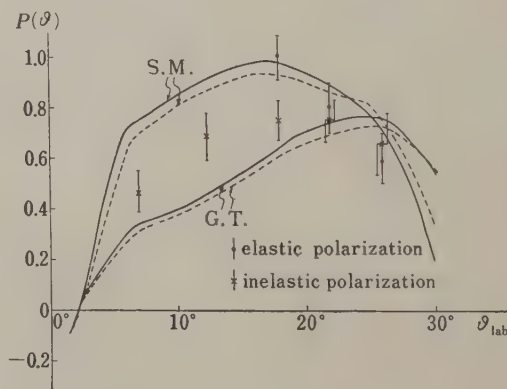


Fig. 3 Elastic polarization and inelastic polarization of 180 Mev protons elastically and inelastically scattered from boron. The solid curves represent the calculated results for elastic polarization and dotted curves represent the calculated results for inelastic polarization. Experimental points are those of Uppsala group.⁶⁾

it does not almost depend on the structure of light nuclei. This seems to suggest that the impulse approximation is valid.

The polarization of inelastically scattered nucleon from low-lying levels of light nuclei depends little on its structure.

It is to be emphasized that the above conclusions are always subject to the limitations contained within the impulse approximation itself, that is, the calculated results have its validity at small angles.

The authors would like to thank Prof. M. Kobayasi for his discussions and encouragement.

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Theory of Classical Fluids: Hyper-Netted Chain Approximation. IIIa

—A New Integral Equation for
Pair Distribution Function—

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October 14, 1959

I will briefly report here on that the pair distribution function in the hyper-netted chain approximation is found to satisfy a new integral equation which is simpler and expected to give better results than the Yvon, Born and Green integral equation.

I have found some errors in my previous calculations of the hyper-netted chain approximation.^{1),2)} In I and II, some graphs that should have been included were neglected and the results were obtained in a solution of a tedious recurrence equation. Those graphs are

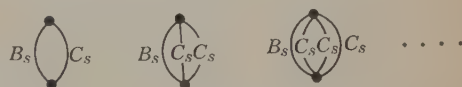


Fig. 1

in Fig. 11 of I. Then we must change the second row of the recurrence equation (30) of I to

$$f^{(n+1)}(r) = \{f^{(n)}(r) + h^{(n-1)}(r) + 1\} \\ \times \exp \{4h^{(n)}(r)\} - 1 - h^{(n)}(r); \quad (1)$$

As the result, our recurrence equation

reduces to the form*

$$f^{(n+1)}(r) = \{f^{(0)}(r) + 1\} \exp h^{(n)}(r) - 1 - h^{(n)}(r) \quad (2)$$

$$H^{(n)}(k) = \rho F^{(n)}(k)^2 / \{1 - \rho F^{(n)}(k)\} \quad (3)$$

and the equation for the free energy is shown to be

$$\begin{aligned} \left(-\frac{A'}{NkT}\right)_{\sum_{s=0}^n (B^{(s)} + R^{(s)})} &= \frac{\rho}{2} F^{(n)}(0) \\ &+ \frac{1}{2\rho V} \sum_k \{-\ln(1 - \rho F^{(n)}(k)) \\ &- \rho F^{(n)}(k) - \frac{1}{2} \rho^2 F^{(n)}(k)^2\} \\ &- \frac{\rho}{2} \int d\mathbf{r} f^{(n)}(r) h^{(n)}(r) \\ &- \frac{\rho}{4} \int d\mathbf{r} h^{(n-1)}(r)^2 \end{aligned} \quad (4)$$

or

$$\begin{aligned} \left(-\frac{A'}{NkT}\right)_{\sum_{s=0}^n (B^{(s)} + R^{(s)}) + B^{(n+1)}} &= \frac{\rho}{2} F^{(n+1)}(0) \\ &+ \frac{1}{2\rho V} \sum_k \{-\ln(1 - \rho F^{(n)}(k)) \\ &- \rho F^{(n)}(k) - \frac{1}{2} \rho^2 F^{(n)}(k)^2\} \\ &- \frac{\rho}{2} \int d\mathbf{r} f^{(n)}(r) h^{(n)}(r) \\ &- \frac{\rho}{4} \int d\mathbf{r} h^{(n)}(r)^2. \end{aligned} \quad (5)$$

The equation for the pair distribution function is

$$[g(r)]_{B^{(n)}*} = \{f^{(0)}(r) + 1\} \exp h^{(n)}(r). \quad (6)$$

Then the equation for the free energy and the pair distribution function in the hyper-netted chain approximation ($n \rightarrow \infty$) are found to be expressible by means of $\mathfrak{f}(r)$ and $\mathfrak{h}(r)$ which are defined by

$$\mathfrak{f}(r) = \lim_{n \rightarrow \infty} f^{(n)}(r), \quad \mathfrak{h}(r) = \lim_{n \rightarrow \infty} h^{(n)}(r). \quad (7)$$

That is,

$$\begin{aligned} \left(-\frac{A'}{NkT}\right)_{\text{HNC}} &= \frac{\rho}{2} \mathfrak{F}(0) \\ &+ \frac{1}{2\rho V} \sum_k \{-\ln(1 - \rho \mathfrak{F}(k)) - \rho \mathfrak{F}(k) \\ &- \frac{\rho^2}{2} [\mathfrak{F}(k) + \mathfrak{G}(k)]^2\} \end{aligned} \quad (8)$$

and

$$[g(r)]_{\text{HNC}} = \{f^{(0)}(r) + 1\} \exp \mathfrak{h}(r). \quad (9)$$

Moreover, the $\mathfrak{f}(r)$ and $\mathfrak{h}(r)$ satisfy a set of equations,

$$\mathfrak{f}(r) = \{f^{(0)}(r) + 1\} e^{\mathfrak{h}(r)} - 1 - \mathfrak{h}(r) \quad (10)$$

$$\mathfrak{G}(k) = \rho \mathfrak{F}(k)^2 / \{1 - \rho \mathfrak{F}(k)\}. \quad (11)$$

These are the integral equation for the pair distribution function in the hyper-netted chain approximation.

Now, if we neglect the second and higher powers of $\mathfrak{h}(r)$ in (10) :

$$\mathfrak{f}(r) = f^{(0)}(r) \{1 + \mathfrak{h}(r)\} \quad (12)$$

and replace $\mathfrak{h}(r)$ in this equation by a suitable mean value :

$$\varepsilon \equiv 1 + \overline{\mathfrak{h}(r)}, \quad (13)$$

then this integral equation is linearized and solved as

$$\mathfrak{G}(k) = \rho \varepsilon^2 F^{(0)}(k)^2 / \{1 - \rho \varepsilon F^{(0)}(k)\}. \quad (14)$$

* The notation and terminology follow those used in I. Especially, $F^{(n)}(k)$, $H^{(n)}(k)$, $\mathfrak{F}(k)$ and $\mathfrak{G}(k)$ are Fourier transforms of $f^{(n)}(r)$, $h^{(n)}(r)$, $\mathfrak{f}(r)$ and $\mathfrak{h}(r)$, respectively.

This is just the linearized Born-Green integral equation for the pair distribution function.³⁾

The comparison with the Born-Green integral equation³⁾ shows that more graphs are correctly considered in our formula than in the Born and Green one.

Because of its simple analytical form, this integral equation will be convenient for practical computation of the distribution function and thermodynamical functions, especially when some computational machine is available for the Fourier transformations.

In conclusion, it should be noticed that Mayer's cluster expansion formulae in terms of $f_{ij}^{(0)} = \exp\{-\phi(r_{ij})/kT\}^{-1}$ have been reduced to cluster expansion formulae in terms of the hyper-netted chains: $\hat{f}_{ij} + \hat{h}_{ij} = [g(r)]_{\text{HNC}} - 1$. The leading terms of these expansion formulae are those in the hyper-netted chain approximation and each of the remaining terms is corresponded to a graph which has no identifiable part, among the graphs in Mayer's expansion formulae.⁴⁾

Detailed accounts of this note and the errata for I and II will be published shortly.

The situations are similar in the case of multi-component systems: the discussions on this case will be retained to another occasion.

The New Viewpoint of the High Energy Elastic Scattering of Nucleons from Nuclei

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October 22, 1959

In the previous notes,¹⁾ we showed that the scattering of nucleon by deuteron or He^4 can be obtained by the superposition of the nucleon-nucleon scattering amplitudes multiplied by the sticking factor. Others²⁾⁻⁶⁾ also reached this conclusion.

This important conclusion is obtained by the experimental checks of light nuclei, lighter than C. In heavy nuclei, however, the multiple scattering effect will be important. And it is supposed that we cannot derive the same conclusion from heavy nuclei as got by the light nuclei. Then we must perform tedious calculations to reproduce the experimental data of the scattering by heavy nuclei. In fact, if we calculate the high energy elastic scattering under the assumption of the direct two-body collision, we get the results somewhat larger than the experimental values, though we can reproduce the main feature, as stated in the previous note.⁷⁾ This is seen in the dotted line of Fig. 1, in the case of the elastic scattering of 95 Mev protons from typical nuclei.

We take the Saxson-type density distribution as described in the previous note.⁷⁾ The parameters are $R=1.32 \times A^{1/3} \times 10^{-13}$ cm, $a=0.49 \times 10^{-13}$ cm.

- 1) T. Morita, Prog. Theor. Phys. **20** (1958), 920; to be referred to as I.
- 2) T. Morita, Prog. Theor. Phys. **21** (1959), 361; to be referred to as II.
- 3) H. S. Green, *The Molecular Theory of Fluids* (North-Holland Publishing Co., 1952), Chap. III, § 6.
- 4) Ref. 1), § 4.

In this calculation, the Coulomb effect is neglected. But it is proved that the Coulomb effect cannot reduce the calculated value to the experimental value, except in the case of carbon.

The purpose of the present paper is to point out the following interesting theoretical finding, in relation to the above discussion.

$$\lambda = \frac{\sum_c (\hat{\xi}_i | \varphi_c) (\varphi_c | \sum_p V_p | \psi_c(p)) (\varphi_c | \hat{\xi}_i) + \sum_{mn} (\hat{\xi}_i | \varphi_m) (\varphi_m | \sum_p V_p | \psi_n(p)) (\varphi_n | \hat{\xi}_i)}{(\varphi_i | \sum_{p'} V_{p'} | \psi_i(p')) + \sum_{mn} (\hat{\xi}_i | \varphi_m) (\varphi_m | \sum_{p'} V_{p'} | \psi_n(p')) (\varphi_n | \hat{\xi}_i)} \quad (1)$$

Here V_p represents the interaction in free space of the incident and the target nucleon specified by p . $1/N \cdot (\varphi_i + \hat{\xi}_i)$ is the wave function of the initial state. N is the normalization factor. $\hat{\xi}_i$ represents the product of the incident free nucleons and the correlations between the nucleons in the target nucleus in its ground state. We assume that the correlation in the target nucleus is described at least by two-particles-jump from the Fermi sea. φ_i is the product of the wave function of the incident (free) nucleon and that of the correlation free part (under the Fermi sea) of the target nucleons. Here the target nucleons are distorted by the average potential, but the incident particle is not distorted. $|\psi_n(p)\rangle$ describes the collision of the incident and the target nucleon p . $|\psi_n(p)\rangle$ is the solution of the following equation,

$$|\psi_n(p)\rangle = |\varphi_n\rangle + \frac{1}{E_n + i\eta - (K + \tilde{U})} V_p |\psi_n(p)\rangle. \quad (2)$$

K : the total kinetic energy operator.

\tilde{U} : the sum of the single particle po-

Theoretically, we can derive the fact that in the high energy elastic scattering where the effect of the binding can be neglected in the multiple scattering processes, the main terms of the multiple scattering together with the direct two-body interaction can be represented as if it were the direct interaction with the reduced strength.⁸⁾ After a long calculation, the reduction factor λ is found to be

tential that the target nucleons feel in the nucleus.

It is expected that the value of λ becomes smaller and reaches some constant value with the increasing mass number of the target nucleus, because in light nuclei the target is appreciably different from the Fermi gas and because in heavy nuclei it is expected that only a limited and a definite number of particles can have an appreciable correlation to the representative particle p . Thus in heavy nuclei the mass number dependence of both the numerator and the denominator of λ is linear to A . Then λ is independent of mass number in heavy nuclei.

In comparison with the experiments we can confirm this expectation. In fact, we get that the value of $\sqrt{|\lambda|^2}$ is 1.00 for C, 0.75 for Al, 0.55 for heavier nuclei than Cu. With this reducing factor we get the solid curves in Fig. 1.

This result will present the very important knowledge as we treat the other high energy nuclear reactions. The details of this note will be presented in this journal.

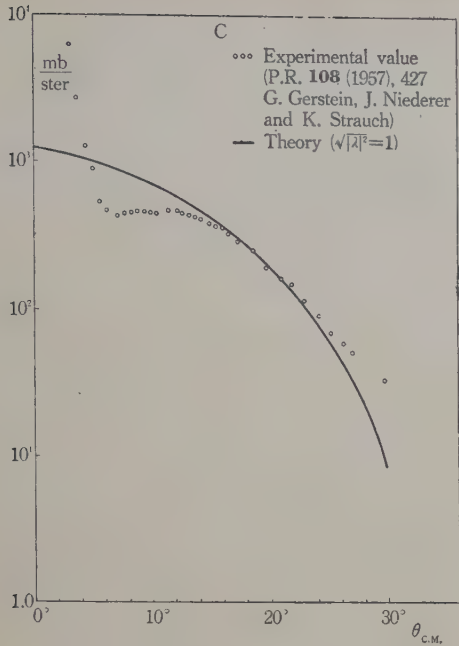


Fig. 1a

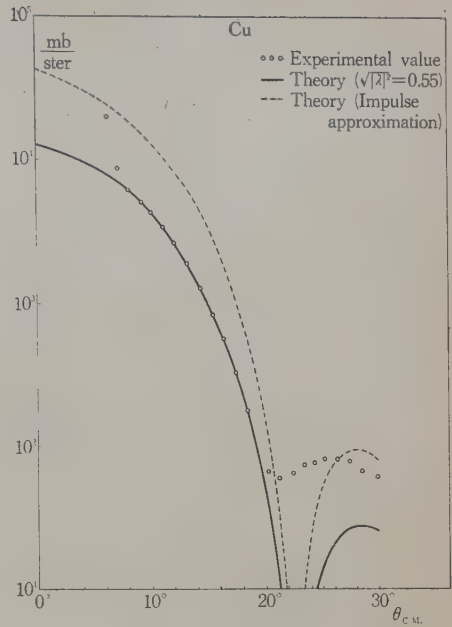


Fig. 1c

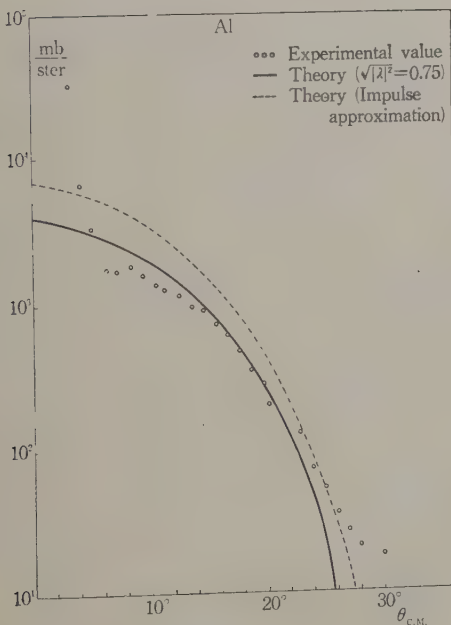


Fig. 1b

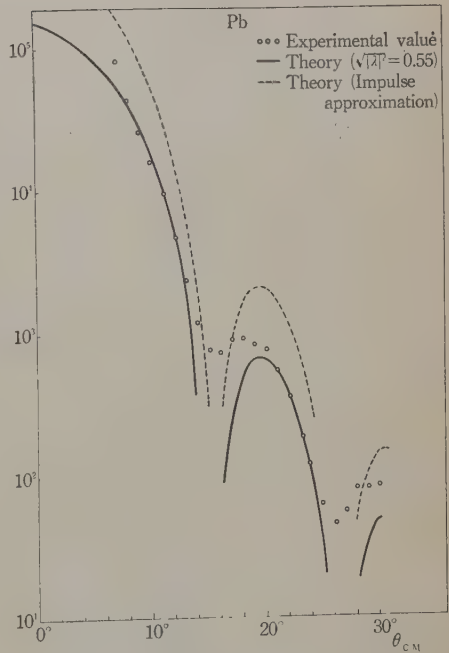


Fig. 1d

Comparison with experiment. The small circles are the experimental values presented by Gerstein, Niederer and Strauch (Phys. Rev. **108** (1957), 1427). (The author thanks for their sending experimental data.) The solid curve represents the theoretical value with reducing factor as indicated in the text. The dotted curve shows the calculated value with use of the impulse approximation. (We calculated the t -matrix using the pion theoretical phase shifts.)

The author wishes to express his gratitude to Professor M. Kobayasi for his encouragement. Thanks are also due to Mr. Y. Sakamoto who aided the work in some numerical calculations.

- 1) Y. Sakamoto and T. Sasakawa, *Prog. Theor. Phys.* **19** (1958), 745; **21** (1959), 879; **22** (1959), 299.
- 2) J. Sawicki and S. Watanabe, *Nucl. Phys.* **10** (1959), 151.
- 3) L. Gattillejo and L. S. Singh, *Nuovo Cimento* **XI** (1959), 131.
- 4) H. A. Bethe, *Ann. Phys. N. Y.* **3** (1958), 190.
- 5) S. Ohnuma, *Phys. Rev.* **111** (1958), 1173.
- 6) A. H. Cromer, *Phys. Rev.* **113** (1953), 1607.
- 7) T. Sasakawa, *Prog. Theor. Phys.* **22** (1959), (to be published)
- 8) T. Sasakawa, *Prog. Theor. Phys.* (to be published).

A Variation Principle in the Theory of Transport Phenomena

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October 22, 1959

It is possible to make use of the variation principle recently suggested in the problem of calculating the electrical conductivity¹⁾ also for the purpose of calculating various kinds of transport coefficients in various irreversible processes. We shall here consider as an example the case of the thermoelectric phenomena, taking for brevity the system that has one kind of charge carriers.

In order to investigate the system in the presence of the electric field and the temperature gradient, it is effectual

to solve the Schrödinger equation for the density matrix ρ of the system in the electric field in regard of $\rho_1 = \rho - \rho_L$, where ρ_L represents the local equilibrium and expressed as

$$\rho_L = K \exp \left[- \int \{ \tilde{\xi}(\mathbf{r}, t) n(\mathbf{r}) + \beta(\mathbf{r}, t) u(\mathbf{r}) \} dV \right] \quad (1)$$

in terms of the operators of local densities $n(\mathbf{r})$ and $u(\mathbf{r})$ of number and energy of charge carrying particles. K is a normalization constant, dV represents the volume element in the system, the integration is taken over the total volume of the system we take as unity and use is made as usual of the local parameters $\tilde{\xi}(\mathbf{r}, t)$ and $\beta(\mathbf{r}, t)$ in place of the local temperature and local chemical potential. The Schrödinger equation can be written in the linear approximation of the external disturbances as

$$\begin{aligned} i \frac{\partial \rho_1}{\partial t} = & [H, \rho_1] + i \rho_0 \int_0^\beta d\lambda \int dV e^{\lambda H} \\ & \times \{ X_1(\mathbf{r}, t) \cdot \mathbf{J}(\mathbf{r}) \\ & + X_2(\mathbf{r}, t) \cdot \mathbf{W}(\mathbf{r}) \} e^{-\lambda H}, \end{aligned} \quad (2)$$

with use of the equilibrium density matrix ρ_0 representing the grand canonical distribution, the Hamiltonian H of the system free from any disturbance, the operators of local current densities $\mathbf{J}(\mathbf{r})$ and $\mathbf{W}(\mathbf{r})$ of charge and energy of carriers and the generalized forces $X_1(\mathbf{r}, t)$ and $X_2(\mathbf{r}, t)$, and by making use of the conservation law $edn(\mathbf{r})/dt = -\text{div } \mathbf{J}(\mathbf{r})$ and $du(\mathbf{r})/dt = -\text{div } \mathbf{W}(\mathbf{r})$. The generalized forces are expressed as

$$\begin{aligned} X_1(\mathbf{r}, t) = & \mathbf{E} - \frac{T}{e} \text{grad} \left(\frac{\xi}{T} \right), \\ X_2(\mathbf{r}, t) = & -\frac{1}{T} \text{grad } T, \end{aligned} \quad (3)$$

where E , T and ζ are the electric field, the local temperature and the local chemical potential, e is the electric charge of the carrier particles and κ is the Boltzmann constant. By assuming in the same way as in I,

$$X_A(\mathbf{r}, t) = X_A(\mathbf{r}) \exp(\pm st) \quad (A=1, 2) \quad (4)$$

and

$$\rho_1(t) = \exp(\pm st) \rho_c \int_0^\beta d\lambda e^{\lambda H} \Psi^{(\pm)} e^{-\lambda H} \quad (5)$$

respectively in $t < 0$ and $t > 0$ with a parameter s which real part is positive, Eq. (2) reduces to

$$\begin{aligned} & \rho_c \int_0^\beta d\lambda e^{\lambda H} \{ \pm s \Psi^{(\pm)} + i[H, \Psi^{(\pm)}] \\ & - \int \{ \mathbf{J}(\mathbf{r}) \cdot \mathbf{X}_1(\mathbf{r}) \\ & + \mathbf{W}(\mathbf{r}) \cdot \mathbf{X}_2(\mathbf{r}) \} dV \} e^{-\lambda H} = 0. \end{aligned} \quad (6)$$

We can now suggest the variation principle, which is as follows.

Let Ψ be the solutions of Eq. (6). Then of all state operators $\Phi^{(\pm)}$, $\Psi^{(\pm)}$ are those which make the functional of unknown operators $\Phi^{(-)}$ and $\Phi^{(+)}$ an extremum:

$$\begin{aligned} \theta(\Phi_0^{(-)} \Phi^{(+)}) &= \int_0^\beta d\lambda \text{Tr} \{ \Phi^{(-)} \rho_c e^{\lambda H} \\ & \times (s \Phi^{(+)} + i[H, \Phi^{(+)}]) e^{-\lambda H} \\ & + (\Phi^{(+)} - \Phi^{(-)}) \rho_c e^{\lambda H} \int \{ \mathbf{X}_1(\mathbf{r}) \cdot \mathbf{J}(\mathbf{r}) \\ & + \mathbf{X}_2(\mathbf{r}) \cdot \mathbf{W}(\mathbf{r}) \} dV e^{-\lambda H} \}, \end{aligned} \quad (7)$$

and the extremum value $\theta = \theta(\Psi^{(-)}, \Psi^{(+)})$ is equal to the entropy production in the system.

In the case that the generalized forces are constant over the whole space in

the system, the expression (7) reduces to

$$\begin{aligned} \theta(\Phi^{(-)}, \Phi^{(+)}) &= \int_0^\beta d\lambda \text{Tr} \{ \Phi^{(-)} \rho_c e^{\lambda H} \\ & \cdot (s \Phi^{(+)} + i[H, \Phi^{(+)}]) e^{-\lambda H} + (\Phi^{(+)} \\ & - \Phi^{(-)}) \rho_c e^{\lambda H} (\mathbf{X}_1 \cdot \mathbf{J} + \mathbf{X}_2 \cdot \mathbf{W}) e^{-\lambda H} \}, \end{aligned} \quad (8)$$

where \mathbf{X}_1 and \mathbf{X}_2 are the constant values of generalized forces and \mathbf{J} and \mathbf{W} are the total currents or the current densities

$$\mathbf{J} = \int \mathbf{J}(\mathbf{r}) dV, \quad \mathbf{W} = \int \mathbf{W}(\mathbf{r}) dV. \quad (9)$$

Let us take, in place of the expression (8), the dyadics which is the functional of unknown vector operators $\Phi^{(-)}$ and $\Phi^{(+)}$

$$\begin{aligned} L_{AB}(\Phi^{(-)}, \Phi^{(+)}) &= \int_0^\beta d\lambda \text{Tr} \{ \Phi^{(-)} \rho_c e^{\lambda H} \\ & \cdot (s \Phi^{(+)} + i[H, \Phi^{(+)}] - \mathbf{I}_B) e^{-\lambda H} \\ & + \mathbf{I}_A \rho_c e^{\lambda H} e^{\lambda H} \Phi^{(+)} e^{-\lambda H} \}, \\ & (A=1, 2; B=1, 2) \end{aligned} \quad (10)$$

where the notations \mathbf{I}_1 and \mathbf{I}_2 are used for the currents \mathbf{J} and \mathbf{W} in (9), and then as its extremum as to $\Phi^{(-)}$ and $\Phi^{(+)}$ we get the dyadics of transport coefficients in the limit $s \rightarrow +0$

$$\begin{aligned} L_{AB} &= \int_0^\infty dt \int_0^\beta d\lambda \text{Tr} \{ \mathbf{I}_A \rho_c \exp(\lambda H - iHt) \\ & \cdot \mathbf{I}_B \exp(-\lambda H + iHt) \}. \end{aligned} \quad (11)$$

The entropy production which is obtained as the extremum of (8) is written in terms of L_{AB} in (11) as $\theta = \sum_{A,B} \mathbf{X}_A \cdot L_{AB} \cdot \mathbf{X}_B$. In the case that the description is possible in the one-body picture and in effect the operators are ex-

pressed in the form as in (9a), (9b) in I,* we can rewrite (8) and (10) as

$$\begin{aligned} \theta(\phi^{(-)}, \phi^{(+)}) = & \frac{i}{2} \lim_{\epsilon \rightarrow 0} \int_0^{\infty} dt e^{-\epsilon t} \\ & \times \text{Tr} \{ f_c([\phi^{(-)}(t) - \phi^{(-)}(-t), s\phi^{(+)} \\ & + i[h, \phi^{(+)}]] + [\phi^{(+)}(t) - \phi^{(+)}(-t) \\ & - \phi^{(-)}(t) + \phi^{(-)}(-t), \mathbf{X}_1 \cdot \mathbf{J} + \mathbf{X}_2 \cdot \mathbf{W}] \}, \end{aligned} \quad (12)$$

$$\begin{aligned} L_{AB}(\phi^{(-)}, \phi^{(+)}) = & \frac{i}{2} \lim_{\epsilon \rightarrow 0} \int_0^{\infty} dt e^{-\epsilon t} \\ & \times \text{Tr} \{ f_c([\phi^{(-)}(t) - \phi^{(-)}(-t), s\phi^{(+)} \\ & + i[h, \phi^{(+)}] - \mathbf{i}_B] \\ & - [\mathbf{i}_A, \phi^{(+)}(t) - \phi^{(+)}(-t)] \}, \end{aligned} \quad (13)$$

where the small letters represent the respective matrices in the one-body problem and $\phi^{(\pm)}(t)$ means $\exp(iht)\phi^{(\pm)}\exp(-iht)$.

- 1) H. Nakano, Prog. Theor. Phys. **22** (1959), 453, referred to as I hereafter.

A Variation Principle for Calculating General Susceptibility Tensors

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We have recently investigated some

* The expression (9c) in I is erroneous and it must be read as $\langle m|f_c|n\rangle = \text{Tr}(\rho_c a_n^* a_m)$. Here we should like to correct another mistake in I as to the charge of carriers; viz. e in the third line in p. 453 is to be read as $-e$.

variation principles in the problem of transport phenomena.¹⁾ We can make use of the variation principle of the same sort in calculating different kinds of susceptibility tensors, e. g. the electric and the magnetic susceptibility tensors.

For the present we here take the electric case. As we have seen in I the Schrödinger equation for the density matrix,

$$i \frac{\partial}{\partial t} \rho(t) = [H, \rho(t)] - [\mathbf{E}(t) \cdot \mathbf{P}, \rho(t)] \quad (1)$$

is to be investigated under the conditions that in the one case the applied electric field $\mathbf{E}(t)$ goes in from $t = -\infty$ to $t = 0$ as $\mathbf{E} \exp(st)$ and in the other case it fades away in $t > 0$ as $\mathbf{E} \exp(-st)$ where s is a complex number with positive real part. In Eq. (1) \mathbf{P} is the operator of electric polarization vector. The solution of (1) is obtained in the form

$$\rho(t) = \rho_c + \mathbf{F}^{(\pm)} \exp(\pm st) \quad (2)$$

in each case, where ρ_c represents the equilibrium or the grand canonical distribution

$$\rho_c = K \cdot \exp(-\beta H - \xi N).$$

By substituting (2) into (1) and taking the direction of \mathbf{E} as ν -axis, we get

$$\begin{aligned} \pm is \mathbf{F}^{(\pm)} = & [H, \mathbf{F}^{(\pm)}] - E[P_\nu, \rho_c] \\ (\nu = x, y, z) \end{aligned} \quad (3)$$

in the linear approximation concerning the electric field. Instead of taking the form (5) in I or (5) in II we put $\mathbf{F}^{(\pm)}$ as

$$\mathbf{F}^{(\pm)} = -iE[\rho_c, \Phi^{(\pm)}] \quad (4)$$

and the variation principle is suggested in regard to this state operator $\Phi^{(\pm)}$,

which is equal to the time derivative $\mathcal{P}_v^{(\pm)}$ in Eq. (5) of I.

Let $\Phi^{(\pm)}$ be the solution of (3) substituted by (4), they make the functional of unknown operators $\Xi^{(\pm)}$ an extremum:

$$\chi_{\mu\nu}(\Xi^{(-)}, \Xi^{(+)}) \\ = \text{Tr} \{ \Xi^{(-)} [\rho_c, [H, \Xi^{(+)}] - i s \Xi^{(+)}] \\ + i \Xi^{(+)} [\rho_c, P_\mu] + i \Xi^{(-)} [\rho_c, P_\nu] \}, \quad (5)$$

and the extremum value $\chi_{\mu\nu}(\Phi^{(-)}, \Phi^{(+)})$ is equal to the electric susceptibility of the system.

If we take as $s = +0 + i\omega$, we can investigate the frequency-dependent susceptibility $\chi_{\mu\nu}(\omega) = \chi'_{\mu\nu}(\omega) - i\chi''_{\mu\nu}(\omega)$. We can also discuss the magnetic susceptibility by taking the operator of magnetization vector \mathbf{M} instead of that of electric polarization \mathbf{P} . If we take the functional

$$I'(\Xi^{(-)}, \Xi^{(+)}) \\ = \text{Tr} \{ \Xi^{(-)} [\rho_c, i s \Xi^{(+)} - [H, \Xi^{(+)}]] \\ - i (\Xi^{(+)} + \Xi^{(-)}) [\rho_c, \mathbf{E} \cdot \mathbf{P}] \}, \quad (6)$$

the variation principle is that the extremum of (6) is equal to the electric internal energy.

In the case that it is possible to discuss in terms of the one-body problem, the functional (5) reduces to

$$\chi_{\mu\nu}(\hat{\xi}^{(-)}, \hat{\xi}^{(+)}) \\ = \text{Tr} \{ \hat{\xi}^{(-)} [f_c, [h, \hat{\xi}^{(+)}] - i s \hat{\xi}^{(+)}] \\ + i \hat{\xi}^{(+)} [f_c, p_\mu] + i \hat{\xi}^{(-)} [f_c, p_\nu] \} \quad (7)$$

and (6) also does to the similar one, where operators appearing in these are all in regard to the one-body problem, viz. they represent the matrices in the right-hand sides of the expressions

$$H = \sum_{m,n} (m|h|n) a_m^* a_n, \\ \Xi^{(\pm)} = \sum_{m,n} (m|\hat{\xi}^{(\pm)}|n) a_m^* a_n, \\ P_\nu = \sum_{m,n} (m|p_\nu|n) a_m^* a_n \\ (\nu = x, y, z)$$

and

$$(m|f_c|n) = \text{Tr}(\rho_c a_n^* a_m)$$

and Tr means the traces of the matrix of this kind.

The extremum of (5) and that of (7), which are both proved to be the susceptibility tensor, are

$$\chi_{\mu\nu}(\Phi^{(-)}, \Phi^{(+)}) = \chi_{\mu\nu} \\ \equiv i \int_0^m dt \text{Tr} \{ [\rho_c, P_\mu(t)] P_\nu \}, \\ \chi_{\mu\nu}(\phi^{(-)}, \phi^{(+)}) = \chi_{\mu\nu} \\ \equiv i \int_0^\infty dt \text{Tr} \{ [\rho_c, p_\mu(t)] p_\nu \}.$$

- 1) H. Nakano, Prog. Theor. Phys. **22** (1959), 453, referred to as I hereafter; *ibid.* **23** (1960), 13, referred to as II hereafter. These letters I and II and the present letter are abstracted from the papers in Japanese, Busseiron-Kenkyu (mimeographed circular) 2nd Series, **5** (1959), 725; **6** (1959), 34, 164, 174.

On the High Energy Protons Inelastically Scattered from C^{12} and O^{16}

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Tyrén and Maris¹⁾ measured the ener-

gy distributions of the high energy protons inelastically scattered from C^{12} , O^{16} , etc., and observed the broad resonances with several Mev width in the neighbourhood of excitation energy 19~22 Mev. And they concluded that these 'giant resonances are related to the excitation modes of photo-nuclear reactions through the Coulomb interactions between the incident proton and the protons in the target nucleus. Kawai and Terasawa²⁾ used the Born approximation for the electrostatic interactions between these protons, and they replaced the reduced matrix elements by the experimental values of photo-nuclear reactions in the course of their treatments. Then, they could compare their calculated results with the experimental data having the characteristic peak of photo-nuclear reactions without referring to any special model of photo-nuclear mechanism. And they obtained the results which are smaller than the observed by a few factor for various target nuclei. Then they concluded that these discrepancies are owing to the neglect of nuclear interactions between the incident proton and the nucleons in the target nucleus, or to the ambiguity of experimental data for the photo-nuclear reactions.

In the present note, we use the t -matrix in order to take into account that the incident proton acts on the nucleons in the target nucleus through the nuclear interactions and these interactions contribute to the excitation mode of 19~22 Mev regions.

For brevity, we consider the even-even target nuclei with spin 0 and isotopic spin 0 at the ground state such

as C^{12} and O^{16} . If one uses the t -matrix expressed by³⁾

$$t(i) = \alpha + \beta(\sigma_0 + \sigma_i) \hat{n} + \gamma(\sigma_0 \hat{n})(\sigma_i \hat{n}) \\ + \delta(\sigma_0 \hat{k})(\sigma_i \hat{k}) + \epsilon(\sigma_0 \hat{p})(\sigma_i \hat{p}),$$

where \hat{n} , \hat{p} and \hat{k} are three mutually perpendicular unit vectors, chosen so that \hat{n} is normal to the scattering plane, the energy distribution of the high energy proton inelastically scattered from the levels of 19~22 Mev region is, after some calculations, given by

$$\frac{d^2\sigma}{d\Omega dE} = \frac{2}{\pi^2 E_{\text{exc}}} \frac{k_i}{k_f} \{ |\bar{\alpha}|^2 + |\bar{\beta}|^2 \\ + \lambda(|\bar{\beta}|^2 + |\bar{\gamma}|^2 + |\bar{\epsilon}|^2) \} k^2 \left(\frac{e^2}{\hbar c} \right)^{-1} \sigma_{ph}(E_{\text{exc}}) \quad (1)$$

where k_i and k_f are the initial and final wave numbers of the proton, respectively, and $\bar{\alpha} = \frac{1}{2}(\alpha_{pp} - \alpha_{pn})$, $\bar{\beta} = \frac{1}{2}(\beta_{pp} - \beta_{pn})$ etc., here α_{pp} , β_{pp} , etc., and α_{pn} , β_{pn} , etc., are the coefficients of p - p and p - n scattering matrices, respectively. λ is the ratio of the spin flip to the non-spin flip reduced matrix elements, and does not depend on the momentum transfer $\mathbf{k}_f - \mathbf{k}_i = \mathbf{k}$. $\sigma_{ph}(E_{\text{exc}})$ is the cross section for excitation of the nucleus by energy E_{exc} through the absorption of electric dipole γ -ray.

The value of λ is determined by the measurements of the polarization, because the polarizations of the nucleons inelastically scattered from the levels of 19~22 Mev regions are very sensitive to the values of λ . Thaler et al.⁴⁾ obtained $\lambda=0.1$ for C^{12} and $\lambda=1$ for O^{16} in order to explain the polarizations of the nucleons, for the nucleon of incident energy 156 Mev. It would be expected

that the value of λ does not change rapidly in the difference of the incident energy of the nucleon between 156 Mev and 180 Mev.

Recently, many sets of the nucleon-nucleon scattering phase shifts have been available⁵⁾, then one can obtain the amplitudes of the scattered nucleon from various levels of the nucleus by the use of the direct interaction model for nuclear reactions.

When $\bar{\alpha}$, $\bar{\beta}$, etc., are determined by

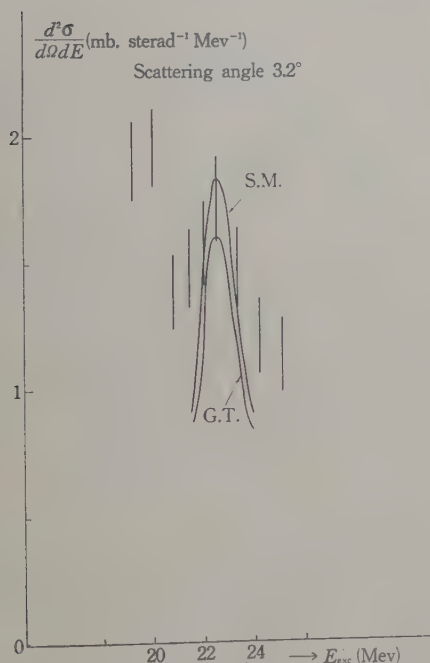


Fig. 1 Energy distribution of 182 Mev proton scattered from the levels of dipole γ -giant resonance region of C^{12} .

In the calculations of $\bar{\alpha}$, $\bar{\beta}$, etc., we use Gammel-Thaler and Signell-Marshak phase shifts at 180 Mev.⁵⁾ For $\sigma_{ph}(E_{exc})$, we use $\sigma_{ph}(E_{exc}) = \sigma(\gamma, n) + \sigma(\gamma, p)$, and the data of Barber et al.⁶⁾ for $C^{12}(\gamma, n)$, and the data of Cohen et al.⁷⁾ for $C^{12}(\gamma, p)$.

In present note we assume that the contributions of $\sigma(\gamma, np)$ and others to $\sigma_{ph}(E_{exc})$ are very small.

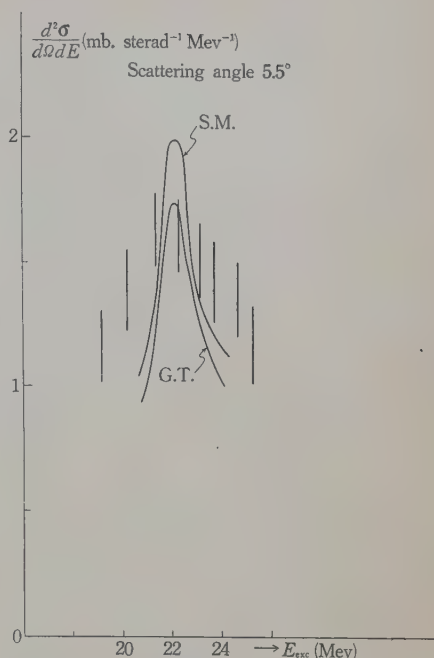


Fig. 2 Energy distribution of 177 Mev proton scattered from the dipole γ -giant resonance levels of O^{16} .

In the estimation of $\sigma_{ph}(E_{exc})$, we use the data of Katz et al.⁸⁾ for $O^{16}(\gamma, n)$ and the data of Stephens et al.⁹⁾ for $O^{16}(\gamma, p)$.

the use of the nucleon-nucleon scattering phase shifts, the magnitude of the energy distribution of the proton inelastically scattered from the levels of the dipole giant γ -resonance region is a function of $\sigma_{ph}(E_{exc})$ and λ . Therefore one can determine the value of λ through the experimental data of the energy distribution of the proton, if $\rho_{ph}(E_{exc})$ is measured. Or, one can determine the magnitude of $\sigma_{ph}(E_{exc})$, if λ is determined.

Also, the measurements of the energy distributions of the inelastically scattered protons provide a tool to decide the excitation energy E_{exc} which gives the maximum value of $\sigma_{ph}(E_{exc})$ in the

dipole giant γ -resonance.

If experimental values such as $d^2\sigma/d\Omega dE$, $\sigma_{ph}(E_{exc})$, and λ are available, one can verify the validity of Eq. (1) by making use of these experimental data. And, the fair agreements of the calculated results with the experimental data for C^{12} and O^{16} show the validity of Eq. (1). (See Figs. 1 and 2) Then one can determine another value through the other two experimental values by Eq. (1) for even-even nucleus such as Ne^{20} .

In a forthcoming paper, we shall give the details of the above discussions, other results and estimates of the contributions of $\sigma(\gamma, np)$, $\sigma(\gamma, 2n)$, etc., to $\sigma_{ph}(E_{exc})$, of the effects of the nuclear distortions for the incident proton, and of the contributions from the levels in 19~22 Mev region which are not excited by the dipole γ -ray, etc.

The author would like to express his gratitude to Prof. M. Kobayasi for his kind encouragements.

The author would like to express his sincere thanks to Dr. Y. Nishida for many discussions. Thanks are also due to Dr. T. Terasawa for discussions.

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Ground State of a System Consisting of Two Oppositely Charged Particles in Coulomb Field

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Whether two oppositely charged particles form a bound state in the Coulomb field is an important problem when we discuss the annihilation process of the positron in matter¹⁾ or the interaction of the exciton with a charged imperfection in crystalline solid.

Let \mathbf{r}_1 be the position of the particle 1 with charge $-e$ and mass m_1 and \mathbf{r}_2 be that of the particle 2 with charge $+e$ and mass m_2 , then the Hamiltonian of the system consisting of the two particles in the Coulomb field generated by a static charge $+e$ at the origin is

$$\mathcal{H} = -\frac{A_1}{2} - \frac{A_2}{2\kappa} - \frac{1}{r_1} + \frac{1}{r_2} - \frac{1}{r_{12}}, \quad (1)$$

where we use the unit $\hbar = e = m_1 = 1$ and put $\kappa = m_2/m_1$, $r_{12} = |\mathbf{r}_1 - \mathbf{r}_2|$. We choose the following trial function for the ground state to perform variational calculation, i.e.,

$$\psi = \exp(-\zeta r_1) r_2^n \exp(-\eta r_2) \times \exp(-\beta r_{12}), \quad (2)$$

where ζ , η , β and n are parameters to be determined so as to minimize the energy $E = (\psi, \mathcal{H}\psi) / (\psi, \psi)$. Since the energy of the state where only the particle 1 is near the origin while the particle 2 is at infinity amounts just to $-1/2$, the binding energy B of the two particles at the static charge is $B = -1/2 - E$. Evidently a positive value of B corresponds to a bound state. All the integrals appearing in the expression for the energy E can be evaluated analytically as far as $2n$ is a positive integer. Numerical calculation has been carried out exclusively on the TAC, an electronic computer.

For $\kappa = 1$, no positive B was obtained. In other words, the positron affinity of the hydrogen atom turned out to be negative as far as we adopt the trial function (2). This result naturally was unaffected when we replaced the positive static charge at the origin by the proton with a finite mass in order to consider the effect of its motion, which proved to contribute less than a percent to the energy.

Further we estimated the minimum value κ_m of κ which can give a bound state. The result was $\kappa_m = 7.8$. It is

a remarkably rigorous estimate as compared with Ore's value, $\kappa_m \approx 20^{(2)}$, obtained by a simplified treatment. For example, the effective masses of a conduction electron (m_1) and a hole (m_2) in InSb crystal are known to be $m_1 = 0.02 m_e$ and $m_2 = 0.3 m_e$ respectively, where m_e is the free electron mass⁽³⁾. Since $\kappa = m_2/m_1 = 15 > \kappa_m$, we conclude in the approximation of the effective mass theory that the exciton in InSb, if it ever exists, can form a bound state localized at a positively charged imperfection. In fact, $B = 0.02099$ in our unit for $\kappa = 15$, the values of variational parameters being $\zeta = 2.19765$, $\eta = 0.728095$, $\beta = 0.566542$ and $n = 4.5$. When we use the dielectric constant 16.8 for InSb,⁽⁴⁾ this binding energy corresponds to 4.04×10^{-5} eV.

Extension of the trial function to a linear combination of the terms like (2) is now being attempted. The result will be reported in a forthcoming paper together with a fuller account of the present work.

Finally, we appreciate the kindness of the members of the Electronic Computer Laboratory of the Faculty of Engineering who gave us facilities for numerical calculations.

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Electric Multiple Transitions in the $D(\gamma p)n$ Reaction at High Energy

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Recently the photodisintegration of the deuteron at medium energies (20~100 Mev) has been investigated by many authors.¹⁻⁵⁾ It has become rather convincing that one can explain the main features of the photodisintegration at the medium energies in terms of the electric transitions, if one assume a large D -state mixing in the deuteron and the strong positive tensor potential in the triplet odd final state. Both of them are the natural consequences of the pion theory of nuclear forces.⁶⁾

Most of the authors, however, confined themselves to the calculatson of $E1$ (and $E2$) transition, and, further, they took up only the first term in the power series expansion in terms of $\kappa_r r$ of each multipole interaction. The present author and his collaborator have performed exact calculations for $E1$ and $E2$ transitions, and found that the breaking up of the power series is not justified even at the energy as low as 80 Mev.^{5)*} Therefore, it seems worthwhile to investigate the effects of the higher power of $\kappa_r r$ in a more general way.

Concerning the disintegration at higher energies, it is clear that one cannot reproduce the hump in the observed excitation function around 280 Mev

in terms of the electric transitions, because the latter predicts a smooth energy dependence of the excitation function. Of course, the hump is due to the resonance production of the virtual meson by the incident photon. The recent investigation,⁷ however, suggests that the virtual meson effects give appreciable contributions only to the spin-flip $M1$ transition within the narrow range of energy around the observed hump.⁷⁾ Then, it is rather interesting to see whether one can reproduce all the data by just adding the contributions (the meson effects and so on) to the excitation function due to the electric transitions.

In the present note, we shall show the results of our calculations in the energy range of 80-300 Mev in which the complete expression for the electric interaction is used. One can reduce the full electric interactions into the following form.⁸⁾

$$\int_0^1 ds \frac{1}{2} e\mathbf{r} \left[\exp\left(i\frac{s}{2}\kappa_T\mathbf{r}\right) \tau_p^{(1)} + \tau_p^{(2)} \exp\left(-i\frac{s}{2}\kappa_T\mathbf{r}\right) \right]. \quad (1)$$

Here the integral over s takes account of the so-called retardation effects and if one simply puts $s=0$ in (1), then one will get the conventional expression for the electric dipole interaction. Since $\kappa_r R$ (R is the deuteron radius) is almost unity for 90 Mev of photon energy, it is not legitimate to invoke such an approximation in high energy region.

In our calculations, we kept the full expression of (1). For the sake of simplifying the calculations, we approxi-

* For the importance of the multipole of $E1$ transition, it has been reported by Hsieh.

mated the final state wave function by a plane wave which may be justified at high energies. For the deuteron state, we adopted the poin theoretical wave function,⁶⁾ which has about 7.1% of *D*-state mixture and reproduces the quadrupole moment of $2.6 \times 10^{-27} \text{cm}^2$.

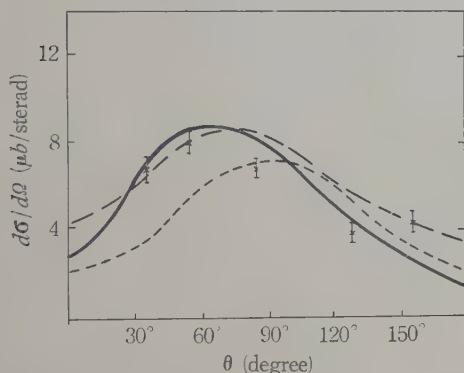


Fig. 1 Angular distribution of the $D(\gamma p)n$ reaction at $E_\gamma=80$ Mev. The solid curve represent our exact calculation. The dotted and dashed curve represent the case of $s=0$, $E1+E2$ respectively.

In Fig 1. we have plotted the differential cross sections given by our exact calculations and $s=0$ approximation at 80 Mev photon energy. For the sake of comparison, we also plotted the differential cross section given by our previous calculations⁵⁾ in which we took into account the final state interactions making use of the full expression for $E1$ and $E2$ interactions. It is readily seen from Fig. 1 that the $s=0$ approximation gives rather poor results. This means that the $\kappa_\gamma r$ expansion is not justified in the energy region $E_\gamma \gtrsim 80$ Mev and gives an underestimate for $E1$ cross section.* ** The comparison of our exact curve with $E1+E2$ curve shows that the effects of the higher

multipoles than $E2$ are not appreciable and that the approximation of the final states wave function by a plane wave is not too serious. In Fig. 2 we have plotted the excitation functions given

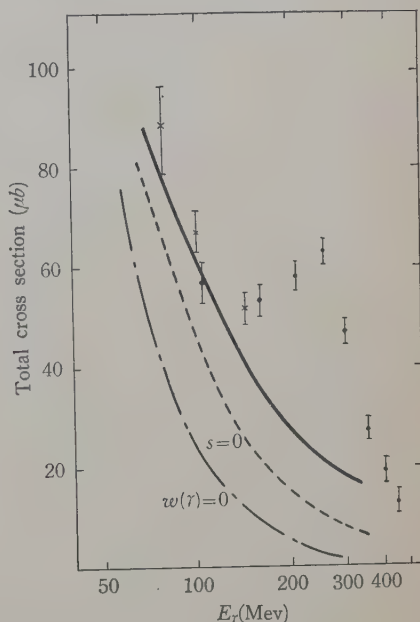


Fig. 2 Excitation function of the $D(\gamma p)n$ reaction. The solid curve is our exact calculation.
----- is the case of $s=0$.
- · - · - is the case of the deuteron d-state $w(\gamma)=0$.

by our exact calculations, the $s=0$ approximation and the case of the zero D -state mixture of the deuteron.

The prediction by the exact calculations follows up the observed results fairly

* This conclusion is in contradiction with that of Nicholson and Brown.⁹⁾ This might be due to their rough estimation of the retardation effects.

** These conclusions have been reached also in reference 3) on the basis of less extensive calculation.

well except in the hump region.

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A Note on the Polarizations for p -He³ and p -T³ Scattering

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November 7, 1957

In a previous note¹⁾ we reported the polarizations of the high energy protons scattered from Li⁶, Be⁹ and B¹¹. The

amplitudes of these scattered protons consist of the contributions of non spin flip and spin flip terms of the target nucleons. However, the contributions of spin flip terms are much smaller than those of non spin spin flip ones, and the former are masked by the latter. And, generally, when there are no contributions of spin flip terms to the amplitudes of the nucleon scattered from the nucleus, these amplitudes give rise to a large polarization.²⁾ Then, the polarizations are expected to be large for the nucleons scattered from nuclei such as Li⁶, Be⁹ and B¹¹, in agreement with observations.³⁾

Now, the most interesting ratios of the contributions of spin flip terms to those of non spin flip ones would be those in nucleon-He³ and nucleon-T³ scattering amplitudes. That is, in the polarizations of nucleons scattered from He³ and T³, the contributions of spin flip terms to the scattering amplitudes play important roles since those of non spin flip ones are relatively small.

In the calculations in the present note we use the impulse approximation and neglect D -states of He³ and T³, as the probabilities of those states are very small, i.e., 2~3%.

We express the two-body scattering spin matrix as follows,⁴⁾

$$\begin{aligned}
 M_{1i} = & BS + [C(\sigma_1 + \sigma_i)n \\
 & + N(\sigma_1 n)(\sigma_i n) \\
 & + (1/2)G\{(\sigma_1 k)(\sigma_i k) \\
 & + (\sigma_1 p)(\sigma_i p)\} \\
 & + (1/2)H\{(\sigma_1 k)(\sigma_i k) \\
 & - (\sigma_1 p)(\sigma_i p)\}]T
 \end{aligned} \tag{1}$$

where the coefficients B , C , N , G and H , depend on energy and scattering angle θ in the center of mass system, and S and T are the respective singlet and triplet spin projection operators. \mathbf{n} , \mathbf{k} , \mathbf{p} , are unit vectors in the direction of $\mathbf{k}_i \times \mathbf{k}_f$, $\mathbf{k}_f - \mathbf{k}_i$ and $\mathbf{k}_f + \mathbf{k}_i$, respectively, here \mathbf{k}_f and \mathbf{k}_i are the final

and initial center of mass momenta of the particles. For the sake of clarity, for the singlet and triplet state interactions of two-body we use the scattering spin matrix containing the projection operators S and T .

The expression for the polarization of proton elastically scattered from He^3 is given by, after some calculations,

$$P(\theta) = \frac{2\text{Re}\{(2\alpha_{pp} + \alpha_{pn})^*(2\gamma_{pp} + \gamma_{pn}) + \beta_{pn}^* \gamma_{pn}\}}{|2\alpha_{pp} + \alpha_{pn}|^2 + |2\gamma_{pp} + \gamma_{pn}|^2 + |\gamma_{pn}|^2 + |\beta_{pn}|^2 + |\delta_{pn}|^2 + |\epsilon_{pn}|^2} \quad (2)$$

and also that for T^3

$$P(\theta) = \frac{2\text{Re}\{(2\alpha_{pn} + \alpha_{pp})^*(2\gamma_{pn} + \gamma_{pp}) + \beta_{pp}^* \gamma_{pp}\}}{|2\alpha_{pn} + \alpha_{pp}|^2 + |2\gamma_{pn} + \gamma_{pp}|^2 + |\gamma_{pp}|^2 + |\beta_{pp}|^2 + |\delta_{pp}|^2 + |\epsilon_{pp}|^2} \quad (3)$$

where α , β , etc., are related to

$$\begin{aligned} \alpha &= (1/4)(B + N + G), \\ \beta &= (-1/4)(B + G - 3N) \\ \gamma &= C \\ \delta &= (-1/4)(B + N - G - 2H) \\ \epsilon &= (-1/4)(B + N - G + 2H) \end{aligned} \quad (4)$$

and α_{pp} , β_{pp} , etc., and α_{pn} , β_{pn} etc., are the coefficients of p - p and p - n scattering spin matrices given by Eq. (1) through Eq. (4), respectively.

When the polarization of p - p scattering is compared with that of p - n scattering, it is evident that the latter is larger than the former, so that it would be expected that the polarization of proton is large for scattering by T^3 than by He^3 . However, if the calculations on Eqs. (2) and (3) are carried out, the polarization may be obtained as being large for the scattering by He^3 than by T^3 .

Since the two neutrons in T^3 at the ground state will be bound in the singlet spin state, these neutrons do not give rise to the spin flip, then the am-

plitudes of protons scattered from these neutrons are related to $2\alpha_{pn}$ and $2\gamma_{pn}$, and other terms in p - n scattering spin matrix cancel out. The proton in T^3 gives rise to the spin flip, so that all the coefficients of p - p scattering spin matrix contribute to the amplitude of proton scattered from T^3 . On the contrary, in p - He^3 scattering, the roles of two protons and a neutron are contrary to the roles of two neutrons and a proton in T^3 . The contributions of non spin flip terms to the amplitudes of protons scattered from T^3 and He^3 are related to $2\alpha_{pn} + \alpha_{pp}$, $2\gamma_{pn} + \gamma_{pp}$ and $2\alpha_{pp} + \alpha_{pn}$, $2\gamma_{pp} + \gamma_{pn}$, respectively. On the other hand, the contributions of spin flip ones to p - T^3 scattering amplitude are related to γ_{pp} , δ_{pp} , ϵ_{pp} , and β_{pp} , while those to p - He^3 to β_{pn} , γ_{pn} , δ_{pn} and ϵ_{pn} . These contributions of spin flip terms play the relatively more important role than those of non spin flip ones. This is a main reason why the polarization for p - He^3 scattering is larger

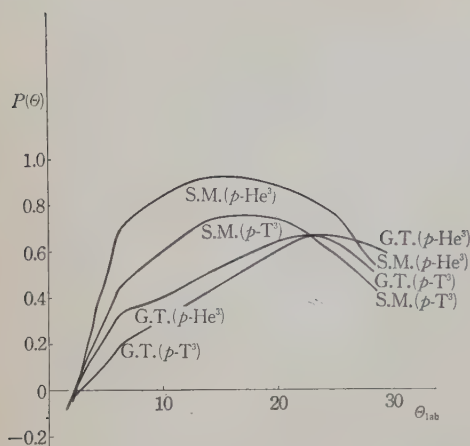


Fig. 1 Polarizations of 180 Mev protons elastically scattered from He^3 and T^3 .

G. T. and S. M. indicate Gammel-Thaler⁵⁾ and Signell-Marshak⁶⁾ phase shifts for nucleon-nucleon scattering, respectively.

than that for $p\text{-T}^3$ scattering.

The calculated results shown in Fig. 1 have been carried out with use of Gammel-Thaler⁵⁾ and Signell-Marshak⁶⁾ phase shifts at 180 Mev.

Since very different results for the absolute and the relative magnitudes of polarizations of nucleons scattered from He^3 and T^3 are obtained by using available sets of nucleon-nucleon scattering phase shifts, it seems that there are some possibility of choosing the reasonable set of phase shifts if it is measured, the absolute magnitudes of polarizations of nucleons scattered from T^3 and He^3 and also the magnitudes of differences of these polarizations.

The authors would like to express their sincere appreciation of kind encouragement given by Prof. M. Kobayashi.

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On the Charge Distribution of the Proton

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It seems to be generally believed that the experiments on high-energy electron-proton and electron-deuteron scatterings and low-energy neutron-atom scattering indicate

$$\langle r^2 \rangle_{1,n} \simeq 0 \quad (1)$$

$$\langle r^2 \rangle_{1,p} = [(0.80 \pm 0.04) \times 10^{-13} \text{ cm}]^2 \quad (2)$$

$$\simeq \langle r^2 \rangle_{2,p} \simeq \langle r^2 \rangle_{2,n}$$

for the mean square radii of charge and a. m. m. of the nucleon.¹⁾ But from our previous theoretical investigations,²⁾ we feel it rather unlikely that the current meson theory predicts such large mean square radii as shown by (2). However, before we doubt the validity of meson theory or quantum electrodynamics at short distances, we must of course examine the adequacy of deriving the results (1) and (2) from the

experimental data. (1) is the result which was obtained by a shape-independent method and also it is free from the ambiguity of the a. m. m. form factor, while (2) is what was estimated by using very special models for the proton under the assumption $F_1=F_2$. Therefore we will for the present rely only upon (1), i.e.,

$$\langle r^2 \rangle_1^s \simeq \langle r^2 \rangle_1^v. \quad (3)$$

We also believe in the cross sections obtained by the Stanford experiments and the validity of meson theory in the outer region $r \gtrsim 1/\mu$. We will furthermore assume that

- i) the charge distribution of the isovector part is positive definite,
- ii) that of the isoscalar part is negative in the inner region and positive in the outer region, and
- iii) the a. m. m. distribution of the proton is positive definite (see Fig. 1 (b)),

these being conjectured from our previous analyses.²⁾ If we assume that the radial distance for the region mainly contributing to $\langle r^2 \rangle$ is roughly proportional to the range of the distribution (i.e., the reciprocal of the threshold energy), we shall roughly have

$$\frac{\langle r^2 \rangle_1^v}{\langle r^2 \rangle_1^s} \simeq \frac{(1/2\mu)^2 \cdot e/2}{(1/3\mu)^2 Qe}, \quad (4)$$

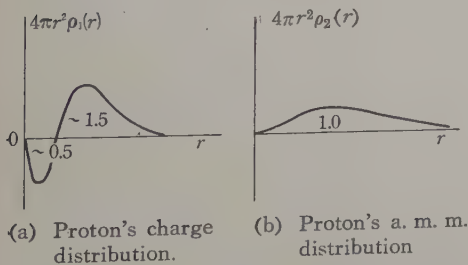
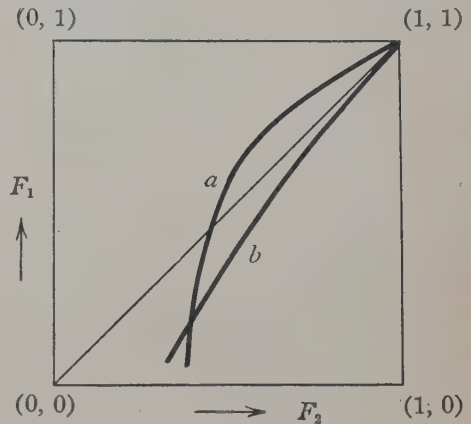


Fig. 1

where Qe stands for the amount of the positive charge in the isoscalar part. We get $Q \simeq 1.0$ from (3) and (4). Then the neutron's charge distribution will take the form suggested by Schiff,³⁾ and the proton's charge distribution will be such a form as in Fig. 1(a).* (For simplicity we have assumed $Z_{2p}=0$, but this is not essential.) It may be imagined that the inner negative part would be owing to virtual anti-protons. At any rate, if it is true that the proton's charge distribution is negative in the inner region, then we can expect that F_1/F_2 will behave like Fig. 2, as is



- (a) The case $\langle r^2 \rangle_{1,p} < \langle r^2 \rangle_{2,p}$
- (b) The case $\langle r^2 \rangle_{1,p} \geq \langle r^2 \rangle_{2,p}$

Fig. 2 F_1/F_2 diagram

easily seen by comparing Fig. 1(a) with Fig. 1(b). It is of some interest to notice that Karplus' analysis⁵⁾ of F_1/F_2 seems to exhibit somewhat such a tendency as the case (a) of Fig. 2.

* Such a possibility was suggested by one of the authors (K. H.) in the preliminary meeting (in Japan) for Kiev Conference. Independently, a similar form was considered as a trial model by Katayama et al.⁴⁾

Therefore the experiment more precisely determining F_1/F_2 is highly desirable.

Finally, we remark on the value (2) for $\langle r^2 \rangle_{1,p}$. If we assume such a charge distribution for the proton as in Fig. 1(a), we can construct a simple model which is consistent with all the Stanford data and gives a rather smaller value for $\langle r^2 \rangle_{1,p}$ than (2).

Detailed accounts will be reported before long.

The authors wish to express their sincere thanks to Prof. H. Yukawa for his kind interest.

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Proton-Antiproton Annihilation and Nucleon Structure

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As is well known, the observed multiplicity of pions in proton-antiproton annihilation process, $\langle n_\pi \rangle = 4.7 \pm 0.4$, is

in contradiction with the theoretical value obtained by a simple straightforward application of Fermi's statistical model. Recently Koba and Takeda¹⁾ have obtained the result $\langle n_\pi \rangle \cong 4.8$ by the calculation based on their model. In this note we also intend to study this problem with the object of pursuing some knowledge about the dynamics in the region smaller than 10^{-13} cm.

Let us now consider the case in which the proton-antiproton annihilation process takes place at the distance r from the center of the proton and r' from that of the antiproton.* The produced pions will undergo both $\pi\pi$ interaction and πN interaction until they will be observed. In this case we assume the following relation between the magnitude of mean momentum of the observed pions $\langle k \rangle$ and the place where the collision takes place.

$$\langle k \rangle = 1/\langle r \rangle, \quad \text{where } \langle r \rangle = (r + r')/2. \quad (1)$$

Then the average multiplicity of pions $n_\pi(\langle r \rangle)$ can be estimated by the following relation (2) if the kinetic energies of both proton and antiproton are neglected.

$$n_\pi(\langle r \rangle) \cong 2M/\langle \epsilon_r \rangle, \quad (2)$$

$$\langle \epsilon_r \rangle = \sqrt{\langle k \rangle^2 + \mu^2},$$

where M and μ are the masses of nucleon and pion respectively.

Now let us take into account the effect of nucleon structure. We show in

* Hereafter, as the unit of length we adopt $1/M$ instead of $1/\mu$ from the viewpoint that the former may probably play the more important role than the latter in our approach to the dynamics.

Fig. 1 the charge distribution for proton which has been obtained by Hofstadter.²⁾ This distribution may be interpreted in such a meaning that it represents

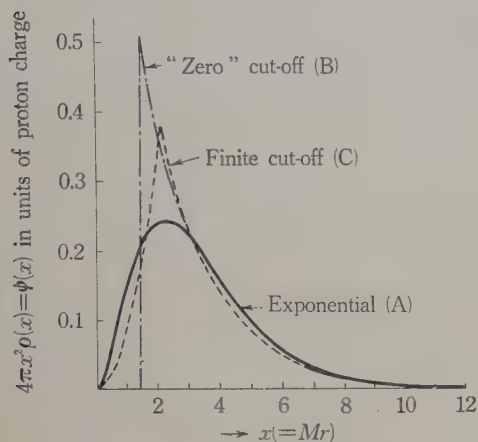


Fig. 1

the probability amplitude with which the electromagnetic interaction is realized. Basing on the charge independence of the interaction in π - N system, we assume that the situation in π - N interaction is the same as that in electromagnetic interaction. Namely the distribution function $4\pi x^2 \rho(x) = \psi(x)$ shown in Fig. 1 is also regarded as an expression of the probability amplitude with which the interaction in π - N system is realized in the place r , where $x = Mr$. In the case of antiproton there exists the same situation as in the case of proton. Therefore, when we denote the distribution function for antiproton as $\psi'(x')$, the function $\psi(x) \psi'(x')$ means the probability amplitude with which the annihilation process takes place in the distance r from the center of the proton and r' from that of the antiproton. Thus the average multiplicity of pions is expressed by using the

notations of (1) and (2) as follows,

$$\langle n_\pi \rangle = \frac{\iint n_\pi(\langle r \rangle) \cdot P(x, x') dx dx'}{\iint P(x, x') dx dx'},$$

$$P(x, x') = [\psi(x) \cdot \psi'(x')]^2. \quad (3)$$

In the discussion about the pion multiplicity, we can use the following relation,

$$\text{at } x = x', \quad \psi'(x) = \psi(x), \quad (4)$$

because the structure of the antiproton is identical with that of the proton except for the sign of charge.

Hofstadter has given the following three forms as the charge distribution for proton (cf. Fig. 1), (i) Exponential model (A), (ii) "Zero" cut-off model (B), (iii) Finite cut-off model (C). Adopting these forms, we calculate $\langle n_\pi \rangle$ by Eq. (3) and obtain the following results.

$$\text{In case (A) ; } \langle n_\pi \rangle \cong 5.0$$

$$\text{In case (B) ; } \langle n_\pi \rangle \cong 4.3 \quad (5)$$

$$\text{In case (C) ; } \langle n_\pi \rangle \cong 4.9$$

Every value of these agrees fairly well with the experimental one $\langle n_\pi \rangle = 4.7 \pm 0.4$

Finally let us examine our assumptions recalling the experimental results for the other pion phenomena. The most remarkable fact in low energy pion phenomena is the $(3/2, 3/2)$ -resonance in π - N interaction. Since the pion momentum in π - N scattering at 190 Mev is $k \cong 1.15 \times 10^{13} \text{ cm}^{-1}$, the impact parameter r which is responsible to the p -wave resonance turns out to be nearly equal to the mean radius of the charge distribution. The phenomena for the

(3/2, 3/2)-resonance may be closely connected with the nucleon structure.

We now pay our attention to the pion phenomena in high energy region. In our study for the proton-antiproton annihilation process, we have seen $\langle\epsilon\rangle = \sqrt{\langle\vec{k}\rangle^2 + \mu^2} \simeq 380$ Mev, where $\langle\epsilon\rangle$ is the mean energy of emitted pions. When π - N collision* or N - N collision at high energy is examined along the same line with this, we may also expect that the value of mean energy of pions will be a similar one with the $\langle\epsilon\rangle$ owing to the effect of the nucleon structure. In other words, the pions of energy $\langle\epsilon\rangle$ will be emitted with

large probability. Then the cross sections for π - N scattering will have the maxima at the energies where some number of pions of energy $\langle\epsilon\rangle$ are produced. These maxima may correspond to the ones at 0.9 Bev and 1.4 Bev which have been established by experiments.³⁾

Thus it may be expected that the pion phenomena at high energy as well as those at low energy can be explained to some extent by taking into account both the nucleon structure and the assumption (1).

The author should like to express his thanks to Dr. Y. Munakata and Dr. K. Hiida for their helpful comments.

* In π - N collision the $\langle r \rangle$ in Eq. (1) must be reinterpreted as the distance between the center of the nucleon and the place where the collision takes place.

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A Method of Approximate Second Quantization in the Theory of Superconductivity

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The Hamiltonian of a dynamical system of Fermi-particles is transformed by means of the general unitary transformation proposed by N. N. Bogolubov. From this Hamiltonian we obtain the approximate second quantization (a. s. q.) Hamiltonian, introducing the Bose-amplitudes with two indices. This Hamiltonian is diagonalized and the collective oscillations considered, especially for the pairs of particles with parallel spins. For the forced collective oscillations the paramagnetic term in the Hamiltonian is also taken into account. This term leads to the additional "spin" current, connected with the elementary excitations with spin-moment ± 1 . If the transfer momentum tends to zero, this current vanishes as the terms omitted in obtaining the Meissner-Ochsenfeld effect.

§ 1. Introduction

In the paper of N. N. Bogolubov¹⁾ using the self-consistent field method, among others, the problem of collective oscillations of a dynamical system of Fermi-particles and the problem of electrodynamics of superconducting state are studied. It is proved that in the approximation used, the collective oscillations split into two independent branches: *a)* for the elementary excitations with spin moment 0, *b)* for the elementary excitations with spin moment ± 1 . The collective oscillations of the first branch are investigated (see also the monography of the theory of superconductivity N. N. Bogolubov, V. V. Tolmachov, D. V. Shirkov²⁾), and the results are applied to the electrodynamics of superconducting state, obtaining the Meissner effect.

In this paper the problems investigated in the paper quoted above are considered with approximate second quantization method^{2),3)}. The collective oscillations of the second branch, connected with the elementary excitations with spin moment ± 1 are studied. Moreover, the additional term in the Hamiltonian which gives the energy of magnetic spin moment in a magnetic field is taken into account. This leads to the additional "spin" current.

In order to use the approximate second quantization method, the products of Fermi-amplitudes α, α_μ are replaced by the Bose-amplitudes $\beta_{\nu\mu}$. It is proved that the secular equation for the energy of collective oscillations obtained by means of this method is identical with the equation obtained in paper 1). Furthermore the

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forced collective oscillations for the case of weak external electromagnetic field are considered. In the Hamiltonian besides the terms that depend linearly on the vector-potential of the external field (see also 1)) the paramagnetic term proportional to the product $\mathbf{s} \cdot \mathbf{H}$ (where \mathbf{s} is the spin vector of the particle, and \mathbf{H} the magnetic field vector) is also taken into account. This additional term of the Hamiltonian depends linearly on the operators $\beta_{\nu\mu}$. So we introduce new operators $\beta_{\nu\mu} + C(\nu, \mu)$, where $C(\nu, \mu)$ are c -numbers. We determine them from the condition that in the transformed Hamiltonian, the terms linear in the operators $\beta_{\nu\mu}$ must vanish. It is shown that current and particle density are expressed by the functions $C(\nu, \mu)$. Thereby, from the approximate second quantization method like in the self-consistent field method¹⁾ we obtain the electrodynamics of superconducting state and especially the Meissner effect.—The paramagnetic term in the Hamiltonian leads to the "spin" current, connected with the elementary excitations with spin moment ± 1 .

§ 2. Approximate second quantization Hamiltonian

Consider a dynamical system of Fermi-particles with a Hamiltonian

$$H = \sum_{f, f'} T(f, f') a_f^\dagger a_{f'} + \frac{1}{2} \sum_{\substack{f_1, f_2 \\ f'_1, f'_2}} U(f_1, f_2; f'_1, f'_2) a_{f_1}^\dagger a_{f_2}^\dagger a_{f'_2} a_{f'_1} = H_1 + H_2, \\ T(f, f') = I(f, f') - \lambda \delta(f - f') \quad (1)$$

where I is the Hamiltonian of the particle, U —the interaction energy, λ —the chemical potential, a_f, a_f^\dagger the Fermi-amplitudes and f is a set of indices characterising one-particle states; in particular $f = (\mathbf{p}, \sigma)$, where \mathbf{p} is a wave vector and σ a spin index.

Similarly as in the paper 2) we transform the Hamiltonian changing to new amplitudes

$$a_f = \sum_{\nu} (u_{f\nu} \alpha_{\nu} + v_{f\nu} \alpha_{\nu}^\dagger). \quad (2)$$

To secure the canonical character of transformation (2) the functions $\{u, v\}$ must be connected by orthonormality relations:

$$\sum_{\nu} \{u_{f\nu} u_{f'\nu}^* + v_{f\nu} v_{f'\nu}^*\} = \delta(f - f'), \\ \sum_{\nu} \{u_{f\nu} v_{f'\nu} + u_{f'\nu} v_{f\nu}\} = 0. \quad (3)$$

We find the functions $\{u, v\}$ from the additional equations obtained from the compensation principle of dangerous graphs,¹⁾

$$\langle \alpha_{\nu_1} \alpha_{\nu_2} H \rangle_0 = 0. \quad (4)$$

The expectation value corresponds to the vacuum state C_0 in the α -representation:

$$\alpha_{\nu} C_0 = 0, \quad C_0^* \alpha_{\nu}^\dagger = 0. \quad (5)$$

In the paper 1) it has been shown that the equations (4) are equivalent to the equation

$$\sum_f \{ \zeta(f_1, f) \phi(f, f_2) + \zeta(f_2, f) \phi(f_1, f) \} + S(f_1, f_2) - \sum_f \{ F(f, f_1) S(f, f_2) + F(f, f_2) S(f_1, f) \} = 0 \quad (6)$$

where

$$\begin{aligned} \zeta(f_1, f) &= T(f_1, f) + \sum_{f', f''} \{ U(f_1, f''; f', f) - U(f_1, f''; f, f') \} F(f'', f'), \\ S(f_1, f_2) &= \sum_{f_1', f_2'} U(f_1, f_2; f_1', f_2') \phi(f_1', f_2'), \\ F(f, f') &= \sum_v v_{fv}^* v_{f'v}, \quad \phi(f_1, f_2) = \sum_v u_{f_1v} v_{f_2v}. \end{aligned} \quad (7)$$

The transformed Hamiltonian has the form

$$H = H_1 + H_2^{(A)} + H_2^{(B)} + H_2' \quad (8)$$

where

$$\begin{aligned} H_1 &= \sum_{f, f'} T(f, f') \sum_{\mu, \nu} (u_{f\mu}^* u_{f'\nu} \alpha_\mu^+ \alpha_\nu + v_{f\mu}^* v_{f'\nu} \alpha_\mu \alpha_\nu^+ + u_{f\mu}^* v_{f'\nu} \alpha_\mu^+ \alpha_\nu^+ + v_{f\mu}^* u_{f'\nu} \alpha_\mu \alpha_\nu), \\ H^{(A)} &= \frac{1}{2} \sum_{\substack{f_1, f_2 \\ f_1', f_2'}} U(f_1, f_2; f_2', f_1') \sum_{\substack{\mu, \nu \\ \rho, \delta}} \{ u_{f_1\mu}^* u_{f_2\nu}^* u_{f_2'\rho} u_{f_1'\delta} \alpha_\mu^+ \alpha_\nu^+ \alpha_\rho \alpha_\delta \\ &\quad + v_{f_1\mu}^* v_{f_2\nu}^* v_{f_2'\rho} v_{f_1'\delta} \alpha_\mu \alpha_\nu \alpha_\rho^+ \alpha_\delta^+ + u_{f_1\mu}^* v_{f_2\nu}^* v_{f_2'\rho} u_{f_1'\delta} \alpha_\mu^+ \alpha_\nu \alpha_\rho^+ \alpha_\delta \\ &\quad + v_{f_1\mu}^* u_{f_2\nu}^* u_{f_2'\rho} v_{f_1'\delta} \alpha_\mu \alpha_\nu^+ \alpha_\rho \alpha_\delta^+ + u_{f_1\mu}^* v_{f_2\nu}^* u_{f_2'\rho} v_{f_1'\delta} \alpha_\mu^+ \alpha_\nu \alpha_\rho \alpha_\delta^+ \\ &\quad + v_{f_1\mu}^* u_{f_2\nu}^* v_{f_2'\rho} u_{f_1'\delta} \alpha_\mu \alpha_\nu^+ \alpha_\rho^+ \alpha_\delta \}, \\ H_2^{(B)} &= \frac{1}{2} \sum_{\substack{f_1, f_2 \\ f_1', f_2'}} U(f_1, f_2; f_2', f_1') \sum_{\substack{\mu, \nu \\ \rho, \delta}} \{ u_{f_1\mu}^* u_{f_2\nu}^* v_{f_2'\rho} v_{f_1'\delta} \alpha_\mu^+ \alpha_\nu^+ \alpha_\rho^+ \alpha_\delta^+ \\ &\quad + v_{f_1\mu}^* v_{f_2\nu}^* u_{f_2'\rho} u_{f_1'\delta} \alpha_\mu \alpha_\nu \alpha_\rho \alpha_\delta \}, \\ H_2' &= \frac{1}{2} \sum_{\substack{f_1, f_2 \\ f_1', f_2'}} U(f_1, f_2; f_2', f_1') \sum_{\substack{\mu, \nu \\ \rho, \delta}} \{ u_{f_1\mu}^* u_{f_2\nu}^* v_{f_2'\rho} v_{f_1'\delta} \alpha_\mu^+ \alpha_\nu^+ \alpha_\rho \alpha_\delta^+ \\ &\quad + u_{f_1\mu}^* v_{f_2\nu}^* v_{f_2'\rho} v_{f_1'\delta} \alpha_\mu^+ \alpha_\nu \alpha_\rho^+ \alpha_\delta^+ + v_{f_1\mu}^* u_{f_2\nu}^* v_{f_2'\rho} v_{f_1'\delta} \alpha_\mu \alpha_\nu^+ \alpha_\rho \alpha_\delta^+ \\ &\quad + u_{f_1\mu}^* u_{f_2\nu}^* v_{f_2'\rho} u_{f_1'\delta} \alpha_\mu^+ \alpha_\nu^+ \alpha_\rho^+ \alpha_\delta + v_{f_1\mu}^* v_{f_2\nu}^* v_{f_2'\rho} u_{f_1'\delta} \alpha_\mu \alpha_\nu \alpha_\rho^+ \alpha_\delta \\ &\quad + u_{f_1\mu}^* v_{f_2\nu}^* u_{f_2'\rho} u_{f_1'\delta} \alpha_\mu^+ \alpha_\nu \alpha_\rho \alpha_\delta + v_{f_1\mu}^* u_{f_2\nu}^* u_{f_2'\rho} u_{f_1'\delta} \alpha_\mu \alpha_\nu^+ \alpha_\rho \alpha_\delta \\ &\quad + v_{f_1\mu}^* v_{f_2\nu}^* u_{f_2'\rho} v_{f_1'\delta} \alpha_\mu \alpha_\nu \alpha_\rho \alpha_\delta^+ \}. \end{aligned} \quad (9)$$

Now we want to obtain from the Hamiltonian (8) the approximate second quantization Hamiltonian (*a. s. q.* Hamiltonian). Similary as in the particular case²⁾ we make use of the results of papers of Gell-Mann, Brueckner, Sawada, Brout and Fukuda⁴⁾ This means in order to obtain the fundamental approximation, one may restrict oneself to the summation of only those graphs of the form of the complex, given in Fig. 1.



Fig. 1

All more complicated graphs are of the form represented in Fig. 2 (see also 2)).

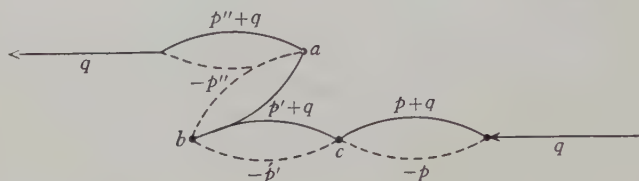


Fig. 2

These graphs contain the vertex parts of Fig. 3.

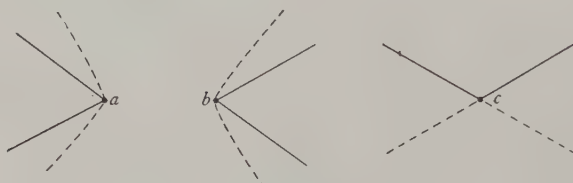


Fig. 3

In the exact Hamiltonian (8) the terms grouped in $H_2^{(A)}$ and $H_2^{(B)}$ correspond to these vertex parts. In the approximate Hamiltonian thus obtained we omit the contribution from H_2' .

The second term of $H_2^{(B)}$ corresponds to the vertex parts a) (Fig. 3), the first term of $H_2^{(B)}$ to the vertex parts b) and the terms of $H_2^{(A)}$ to the vertex parts c). To obtain the *a. s. q.* Hamiltonian we consider only $H_2^{(A)}$ and $H_2^{(B)}$. Now, instead of the products $\alpha_{v_1}\alpha_{v_2}$ of Fermi-amplitudes, describing the particle-hole complexes, we introduce Bose-amplitudes $\beta_{v_1v_2}$ ($\beta_{v_1v_2} = -\beta_{v_2v_1}$). This means that from the Hamiltonians $H_2^{(A)}$ and $H_2^{(B)}$ we must take out pairs of operators $\alpha_{v_1}\alpha_{v_2}$ and $(\alpha_{v_3}\alpha_{v_4})^+$ and substitute them by Bose-amplitudes $\beta_{v_1v_2}$ and $\beta_{v_3v_4}^+$. Also we must find in H_2 the coefficients of the products $(\alpha_{v_1}\alpha_{v_2})(\alpha_{v_3}\alpha_{v_4})$ and $(\alpha_{v_3}\alpha_{v_4}) + (\alpha_{v_1}\alpha_{v_2})$. These coefficients are given by formulae

$$\langle \alpha_{v_4}^+ \alpha_{v_3}^+ \alpha_{v_2}^+ \alpha_{v_1}^+ H_2 \rangle_0, \quad \langle \alpha_{v_3} \alpha_{v_4} H_2 \alpha_{v_2}^+ \alpha_{v_1}^+ \rangle_0.$$

So we finally get from H_2

$$\begin{aligned} \widehat{H}_2 = & \sum_{\substack{v_1, v_2 \\ v_3, v_4}} A(v_3, v_4; v_1, v_2) \beta_{v_3v_4}^+ \beta_{v_1v_2} + \frac{1}{2} \sum_{\substack{v_1, v_2 \\ v_3, v_4}} B(v_1, v_2; v_3, v_4) \beta_{v_3v_4}^+ \beta_{v_1v_2}^+ \\ & + \frac{1}{2} \sum_{\substack{v_1, v_2 \\ v_3, v_4}} B^*(v_1, v_2; v_3, v_4) \beta_{v_1v_2} \beta_{v_3v_4} \end{aligned} \quad (10)$$

where

$$\begin{aligned}
 A(\nu_3, \nu_4; \nu_1, \nu_2) &= \frac{1}{4} \langle \beta_{\nu_3 \nu_4} H_2 \beta_{\nu_1 \nu_2}^+ \rangle_0 = \frac{1}{4} \langle \alpha_{\nu_3} \alpha_{\nu_4} H_2^{(A)} \alpha_{\nu_2}^+ \alpha_{\nu_1}^+ \rangle_0 = \frac{1}{8} \sum_{\substack{f_1, f_2 \\ f_1', f_2'}} U(f_1, f_2; f_2', f_1') \\
 &\times \{ (u_{f_1 \nu_4}^* u_{f_2 \nu_3}^* - u_{f_1 \nu_3}^* u_{f_2 \nu_4}^*) (u_{f_2' \nu_1} u_{f_1' \nu_2} - u_{f_2' \nu_2} u_{f_1' \nu_1}) + (v_{f_1' \nu_3} v_{f_2' \nu_4} - v_{f_1' \nu_4} v_{f_2' \nu_3}) \\
 &\times (v_{f_1 \nu_1}^* v_{f_2 \nu_2}^* - v_{f_1 \nu_2}^* v_{f_2 \nu_1}^*) + (u_{f_2 \nu_3}^* v_{f_1 \nu_1}^* - u_{f_1 \nu_3}^* v_{f_2 \nu_1}^*) (v_{f_1' \nu_4} u_{f_2' \nu_2} - v_{f_2' \nu_4} u_{f_1' \nu_2}) \\
 &+ (u_{f_2 \nu_4}^* v_{f_1 \nu_2}^* - u_{f_1 \nu_4}^* v_{f_2 \nu_2}^*) (v_{f_1' \nu_3} u_{f_2' \nu_1} - v_{f_2' \nu_3} u_{f_1' \nu_1}) + (u_{f_2 \nu_3}^* v_{f_1 \nu_2}^* - u_{f_1 \nu_3}^* v_{f_2 \nu_2}^*) \\
 &\times (v_{f_2' \nu_4} u_{f_1' \nu_1} - v_{f_1' \nu_4} u_{f_2' \nu_1}) + (u_{f_2 \nu_4}^* v_{f_1 \nu_1}^* - u_{f_1 \nu_4}^* v_{f_2 \nu_1}^*) (v_{f_2' \nu_3} u_{f_1' \nu_2} - v_{f_1' \nu_3} u_{f_2' \nu_2}) \}, \\
 B(\nu_1, \nu_2; \nu_3, \nu_4) &= \frac{1}{4} \langle \beta_{\nu_3 \nu_4} \beta_{\nu_1 \nu_2} H_2 \rangle_0 = \frac{1}{4} \langle \alpha_{\nu_4} \alpha_{\nu_3} \alpha_{\nu_2} \alpha_{\nu_1} H_2^{(B)} \rangle_0 = \frac{1}{8} \sum_{\substack{f_1, f_2 \\ f_1', f_2'}} U(f_1, f_2; f_2', f_1') \\
 &\times \{ (u_{f_1 \nu_2}^* u_{f_2 \nu_1}^* - u_{f_1 \nu_1}^* u_{f_2 \nu_2}^*) (v_{f_2' \nu_4} v_{f_1' \nu_3} - v_{f_2' \nu_3} v_{f_1' \nu_4}) + (u_{f_1 \nu_4}^* u_{f_2 \nu_3}^* - u_{f_1 \nu_3}^* u_{f_2 \nu_4}^*) \\
 &\times (v_{f_2' \nu_2} v_{f_1' \nu_1} - v_{f_2' \nu_1} v_{f_1' \nu_2}) + (u_{f_1 \nu_3}^* u_{f_2 \nu_2}^* - u_{f_1 \nu_2}^* u_{f_2 \nu_3}^*) (v_{f_2' \nu_4} v_{f_1' \nu_1} - v_{f_2' \nu_1} v_{f_1' \nu_4}) \\
 &+ (u_{f_1 \nu_4}^* u_{f_2 \nu_1}^* - u_{f_1 \nu_1}^* u_{f_2 \nu_4}^*) (v_{f_2' \nu_3} v_{f_1' \nu_2} - v_{f_2' \nu_2} v_{f_1' \nu_3}) + (u_{f_1 \nu_1}^* u_{f_2 \nu_3}^* - u_{f_1 \nu_3}^* u_{f_2 \nu_1}^*) \\
 &\times (v_{f_2' \nu_4} v_{f_1' \nu_2} - v_{f_2' \nu_2} v_{f_1' \nu_4}) + (u_{f_1 \nu_4}^* u_{f_2 \nu_2}^* - u_{f_1 \nu_2}^* u_{f_2 \nu_4}^*) (v_{f_2' \nu_3} v_{f_1' \nu_1} - v_{f_2' \nu_1} v_{f_1' \nu_3}) \}. \quad (11)
 \end{aligned}$$

The expectation value corresponds to the vacuum state C_0 defined by (5).

The matrix elements A and B have the following properties,

$$\begin{aligned}
 A(\nu_3, \nu_4; \nu_1, \nu_2) &= A^*(\nu_1, \nu_2; \nu_3, \nu_4), \\
 A(\nu_3, \nu_4; \nu_1, \nu_2) &= A(\nu_4, \nu_3; \nu_2, \nu_1) = -A(\nu_4, \nu_3; \nu_1, \nu_2), \\
 B(\nu_1, \nu_2; \nu_3, \nu_4) &= B(\nu_3, \nu_4; \nu_1, \nu_2), \\
 B(\nu_1, \nu_2; \nu_3, \nu_4) &= B(\nu_2, \nu_1; \nu_4, \nu_3) = -B(\nu_2, \nu_1; \nu_3, \nu_4). \quad (12)
 \end{aligned}$$

(Owing to these properties and to the fact that $\beta_{\nu_1 \nu_2} = -\beta_{\nu_2 \nu_1}$, we should have written the coefficient $1/4$ in the definition of A and B).

Now in order to obtain the complete Hamiltonian H we must add the part \tilde{H}_1 —self-energy of the particle-hole complex. In order to obtain the correct energy denominators, the same as in the correct Hamiltonian, it is necessary to choose

$$\tilde{H}_1 = \frac{1}{2} \sum_{\nu_1, \nu_2} [\mathcal{Q}(\nu_1) + \mathcal{Q}(\nu_2)] \beta_{\nu_1 \nu_2}^+ \beta_{\nu_1 \nu_2} \quad (13)$$

where

$$\begin{aligned}
 \mathcal{Q}(\nu_1) &= \langle \alpha_{\nu_1} H \alpha_{\nu_1}^+ \rangle_0 = \sum_{f, f'} \zeta(f, f') (u_{f \nu_1}^* u_{f' \nu_1} - v_{f \nu_1}^* v_{f' \nu_1}) + \frac{1}{2} \sum_{\substack{f_1, f_2 \\ f_1', f_2'}} U(f_1, f_2; f_2', f_1') \\
 &\times [\phi(f_2, f_1) (u_{f_1' \nu_1}^* v_{f_2' \nu_1}^* - v_{f_1' \nu_1}^* u_{f_2' \nu_1}^*) + \phi^*(f_2, f_1) (u_{f_1 \nu_1} v_{f_2 \nu_1} - u_{f_2 \nu_1} v_{f_1 \nu_1})] + +, \\
 \zeta(f, f') &= T(f, f') + \sum_{f_1, f_1'} [U(f', f_1'; f_1, f) - U(f', f_1'; f, f_1)] F(f_1, f_1'). \quad (14)
 \end{aligned}$$

The terms neglected in (14) do not depend on ν_1 .

Thus the complete Hamiltonian in the method of approximate second quantization has the form

$$H = \frac{1}{2} \sum_{\nu_1, \nu_2} [\mathcal{Q}(\nu_1) + \mathcal{Q}(\nu_2)] \beta_{\nu_1 \nu_2}^+ \beta_{\nu_1 \nu_2} + \sum_{\substack{\nu_1, \nu_2 \\ \nu_3, \nu_4}} A(\nu_3, \nu_4; \nu_1, \nu_2) \beta_{\nu_3 \nu_4}^+ \beta_{\nu_1 \nu_2} \\ + \frac{1}{2} \sum_{\substack{\nu_1, \nu_2 \\ \nu_3, \nu_4}} \beta(\nu_3, \nu_4; \nu_1, \nu_2) \beta_{\nu_3 \nu_4}^+ \beta_{\nu_1 \nu_2}^+ + \frac{1}{2} \sum_{\substack{\nu_1, \nu_2 \\ \nu_3, \nu_4}} B^*(\nu_3, \nu_4; \nu_1, \nu_2) \beta_{\nu_3 \nu_4} \beta_{\nu_1 \nu_2}. \quad (15)$$

This Hamiltonian is a quadratic form in the Bose-operators β .

§ 3. The diagonalization of H and the collective oscillations

In the monography of 3) it was proved that the diagonalization of a quadratic form of the type (15) may be reduced to solving a system of linear homogeneous equations with respect to the c -number quantities $\hat{\xi}(\nu_1, \nu_2)$, $\eta(\nu_1, \nu_2)$

$$E \hat{\xi}(\nu_1, \nu_2) = [\mathcal{Q}(\nu_1) + \mathcal{Q}(\nu_2)] \hat{\xi}(\nu_1, \nu_2) + 2 \sum_{\nu_3, \nu_4} A(\nu_1, \nu_2; \nu_3, \nu_4) \hat{\xi}(\nu_3, \nu_4) \\ + 2 \sum_{\nu_3, \nu_4} B(\nu_1, \nu_2; \nu_3, \nu_4) \eta(\nu_3, \nu_4), \\ -E \eta(\nu_1, \nu_2) = [\mathcal{Q}(\nu_1) + \mathcal{Q}(\nu_2)] \eta(\nu_1, \nu_2) + 2 \sum_{\nu_3, \nu_4} A^*(\nu_1, \nu_2; \nu_3, \nu_4) \eta(\nu_3, \nu_4) \\ + 2 \sum_{\nu_3, \nu_4} B^*(\nu_1, \nu_2; \nu_3, \nu_4) \hat{\xi}(\nu_3, \nu_4), \quad (16)$$

with the normalization condition

$$\sum_{\nu_1, \nu_2} \{ |\hat{\xi}(\nu_1, \nu_2)|^2 - |\eta(\nu_1, \nu_2)|^2 \} = 1. \quad (17)$$

The homogeneous equations (16) lead to the secular equation for determining E . When E_n are the roots of this equation and $\hat{\xi}_n(\nu_1, \nu_2)$, $\eta_n(\nu_1, \nu_2)$ the corresponding functions, we can perform the diagonalization of Hamiltonian (15) changing to new Bose-amplitudes b_n , b_n^+ by means of the canonical transformation

$$\beta(\nu_1, \nu_2) = \sum_n \{ \hat{\xi}_n(\nu_1, \nu_2) b_n + \eta_n(\nu_1, \nu_2) b_n^+ \}. \quad (18)$$

We obtain the Hamiltonian (15) in the form

$$\tilde{H} = - \sum_n E_n \sum_{\nu_1, \nu_2} \eta^*(\nu_1, \nu_2) \eta(\nu_1, \nu_2) + \sum_n E_n b_n^+ b_n. \quad (19)$$

The functions $\hat{\xi}_n$, η_n must obey the following normalization condition,

$$\sum_n \{ |\hat{\xi}_n|^2 - |\eta_n|^2 \} = 1. \quad (20)$$

When in (16) and (11) we denote the coefficients $2A=X$, $2B=-Y$, formulae (16) and (11) are identical with the formulae obtained in paper 1) with the self-consistent field method. The normalization condition for functions $\hat{\xi}$, η gives the positive sign of E . This is the advantage of the method of approximate second quantization over the self-consistent field method.

From equations (16) we obtain

$$E(\hat{\xi}(\nu_1, \nu_2) - \eta(\nu_1, \nu_2)) = [\mathcal{Q}(\nu_1) + \mathcal{Q}(\nu_2)](\hat{\xi}(\nu_1, \nu_2) + \eta(\nu_1, \nu_2)) \\ + 2 \sum_{\nu_3, \nu_4} [A(\nu_1, \nu_2; \nu_3, \nu_4) + B^*(\nu_1, \nu_2; \nu_3, \nu_4)] \hat{\xi}(\nu_3, \nu_4) \\ + 2 \sum_{\nu_3, \nu_4} [A^*(\nu_1, \nu_2; \nu_3, \nu_4) + B(\nu_1, \nu_2; \nu_3, \nu_4)] \eta(\nu_3, \nu_4), \quad (21)$$

$$\begin{aligned}
 E(\xi(\nu_1, \nu_2) + \eta(\nu_1, \nu_2)) &= [\mathcal{Q}(\nu_1) + \mathcal{Q}(\nu_2)](\xi(\nu_1, \nu_2) - \eta(\nu_1, \nu_2)) \\
 &+ 2 \sum_{\nu_3, \nu_4} [A(\nu_1, \nu_2; \nu_3, \nu_4) - B^*(\nu_1, \nu_2; \nu_3, \nu_4)] \xi(\nu_3, \nu_4) \\
 &- 2 \sum_{\nu_3, \nu_4} [A^*(\nu_1, \nu_2; \nu_3, \nu_4) - B(\nu_1, \nu_2; \nu_3, \nu_4)] \eta(\nu_3, \nu_4).
 \end{aligned}$$

We assume that the matrices A and B are real. For the types of interactions investigated, this assumption leads to the requirement, that the functions $\{u, v\}$ must be real, which does not violate the conditions (3). After these assumptions we can introduce new functions

$$\vartheta(\nu_1, \nu_2) = \xi(\nu_1, \nu_2) + \eta(\nu_1, \nu_2), \quad \theta(\nu_1, \nu_2) = \xi(\nu_1, \nu_2) - \eta(\nu_1, \nu_2). \quad (22)$$

For the functions θ, ϑ we obtain the equations

$$\begin{aligned}
 E\theta(\nu_1, \nu_2) &= [\mathcal{Q}(\nu_1) + \mathcal{Q}(\nu_2)]\vartheta(\nu_1, \nu_2) \\
 &+ 2 \sum_{\nu_3, \nu_4} [A(\nu_1, \nu_2; \nu_3, \nu_4) + B(\nu_1, \nu_2; \nu_3, \nu_4)]\vartheta(\nu_3, \nu_4), \\
 E\vartheta(\nu_1, \nu_2) &= [\mathcal{Q}(\nu_1) + \mathcal{Q}(\nu_2)]\theta(\nu_1, \nu_2) \\
 &+ 2 \sum_{\nu_3, \nu_4} [A(\nu_1, \nu_2; \nu_3, \nu_4) - B(\nu_1, \nu_2; \nu_3, \nu_4)]\theta(\nu_3, \nu_4).
 \end{aligned} \quad (23)$$

These equations are more convenient than equations (16), because the right-hand side depends either on functions ϑ or on functions θ . Equations of this form are obtained in a form more general than in paper 1).

Let us consider equations (23) in the case of a simple form of the transformation (2), used in the theory of superconductivity^(1),2). One may determine the corresponding functions $\{u, v\}$, putting

$$\begin{aligned}
 u_{f\nu} &= u(p)\delta(\nu - f), \quad v_{f\nu} = v(p, \sigma)\delta(\nu + f), \\
 v(p, +) &= v(p), \quad v(p, -) = -v(p).
 \end{aligned} \quad (24)$$

Moreover, we take

$$I(f, f') = E(p)\delta(f - f')$$

$$U(f_1, f_2; f_2', f_1') = \frac{1}{V} J(p_1, p_2; p_2', p_1') \delta(p_1 + p_2 - p_1' - p_2') \delta(\sigma_1 - \sigma_1') \delta(\sigma_2 - \sigma_2'). \quad (25)$$

As one can see, the nonvanishing matrix elements of (15) are of the form

$$\begin{aligned}
 A_{-+, -+}(p_1, p_2; p_1', p_2') &= A_{-+, +-}(\dots), \quad A_{++, ++}(\dots) = A_{--, --}(\dots), \\
 B_{-+, -+}(p_1, p_2; p_1', p_2') &= B_{-+, +-}(\dots), \quad B_{++, --}(\dots) = B_{--, ++}(\dots).
 \end{aligned} \quad (26)$$

The spectrum of collective oscillations is divided also into two branches. For the first one the oscillations take place for pairs of particles having opposite spins (elementary excitations with spin 0), for the second one for pairs of particles having parallel spins (elementary excitations with spin ± 1). Let us note that the compensation equations (6), according to (24) and (25), go over into the equations

of compensation for pairs of particles having opposite spins. These compensation equations are used for determining the lowest superconducting energy state in the theory of superconductivity^{2),1)}. The collective oscillations of the first branch are investigated in paper 1). Now let us consider the spectrum of elementary excitations for the second branch. From (11), (24) and (25) we get

$$\begin{aligned} E_{\xi_{++}}^{\tilde{\xi}}(p_1, p_2) &= [\mathcal{Q}(p_1) + \mathcal{Q}(p_2)] \xi_{++}(p_1, p_2) \\ &+ 2 \sum_{p_1', p_2'} A_{++,++}(p_1, p_2; p_1', p_2') \xi_{++}(p_1', p_2') \\ &- 2 \sum_{p_1', p_2'} B_{++,--}(p_1, p_2; p_1', p_2') \eta_{--}(-p_2', -p_1'), \end{aligned} \quad (27a)$$

$$\begin{aligned} -E\eta_{++}(-p_2, -p_1) &= [\mathcal{Q}(p_1) + \mathcal{Q}(p_2)] \eta_{++}(-p_2, -p_1) \\ &+ 2 \sum_{p_1', p_2'} A_{++,++}(p_1, p_2; p_1', p_2') \eta_{++}(-p_2', -p_1') \\ &- 2 \sum_{p_1', p_2'} B_{++,--}(p_1, p_2; p_1', p_2') \xi_{--}(p_1', p_2'), \end{aligned} \quad (27b)$$

$$\begin{aligned} E_{\xi_{--}}^{\tilde{\xi}}(p_1, p_2) &= [\mathcal{Q}(p_1) + \mathcal{Q}(p_2)] \xi_{--}(p_1, p_2) \\ &+ 2 \sum_{p_1', p_2'} A_{--,--}(p_1, p_2; p_1', p_2') \xi_{--}(p_1', p_2') + \\ &- 2 \sum_{p_1', p_2'} B_{--,++}(p_1, p_2; p_1', p_2') \eta_{++}(-p_2', -p_1') \end{aligned} \quad (28a)$$

$$\begin{aligned} -E\eta_{--}(-p_2, -p_1) &= [\mathcal{Q}(p_1) + \mathcal{Q}(p_2)] \eta_{--}(-p_2, -p_1) \\ &+ 2 \sum_{p_1', p_2'} A_{--,--}(p_1, p_2; p_1', p_2') \eta_{--}(-p_2', -p_1') \\ &- 2 \sum_{p_1', p_2'} B_{--,++}(p_1, p_2; p_1', p_2') \xi_{++}(p_1', p_2') \end{aligned} \quad (28b)$$

where

$$\begin{aligned} A_{++,++}(p_1, p_2; p_1', p_2') &= A_{--,--}(\dots) = A(\dots) \\ &= \frac{1}{4V} \delta(p_1 + p_2 - p_1' - p_2') \{ [J(p_2, p_1; p_1', p_2') - J(p_1, p_2; p_1', p_2')] \\ &\times [u(p_1)u(p_2)u(p_1')u(p_2') + v(p_1)v(p_2)v(p_1')v(p_2')] \\ &+ J(p_2, -p_2'; -p_1, p_1') [v(p_1)u(p_2)u(p_1')v(p_2') + u(p_1)v(p_2)v(p_1')u(p_2')] \\ &- J(p_2, -p_1'; -p_1, p_2') [v(p_1)u(p_2)v(p_1')u(p_2') + u(p_1)v(p_2)u(p_1')v(p_2')] \}, \\ B_{++,--}(p_1, p_2; p_1', p_2') &= B_{--,++}(\dots) = B(\dots) \\ &= \frac{1}{4V} \delta(p_1 + p_2 - p_1' - p_2') \{ [J(p_1, p_2; p_1', p_2') - J(p_2, p_1; p_1', p_2')] \\ &\times [u(p_1)u(p_2)v(p_1')v(p_2') + v(p_1)v(p_2)u(p_1')u(p_2')] \\ &+ J(p_2, -p_2'; -p_1, p_1') [v(p_1)u(p_2)v(p_1')u(p_2') + u(p_1)v(p_2)u(p_1')v(p_2')] \\ &- J(p_2, -p_1'; -p_1, p_2') [v(p_1)u(p_2)u(p_1')v(p_2') + u(p_1)v(p_2)v(p_1')u(p_2')] \}. \end{aligned} \quad (29)$$

$\Omega(p)$ is the same as in 1)

$$\Omega(p) = \sqrt{\zeta^2(p) + C^2(p)},$$

$$\zeta(p) = E(p) - \lambda + \frac{1}{V} \sum_{p'} \{2J(p', p; p, p') - J(p, p'; p, p')\} v^2(p'). \quad (30)$$

The function $C(p)$ satisfies the equation

$$C(p) + \frac{1}{V} \sum_{p'} J(p, -p; -p', p') \frac{C(p')}{2\Omega(p')} = 0.$$

For a change of the spin direction from (+) to (-), equations (27) go into equations (28) and vice versa. These equations do not separate into equations for functions $\hat{\xi}_{++}$, γ_{++} and for functions $\hat{\xi}_{--}$, γ_{--} . Moreover, these equations connect only the functions with fixed $\mathbf{p}_1 + \mathbf{p}_2$. Therefore we may put

$$p_1 = p, \quad p_2 = -p + q, \quad p'_1 = p', \quad p'_2 = -p' + q.$$

Then we obtain (29) in the form

$$\begin{aligned} & A(p, -p+q; p', -p'+q) \\ &= \frac{1}{4V} \{ [J(-p+q, p; p', -p'+q) - J(p, -p+q; p', -p'+q)] \\ & \times [u(p)u(p-q)u(p')u(p'-q) + v(p)v(p-q)v(p')v(p'-q)] \\ & + J(-p+q, p'-q; -p, p') \\ & \times [v(p)u(p-q)u(p')v(p'-q) + u(p)v(p-q)v(p')u(p'-q)] \\ & - J(-p+q, -p'; -p, -p'+q) \\ & \times [v(p)u(p-q)v(p')u(p'-q) + u(p)v(p-q)u(p')v(p'-q)] \}, \quad (31) \end{aligned}$$

$$\begin{aligned} & B(p, -p+q; p', -p'+q) \\ &= \frac{1}{4V} \{ [J(p, -p+q; p', -p'+q) - J(-p+q, p; p', -p'+q)] \\ & \times [u(p)u(p-q)v(p')v(p'-q) + v(p)v(p-q)u(p')u(p'-q)] \\ & + J(-p+q, p'-q; -p, p') \\ & \times [v(p)u(p-q)v(p')u(p'-q) + u(p)v(p-q)u(p')v(p'-q)] \\ & - J(-p+q, -p'; -p, -p'+q) \\ & \times [v(p)u(p-q)u(p')v(p'-q) + u(p)v(p-q)v(p')u(p'-q)] \}. \end{aligned}$$

The solution of equations (27) and (28) leads to the diagonalization of the Hamiltonian

$$\begin{aligned} \tilde{H} = & \frac{1}{2} \sum_{p_1, p_2} [\Omega(p_1) + \Omega(p_2)] \beta_{++}^+(p_1, p_2) \beta_{++}(p_1, p_2) \\ & + \sum_{\substack{p_1, p_2 \\ p'_1, p'_2}} A(p_1, p_2; p'_1, p'_2) \beta_{++}^+(p_1, p_2) \beta_{++}(p'_1, p'_2) \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \sum_{p_1, p_2} [\Omega(p_1) + \Omega(p_2)] \beta_{--}^+(p_1, p_2) \beta_{--}(p_1, p_2) \\
& + \sum_{\substack{p_1, p_2 \\ p_1', p_2'}} A(p_1, p_2; p_1', p_2') \beta_{--}^+(p_1, p_2) \beta_{--}(p_1', p_2') \\
& + \sum_{\substack{p_1, p_2 \\ p_1', p_2'}} B(p_1, p_2; p_1', p_2') [\beta_{++}^+(p_1, p_2) \beta_{--}^+(-p_1', -p_2') \\
& + \beta_{++}(p_1, p_2) \beta_{--}(-p_1', -p_2')]. \tag{32}
\end{aligned}$$

$\tilde{\text{H}}$ amiltonian (32) describes the elementary excitations with spin moment ± 1 which interact with each other. Therefore equations (28 a, b) do not split into the equations for the two independent branches. If we group equation (27 b) with (28 a) and (27 a) with (28 b) we obtain two independent systems of equations, but for the functions with sign (+) and (-) simultaneously. Then we define new functions:

$$\begin{aligned}
\tilde{\xi}_{--}(p_1, p_2) - \eta_{++}(-p_2, -p_1) &= \theta_q^{(+)}(p) = \theta_q(p), \\
\tilde{\xi}_{--}(p_1, p_2) + \eta_{++}(-p_2, -p_1) &= \vartheta_q^{(+)}(p) = \vartheta_q(p), \quad (p_1 + p_2 = p_1' + p_2' = q). \tag{33}
\end{aligned}$$

For the functions θ_q and ϑ_q we obtain equations in the some convenient form as (23)

$$\begin{aligned}
E\theta_q(p) &= [\Omega(p) + \Omega(p-q)] \vartheta_q(p) \\
&+ 2 \sum_{p'} [A(p, -p+q; p', -p'+q) + B(p, -p+q; p', -p'+q)] \vartheta_q(p'), \\
E\vartheta_q(p) &= [\Omega(p) + \Omega(p-q)] \theta_q(p) \\
&+ 2 \sum_{p'} [A(p, -p+q; p', -p'+q) - B(p, -p+q; p', -p'+q)] \theta_q(p'). \tag{34}
\end{aligned}$$

We note that the functions $\tilde{\xi}$, η and consequently the functions θ , ϑ are odd functions of p_1, p_2 variables, this means for $q=0$

$$\theta_0(\mathbf{p}) = \theta(\mathbf{p}) = -\theta(-\mathbf{p}), \quad \vartheta_0(\mathbf{p}) = \vartheta(\mathbf{p}) = -\vartheta(-\mathbf{p}). \tag{35}$$

We consider equation (34) for $q=0$. We get from (31)

$$\begin{aligned}
A(p, -p; p', -p') &= \frac{1}{4V} \{ [J(-p, p; p', -p') - J(p, -p; p', -p')] \\
&\times [u^2(p)u^2(p') + v^2(p)v^2(p')] \\
&+ [J(-p, p'; -p, p') - J(p, p'; p, p')] v(p)u(p)v(p')u(p') \}, \\
B(p, -p; p', -p') &= \frac{1}{4V} \{ [J(p, -p; p', -p') - J(-p, p; p', -p')] \\
&\times [u^2(p)v^2(p') + v^2(p)u^2(p')] \\
&+ [J(-p, p'; -p, p') - J(p, p'; p, p')] v(p)u(p)v(p')u(p') \}. \tag{36}
\end{aligned}$$

If we restrict ourselves to the case in which the interaction may be replaced by a constant inside a thin layer in the neighbourhood of the Fermi-sphere ($E_F \pm \omega$), and by zero outside it, we get

$$A=B=0. \quad (37)$$

(Hamiltonian (32) is obtained also in paper 5) where it is stated that in the case of interaction which is constant in a thin layer, we do not obtain the collective oscillations with reversed spin. Then the temperature dependence of paramagnetic susceptibility obtained in paper 6) is not altered and remains inconsistent with Reif's experiment⁷⁾).

For small q from (27), using (37), we get

$$\begin{aligned} E\theta_q(p) &= [\Omega(p) + \Omega(p-q)]\vartheta_q(p), \\ E\vartheta_q(p) &= [\Omega(p) + \Omega(p-q)]\theta_q(p). \end{aligned} \quad (38)$$

To show that $E > 0$, as solutions of (38), we take the antisymmetrical functions

$$\begin{aligned} \theta_q(p) &= S[\delta(p-p_0) - \delta(p-q+p_0)], \\ \vartheta_q(p) &= S[\delta(p-p_0) - \delta(p-q+p_0)]. \end{aligned} \quad (39)$$

Thus we get the continuous spectrum

$$E = \Omega(p_0) + \Omega(p_0 - q) \quad (40)$$

separated by a gap. With the given q the energy E depends continuously on the momentum p_0 . Hence the simple model of interaction of particles leads to elementary excitations which have a continuous energy spectrum. We next consider the case in which the interaction J is effective only in a narrow layer near the Fermi-surface ($E_F \pm \omega$) and has the form

$$J(p_1, p_2; p'_2, p'_1) = -g^2(p_1 - p'_1), \quad p_1 + p_2 = p'_1 + p'_2. \quad (41)$$

After introducing (41) into (36) and taking (35) into account we obtain

$$\begin{aligned} E\vartheta(\mathbf{p}) &= 2\Omega(p)\theta(\mathbf{p}) - \frac{1}{V} \sum_{\mathbf{p}'} g^2(|\mathbf{p}' - \mathbf{p}|)\theta(\mathbf{p}'), \\ E\theta(\mathbf{p}) &= 2\Omega(p)\vartheta(\mathbf{p}) + \frac{1}{V} \sum_{\mathbf{p}'} g^2(|\mathbf{p}' - \mathbf{p}|) \frac{C(p)C(p') - \zeta(p)\zeta(p')}{\Omega(p)\Omega(p')} \vartheta(\mathbf{p}'). \end{aligned} \quad (42)$$

These equations are identical with the equations obtained in 1) and 2) by considering the collective oscillations of the first branch. But in our case the solution must be the odd function of \mathbf{p} whereas in the former case, following the compensation relations, it had to be the even function.

We change to spherical variables taking the direction of the \mathbf{p} vector for the spherical axis. In this case

$$\begin{aligned} g^2(|\mathbf{p}' - \mathbf{p}|) &\cong g^2(p, p', p_F \sqrt{2(1 - \cos \alpha)}), \\ \vartheta(\mathbf{p}) &= \vartheta(p, \cos \alpha), \quad \theta(\mathbf{p}) = \theta(p, \cos \alpha), \end{aligned} \quad (43)$$

where α is the angle between the vectors \mathbf{p} and \mathbf{p}' . If the angle α takes on the value π , the functions ϑ and θ change the sign. Then in the expansions of the functions ϑ and θ we have only the terms with odd spherical harmonical functions:

$$\begin{aligned}\theta &= \sum_{n/0}^{\infty} a_{2n+1}(p) P_{2n+1}(\cos \alpha), \quad \vartheta = \sum_{n/0}^{\infty} b_{2n+1}(p) P_{2n+1}(\cos \alpha), \\ g^2 &= \sum_{n/0}^{\infty} g_n(p, p') P_n(\cos \alpha).\end{aligned}\quad (44)$$

Putting (44) in (42) we obtain the equations for the separate terms of the expansions. We consider the equations for the first term only, writing them in the integral form

$$\begin{aligned}Eb_1(\zeta) &= 2\Omega(\zeta) a_1(\zeta) - \frac{1}{3} \frac{dn}{dE} \int_{-\infty}^{+\infty} g_1(\zeta, \zeta') a_1(\zeta') d\zeta', \\ Ea_1(\zeta) &= 2\Omega(\zeta) b_1(\zeta) + \frac{1}{3} \frac{dn}{dE} \int_{-\infty}^{+\infty} \frac{C(\zeta)C(\zeta') - \zeta \zeta'}{\Omega(\zeta)\Omega(\zeta')} g_1(\zeta, \zeta') b_1(\zeta') d\zeta'\end{aligned}\quad (45)$$

where

$$\frac{dn}{dE} = \frac{1}{2\pi^2} \frac{p_F^2}{E'(p_F)}.$$

At first we consider the collective oscillations in the superconducting state when C is a constant different from zero in a narrow layer near the Fermi-surface. We define

$$\begin{aligned}C_1(\zeta) &= \frac{1}{3} \frac{dn}{dE} \int_{-\infty}^{+\infty} g_1(\zeta, \zeta') a_1(\zeta') d\zeta', \\ C_2(\zeta) &= \frac{1}{3} \frac{dn}{dE} \int_{-\infty}^{+\infty} \frac{\zeta' g_1(\zeta, \zeta')}{\Omega(\zeta')} b_1(\zeta') d\zeta', \\ C_3(\zeta) &= \frac{1}{3} \frac{dn}{dE} C \int_{-\infty}^{+\infty} \frac{g_1(\zeta, \zeta')}{\Omega(\zeta')} b_1(\zeta') d\zeta'.\end{aligned}\quad (46)$$

From (45) and (46) we obtain

$$\begin{aligned}a_1(\zeta) &= \frac{1}{4\Omega^2(\zeta) - E^2} \left[2\Omega(\zeta) C_1(\zeta) - \frac{CE}{\Omega(\zeta)} C_3(\zeta) + \frac{\zeta}{\Omega(\zeta)} EC_2(\zeta) \right], \\ b_1(\zeta) &= \frac{1}{4\Omega^2(\zeta) - E^2} [EC_1(\zeta) - 2CC_3(\zeta) + 2\zeta C_2(\zeta)].\end{aligned}\quad (47)$$

The first and second term on the right-hand side of (47) are even functions of ζ , the third term an odd function. Then we get the following dependence:

$$EC_3(\zeta) = 2CC_1(\zeta) - C \frac{1}{3} \frac{dn}{dE} \int_{-\infty}^{+\infty} \frac{g_1(\zeta, \zeta')}{\Omega(\zeta')} C_1(\zeta') d\zeta'$$

and the following equations for the determination of E :

$$C_1(\zeta) = \frac{1}{3} \frac{dn}{dE} \int_{-\infty}^{+\infty} \frac{2g_1(\zeta, \zeta') \zeta'^2}{\Omega(\zeta') [4\Omega^2(\zeta') - E^2]} C_1(\zeta') d\zeta' + C^2 \left(\frac{1}{3} \frac{dn}{dE} \right)^2 \int_{-\infty}^{+\infty} \frac{g_1(\zeta, \zeta')}{\Omega(\zeta') [4\Omega^2(\zeta') - E^2]} \left[\int_{-\infty}^{+\infty} \frac{g_1(\zeta', \zeta'')}{\Omega(\zeta'')} C_1(\zeta'') d\zeta'' \right] d\zeta', \quad (48)$$

$$C_2(\zeta) = \frac{1}{3} \frac{dn}{dE} \int_{-\infty}^{+\infty} \frac{2g_1(\zeta, \zeta') \zeta'^2}{\Omega(\zeta') [4\Omega^2(\zeta') - E^2]} C_2(\zeta') d\zeta'.$$

For $g_1(\zeta, \zeta') \cong g_1(0, 0) = g_1(p_F)$ equations (48) lead to the expressions

$$\left\{ \tilde{\rho} \ln \frac{2\omega}{c} - \tilde{\rho} \frac{\sqrt{1-\varepsilon^2}}{\varepsilon} \operatorname{arctg} \frac{\varepsilon}{\sqrt{1-\varepsilon^2}} - 1 \right\} = 0, \quad (49)$$

$$\left\{ \left(\tilde{\rho} \ln \frac{2\omega}{c} - 1 + \varepsilon^2 \right) \left(\frac{\tilde{\rho}}{\varepsilon \sqrt{1-\varepsilon^2}} \operatorname{arctg} \frac{\varepsilon}{\sqrt{1-\varepsilon^2}} + 1 \right) - \varepsilon^2 \right\} = 0$$

where

$$\tilde{\rho} = \frac{1}{3} g_1(p_F) \frac{dn}{dE}, \quad \varepsilon = \frac{E}{2C}.$$

From (49) we see that the stability conditions for the collective excitations of the second branch in the superconducting state are the same as the stability conditions for the first branch for transversed waves, obtained in 2). Eq. (49) has a single root, for $\tilde{\rho}$ in the interval

$$-1 < \tilde{\rho} < \frac{1}{\ln 2\omega/c}. \quad (50)$$

That is, the interaction (41), different from the interaction formerly considered, leads to the spin-wave like elementary excitations. These excitations are stable if the interaction is not too strong.

Let us now consider the collective excitations in the normal state. Putting $C=0$ in (45), we finally obtain

$$C_1(\zeta) = \frac{1}{3} \frac{dn}{dE} \int_0^{\infty} g_1(\zeta, \zeta') \frac{4\zeta'}{4\zeta'^2 - E^2} C_1(\zeta') d\zeta' \quad (51)$$

which leads to the asymptotic formula

$$-E^2 \sim e^{-2/g_1}$$

which means that the normal state is unstable.

§ 4. *A. s. q.* Hamiltonian in the case of weak external fields

Let us consider the dynamical system of Fermi-particles under the action of a weak external fields. We assume that this gives rise only to a variation $I(f, f')$. Thus instead of Hamiltonian (1) we have

$$H + \delta H = H + \sum_{f, f'} \delta I(f, f') a_f^* a_{f'}. \quad (52)$$

Applying the transformation (2) to the additional term in (52) and—in the same way as before—to the Bose-amplitudes $\beta_{\nu\mu}$, we get in the approximate Hamiltonian (15) the additional term $\delta \tilde{H}$

$$\begin{aligned} \tilde{H}' = \tilde{H} + \delta \tilde{H} = \tilde{H} + \frac{1}{2} \sum_{f, f'} \delta I(f, f') \sum_{\nu_1, \nu_2} (u_{f\nu_2}^* v_{f'\nu_1} - u_{f\nu_1}^* v_{f'\nu_2}) \beta_{\nu_1 \nu_2}^+ \\ + \frac{1}{2} \sum_{f, f'} \delta I^*(f, f') \sum_{\nu_1, \nu_2} (u_{f\nu_2} v_{f'\nu_1}^* - u_{f\nu_1} v_{f'\nu_2}^*) \beta_{\nu_1 \nu_2}. \end{aligned} \quad (53)$$

It is seen that the *a. s. q.* Hamiltonian now contains terms linear in the operators $\beta_{\nu\mu}$, which leads to the forced collective oscillations. In the Hamiltonian (15) we do not have terms linear in the operators $\beta_{\nu\mu}$ due to the compensation relation (4).

In order to remove the terms linear in $\beta_{\nu\mu}$ in the Hamiltonian (53) we can use the method of translation of the Bose-amplitudes,

$$\beta_{\nu\mu} \rightarrow \beta_{\nu\mu} + C(\nu, \mu), \quad \beta_{\nu\mu}^+ \rightarrow \beta_{\nu\mu}^+ + C^*(\nu, \mu), \quad (54)$$

where C and C^* are *c*-numbers. They are to be determined from the condition of the vanishing of the linear forms

$$\partial \tilde{H}' / \partial \beta_{\nu\mu}^+ = 0, \quad \partial \tilde{H}' / \partial \beta_{\nu\mu} = 0. \quad (55)$$

We now write the Hamiltonian (53) when the transformation (2) is defined by the formulae (24) and the interaction may be replaced by a constant. In the approximation used, we have the sum of three terms, according to the spin of quasiparticles:

$$\tilde{H}' = \tilde{H}'(-+) + \tilde{H}'(++) + \tilde{H}'(--). \quad (56)$$

We write the perturbation terms caused by the external weak fields in the form

$$-\frac{e}{2m} \{(\mathbf{p}\mathbf{A}(r)) + (\mathbf{A}(r)\mathbf{p})\} - \frac{e}{2m} \boldsymbol{\sigma} \delta \mathfrak{H}(r) \quad (57)$$

where $+e$, m are charge and mass of electron, $\delta \mathfrak{H}$ the magnetic field vector, \mathbf{A} the vector-potential of this field, \mathbf{p} momentum operator of particle, $\boldsymbol{\sigma}$ spin vector, the components of which are the Pauli matrices.

By means of the functions

$$\phi_i(r, s_z) = \frac{1}{\sqrt{V}} \sum_{p, \sigma} a_{p\sigma} e^{ipr} S_\sigma(s_z) \quad (58)$$

and using the formulae

$$\sigma_x S_\alpha = S_{-\alpha}, \quad \sigma_y S_\alpha = i S_{-\alpha} \operatorname{sgn} \alpha, \quad \sigma_z S_\alpha = S_\alpha \operatorname{sgn} \alpha, \quad (59)$$

we change the Hamiltonian (57) to the second quantized form as follows,

$$\delta H_A = -\frac{e}{2m} \sum_{p_1+p_2=q} (\mathbf{p}_2 - \mathbf{p}_1) \mathbf{A}(\mathbf{p}_1 + \mathbf{p}_2) [a_{-p_1+}^\dagger a_{p_2+} + a_{-p_1-}^\dagger a_{p_2-}], \quad (60a)$$

$$\begin{aligned} \delta H_{\mathfrak{S}} = & -\frac{e}{2m} \sum_{p_1+p_2=q} \{ [\delta \mathfrak{S}_x(\mathbf{p}_1 + \mathbf{p}_2) - i \delta \mathfrak{S}_y(\mathbf{p}_1 + \mathbf{p}_2)] a_{-p_1+}^\dagger a_{p_2-} \\ & + [\delta \mathfrak{S}_x(\mathbf{p}_1 + \mathbf{p}_2) + i \delta \mathfrak{S}_y(\mathbf{p}_1 + \mathbf{p}_2)] a_{-p_1-}^\dagger a_{p_2+} + \delta \mathfrak{S}_z(\mathbf{p}_1 + \mathbf{p}_2) [a_{-p_1+}^\dagger a_{p_2+} - a_{-p_1-}^\dagger a_{p_2-}] \}. \end{aligned} \quad (60b)$$

(The Hamiltonian (60b) for $\delta \mathfrak{S} = \delta \mathfrak{S}_x$ has been used in paper 6) in order to obtain the temperature dependence of paramagnetic susceptibility in the B. C. S. theory⁸⁾.)

Finally we get in the approximate Hamiltonian (56) the following terms linear in the β operators:

$$\begin{aligned} \delta \tilde{H}_A(-+) &= -\frac{e}{2m} \sum_{p_1+p_2=q} (\mathbf{p}_2 - \mathbf{p}_1) \mathbf{A}(\mathbf{p}_1 + \mathbf{p}_2) [v(p_1)u(p_2) - v(p_2)u(p_1)] \\ &\quad \times [\beta_{-+}(p_1, p_2) - \beta_{-+}^\dagger(-p_2, -p_1)], \\ \delta \tilde{H}_{\mathfrak{S}}(-+) &= -\frac{e}{2m} \sum_{p_1+p_2=q} \delta \mathfrak{S}_x(\mathbf{p}_1 + \mathbf{p}_2) [v(p_1)u(p_2) - v(p_2)u(p_1)] \\ &\quad \times [\beta_{-+}(p_1, p_2) - \beta_{-+}^\dagger(-p_2, -p_1)], \\ \delta \tilde{H}(++) &= -\frac{e}{4m} \sum_{p_1+p_2=q} [v(p_2)u(p_1) - v(p_1)u(p_2)] \\ &\quad \times \{ [\delta \mathfrak{S}_x(\mathbf{p}_1 + \mathbf{p}_2) + i \delta \mathfrak{S}_y(\mathbf{p}_1 + \mathbf{p}_2)] \beta_{++}(p_1, p_2) \\ &\quad - [\delta \mathfrak{S}_x(\mathbf{p}_1 + \mathbf{p}_2) - i \delta \mathfrak{S}_y(\mathbf{p}_1 + \mathbf{p}_2)] \beta_{++}^\dagger(-p_2, -p_1) \}, \\ \delta \tilde{H}(-) &= -\frac{e}{4m} \sum_{p_1+p_2=q} [v(p_2)u(p_1) - v(p_1)u(p_2)] \\ &\quad \times \{ [\delta \mathfrak{S}_x(\mathbf{p}_1 + \mathbf{p}_2) - i \delta \mathfrak{S}_y(\mathbf{p}_1 + \mathbf{p}_2)] \beta_{--}(p_1, p_2) \\ &\quad - [\delta \mathfrak{S}_x(\mathbf{p}_1 + \mathbf{p}_2) + i \delta \mathfrak{S}_y(\mathbf{p}_1 + \mathbf{p}_2)] \beta_{--}^\dagger(-p_2, -p_1) \}. \end{aligned} \quad (61)$$

Thus the Hamiltonian (56) is the sum of the Hamiltonians

$$\begin{aligned} \tilde{H}'(-+) &= \sum_{p_1, p_2} [\mathcal{Q}(p_1) + \mathcal{Q}(p_2)] \beta_{-+}^\dagger(p_1, p_2) \beta_{-+}(p_1, p_2) \\ &\quad + \sum_{\substack{p_1, p_2 \\ p_1', p_2'}} A_{-+}(p_1, p_2; p_1', p_2') \beta_{-+}^\dagger(p_1, p_2) \beta_{-+}(p_1, p_2) \\ &\quad + \sum_{\substack{p_1, p_2 \\ p_1', p_2'}} B_{-+}(p_1, p_2; p_1', p_2') [\beta_{-+}^\dagger(p_1, p_2) \beta_{-+}^\dagger(-p_2', -p_1')] \\ &\quad + \beta_{-+}(p_1, p_2) \beta_{-+}(-p_2', -p_1')] + \delta \tilde{H}_A(-+) + \delta \tilde{H}_{\mathfrak{S}}(-+), \end{aligned} \quad (62)$$

where

$$A_{-+}(p_1, p_2; p_1', p_2') = -\frac{1}{V} \delta(p_1 + p_2 - p_1' - p_2') \{J(p_1, p_2; p_2', p_1')\}$$

$$\begin{aligned}
& \times [u(p_1)u(p_2)u(p_1')u(p_2') + v(p_1)v(p_2)v(p_1')v(p_2')] \\
& + J(p_1, -p_1'; p_2', -p_2)[u(p_1)v(p_2)v(p_1')u(p_2') + v(p_1)u(p_2)u(p_1')v(p_2')] \\
& + [J(-p_1, p_2'; -p_1', p_2) - J(p_2', -p_1; -p_1', p_2)] \\
& \times [v(p_1)u(p_2)v(p_1')u(p_2') + u(p_1)v(p_2)u(p_1')v(p_2')] \}, \\
B_{-+}(p_1, p_2; p_1', p_2') &= \frac{1}{V} \delta(p_1 + p_2 - p_1' - p_2') \{ -J(p_1, p_2; p_2', p_1') \\
& \times [u(p_1)u(p_2)v(p_1')v(p_2') + v(p_1)v(p_2)u(p_1')u(p_2')] \\
& + J(p_1, -p_1'; p_2', -p_2)[v(p_1)u(p_2)v(p_1')u(p_2') + u(p_1)v(p_2)u(p_1')v(p_2')] \\
& + [J(-p_1, p_2'; -p_1', p_2) - J(p_2', -p_1; -p_1', p_2)] \\
& \times [u(p_1)v(p_2)v(p_1')u(p_2') + v(p_1)u(p_2)u(p_1')v(p_2')] \}, \quad (63)
\end{aligned}$$

$$\tilde{H}'(++) = \frac{1}{2} \sum_{p_1, p_2} [\mathcal{Q}(p_1) + \mathcal{Q}(p_2)] \beta_{++}^+(p_1, p_2) \beta_{++}(p_1, p_2) + \delta \tilde{H}(++), \quad (64)$$

$$\tilde{H}'(--) = \frac{1}{2} \sum_{p_1, p_2} [\mathcal{Q}(p_1) + \mathcal{Q}(p_2)] \beta_{--}^+(p_1, p_2) \beta_{--}(p_1, p_2) + \delta \tilde{H}(--). \quad (65)$$

(The first terms in (62) lead to the secular equation for the collective oscillations of the first branch, investigated in 1) and 2)).

The perturbation in (62), (64), (65) linear in β are given by (61). Thus we perform the translation of β amplitudes

$$\begin{aligned}
\beta_{-+}(p_1, p_2) &\rightarrow \beta_{-+}(p_1, p_2) + C_{-+}(p_1, p_2), \\
\beta_{++}(p_1, p_2) &\rightarrow \beta_{++}(p_1, p_2) + C_{++}(p_1, p_2), \dots
\end{aligned} \quad (66)$$

and according to (55) the equations for the function C are

$$\begin{aligned}
& [\mathcal{Q}(p_1) + \mathcal{Q}(p_2)] C_{-+}^*(p_1, p_2) + \sum_{p_1', p_2'} A_{-+}(p_1, p_2; p_1', p_2') C_{-+}^*(p_1', p_2') \\
& + \sum_{p_1', p_2'} B_{-+}(p_1, p_2; p_1', p_2') C_{-+}(-p_2', -p_1') + \frac{e}{2m} \mathbf{A}(p_1 + p_2) (\mathbf{p}_2 - \mathbf{p}_1) \\
& \times [v(p_2)u(p_1) - v(p_1)u(p_2)] = 0, \\
& [\mathcal{Q}(p_1) + \mathcal{Q}(p_2)] C_{-+}(-p_2, -p_1) + \sum_{p_1', p_2'} A_{-+}(p_1, p_2; p_1', p_2') C_{-+}(-p_2', -p_1') \\
& + \sum_{p_1', p_2'} B_{-+}(p_1, p_2; p_1', p_2') C_{-+}^*(p_1, p_2) - \frac{e}{2m} \mathbf{A}^*(-p_1 - p_2) (\mathbf{p}_2 - \mathbf{p}_1) \\
& \times [v(p_2)u(p_1) - v(p_1)u(p_2)] = 0 \quad (67)
\end{aligned}$$

or

$$\begin{aligned}
& [\mathcal{Q}(p_1) + \mathcal{Q}(p_2)] [C_{-+}^*(p_1, p_2) - C_{-+}(-p_2, -p_1)] \\
& + \sum_{p_1', p_2'} [A_{-+}(p_1, p_2; p_1', p_2') - B_{-+}(p_1, p_2; p_1', p_2')] \\
& \times [C_{-+}^*(p_1', p_2') - C_{-+}(-p_2', -p_1')] = -\frac{e}{m} (\mathbf{p}_2 - \mathbf{p}_1) \mathbf{A}(p_1 + p_2) \\
& \times [v(p_2)u(p_1) - v(p_1)u(p_2)],
\end{aligned}$$

$$\begin{aligned} & [\mathcal{Q}(p_1) + \mathcal{Q}(p_2)][C_{-+}^*(p_1, p_2) + C_{-+}(-p_2, -p_1)] \\ & + \sum_{p_1', p_2'} [A_{-+}(p_1, p_2; p_1', p_2') + B_{-+}(p_1, p_2; p_1', p_2')] \\ & \times [C_{-+}^*(p_1', p_2') + C_{-+}(-p_2', -p_1')] = 0. \end{aligned} \quad (68)$$

Similarly we get

$$\begin{aligned} & C_{++}(p_1, p_2) + C_{++}^*(-p_2, -p_1) = C_{--}(p_1, p_2) + C_{--}^*(-p_2, -p_1) \\ & = -\frac{ei}{m} \delta \mathfrak{S}_y(-p_1 - p_2) f(p_1, p_2), \\ & C_{++}(p_1, p_2) - C_{++}^*(-p_2, -p_1) = -[C_{--}(p_1, p_2) - C_{--}^*(-p_2, -p_1)] \\ & = \frac{e}{m} \delta \mathfrak{S}_x(-p_1 - p_2) f(p_1, p_2) \end{aligned} \quad (69)$$

where

$$f(p_1, p_2) = \frac{1}{\mathcal{Q}(p_1) + \mathcal{Q}(p_2)} [v(p_2)u(p_1) - v(p_1)u(p_2)].$$

Equations (68) connect the functions C, C^* only with fixed $\mathbf{p}_1 + \mathbf{p}_2$. In these equations, for the time being, we do not take into account the term $\partial \hat{H}_{\mathfrak{S}}$. When we denote in (68) the functions

$$\begin{aligned} & [C_{-+}^*(p_1, p_2) - C_{-+}(-p_2', -p_1')] \rightarrow \theta_q(p), \\ & [C_{-+}^*(p_1, p_2) + C_{-+}(-p_2, -p_1)] \rightarrow \vartheta_q(p), \end{aligned} \quad (70)$$

equations (68) are identical with equations (81) and (117) solved in paper 1). We shall apply the results of this paragraph to the investigation of the electrodynamics of superconducting state.

§ 5. Electrodynamics of superconducting state

Let us consider the change of the lowest superconducting state due to the weak external electromagnetic field. We want to obtain a current as a function of vector-potential \mathbf{A} and a magnetic field $\partial \mathfrak{S}$.

Considering the Hamiltonian whose mean value is given by

$$\bar{H} = \int \phi^+(r, s_z) \left[\frac{1}{2m} (\mathbf{p} - e\mathbf{A}(r)) - \frac{e}{2m} \boldsymbol{\sigma} \partial \mathfrak{S}(r) \right] \phi(r, s_z) dV, \quad (71)$$

we obtain the current as coefficient of the variation of \mathbf{A} in the formula

$$\delta \bar{H} = - \int \mathbf{j}' \partial \mathbf{A} dV. \quad (72)$$

Thus the current averaged in addition over the spin variables is

$$\begin{aligned} \mathbf{j}'(r) = \sum_{s_z} \left\{ \frac{ie}{m} (\nabla \phi^+(r, s_z) \phi(r, s_z) - \phi^+(r, s_z) \nabla \phi(r, s_z)) - \frac{e^2}{m} \mathbf{A}(r) \phi^+(r, s_z) \phi(r, s_z) \right. \\ \left. - \frac{e}{2m} \text{curl } \phi^+(r, s_z) \boldsymbol{\sigma} \phi(r, s_z) \right\} = \mathbf{j}_1'(r) + \mathbf{j}_2'(r), \end{aligned} \quad (73)$$

where $\mathbf{j}'_2(r)$ denotes the last term of (73). In the Hamiltonian (71) we do not write the interaction energy of particles because this term gives no contribution to the current.

Let us consider the Fourier-representation for the current $\mathbf{j}'_1(r)$,

$$\begin{aligned} \mathbf{j}'_1(r) = & \frac{e}{2m} \frac{1}{V} \sum_{\mathbf{p}_1, \mathbf{p}_2} (\mathbf{p}_2 - \mathbf{p}_1) e^{i(\mathbf{p}_1 + \mathbf{p}_2)r} (a_{-\mathbf{p}_1+}^+ a_{\mathbf{p}_2+} + a_{-\mathbf{p}_1-}^+ a_{\mathbf{p}_2-}) \\ & - \frac{e^2}{m} \mathbf{A}(r) [\rho_0 + \sum_{\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{q} \neq 0} e^{i(\mathbf{p}_1 + \mathbf{p}_2)r} (a_{-\mathbf{p}_1+}^+ a_{\mathbf{p}_2+} + a_{-\mathbf{p}_1-}^+ a_{\mathbf{p}_2-})] \end{aligned} \quad (74)$$

where

$$\rho_0 = \frac{2}{V} \sum_{\mathbf{p}} v^2(\mathbf{p}).$$

In the same approximation in which we have obtained the Hamiltonian (62), (64), (65) we express the current \mathbf{j}'_1 as a linear function of β_{-+} and β_{+-}^+ . Then we introduce the new translated Bose-amplitudes $\beta_{-+} + C_{-+}$. The functions C_{-+} are given by (68).

Let us consider the current

$$\mathbf{j}_1(r) \doteq \langle \mathbf{j}'_1(r) \rangle_0. \quad (75)$$

Taking into account that

$$\langle \beta_{-+} + C_{-+} \rangle_0 = C_{-+}, \quad (76)$$

we obtain $\mathbf{j}_1(r)$ as a function of C_{-+}

$$\begin{aligned} \mathbf{j}_1(r) = \mathbf{j}_1(-+) = & \frac{e}{2m} \frac{1}{V} \sum_{\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{q}} (\mathbf{p}_2 - \mathbf{p}_1) e^{i(\mathbf{p}_1 + \mathbf{p}_2)r} [v(\mathbf{p}_1)u(\mathbf{p}_2) - v(\mathbf{p}_2)u(\mathbf{p}_1)] \\ & \times [C_{-+}(\mathbf{p}_1, \mathbf{p}_2) - C_{-+}^*(-\mathbf{p}_2, -\mathbf{p}_1)] - \frac{e^2}{m} \left[\sum_{\mathbf{q}} \mathbf{A}(\mathbf{q}) e^{i\mathbf{q}r} \right] \\ & \times \left\{ \rho_0 + \sum_{\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{q} \neq 0} (v(\mathbf{p}_1)u(\mathbf{p}_2) + v(\mathbf{p}_2)u(\mathbf{p}_1)) \right. \\ & \left. \times [C_{-+}(\mathbf{p}_1, \mathbf{p}_2) + C_{-+}^*(-\mathbf{p}_2, -\mathbf{p}_1)] e^{i(\mathbf{p}_1 + \mathbf{p}_2)r} \right\}. \end{aligned} \quad (77)$$

Similarly we obtain $\mathbf{j}_2(r)$ as a function of C_{-+} , C_{++} , C_{--} . In the formula (77) we can use the notation of (70) and make direct use of the results of paper 1):

$$\vartheta_q(\mathbf{p}) = 0, \quad \theta_q(\mathbf{p}) = e\mathfrak{A}(\mathbf{q})\tau(\mathbf{p}, \mathbf{q}) \quad (78)$$

where $\mathfrak{A}(\mathbf{q})$ is the Fourier-component of the transverse part of the vector-potential \mathbf{A} , the q -dependence of the function $\tau(\mathbf{p}, \mathbf{q})$ can be emphasized by writing $\tau(\mathbf{p}, \mathbf{q}) = q\tilde{\tau}(\mathbf{p}, \mathbf{q})$. The Fourier-components of (77) are

$$\begin{aligned} \mathbf{j}_1(\mathbf{q}) = & \frac{e}{2m} \frac{1}{V} \sum_{\mathbf{p}} (2\mathbf{p} - \mathbf{q}) [v(\mathbf{p} - \mathbf{q})u(\mathbf{p}) - v(\mathbf{p})u(\mathbf{p} - \mathbf{q})] \theta_q(\mathbf{p}) \\ & - \frac{2e^2}{mV} \mathbf{A}(\mathbf{q}) \sum_{\mathbf{p}} v^2(\mathbf{p}). \end{aligned} \quad (79)$$

Following 1), for sufficiently small q

$$\mathbf{j}_1(q) = -\frac{e^2 \rho_0}{m} \mathfrak{A}, \quad (80)$$

we obtain the Meissner effect^{(5),(6)}. To obtain (80) we consider the collective excitations of the Bose-type only. In papers 1), 9) it is proved that in order to satisfy the Buckingham relations (or, what is the same, the gauge invariance of the Meissner-Ochsenfeld effect), only the contribution from the Bose-type collective excitations is essential for small q .

Now, let us consider the last term of (73)

$$\mathbf{j}'_2(r) = -\frac{e}{2m} \text{curl} \sum_{s_z} \psi^+(r, s_z) \boldsymbol{\sigma} \psi(r, s_z). \quad (81)$$

$\mathbf{j}'_1(r)$ is the current of elementary excitations with spin 0, and $\mathbf{j}'_2(r)$ is the current of elementary excitations with spin 0, ± 1 . This means that we obtain an additional term to \mathbf{j}_1 and, moreover, the "spin" current, current of elementary excitations with spin $\neq 0$. The components of (81) in Fourier-representation are

$$\begin{aligned} j'_{2x}(-+) &= \frac{ie}{2m} \frac{1}{V} \sum_{p_1+p_2=q} e^{i(p_1+p_2)r} (p_1+p_2)_y (a_{-p_1+}^+ a_{p_2+} - a_{-p_1-}^+ a_{p_2-}), \\ j'_{2y}(-+) &= -\frac{ie}{2m} \frac{1}{V} \sum_{p_1+p_2=q} e^{i(p_1+p_2)r} (p_1+p_2)_x (a_{-p_1+}^+ a_{p_2+} - a_{-p_1-}^+ a_{p_2-}), \\ [j'_2(++) + j'_2(--)]_x &= -\frac{e}{2m} \frac{1}{V} \sum_{p_1+p_2=q} e^{i(p_1+p_2)r} (p_1+p_2)_z (a_{-p_1+}^+ (a_{p_2-} - a_{-p_1-}^+ a_{p_2+}), \\ [j'_2(++) + j'_2(--)]_y &= \frac{ie}{2m} \frac{1}{V} \sum_{p_1+p_2=q} e^{i(p_1+p_2)r} (p_1+p_2)_z (a_{-p_1+}^+ a_{p_2-} + a_{-p_1-}^+ a_{p_2+}), \\ [j'_2(++) + j'_2(--)]_z &= -\frac{ie}{2m} \frac{1}{V} \sum_{p_1+p_2=q} e^{i(p_1-p_2)r} \\ &\quad \times \{i(p_1+p_2)_x (a_{-p_1+}^+ a_{p_2-} - a_{-p_1-}^+ a_{p_2+}) + (p_1+p_2)_y (a_{-p_1+}^+ a_{p_2-} + a_{-p_1-}^+ a_{p_2+})\}. \end{aligned} \quad (82)$$

In a similar way to the formula (77) we get

$$\begin{aligned} j_{2x}(-+) &= \frac{ie}{2m} \frac{1}{V} \sum_{p_1+p_2=q} e^{i(p_1+p_2)r} (p_1+p_2)_y [v(p_1)u(p_2) - v(p_2)u(p_1)] \\ &\quad \times [C_{-+}(p_1, p_2) - C_{-+}^*(-p_2, -p_1)], \\ j_{2y}(-+) &= -\frac{ie}{2m} \frac{1}{V} \sum_{p_1+p_2=q} e^{i(p_1+p_2)r} (p_1+p_2)_x [v(p_1)u(p_2) - v(p_2)u(p_1)] \\ &\quad \times [C_{-+}(p_1, p_2) - C_{-+}^*(-p_2, -p_1)], \\ j_{2x}(++) &= \frac{e}{4m} \frac{1}{V} \sum_{p_1+p_2=q} e^{i(p_1+p_2)r} (p_1+p_2)_z [v(p_2)u(p_1) - v(p_1)u(p_2)] \\ &\quad \times [C_{++}(p_1, p_2) + C_{++}^*(-p_2, -p_1)], \end{aligned} \quad (84)$$

$$\begin{aligned}
j_{2y}(++) &= \frac{ie}{4m} \frac{1}{V} \sum_{p_1+p_2=q} e^{i(p_1+p_2)r} (p_1+p_2)_z [v(p_2)u(p_1) - v(p_1)u(p_2)] \\
&\quad \times [C_{++}(p_1, p_2) - C_{++}^*(-p_2, -p_1)], \\
j_{2x}(++) &= \frac{ie}{4m} \frac{1}{V} \sum_{p_1+p_2=q} e^{i(p_1+p_2)r} \{i(p_1+p_2)_x [v(p_2)u(p_1) - v(p_1)u(p_2)] \\
&\quad \times [C_{++}(p_1, p_2) + C_{++}^*(-p_2, -p_1)] - (p_1+p_2)_y \\
&\quad \times [v(p_2)u(p_1) - v(p_1)u(p_2)] [C_{++}(p_1, p_2) - C_{++}^*(-p_2, -p_1)]\}. \quad (85)
\end{aligned}$$

We do not write the components of $\mathbf{j}_2(--)$ to be equal to those of $\mathbf{j}_2(++)$.

Let us consider the current (84) representing an additional term to $\mathbf{j}_1(-+)$ which arises in the (X, Y) plane. In order to obtain the exact functions $(C_{-+} - C_{-+}^*)$ one must in equations (68) take into account the term $\delta\tilde{H}_{\tilde{5}}$ too. This gives also a small correction to \mathbf{j}_1 . We want to investigate the current $\mathbf{j}_2(-+)$ in the zero approximation only. We take equations (68) as our basic equations and consider the term arising from $\delta\tilde{H}_{\tilde{5}}$ as a perturbation. According to 1) the equation without perturbation has the solution of the form

$$C_{-+}(p_1, p_2) - C_{-+}^*(-p_2, -p_1) = e\mathfrak{A}_z(q)q\tilde{\tau}(p, q). \quad (86)$$

Putting this solution in (84) we obtain $\mathbf{j}_2(-+)$ in the zero approximation:

$$\begin{aligned}
j_{2x}(-+) &= \frac{ie^2}{2m} \frac{1}{V} \sum_q e^{iqr} q_y \tilde{\mathfrak{A}}_z(q) = \frac{e^2}{2m} \frac{\partial}{\partial y} \tilde{\mathfrak{A}}_z(r), \\
j_{2y}(-+) &= -\frac{ie^2}{2m} \frac{1}{V} \sum_q e^{iqr} q_x \tilde{\mathfrak{A}}_z(q) = -\frac{e^2}{2m} \frac{\partial}{\partial x} \tilde{\mathfrak{A}}_z(r)
\end{aligned} \quad (87)$$

where

$$\begin{aligned}
\tilde{\mathfrak{A}}_z(q) &= \mathfrak{A}_z(q) q \sum_p \tilde{\tau}(p, q) [v(p)u(p-q) - v(p-q)u(p)], \\
\tilde{\mathfrak{A}}_z(r) &\equiv \sum_q \tilde{\mathfrak{A}}_z(q) e^{iqr}.
\end{aligned}$$

Introducing the vector $\vec{\mathfrak{A}}(r) = (0, 0, \tilde{\mathfrak{A}}_z)$, we get (87) in the form

$$\mathbf{j}_2(-+) = \frac{e^2}{2m} \frac{1}{V} \text{curl } \vec{\mathfrak{A}} = \frac{e^2}{2m} \frac{1}{V} \vec{\mathfrak{S}}. \quad (88)$$

As we see from (87), for small q $\mathbf{j}_2(-+)$ vanishes like q^2 , that is, like the terms neglected in obtaining the formula (80). Thus $\mathbf{j}_2(-+)$ does not give rise to the Meissner effect.

Let us consider the "spin" current of elementary excitations with spin moment $+1$. From (85) and (69) we get

$$\begin{aligned}
j_{2x}(++) &= \frac{ie^2}{4m^2} \frac{1}{V} \sum_q e^{-iqr} q_z \tilde{\mathfrak{S}}_y(q) = -\frac{e^2}{4m^2} \frac{1}{V} \frac{\partial}{\partial z} \tilde{\mathfrak{S}}_y(r), \\
j_{2y}(++) &= -\frac{ie^2}{4m^2} \frac{1}{V} \sum_q e^{-iqr} q_z \tilde{\mathfrak{S}}_x(q) = \frac{e^2}{4m^2} \frac{1}{V} \frac{\partial}{\partial z} \tilde{\mathfrak{S}}_x(r),
\end{aligned}$$

$$j_{2z}(++) = -\frac{ie^2}{4m^2} \frac{1}{V} \sum_q e^{-iqr} [q_x \bar{\mathfrak{S}}_y(q) - q_y \bar{\mathfrak{S}}_x(q)] \\ = -\frac{e^2}{4m^2} \frac{1}{V} \left[\frac{\partial}{\partial x} \bar{\mathfrak{S}}_y(r) - \frac{\partial}{\partial y} \bar{\mathfrak{S}}_x(r) \right] \quad (89)$$

where

$$\vec{\mathfrak{S}} = (\bar{\mathfrak{S}}_x, \bar{\mathfrak{S}}_y, 0), \quad \vec{\mathfrak{S}}(r) \equiv \sum_q \vec{\mathfrak{S}}(q) e^{-iqr}, \\ \bar{\mathfrak{S}}_x(q) = \bar{\mathfrak{S}}_x(q) \sum_p \frac{[v(p)u(p+q) - v(p+q)u(p)]^2}{\Omega(p) + \Omega(p+q)} = \bar{\mathfrak{S}}_x(q) \frac{V}{2} F(q). \quad (90)$$

The factor $F(q)$ is obtained in paper 2) in connection with screening of the Coulomb interaction,

$$F(q) = -\frac{4}{V} \sum_{k'-k=q} \frac{\theta_G(k) \theta_F(k')}{\tilde{E}(k) - \tilde{E}(k')},$$

where $\tilde{E}(k)$ is the energy of the electron elementary excitation and θ_G, θ_F the solution of the compensation equations for the normal state.

From (89) we see that the x -component of "screened" magnetic field $\vec{\mathfrak{S}}$ produces the currents in the (Y, Z) -plane and the y -component in the (X, Z) -plane. After introducing the vector $\vec{\mathfrak{S}}$ by (90), we get (89) in the form

$$j_2(++) = \frac{1}{V} \frac{e^2}{4m^2} \text{curl } \vec{\mathfrak{S}}(r) = \frac{1}{V} \frac{e^2}{4m^2} \text{curl curl } \vec{\mathfrak{A}}(r). \quad (91)$$

The same formula we obtain for $j_2(--)$. After changing in (91) to Fourier expansion we obtain the Fourier coefficients vanishing like q^2 . Then the "spin" current does not give rise to the Meissner effect.

It is a pleasant duty to thank Prof. N. N. Bogolubov for proposing this problem and giving helpful advice and D. V. Shirkov, V. V. Tolmachov and V. G. Soloviev for valuable discussions.

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Removal of Ghost-Pole and Unitarity of S -Matrix

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Consistency of the method of removing the ghost-pole proposed by Redmond and Bogoliubov *et al.* is investigated. In conclusion, it seems to us that their method contradicts with the unitarity condition for the S -matrix or is nothing but to show the well-known fact that it is possible to remove the ghost-pole of the propagator by using a suitable cut-off.

I

Recently Redmond¹⁾ and Bogoliubov *et al.*²⁾ proposed a curious method by means of which it is possible to get the expression for the propagator which does not contain the so-called ghost-pole. The propagator acquired by them has the following interesting properties:

(1) As the function of the square of momentum, it has the same analytical behavior as that given by Lehmann³⁾ and others.

(2) It has an essential singularity at $g^2=0$, when considered as a function of the coupling constant.

(3) The constant of renormalization of the wave function is finite.

It may come into question, however, whether their method is self-consistent within the framework of their theory.

We shall investigate this consistency by using the Lee model. Although there are such special situations in Lee's model that the model has an exact solution which contradicts with the result obtained by Redmond *et al.*, our argument will have nothing to do with these special situations in Lee's model. We shall be able to give the similar proof to the actual case, though it will become much more complicated.

II

Let us first evaluate the propagator $\tilde{S}_{rc}(E)$ of the V particle according to Redmond *et al.*'s line of thought. The new propagator is of the form

$$\tilde{S}_{rc}(E) = \frac{1}{M_v - E} + \int_m^\infty d\omega \frac{I(\omega)}{\omega - E - i\epsilon}, \quad (1)$$

$$I(\omega) = \frac{1}{\pi} \text{Im} \left[\frac{1}{M_v - \omega} \left\{ 1 - (M_v - \omega) g^2 \int_m^\infty dz \frac{f^2(z)}{(z - M_v)^2 (z - \omega - i\epsilon)} \right\}^{-1} \right], \quad (2)$$

where M_v is the observable mass and $f(z)$ is the cut-off function. The expression (1) together with (2) also immediately represents the function

$$\tilde{S}_{ve}(E) = \frac{1}{M_v - E} \left[1 - (M_v - E) g^2 \int_m^\infty d\omega \frac{f^2(\omega)}{(\omega - M_v)^2 (\omega - E - i\epsilon)} \right]^{-1} - \frac{Z}{\lambda + E}. \quad (3)$$

Here $-\lambda$ is the zero of the square bracket and Z is the residue of the first term at $E = -\lambda$. The propagator (1) or (3) has no ghost-pole. Now we calculate the S -matrix for the N - θ scattering:

$$S_{fi} = \delta_{fi} + 2\pi i \delta(E_f - E_i) f(E_f) f(E_i) g^2 \tilde{S}_{ve}(E_i). \quad (4)$$

Inserting (3) into (4), we can easily examine the validity of the unitarity condition for the S -matrix, and that the condition is really violated, therefore it is not possible to eliminate the ghost-pole by their method in a consistent way.

III

It should be emphasized that our argument is concerned essentially with the method of derivation of non-pathological solution from the solution obtained in the customary fashion in the case of some given cut-off function (or some explicitly given Hamiltonian). On the contrary, as Medvedev and Polivanov pointed out,⁴⁾ the result given by Redmond *et al.* can be reproduced by suitably changing the cut-off function and making use of the familiar method in place of the somewhat tedious method of Redmond *et al.* From this fact, we can interpret the method proposed by Redmond *et al.* to be equivalent to eliminating the ghost-pole by changing the cut-off factor. If we take this standpoint, all the cut-off functions appearing in the S -matrix should be replaced with the modified $\tilde{f}(E)$. In fact, it is possible, in the case of Lee's model, to determine the cut-off function so as to retain the unitarity condition for the S -matrix.⁴⁾ Consequently this view-point becomes to show that Redmond's method is nothing but to prove by an indirect way the well-known removability of the ghost-pole of the propagator for some well-behaved cut-off function. In addition, if we apply this standpoint to the actual covariant local theory, the local interaction may be replaced with the non-local one (see Appendix), and this will be unfavourable, as long as we do not know how to deal with the covariant non-local theory.

In conclusion, it may be said that the method of Redmond *et al.* will not give any new progress in the theory of quantized fields.

We wish to express our very sincere thanks to Prof. R. Utiyama for his guidance and encouragement and to Mr. N. Mugibayashi for showing us reference 4 and for valuable discussions. Thanks are also due to Mr. G. Konisi for critical comments.

Appendix

As an illustration, let us calculate the new meson propagator by considering only the lowest order proper diagram, a nucleon-antinucleon loop, and show that the unfavourable feature mentioned in the end of III really occurs. In this approximation, the ordinary perturbation theory leads to the usual expression involving a ghost-pole :

$$\Delta_F'(p^2) = (p^2 + \mu^2 - i\epsilon)^{-1} \left[1 - (p^2 + \mu^2) \int_{4M^2}^{\infty} dm^2 \frac{\rho_0(m^2)}{p^2 + m^2 - i\epsilon} \right]^{-1}, \quad (5)$$

$$\begin{aligned} \rho_0(m^2) &= \frac{g^2}{4\pi^2} \frac{m(m^2 - 4M^2)^{1/2}}{(m^2 - \mu^2)^2}, \quad m^2 > 4M^2, \\ &= 0, \quad m^2 \leq 4M^2. \end{aligned} \quad (6)$$

Let us now give the new propagator $\tilde{\Delta}_F'(p^2)$ which has no ghost-pole, according to Redmond *et al.* :

$$\tilde{\Delta}_F'(p^2) = \frac{1}{p^2 + \mu^2 - i\epsilon} + \int_{4M^2}^{\infty} dm^2 \frac{I(m^2)}{p^2 + m^2 - i\epsilon}, \quad (7)$$

where

$$I(m^2) = \frac{1}{\pi} \text{Im} \Delta_F'(m^2) = \rho_0(m^2)/D_0(m^2), \quad (8)$$

$$D_0(m^2) = 1 + \pi^2(m^2 - \mu^2)\rho_0^2(m^2) - (m^2 - \mu^2)P \int_{4M^2}^{\infty} d\ell^2 \frac{\rho_0(\ell^2)}{m^2 - \ell^2}. \quad (9)$$

The expression (7) is also obtained by replacing $\rho_0(m^2)$ in (5) with $\tilde{\rho}_0(m^2)$ given by (10):

$$\begin{aligned} \tilde{\rho}_0(m^2) &= \rho_0(m^2) \cdot \frac{1}{D_0(m^2)} \\ &\times \frac{1}{\left[1 + (m^2 + \mu^2)P \int_{4M^2}^{\infty} d\ell^2 \frac{\rho_0(\ell^2)/D_0(\ell^2)}{m^2 + \ell^2} \right]^2 + \pi^2[\rho_0(m^2)/D_0(m^2)]^2}. \end{aligned} \quad (10)$$

As the term corresponding to the lowest order proper diagram of the self-energy of the meson is $\rho_0(m^2)$ in the case of the point interaction, we see that in order to obtain the expression $\rho_0(m^2)$ from the above diagram the interaction must be non-local.

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Selection Rules for Interaction Types in Quantum Field Theory

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The Dirac equation is generalized to the form involving the γ_5 term, which is reduced to the Klein-Gordon equation when it is iterated. Such a generalized quantum electrodynamics is proved to be equivalent to the usual quantum electrodynamics so far as we are concerned only with the usual vector coupling. If we adopt the principle that the Fermion must obey the generalized Dirac equation and that the equation including interaction is invariant under the similarity transformation, by which it is reduced to the original Dirac equation, it is prohibited to introduce the Pauli term into quantum electrodynamics, because the Pauli term in the generalized Dirac equation violates the parity inversion and time reversal invariance. It is also found that only the interaction terms of types A and V are selected in the universal Fermi-interaction and in the case of hyperonic decays into a nucleon and a π -meson only the interaction types of A and V are allowed by using the principle mentioned above, but in these cases the parity inversion and particle-antiparticle conjugation invariance do not hold. If we adopt the invariance under P , T and C transformation in the case of strong interactions, only one of the interaction types A and V remains for Fermion-Fermion, or Fermion-Boson interacting systems.

§ 1. Introduction

In quantum field theory, the interaction Hamiltonians are usually introduced so as to fulfill the proper and improper Lorentz invariance, Hermiticity and particle-antiparticle conjugation invariance. In quantum electrodynamics, we further require the gauge invariance and charge-current conservation laws more severely than in other cases in addition to the conditions mentioned above. Recently, in the case of weak interactions¹⁾ the violation of parity inversion (P) and particle-antiparticle conjugation (C) law was verified experimentally, while in the case of strong interactions the conservation laws for P , C and T transformations are believed to hold within the limit of the present experimental accuracy. From this point of view, we investigated the possibility of generalizing the Dirac equation so as to include the P and T non-invariant term and also to be reduced to the Klein-Gordon equation when it is iterated. If we construct a generalized quantum electrodynamics with use of such a generalized Dirac equation, the equations of motion for the electron and photon field and their commutation relations can reduce to the usual one by a similarity transformation which means to reduce these equations to the original one. In this case, the introduction of vector coupling violating the P and T invariance, i.e., $ie\gamma_\mu \exp(c\gamma_5)A_\mu$, breaks the charge-current conservation law²⁾ even in the generalized quantum electrodynamics. Of course, there exists another

method found by M. Sachs³⁾ to introduce the P and T non-invariant vector potential without any change of the Dirac equation, which satisfies both the charge-current conservation law and Lorentz condition. Although the theory can be constructed so as to introduce the Pauli term which fulfills the charge-current conservation law, this Pauli term does not obey a similarity transformation. Therefore, since the Pauli term violates P and T invariance, we cannot introduce the Pauli term into quantum electrodynamics so far as we believe in the P and T invariance law within the limit of the present experimental accuracy. This theory which requires the invariance of equations of motion for the similarity transformation corresponds to the so-called Gell-Mann's minimal electromagnetic interaction rule.⁴⁾ Adopting this principle, the selection rules for the interaction types in the case of weak interactions are obtained. Namely, as the P and C invariance does not hold in the case of weak interactions, only A and V type interactions are possible due to the maintenance of the similarity transformation. These results are consistent with the recent experimental results concerning the Fermi-interactions for β decay of nuclei and the μ -meson decay and also concerning the hyperonic decay. Since the types of interactions are not yet well established in the case of strong interactions except in quantum electrodynamics and mesodynamics, the question whether baryons obey the generalized Dirac equation shall be postponed to future.

§ 2. Generalized Dirac equation and commutation relations

The general linearization of the Klein-Gordon equation can be achieved as follows. If we express the Klein-Gordon equation as the product of two operators,

$$d(\partial)A(\partial)\phi=0, \quad (1)$$

where

$$A(\partial)=\Gamma_1\gamma_\mu\partial_\mu+\kappa A_1, \quad d(\partial)=\Gamma_2\gamma_\mu\partial_\mu-\kappa A_2, \quad (2)$$

we have the following conditions:

$$\Gamma_1\gamma_\mu\Gamma_2\gamma_\nu\partial_\mu\partial_\nu=\square, \quad \Gamma_2\gamma_\mu A_1=A_2\Gamma_1\gamma_\mu, \quad A_2A_1=1. \quad (3)$$

If we put $\Gamma_1=\Gamma_2=1$, the conditions (3) are satisfied by the following A_1 and A_2 :

$$A_1=\exp(-a\gamma_5), \quad A_2=\exp(a\gamma_5), \quad (4)$$

where a is an arbitrary c -number.

Then the generalized Dirac equation is written as follows:

$$\{\gamma_\mu\partial_\mu+\kappa\exp(-a\gamma_5)\}\phi=0, \quad (5)$$

$$d(\partial)=\gamma_\mu\partial_\mu-\kappa\exp(a\gamma_5). \quad (6)$$

If the condition $A_1=A_2=1$ is adopted, Γ_1 and Γ_2 are given by the following expression:

$$\Gamma_1=\Gamma_2=\exp(a\gamma_5). \quad (7)$$

Then the Dirac equation is generalized to the following form :

$$\{\exp(a\gamma_5)\gamma_\mu\partial_\mu+\kappa\}\psi=0, \quad (8)$$

$$d(\partial)=\exp(a\gamma_5)\gamma_\mu\partial_\mu-\kappa. \quad (9)$$

Even if we have another case different from (4) or (7), the generalized Dirac equation thus obtained can be reduced to (5) ; e.g., Eq. (8) can be obtained from (5) by multiplying it by a factor $\exp(a\gamma_5)$. Therefore Eq. (8) is not independent of (5) and the important point is to include a P and T non-invariant γ_5 term. The equation to which $\bar{\psi}$ obeys is obtained by the requirement that the Lagrangian density is Hermitian or the equation for $\bar{\psi}$ is the same as the one for ψ except for the sign of κ . In the case of (5),

$$\partial_\mu\bar{\psi}\gamma_\mu-\kappa\bar{\psi}\exp(-a\gamma_5)=0, \quad \bar{\psi}=\psi^*\gamma_4\exp[\frac{1}{2}(a+a^*)\gamma_5], \quad (10)$$

while in the case of (8),

$$\partial_\mu\bar{\psi}\exp(a\gamma_5)\gamma_\mu-\kappa\bar{\psi}=0, \quad \bar{\psi}=\psi^*\gamma_4\exp[\frac{1}{2}(a^*-a)\gamma_5]. \quad (11)$$

Eq. (11) can be obtained from (10) by redefining $\bar{\psi}$. The commutation relations for $\psi(x)$ are set up as follows. In the case of (5),

$$\{\psi_\alpha(x), \bar{\psi}_\beta(x')\} = -i\{\gamma_\mu\partial_\mu - \kappa\exp(a\gamma_5)\}_{\alpha\beta}\Delta(x-x'), \quad (12)$$

while in the case of (8),

$$\{\psi_\alpha(x), \bar{\psi}_\beta(x')\} = -i\{\exp(a\gamma_5)\gamma_\mu\partial_\mu - \kappa\}_{\alpha\beta}\Delta(x-x'). \quad (13)$$

The commutation relation (13) can be obtained by multiplying (12) by a factor $[\exp(-a\gamma_5)]_{\beta\gamma}$ and summing up over β .

$$\{\psi_\alpha(x), [\psi^+(x')\exp\frac{1}{2}(a^*-a)\gamma_5]_\gamma\} = -i\{\exp(a\gamma_5)\gamma_\mu\partial_\mu - \kappa\}_{\alpha\gamma}\Delta(x-x').$$

Since the cases (5) and (8) can be proved to be equivalent to each other including the commutation relations, we need not discuss both cases separately.

The equivalence of the generalized quantum electrodynamics to the usual one can be easily proved as follows by using the relation

$$\exp(a\gamma_5)\gamma_\mu\exp(a\gamma_5)=\gamma_\mu. \quad (14)$$

Putting $\alpha=a/2$, we get

$$\exp(a\gamma_5)\exp(-a\gamma_5)\exp(a\gamma_5)=1. \quad (15)$$

By performing such a similarity transformation, the following equations for electron and photon fields,⁶⁾ which include the apparently P and T non-invariant terms,

$$\{\gamma_\mu(\partial_\mu - ieA_\mu) + \kappa\exp(-a\gamma_5)\}\psi=0, \quad (16)$$

$$\{\gamma_\mu^T(\partial_\mu + ieA_\mu) - \kappa\exp(-a\gamma_5^T)\}\bar{\psi}=0, \quad (17)$$

$$(\square\partial_{\mu\nu} - \partial_\mu\partial_\nu)A_\nu = -\frac{ie}{2}[\bar{\psi}, \gamma_\mu\psi], \quad (18)$$

where

$$\bar{\psi} = \psi^* \gamma_4 \exp[\tfrac{1}{2}(a + a^*)\gamma_5] = \psi^+ \exp[\tfrac{1}{2}(a + a^*)\gamma_5],$$

can be reduced to the usual ones :

$$\{\gamma_\mu(\partial_\mu - ieA_\mu) + \kappa\}\varphi = 0, \quad (19)$$

$$\{\gamma_\mu^T(\partial_\mu + ieA_\mu) - \kappa\}\bar{\varphi} = 0, \quad (20)$$

$$(\square\partial_{\mu\nu} - \partial_\mu\partial_\nu)A_\nu = -\frac{ie}{2}[\bar{\varphi}, \gamma_\mu\varphi], \quad (21)$$

respectively, where

$$\varphi = \exp\left[-\frac{a}{2}\gamma_5\right]\psi. \quad (22)$$

Furthermore, the commutation relation (12) transformed to the interaction representation for spinor fields

$$\{\psi_\alpha(x), [\psi^+(x') \exp \tfrac{1}{2}(a + a^*)\gamma_5]_\beta\} = -i\{\gamma_\mu\partial_\mu - \kappa \exp(a\gamma_5)\}_{\alpha\beta}\Delta(x - x') \quad (23)$$

can be reduced to the usual one by multiplying (23) by factors $[\exp(-a\gamma_5/2)]_{\rho\alpha}$ and $[\exp(-a\gamma_5/2)]_{\beta\sigma}$ and summing up over α and β . Namely, we get

$$\{\varphi_\alpha(x), \varphi^+_\beta(x')\} = -i(\gamma_\mu\partial_\mu - \kappa)_{\alpha\beta}\Delta(x - x').$$

§ 3. Selection rules of the interaction types in quantum electrodynamics

S. N. Gupta⁶⁾ proved that the impossibility of introducing P and T non-invariant vector coupling into quantum electrodynamics is due to the incompatibility of Lorentz condition and charge-current conservation law. M. Sachs⁷⁾ proposed a P and T non-invariant quantum electrodynamics, in which the P and T non-invariant vector potential fulfills both conditions compatibly. However, even if we introduce P and T non-invariant vector coupling term into the generalized Dirac equation developed in section 1, it is proved that such a P and T non-invariant vector coupling term does not satisfy both conditions compatibly.

The interaction Lagrangian including a P and T non-invariant vector potential is written as

$$\bar{L}' = -ie \int \psi^+ \exp[\tfrac{1}{2}(a + a^*)\gamma_5] \gamma_\mu \exp(b\gamma_5) A_\mu \psi d^4x, \quad (24)$$

which is Hermitian if we take b as real. The equations for electron and photon fields are written down as follows :

$$\{\gamma_\mu(\partial_\mu - ie \exp(b\gamma_5) A_\mu) + \kappa \exp(-a\gamma_5)\}\psi = 0, \quad (25)$$

$$(\square\partial_{\mu\nu} - \partial_\mu\partial_\nu)A_\nu = -\frac{ie}{2}[\psi^+, \exp[\tfrac{1}{2}(a + a^*)\gamma_5] \gamma_\mu \exp(b\gamma_5) \psi]. \quad (26)$$

By making the transformations in (14) and (15), (25) and (26) can be transformed into the following forms:

$$\{\gamma_\mu(\partial_\mu - ie \exp(b\gamma_5)A_\mu) + \kappa\}\varphi = 0, \quad (27)$$

$$\{\gamma_\mu^T(\partial_\mu + ie \exp(b\gamma_5^T)A_\mu) - \kappa\}\varphi^+ = 0, \quad (28)$$

$$(\square \partial_{\mu\nu} - \partial_\mu \partial_\nu)A_\nu = -\frac{ie}{2}[\varphi^+, \gamma_\mu \exp(b\gamma_5)\varphi]. \quad (29)$$

Multiplying (27) by φ^+ from the left-hand side, and (28) by φ from the right-hand side and summing up both equations, we get

$$\partial_\mu(\varphi^+ \gamma_\mu \varphi) = 0. \quad (30)$$

If we apply the differentiation ∂_μ to (29), the left-hand side vanishes identically, while the right hand side does not due to (30).

Next, we shall introduce the Pauli term into the generalized quantum electrodynamics. However, the interaction Lagrangian involving the Pauli term can be Hermitian only in the case of imaginary a , because

$$-\frac{e}{2\kappa}\xi[\bar{\psi}\sigma_{\mu\nu}F_{\mu\nu}\psi]^* = \frac{e}{2\kappa}\xi\bar{\psi}\exp[-(a+a^*)\gamma_5]\sigma_{\mu\nu}F_{\mu\nu}\psi. \quad (31)$$

Then if we put $a=ib$, where b is real, we get the generalized quantum electrodynamics involving the Pauli term as follows:

$$\left\{\gamma_\mu(\partial_\mu - ieA_\mu) - \frac{e}{2\kappa}\xi\sigma_{\mu\nu}F_{\mu\nu} + \kappa\exp(-ib\gamma_5)\right\}\psi = 0, \quad (32)$$

$$\left\{\gamma_\mu^T(\partial_\mu + ieA_\mu) + \frac{e}{2\kappa}\xi\sigma_{\mu\nu}^TF_{\mu\nu} - \kappa\exp(-ib\gamma_5^T)\right\}\bar{\psi} = 0, \quad (33)$$

$$(\square \partial_{\mu\nu} - \partial_\mu \partial_\nu)A_\nu = j_\mu^{(1)} + j_\mu^{(2)}, \quad (34)$$

where

$$j_\mu^{(1)} = -\frac{ie}{2}[\bar{\psi}, \gamma_\mu\psi], \quad \partial_\mu j_\mu^{(1)} = 0, \quad (35)$$

$$j_\mu^{(2)} = \frac{e}{2\kappa}\xi\partial_\nu[\bar{\psi}, \sigma_{\mu\nu}\psi], \quad \partial_\mu j_\mu^{(2)} = 0, \quad (36)$$

and ξ is any real parameter.

Here, by performing the similarity transformation in (14) and (15), the above equations are transformed into

$$\left\{\gamma_\mu(\partial_\mu - ieA_\mu) - \frac{e}{2\kappa}\xi\exp(ib\gamma_5)\sigma_{\mu\nu}F_{\mu\nu} + \kappa\right\}\varphi = 0, \quad (37)$$

$$\left\{\gamma_\mu^T(\partial_\mu + ieA_\mu) + \frac{e}{2\kappa}\xi\exp(ib\gamma_5^T)\sigma_{\mu\nu}^TF_{\mu\nu} - \kappa\right\}\bar{\varphi} = 0, \quad (38)$$

$$j_\mu^{(1)} = -\frac{ie}{2} [\bar{\varphi}, \gamma_\mu \varphi], \quad j_\mu^{(2)} = \frac{e}{2\kappa} \xi \partial_\nu [\bar{\varphi}, \exp(ib\gamma_5) \sigma_{\mu\nu} \varphi]. \quad (39)$$

These equations are not evidently invariant for P and T transformations except for the case $b=0$, while C invariance holds, because by transforming the electron field function ψ in (33) as

$$\begin{aligned} \psi' &= C\bar{\psi}, \quad C^T = -C, \quad \gamma_\mu^T = -C^{-1}\gamma_\mu C, \\ C\gamma_5^T C^{-1} &= \gamma_5, \quad C\sigma_{\mu\nu}^T C^{-1} = -\sigma_{\mu\nu}, \end{aligned}$$

we have from (33)

$$\left\{ \gamma_\mu (\partial_\mu + ieA_\mu) + \frac{e}{2\kappa} \xi \sigma_{\mu\nu} F_{\mu\nu} + \kappa \exp(-ib\gamma_5) \right\} \psi' = 0. \quad (40)$$

As far as we believe in P and T invariance laws within the limit of the present experimental accuracy and accept the principle mentioned in § 1, the introduction of the Pauli term into quantum electrodynamics is not allowed. This is another principle of the so-called Gell-Mann's minimal electromagnetic interaction rule.⁸⁾

§ 4. Selection rules of the interaction types in the case of weak interactions and quantum mesodynamics

In the case of Fermi-interaction, for example, in β decay of a nucleus and μ decay into an electron and neutrinos, it is well-known that the P and C invariance breaks down. The similarity transformation as (14) holds only in the case of vector and pseudovector γ matrices:

$$\exp(\alpha\gamma_5) \Gamma_i \exp(\alpha\gamma_5) = \Gamma_i, \quad \text{for } \gamma_\mu \text{ and } \gamma_5\gamma_\mu, \quad (41)$$

while for 1 , γ_5 and $\gamma_\mu\gamma_\nu$,

$$\exp(\alpha\gamma_5) \Gamma_i \exp(\alpha\gamma_5) = \exp(2\alpha\gamma_5) \Gamma_i. \quad (42)$$

If we accept the principle mentioned above, by which the similarity transformation of the generalized Dirac equation including interaction restores the original Dirac equation, it is shown that only the vector and pseudovector interactions remain in Fermi-interactions. Namely, the following equation,

$$\{\gamma_\mu \partial_\mu + \kappa \exp(-\alpha\gamma_5)\} \psi_P(\psi_\mu) + \sum_i g_i \Gamma_i \psi_N \bar{\psi}_e \Gamma_i \psi_\nu (g_i \Gamma_i \psi_e \bar{\psi}_\nu \Gamma_i \psi_\nu) = 0, \quad (43)$$

is transformed into the following form by (41) and (42):

$$\begin{aligned} \{\gamma_\mu \partial_\mu + \kappa\} \psi_P + g_1 \psi_N \gamma_\mu \bar{\psi}_e \gamma_\mu \psi_\nu + g_2 \psi_N \gamma_5 \bar{\psi}_e \gamma_5 \gamma_\mu \psi_\nu + g_3 \psi_N \exp(\alpha\gamma_5) \bar{\psi}_e \psi_\nu \\ + g_4 \psi_N \exp(\alpha\gamma_5) \gamma_5 \bar{\psi}_e \gamma_5 \psi_\nu + g_5 \psi_N \exp(\alpha\gamma_5) \gamma_\mu \gamma_\nu \bar{\psi}_e \gamma_\mu \gamma_\nu \psi_\nu = 0. \end{aligned}$$

The allowed interaction Hamiltonian H' is rewritten as follows with use of wave function $(1+\gamma_5)\psi$ for a neutrino and the universal Fermi-interaction constant $g_1=g_2$:

$$\begin{aligned}
 H' &= g_1 \bar{\psi}_N \gamma_\mu \phi_P \bar{\psi}_e \gamma_\mu (1 + \gamma_5) \psi_\nu + g_2 \bar{\psi}_N \gamma_5 \gamma_\mu \phi_P \bar{\psi}_e \gamma_5 \gamma_\mu (1 + \gamma_5) \psi_\nu \\
 &= g_1 \bar{\psi}_N \gamma_\mu (1 + \gamma_5) \phi_P \bar{\psi}_e \gamma_\mu (1 + \gamma_5) \psi_\nu.
 \end{aligned}
 \tag{44}$$

These results are well established by recent experiments.⁹⁾ Also in the case of Fermion-Boson system, i.e., a hyperonic decay into a nucleon and a π -meson and μ -mesic decay of $\pi(K)$ meson, only the vector and pseudovector couplings are selected on the basis of above principle and this result seems to be consistent with the recent experiment.¹⁰⁾

In quantum mesodynamics, the pseudoscalar (PS) and the pseudovector (PV) couplings of the pseudoscalar π -meson wave field function seem to be adequate at the present stage. Although particularly, PS coupling is accepted by many workers on account of its renormalizability,¹¹⁾ the renormalizable condition is not the final selection rules determining the interaction types. Presumably, the future theory will give the finite renormalizable mass and coupling constants. Judging from this point of view, if P , T and C invariance holds, the interaction type of quantum mesodynamics seem to be the PV coupling from the principle presented here.

As to another interaction types of baryons, we have no definite knowledge so far.

§ 5. Conclusions

By generalizing the Dirac equation so as to include the P and T non-invariant term and retaining the original form by the similarity transformation even if we introduce the interaction with another Fermion or Boson, we get the selection rules of various kinds of interactions which satisfy only Lorentz invariance, Hermiticity and particle-antiparticle conjugation law. First of all, since the severe conditions are required in quantum electrodynamics, so far as P and T invariances are accepted, the introduction of the Pauli term is prohibited. This is the selection principle obtained by the view-point different from Gell-Mann's minimal electromagnetic interaction law. In the weak interactions, only A and V interaction types are selected from others by using P and C non-invariance law. But the profound reason why P and C invariances are violated in the weak interactions will be postponed to future. Furthermore, if we take the point of view that the future theory will give the finite renormalizable constants, the result is obtained that only the PV coupling of PS π -meson is admitted in quantum mesodynamics. However, these selection rules in the case of strong interactions may not hold if P , T or C invariances break down in future experiments. As for these points, another paper will be written by the author and S. Sasaki later on. Especially, a substantialistic explanation of the phenomenological selection rules, e.g., the strangeness conserving or non-conserving law for baryon and heavy meson systems may require a more profound foundation than the one presented here.

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Note added in proof : It can be proved that the Sachs type interaction⁹⁾ is prohibited in the case of quantized field. The succeeding paper concerning these points will be published in Prog. Theor. Phys. in near future.

Theory of Line-Shapes of Interband Magneto-Optical Absorption in Semiconductors

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A theory of line shape of I. M. O.-absorption is developed with use of the two complementary approximations, one of which is an approach from the strong magnetic field and the other is that from the weak field. The interaction between carrier and phonon is taken into account after the method of generating function. The results for the scattering probability and the self-energy are shown to be in good agreement with the observed values. The calculated line shape which corresponds to the minimum photon energy can fairly explain the observed behavior.

§ 1. Introduction

Burstein *et al.*¹⁾ of N. R. group and Zwerdling *et al.*²⁾ of M. I. T. group showed experimentally that in the presence of a magnetic field there appeared many peaks as well as the shift of absorption edge in the line shape of light absorption which corresponds to the electronic transition from v - to c -band of semiconductors. By analyzing the inter-peak distance and the shift of absorption edge, we can determine the energy of band gap of direct transition between v - and c -bands and the effective masses of carriers near the top of v -band and near the bottom of c -band, which is just above the v -band. However, it is needed to analyze the line shape of the absorption peak when we determine the corresponding cyclotron resonance frequency from the positions of the peaks. Our aim of the present study is focused to this point of analyzing the line shape.

We have studied the dependency of line shape upon the magnetic field and sought the scattering probability of carrier by phonon as well as the self-energy in the presence of a magnetic field. We have discussed also whether or not the observed behavior of the line shape which corresponds to the minimum photon energy can be explained by our calculated results.

In order to treat the carriers near the extrema of the bands, we assume the simple parabolic model for both c - and v -band and neglect the band degeneracy as well as the spin-orbital interaction. However, it is shown that these simplifications have no essential effect on the line shape in so far as the first absorption peak concerns.

In § 1, the field dependency of the absorption edge and the selection rules are studied, neglecting the phonon perturbation. In § 2, the general formula of

absorption coefficient for I. M. O. is derived with use of Toyozawa's³⁾ method of generating function in taking the interaction of carrier with phonon. In § 3, we calculate the scattering probability and the self-energy, applying the two complementary approximations, one of which is an approach from a strong field and the other is that from a weak field and the results obtained there are compared with the reduced value from the experimental data in § 4. The results of our calculation have been reported briefly in this journal.⁴⁾

§ 2. General considerations on the no-phonon system

In the presence of a magnetic field H , the energies of the carriers become quantized in the plane perpendicular to the magnetic field but remain quasi-continuous in the direction of the field which is taken as z -direction hereafter, and form a series of Landau subbands (Fig. 1). By the use of the model of parabolic band—neglecting spin—the photon energy ε for transition between a Landau level of magnetic quantum number n_v in the v -band and a level of n_c in the c -band is

$$\varepsilon = \hbar^2 k_z^2 / 2\mu + \hbar\omega_c([n] + \frac{1}{2}) + E_g, \quad (2.1)$$

where $\omega_c = eH/\mu c$ is a reduced cyclotron frequency, $\mu = 1/(m_e^{-1} + m_v^{-1})$ is a reduced effective mass of an electron (m_e) and a hole (m_v), $E_g = E_c - E_v$ is the gap energy between the top of v -band (E_v) and the bottom of c -band (E_c), and $[n]$ is a reduced magnetic quantum number defined by

$$[n] = (m_e n_v + m_v n_c) / (m_e + m_v). \quad (2.2)$$

There are selection rules between n_v and n_c , which will be studied later.

The position of the center of cyclotron motion r_0 measured from a fixed point in the crystal is written as⁵⁾

$$r_0^2 = \lambda^2 (2l + 1), \quad \lambda^2 = \hbar c / eH, \quad (2.3)$$

where l is an azimuthal quantum number.

The density of states $N(\varepsilon)$ is thus given by

$$N(\varepsilon) = \frac{2V^{1/3}}{2\pi} \frac{(2\mu)^{1/2}}{\hbar} \left(\frac{V^{2/3}}{2\pi\lambda^2} \right) \sum_{[n]=0}^{[\varepsilon - E_g / \hbar\omega_c - 1/2]} [\varepsilon - \hbar\omega_c([n] + \frac{1}{2}) - E_g]^{-1/2}, \quad (2.4)$$

where V is the crystal volume and the factor $(V^{2/3}/2\pi\lambda^2)$ comes from the degene-

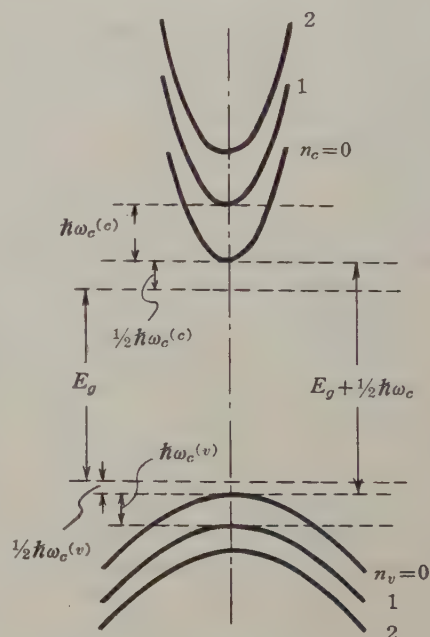


Fig. 1. Landau levels for a simple semiconductor.

racy of the positions of centers. As the summation can be replaced by the integration in the limit of $H \rightarrow 0$, $N(\varepsilon)$ becomes

$$N_0(\varepsilon) = \frac{V}{2\pi^2 \hbar^3} (2\mu)^{3/2} (\varepsilon - E_g)^{1/2}, \quad (2.5)$$

which is nothing but the density of states in the absence of a magnetic field. It should be noted that the density of states at the edge of each Landau band is infinite, while it is zero at the band edge in the absence of a magnetic field. The infinity must be removed by the level broadening caused by the phonon scattering.

After Hall *et al.*⁽⁶⁾ the coefficient of light absorption is given by

$$A(\varepsilon) = \frac{4\pi \hbar e^2}{n_0 c m^2 \varepsilon} \cdot N(E_f - E_i) \cdot |P_{if}|^2, \quad (2.6)$$

where n_0 is the refractive index of the crystal, c is the light velocity, m is the electronic mass and suffixes i and f denote an initial and a final state, respectively.

The matrix-element P_{if} is

$$P_{if} = \int \phi_f^* e_q \left(-i\hbar \nabla + \frac{e}{c} \mathbf{A} \right) \phi_i d\mathbf{r}, \quad (2.7)$$

where e_q is a polarization vector of photon and \mathbf{A} is a vector potential. In order to calculate the matrix element we use the approximate effective mass equation. Neglecting the interband coupling of wave functions,

$$\begin{aligned} \phi_i(\mathbf{r}) &= \sum_m \phi_v(\mathbf{R}_m) a_v(\mathbf{r} - \mathbf{R}_m), \\ \phi_f(\mathbf{r}) &= \sum_{m'} \phi_c(\mathbf{R}_{m'}) a_c(\mathbf{r} - \mathbf{R}_{m'}), \end{aligned} \quad (2.8)$$

where $a_{c,v}$ are the Wannier functions of c - and v -bands, respectively, \mathbf{R}_m denotes the position vector of m -th lattice point and the coefficient functions ϕ_v and ϕ_c satisfy the following equations, respectively.

$$\begin{aligned} -\frac{1}{2m_c} \left(\mathbf{P} + \frac{e}{c} \mathbf{A} \right)^2 \phi_c(\mathbf{R}_{m'}) &= (E_c - \varepsilon) \phi_c(\mathbf{R}_{m'}), \\ -\frac{1}{2m_v} \left(\mathbf{P} + \frac{e}{c} \mathbf{A} \right)^2 \phi_v(\mathbf{R}_m) &= -(E_v - \varepsilon) \phi_v(\mathbf{R}_m). \end{aligned} \quad (2.9)$$

Putting (2.8) into (2.7); we get

$$\begin{aligned} P_{if} &= N^{-1} \sum_{m,m'} \phi_c^*(\mathbf{R}_{m'}) \phi_v(\mathbf{R}_{m'}) \sum_{\mathbf{k}, \mathbf{k}'} e^{i\mathbf{k} \cdot \mathbf{R}_{m'} - i\mathbf{k}' \cdot \mathbf{R}_m} \\ &\quad \times \int u_{c\mathbf{k}}^* e^{-i\mathbf{k}' \cdot \mathbf{r}} \left(-i\hbar \nabla + \frac{e}{c} \mathbf{A} \right) u_{v\mathbf{k}'} e^{i\mathbf{k}' \cdot \mathbf{r}} d\mathbf{r}, \end{aligned} \quad (2.10)$$

where N is the number of unit cells in the crystal and \mathbf{k} is a wave vector of a carrier. Replacing $u_{c\mathbf{k}}$ and $u_{v\mathbf{k}}$ by u_{c0} and u_{v0} , respectively—this is equivalent to the first approximation by Luttinger and Kohn's⁽⁷⁾ method—we get

$$\int u_{\alpha 0}^* e^{-i\mathbf{k}\cdot\mathbf{r}} \left(-i\hbar \nabla_q + \frac{e}{c} A_q(\mathbf{r}) \right) u_{\beta 0} e^{i\mathbf{k}'\cdot\mathbf{r}} d\mathbf{r} \\ = \frac{(2\pi)^3}{V} \left[p_{\alpha\beta}^q(0) + \left\{ \hbar k_q' + \frac{e}{c} A_q(-i\nabla_{\mathbf{k}'}) \right\} \delta_{\alpha\beta} \right] \delta(\mathbf{k}-\mathbf{k}'), \quad (2.11)$$

where

$$p_{\alpha\beta}^q(0) = \int_{\mathbf{r}} u_{\alpha 0}^* (-i\hbar \nabla_q) u_{\beta 0} d\mathbf{r}, \quad (2.12)$$

and α, β are the band suffixes. In the case of I. M. O., the second and third terms of right-hand side of (2.11) vanish, which is in contrast to the vanishing of the first term in cyclotron resonance case. Thus we get

$$P_{ij} = \frac{N}{V} p_{cv}^q(0) \int \phi_c^*(\mathbf{r}) \phi_v(\mathbf{r}) d\mathbf{r}. \quad (2.13)$$

An explicit form for $\phi_{c,v}$ can be written down by the use of the cylindrical coordinate (ρ, θ, z) and the gauge, $\mathbf{A} = \frac{1}{2} \mathbf{H} \times \mathbf{r}$,⁽⁵⁾

$$\phi_{nlk_0}(\rho, \theta, z) \\ = \frac{V^{1/3}}{N^{1/2}} (-i)^n \left(2^{1-\nu} \frac{\pi l!}{n!} \right)^{-1,2} \lambda^{-1} \left(\frac{\rho}{2\lambda} \right)^\nu \exp \left(-\frac{\rho^2}{4\lambda^2} - i\nu\theta + ik_0 z \right) L_n^{(\nu)} \left(-\frac{\rho^2}{2\lambda^2} \right), \quad (2.14)$$

where $\nu = l - n$, $L_n^{(\nu)}$ is a Laguerre's associated polynomial, and k_0 is the wave number of a Landau carrier along z -direction.

Applying (2.14) to (2.13), we get

$$\Delta n = 0, \quad \Delta l = 0, \quad \Delta k_0 = 0 \quad (2.15)$$

as the selection rules. Taking into account the degeneracy of v -band, the additional selection rules must be derived. For example, in germanium the existence of V_1 and V_2 bands lets P_{ij} survive even when $\Delta n = 0$ and another selection rule, $\Delta n = -2$, is derived.^{(1), (8)}

In the next section, we use the Fourier transform of ϕ_{nlk_0} :

$$\chi_{nlk_0}(\mathbf{k}) = N^{-1/2} \sum_m \phi_{nlk_0}(\mathbf{R}_m) e^{-i\mathbf{k}\cdot\mathbf{R}_m}. \quad (2.16)$$

When $p_{cv}^q(0)$ in (2.13) vanishes, we obtain the following expression for the second order matrix element.

$$-\frac{N}{V} \sum_s \frac{i\omega_{cv}}{\hbar} \left[\sum_{\alpha \neq v} \frac{p_{c\alpha}^q(0) p_{\alpha v}^s(0)}{\omega_{\alpha v}} + \sum_{\alpha \neq c} \frac{p_{c\alpha}^s(0) p_{\alpha v}^q(0)}{\omega_{\alpha c}} \right] \cdot \int d\mathbf{r} \phi_c^*(\mathbf{r}) x_s \phi_v(\mathbf{r}) \\ - \frac{N}{V} \frac{e}{imc} \sum_s \frac{p_{cv}^s(0)}{\omega_{cv}} \int d\mathbf{r} \phi_c^*(\mathbf{r}) \left(\frac{\partial A_s}{\partial x_q} - \frac{\partial A_q}{\partial x_s} \right) \phi_v(\mathbf{r}), \quad (2.17)^*$$

* This is different from the one given by Burstein *et al.*⁽¹⁾ in the sign of the second term. Elliot *et al.*⁽⁸⁾ arrived at a different result because they dropped $e/c \cdot A_q$ from the photon perturbation.

where $\hbar\omega_{\alpha\beta} = E_{\alpha} - E_{\beta}$, and the summation over s stands for the summation over x -, y - and z -components of each quantities.

In the case of forbidden transition which is one of the second order effects, we get in the same way as the above,

$$\left\langle f | \mathbf{K} \mathbf{r} \left(-i\hbar \nabla_q + \frac{e}{c} A_q \right) | i \right\rangle = \frac{N}{V} p_{cr}^q(0) \sum_s K_s \int d\mathbf{r} \phi_c^*(\mathbf{r}) x_s \phi_v(\mathbf{r}), \quad (2.18)$$

where \mathbf{K} is the wave vector of photon. (2.18) is very small compared with the first order term expressed by (2.13) when $p_{cr}^q(0) \neq 0$.

§ 3. Absorption coefficient for I. M. O.

The total Hamiltonian of the system of electron, hole, lattice vibration and radiation field is

$$\mathcal{H}_{\text{tot}} = \mathcal{H}_e + \mathcal{H}_L + \mathcal{H}_R + \mathcal{H}_{eL} + \mathcal{H}_{eR} + \mathcal{H}_{RL}. \quad (3.1)$$

The terms on the right-hand side are Hamiltonians of electrons, phonons, photons, interaction Hamiltonians of electron-phonon, electron-photon, and photon-lattice systems, respectively.

We denote creation and destruction operators for an electron in the (nlk_0) state, for a phonon of mode μ , wave number vector \mathbf{w} , and for a photon with polarization vector \mathbf{e}_{σ} and wave number vector \mathbf{K} by $a_{nlk_0}^*$, a_{nlk_0} , $b_{\mu w}^*$, $b_{\mu w}$, $c_{\sigma \mathbf{K}}^*$ and $c_{\sigma \mathbf{K}}$, respectively.

To distinguish electron states from hole states we use primes on electron quantum numbers here, so in this notation (nlk_0) represents a hole in a state with n , l (z -component of angular momentum being $\hbar(n-l)$), and z -component of wave vector k_0 , and $(n'l'k'_0)$ represents an electron state.

\mathcal{H}_e , \mathcal{H}_L and \mathcal{H}_R can be written in the form,

$$\begin{aligned} \mathcal{H}_e = & \sum_{n'l', k'_0} \left\{ \hbar\omega_0^{(e)}(n' + \frac{1}{2}) + \frac{\hbar^2}{2m_e} k_0'^2 + E_g \right\} a_{n'l', k'_0}^* a_{n'l', k'_0} \\ & + \sum_{nlk_0} \left\{ \hbar\omega_0^{(e)}(n + \frac{1}{2}) + \frac{\hbar^2}{2m_v} k_0^2 \right\} a_{nlk_0}^* a_{nlk_0}, \end{aligned} \quad (3.2)$$

$$\mathcal{H}_L = \sum_{\mu w} \hbar\omega_{\mu w} b_{\mu w}^* b_{\mu w}, \quad (3.3)$$

$$\mathcal{H}_R = \sum_{\sigma \mathbf{K}} \hbar\omega_{\sigma \mathbf{K}} c_{\sigma \mathbf{K}}^* c_{\sigma \mathbf{K}}, \quad \omega_{\sigma \mathbf{K}} = \frac{c}{n_0} \mathbf{K}. \quad (3.4)$$

We ignore interaction of electrons with holes so we do not discuss exciton states here. Also we neglect \mathcal{H}_{RL} hereafter.

For simplicity, we write electron and hole states $(n'l'k'_0)$, (nlk_0) , respectively, as P and Q and a state of total electronic system (P, Q) as s .

Denoting the number of $(\sigma \mathbf{K})$ photons by $m_{\sigma \mathbf{K}}$, the matrix element for elec-

tronic transition from the ground state 0 to an excited state s with simultaneous absorption of a $(\sigma\mathbf{K})$ photon is

$$\begin{aligned} & \langle \Psi_s, \dots, m_{\sigma\mathbf{K}}-1, \dots | \mathcal{H}_{eR} | \Psi_0, \dots, m_{\sigma\mathbf{K}}, \dots \rangle \\ &= \frac{ie\hbar}{mc} (2\pi\hbar c m_{\sigma\mathbf{K}} / v_0 n_0 K)^{1/2} (\mathbf{e}_{\sigma\mathbf{K}} \cdot \mathbf{g}_s), \end{aligned} \quad (3.5)$$

where \mathbf{g}_s is defined by the use of P_{if} ((2.13)) through the relation

$$\mathbf{e}_q \mathbf{g}_s = N^{-1/2} P_{if}, \quad (\mathbf{q} = \sigma\mathbf{K}). \quad (3.6)$$

We use the notation

$$F_{ss'}(\mathbf{K}) = \frac{1}{2} \sum_{\sigma} (\mathbf{e}_{\sigma\mathbf{K}} \cdot \mathbf{g}_s^*) (\mathbf{e}_{\sigma\mathbf{K}} \cdot \mathbf{g}_{s'}). \quad (3.7)$$

Though there are several discussions about the role of optical modes of lattice vibration on the scattering of carriers in germanium, we consider here only the interaction of electrons and holes with longitudinal acoustical phonons.

$$\begin{aligned} \mathcal{H}_{eL} &= \sum_{\mu, \nu} i \sum_{ss'} \beta_{\mu, \nu ss'}^e (b_{\mu, \nu} - b_{\mu, \nu}^*) a_p^* a_{p'} + \sum_{\mu, \nu} i \sum_{ss'} \beta_{\mu, \nu ss'}^h (b_{\mu, \nu} - b_{\mu, \nu}^*) a_{q'}^* a_{q''} \\ &= \left(\frac{2\hbar}{9NMu} \right)^{1/2} i \sum_w \omega^{1/2} (b_w - b_w^*) \left\{ \sum_{pp'} C_e q_{pp'}^e(\omega) a_p^* a_{p'} - \sum_{qq'} C_h q_{qq'}^h(\omega) a_{q'}^* a_{q''} \right\}, \end{aligned} \quad (3.8)$$

where N, M, u, C_e and C_h are, respectively, number of unit cells in the crystal, atomic mass, velocity of sound, phonon-electron and phonon-hole coupling constants. $q_{pp'}^e(\omega)$ and $q_{qq'}^h(\omega)$ are the so-called effective charges for an electron and a hole (see the next section).

The absorption coefficient is written in the form,

$$A(\omega) = \frac{2\pi\hbar e^2}{m^2 v_0 n_0 c} \sum_{ss'} \frac{F_{ss'}(K)}{\omega_K} \int_{-\infty}^{\infty} dt \exp(i\omega t - i\varepsilon_s t/\hbar) \sum_{n=0}^{\infty} U_n(t; ss', K), \quad (3.9)$$

$$U_n(t; ss', K) = (i\hbar)^{-n} \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n \langle s | \mathcal{H}'(t_1) \cdots \mathcal{H}'(t_n) | s' \rangle_{AV}, \quad (3.10)$$

$$\mathcal{H}'(t) = \exp\{i(\mathcal{H}_e + \mathcal{H}_L)t/\hbar\} \mathcal{H}_{eL} \exp\{-i(\mathcal{H}_e + \mathcal{H}_L)t/\hbar\}, \quad (3.11)$$

where ε_s is the energy of total electronic system in the s -state relative to the ground state, and the average is taken over initial phonon states. $U_0 = \delta_{ss'}$, and U_n 's are zero for odd n .

In the calculation of

$$U_2 = -\hbar^{-2} \int_0^t dt_1 \int_0^{t_1} dt_2 \sum_{s_1 \nu_0} \{ \langle s | \mathcal{H}'(t_1) | s_1 \nu \rangle \langle s_1 \nu | \mathcal{H}'(t_2) | s' \rangle \}_{AV}, \quad (3.12)$$

states s and s' have energies varying almost continuously with respect to k_0 —the

restriction imposed by the selection rule is $k_0=k_0'$ —and we must be careful in separating diagonal ($s=s'$) and non-diagonal ($s\neq s'$) parts.

We define as

$$f_{ss'}^{s'}(E) = \sum_{\omega s_1} \beta_{\omega s_1 s}^* \beta_{\omega s_1 s'} [(n_w + 1) \delta(\varepsilon_{s_1} - \varepsilon_{s'} + \hbar\omega_w - E) + n_w \delta(\varepsilon_{s_1} - \varepsilon_{s'} - \hbar\omega_w - E)], \quad (3 \cdot 13)$$

$$f_{ss'}^s(E) = \sum_{\omega s_1} \beta_{\omega s_1 s}^* \beta_{\omega s_1 s} [(n_w + 1) \delta(\varepsilon_{s_1} - \varepsilon_s + \hbar\omega_w - E) + n_w \delta(\varepsilon_{s_1} - \varepsilon_s - \hbar\omega_w - E)], \quad (3 \cdot 13')$$

$$\Gamma_{ss'}^{s'} = \frac{2\pi}{\hbar} f_{ss'}^{s'}(0), \quad (3 \cdot 14)$$

$$\Delta_{ss'}^{s'} = -\oint f_{ss'}^{s'}(E)/E \cdot dE, \quad (3 \cdot 15)$$

$$\tau_c^s = \hbar (\partial f_{ss}^s(E)/\partial E)/f_{ss}^s(E), \quad (3 \cdot 16)$$

where \oint means the principal value of integral, n_w is the phonon population with wave number w .

For $|t| > \tau_c^s$,

$$U_2(t; ss', K) = \frac{i}{\varepsilon_s - \varepsilon_{s'}} \left[\exp\{i(\varepsilon_s - \varepsilon_{s'})t/\hbar\} \left(\pm \frac{\hbar}{2} \Gamma_{ss'}^{s'} + i\Delta_{ss'}^{s'} \right) - \left(\pm \frac{\hbar}{2} \Gamma_{ss'}^s + i\Delta_{ss'}^s \right) \right], \quad s' \neq s \quad (3 \cdot 17)$$

$$U_2(t; ss, K) = -(\pm \Gamma_s/2 + i\Delta_s/\hbar)t, \quad s' = s. \quad (3 \cdot 18)$$

\pm corresponding to $t \leq 0$.

As it will be shown in the next section the off-diagonal terms $\Delta_{ss'}^{s'}$ and $\Gamma_{ss'}^{s'}$ are much smaller than Δ_{ss}^s and Γ_{ss}^s , respectively, it may be allowed to take the interaction between sub-peaks into account by the use of correction factors,

$$\eta_s = 2 \sum_{s'} \text{Re} \{F_{ss'}(K) \Delta_{ss'}^s\} / F_{ss}(K) (\varepsilon_s - \varepsilon_{s'}), \quad (3 \cdot 19)$$

$$\mathcal{A}_s = \frac{1}{2} \sum_{s'} \text{Re} \{F_{ss'}(K) \hbar \Gamma_{ss'}^s\} / F_{ss}(K) (\varepsilon_s - \varepsilon_{s'}). \quad (3 \cdot 20)$$

Finally, the absorption coefficient is given by

$$A(\omega) = \frac{4\pi\hbar e^2}{3m^2 c v_0 n_0} \sum_s |g_s|^2 (1 + \eta_s) \frac{1}{\omega_k} \frac{\Gamma_s/2 + 2\mathcal{A}_s \Omega_s}{\Omega_s^2 + (\Gamma_s/2)^2}, \quad (3 \cdot 21)$$

$$\Omega_s = \omega - \varepsilon_s/\hbar - \Delta_s/\hbar, \quad (3 \cdot 22)$$

$$\Delta_s = \Delta_{ss}^s, \quad \Gamma_s = \Gamma_{ss}^s. \quad (3 \cdot 23)$$

§ 4. Scattering probability and self-energy

(A) Approximation of strong magnetic field

When the magnetic field is so strong that the energy distance between Landau levels becomes too large to allow the transition of carrier from a level to another one by the phonon perturbation, we can assume that the summation over intermediate states is restricted to the states within one Landau level. However, not only all the states of which orbital centers of the cyclotron motion are different from the initial one but also the phonon energy and the recoil energy of carriers are taken into consideration.

By the use of the approximation of high temperature for phonon population and the acoustical mode of phonon wave, we get, from (3.8) and (3.13),

$$f_{ss'}^{st}(E) = \frac{2\kappa T}{9NMu} \left\{ C_c^2 \sum_{i,w} q_{P_i P'}^*(w) q_{P_i P'}(w) [\delta(\mathcal{J}E_i + \hbar\omega_{uw}) + \delta(\mathcal{J}E_i - \hbar\omega_{uw})] \right. \\ \left. + C_v^2 \sum_{i,w} q_{Q_i Q'}^*(w) q_{Q_i Q'}(w) [\delta(\mathcal{J}E_Q + \hbar\omega_{uw}) + \delta(\mathcal{J}E_Q - \hbar\omega_{uw})] \right. \\ \left. - 2C_c C_v \sum_w q_{P_i P'}(w) q_{Q_i Q'}(w) [\delta(\mathcal{J}E_{PQ} + \hbar\omega_{uw}) + \delta(\mathcal{J}E_{PQ} - \hbar\omega_{uw})] \right\}. \quad (4.1)$$

Suffix i shows the intermediate state and the prime refers to the final state, thus the initial state is shown without suffix in (4.1).

$$\begin{aligned} \mathcal{J}E_P &= \alpha' (k_{01}^2 - k_0'^2) + \Delta n \hbar \omega_c^{(v)} - E, \\ \mathcal{J}E_Q &= \alpha (k_{01}^2 - k_0''^2) + \Delta n \hbar \omega_c^{(c)} - E, \\ \mathcal{J}E_{PQ} &= \alpha' (k_0'^2 - k_0''^2) + \alpha (k_0^2 - k_0''^2) + \Delta n \hbar \omega_c^{(c)} + \Delta n' \hbar \omega_c^{(v)} - E, \\ \alpha' &= \hbar^2/2m_c, \quad \alpha = \hbar^2/2m_v. \end{aligned} \quad (4.2) \quad (4.3)$$

E is hereafter put to zero because we are seeking for $\Gamma_{ss'}^{st}$.

In order to treat the line shape of the first peak which appears in I. M. O. absorption, we take each state defined, respectively, by

$$\begin{aligned} \text{initial state: } S(P, Q) &= S\{P(n=0, l=l', k_z=k_0'), Q(n=0, l=l, k_z=k_0)\}, \\ \text{intermediate state: } S_i(P_i, Q_i) &= S_i\{P_i(n=0, l=l_1, k_z=k_{01}), \\ &\quad Q_i(n=0, l=l_1, k_z=k_{01})\}, \\ \text{final state: } S'(P', Q') &= S'\{P'(n=0, l=l', k_z=k_0'''), \\ &\quad Q'(n=0, l=l, k_z=k_0'')\}. \end{aligned}$$

The transition matrix element, which is the so-called effective charge, between two states within a Landau level of $n=0$ is given by

$$q_{s1s'}^{(\pm)}(w) = \sum_{\mathbf{k}} \chi_{s1}^*(\mathbf{k}') \chi_{s'}(\mathbf{k}) \delta_{\mathbf{k}', \mathbf{k} \pm \mathbf{w}}. \quad (4.4)$$

Applying (2.14) and (2.16) to (4.4) and replacing Kronecker's delta by the delta function, we get

$$\begin{aligned}
 q_{s_1 s'}^{(\pm)}(w) &= 2\pi V^{-1/3} \delta(k_{01} - k_0'' \mp w_z) \exp(-\lambda'^2 w_{\perp}^2) \{\text{polynomial in } \lambda' w_{\perp}\}, \\
 q_{s_1 s}^{(\pm)}(w) &= 2\pi V^{-1/3} \delta(k_{01} - k_0 \mp w_z) \exp(-\lambda'^2 w_{\perp}^2) \{\text{polynomial in } \lambda' w_{\perp}\},
 \end{aligned} \quad (4.5)^*$$

and

$$\sum_{l_1} q_{s_1 s'}^{(\pm)*}(w) q_{s_1 s}^{(\pm)}(w) = 4\pi^2 V^{-2/3} \delta(k_{01} - k_0 \mp w_z) \delta(k_{01} - k_0'' \mp w_z) \exp(-\lambda'^2 w_{\perp}^2),$$

where $\lambda' = \lambda/V^{1/2}$. We can derive (4.5) also by the use of an operator calculus (see Appendix I). The factor placed in the curled bracket is due to the transition of the center of cyclotron motion and if this factor were not present, the effective charge for very strong field ($\lambda'^2 w_{\perp}^2 \approx 0$) may be taken as unity which is a similar case to exciton.

Putting (4.5) into (4.1), the first term of the right-hand side of (4.1) is

$$\begin{aligned}
 I_f &= \frac{2\kappa TC_c^2}{9NMu^2} \cdot \frac{V}{(2\pi)^3} \cdot \frac{1}{(\hbar u)^3} \sum_l \int q_{P_i P'}^*(w) q_{P_i P'}(w) [\delta(\Delta E_P + \hbar\omega_w) + \delta(\Delta E_P - \hbar\omega_w)] \\
 &\quad \times (\hbar u w)^2 d(\hbar u w) \sin\theta d\theta d\varphi.
 \end{aligned} \quad (4.6)$$

The summation must be taken over k_{01} and l_1 but the summation over l_1 is easily taken and the resultant formula is given by

$$\begin{aligned}
 I_f &= \frac{2\kappa TVC_c^2}{9NMu^2(\hbar u)^2} (2\pi) \sum_{k_{01}} \Delta E_P^2 \int_0^\pi (2\pi V^{-1/3})^2 \exp\{-\lambda'^2 (\Delta E_P/\hbar u)^2 \sin^2\theta\} \sin\theta d\theta \\
 &\quad \times \left[\delta\left(k_{01} - k_0' - \frac{\Delta E_P}{\hbar u} \cos\theta\right) \delta\left(k_{01} - k_0''' - \frac{\Delta E_P}{\hbar u} \cos\theta\right) \right. \\
 &\quad \left. + \delta\left(k_{01} - k_0' + \frac{\Delta E_P}{\hbar u} \cos\theta\right) \delta\left(k_{01} - k_0''' + \frac{\Delta E_P}{\hbar u} \cos\theta\right) \right] \\
 &= \frac{2\kappa TV^{1/3} C_c^2 \alpha'^2}{9NMu^2(\hbar u)^3} \cdot \delta(k_0' - k_0''') \sum_{k_{01}} (k_{01}^2 - k_0'''^2)^2 \left(\frac{\hbar u}{\Delta E_P}\right) \\
 &\quad \times \int_{-\Delta E_P/\hbar u}^{\Delta E_P/\hbar u} \exp\{\lambda'^2 x^2 - \lambda'^2 (\Delta E_P/\hbar u)^2\} dx \\
 &\quad \times [\delta(k_{01} - k_0' - x) + \delta(k_{01} - k_0' + x)] \\
 &= \frac{2\kappa TC_c^2 V^{2/3} \alpha'}{9NMu^2(\hbar u)^3} \cdot \frac{1}{2\pi} \delta_{k_0', k_0'''} \cdot \frac{V^{1/3}}{2\pi} \cdot 2 \int_{-\infty}^{\infty} dk_{01} (k_{01}^2 - k_0'^2) \\
 &\quad \times \exp\{-\gamma^2 (k_{01}^2 - k_0'^2)^2 + \lambda'^2 (k_{01} - k_0')^2\} \\
 &= \frac{4\kappa TC_c^2 \Omega_0 \alpha'}{9Mu^2(\hbar u)^2 \lambda'^3} \delta_{k_0', k_0'''} \int_{-\infty}^{\infty} dx (x^2 + 2\lambda' k_0' x) \exp\{-\gamma^2 (x^2 + 2\lambda' k_0' x)^2 + x^2\},
 \end{aligned} \quad (4.7)$$

where Ω_0 is the volume of unit cell and

* Concerning the explicit forms of the polynomials, see (A.4) and (4.32).

$$\gamma^2 = (\alpha' / \hbar u \lambda')^2. \quad (4.8)$$

The numerical values of the integration

$$J_{\alpha}(k_0', H) = \int_{-\infty}^{\infty} dx (x^2 + 2\lambda' k_0' x) \times \exp \{ -\gamma^2 (x^2 + 2\lambda' k_0' x)^2 + x^2 \} \quad (4.9)$$

are shown in Fig. 2 for $k_0' = 0$ as function of the strength of field and effective masses.

Because of the underlying assumption of neglecting the virtual transitions between different Landau levels in taking the intermediate states, the present result is valid only near $k_0' = k_0 = 0$. (see § 5).

In the same way the second term of the right-hand side of (4.1) is given by

$$II_f = \frac{4\kappa TC_v^2 \Omega_0 \alpha \delta_{k_0, k_0''}}{9Mu^2 (\hbar u)^2 \lambda'^3} J_{\alpha}(k_0, H). \quad (4.10)$$

The third term—the cross term—of the right-hand side of (4.1) is

$$\begin{aligned} III_f &= -\frac{4\kappa TC_v C_v}{9NMu^2} \cdot \frac{V}{(2\pi)^3} \cdot \frac{1}{(\hbar u)^3} \int (\hbar u w)^2 d(\hbar u w) \sin \theta d\theta d\varphi q_{P'P'}(w) q_{Q'Q'}(w) \\ &\quad \times [\delta(\Delta E_{PQ} + \hbar \omega_w) + \delta(\Delta E_{PQ} - \hbar \omega_w)] \\ &\approx -\frac{4\kappa TC_v C_v}{9NMu^2} \cdot \frac{V^{1/3}}{2\pi} \cdot \frac{(\Delta E_{PQ})^2}{(\hbar u)^3} \int_0^{\pi} \sin \theta d\theta \exp \{ -(\lambda' / \hbar u)^2 \Delta E_{PQ}^2 \sin^2 \theta \} \\ &\quad \times \left[\delta \left(k_0 - k_0'' - \frac{\Delta E_{PQ}}{\hbar u} \cos \theta \right) \delta \left(k_0' - k_0''' - \frac{\Delta E_{PQ}}{\hbar u} \cos \theta \right) \right. \\ &\quad \left. + \delta \left(k_0 - k_0'' + \frac{\Delta E_{PQ}}{\hbar u} \cos \theta \right) \delta \left(k_0' - k_0''' + \frac{\Delta E_{PQ}}{\hbar u} \cos \theta \right) \right] \\ &= -\frac{8\kappa TC_v C_v}{9NMu^2} \cdot \frac{V^{1/3}}{2\pi} \cdot \frac{\Delta E_{PQ}}{(\hbar u)^2} \cdot \delta(k_0 - k_0'' + k_0' - k_0''') \\ &\quad \times \exp \left\{ -\left(\frac{\lambda'}{\hbar u} \Delta E_{PQ} \right)^2 + \lambda'^2 (k_0 - k_0'')^2 \right\} \end{aligned}$$

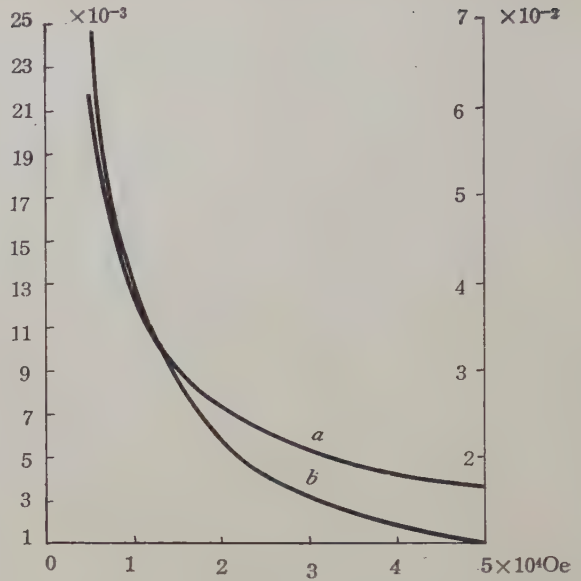


Fig. 2. $J_{\alpha}(0, H)$ versus H . Curve a is plotted for $m^*/m=0.05$ and refers to the left side ordinate. Curve b is plotted for $m^*/m=0.1$ and refers to the right side ordinate.

$$\begin{aligned}
 &= -\frac{8\kappa TC_c C_v \Omega_0}{9Mu^2 (\hbar u)^2} V^{-1/3} \delta_{k_0-k_0'', k_0'''-k_0'} \{ \alpha' (k_0'^2 - k_0'''^2) + \alpha (k_0'^2 - k_0''^2) \} \\
 &\quad \times \exp \left\{ -\left(\frac{\lambda'}{\hbar u} \right)^2 \{ \alpha' (k_0'^2 - k_0'''^2) + \alpha (k_0'^2 - k_0''^2) \}^2 + \lambda'^2 (k_0 - k_0'')^2 \right\}.
 \end{aligned} \quad (4.11)$$

This cross term is very much smaller than I_f and II_f by the ratio of $V^{-1/3}$ to λ^{-1} , therefore we get

$$\Gamma_{ss'} = \frac{2\pi}{\hbar} \cdot \frac{8\kappa T \Omega_0}{9Mu^2 (\hbar u)^2 \lambda'^3} [\alpha' C_c^2 \delta_{k_0', k_0''' } J_{\alpha'}(k_0', H) + \alpha C_v^2 \delta_{k_0, k_0'' } J_{\alpha}(k_0, H)]. \quad (4.12)$$

Substituting the numerical values of $\kappa T \approx 10^{-2}$ eV (room temperature), $M(\text{Ge}) \approx 1.4 \times 10^5 m$, $\Omega_0(\text{Ge}) \approx 2.2 \times 10^{-23}$ c. c., $u \approx 5 \times 10^5$ cm sec $^{-1}$ and $H = 5 \times 10^4$ Oe into (4.12), we obtain for the half-width of the sub-peak near edge

$$\hbar \Gamma_s \approx 0.127 \times 10^{-4} \left[\frac{m}{m_0} C_c^2 J_{\alpha'}(0, H) + \frac{m}{m_v} C_v^2 J_{\alpha}(0, H) \right] \text{eV}. \quad (4.13)$$

If we neglect the phonon energy in the arguments of the delta functions appearing in the $f(E)$ of (4.1), the treatment becomes very simple and the collision frequency is readily obtained, *e. g.*, for the electron

$$\begin{aligned}
 \frac{1}{\tau} &= \frac{4\kappa TC_c^2 \Omega_0}{9Mu^2} \cdot \frac{1}{\lambda^2} \cdot \frac{1}{4\pi \hbar^2} \sum_n \int d\omega_z [\delta(\Delta\omega_p) \delta_{k_{01}, k_0''' + \omega_z} \delta_{k_{01}, k_0' + \omega_z} \\
 &\quad + \delta(\Delta\omega_p) \delta_{k_{01}, k_0''' - \omega_z} \delta_{k_{01}, k_0' - \omega_z}],
 \end{aligned} \quad (4.14)$$

which is in exact agreement with the scattering probability shown in the paper on the quantum theory of galvanomagnetic effects by Argyres⁹⁾.

Next, we proceed to seek the self-energy Δ_s with use of the same assumptions for phonon population and phonon mode as was used in the above.

$$\begin{aligned}
 \Delta_{ss'} &= -\frac{2\kappa T}{9NMu^2} \left\{ C_c^2 \sum_{i, \omega} q_{P_i P'}^*(\omega) q_{P_i P'}(\omega) \left[\frac{\phi}{\Delta E_P - \hbar \omega_{\omega}} + \frac{\phi}{\Delta E_P + \hbar \omega_{\omega}} \right] \right. \\
 &\quad + C_v^2 \sum_{i, \omega} q_{Q_i Q'}^*(\omega) q_{Q_i Q'}(\omega) \left[\frac{\phi}{\Delta E_Q - \hbar \omega_{\omega}} + \frac{\phi}{\Delta E_Q + \hbar \omega_{\omega}} \right] \\
 &\quad \left. - 2C_c C_v \sum_{\omega} q_{P_i P'}(\omega) q_{Q_i Q'}(\omega) \left[\frac{\phi}{\Delta E_{PQ} - \hbar \omega_{\omega}} + \frac{\phi}{\Delta E_{PQ} + \hbar \omega_{\omega}} \right] \right\}.
 \end{aligned} \quad (4.15)$$

The first term of the right-hand side of the above equation is

$$\begin{aligned}
 I_g &= -\frac{4\pi\kappa TC_c^2}{9NMu^2} V^{-1/3} \sum_{\omega} \exp(-\lambda'^2 \omega_{\perp}^2) \int dk_{01} \left[\delta(k_{01} - k_0' - \omega_z) \delta(k_{01} - k_0''' - \omega_z) \right. \\
 &\quad \left. \times \frac{\phi}{\Delta E_P - \hbar \omega_{\omega}} + \delta(k_{01} - k_0' + \omega_z) \delta(k_{01} - k_0''' + \omega_z) \frac{\phi}{\Delta E_P + \hbar \omega_{\omega}} \right]
 \end{aligned}$$

$$\begin{aligned}
&= -\frac{4\pi\kappa TC_c^2 V^{-1/3}}{9NMu^2} \cdot \delta(k_0' - k_0''') \cdot \frac{V}{(2\pi)^3} \int_0^{2\pi} d\varphi \int_0^\infty w_\perp dw_\perp \exp(-\lambda'^2 w_\perp^2) \int_{-\infty}^\infty dw_z \\
&\quad \times \left[\frac{\phi}{\alpha' \{ (k_0' + w_z)^2 - k_0'''^2 \} - \hbar u \{ w_\perp^2 + w_z^2 \}^{1/2}} \right. \\
&\quad \left. + \frac{\phi}{\alpha' \{ (k_0' + w_z)^2 - k_0'''^2 \} + \hbar u \{ w_\perp^2 + w_z^2 \}^{1/2}} \right]. \quad (4.16)
\end{aligned}$$

The integration over w_z is carried out in the case of $k_0' = k_0''' = 0$ and we get (see Appendix II)

$$I_g = \frac{2\pi\kappa T \Omega_0 C_c^2 \delta_{k_0', k_0'''}}{9M(2\pi u)^2 \alpha' \lambda'} \int_0^\infty dx e^{-x^2} \frac{1}{(1+X'^2)^{1/2} \{2X'^2 + 2X'(1+X'^2)^{1/2}\}^{1/2}}, \quad (4.17)$$

$$X' = m_e u \lambda' / \hbar x, \quad x = 2\lambda' w_\perp. \quad (4.18)$$

In the same way, the second term of the right-hand side of (4.15) is given by

$$II_g = \frac{2\pi\kappa T \Omega_0 C_v^2 \delta_{k_0, k_0''}}{9M(2\pi u)^2 \alpha' \lambda'} \int_0^\infty dx e^{-x^2} \frac{1}{(1+X^2)^{1/2} \{2X^2 + 2X(1+X^2)^{1/2}\}^{1/2}}, \quad (4.19)$$

$$X = \frac{m_e u \lambda'}{\hbar x}.$$

The third term—cross term—is

$$\begin{aligned}
III_g &\approx -\frac{2\kappa TC_c C_v \Omega_0 \delta_{k_0 - k_0'', k_0''' - k_0'}}{9Mu^2 \pi V^{1/3}} \int_0^\infty w_\perp dw_\perp \exp(-\lambda'^2 w_\perp^2) \\
&\quad \times \left[\frac{\phi}{\alpha(k_0'''^2 - k_0^2) + \alpha'(k_0'''^2 - k_0'^2) - \hbar u \{ (k_0' - k_0''')^2 + w_\perp^2 \}^{1/2}} \right. \\
&\quad \left. + \frac{\phi}{\alpha(k_0'''^2 - k_0^2) + \alpha'(k_0'''^2 - k_0'^2) + \hbar u \{ (k_0' - k_0''')^2 + w_\perp^2 \}^{1/2}} \right], \quad (4.20)
\end{aligned}$$

which is smaller than I_g , II_g by the ratio of $V^{-1/3}$ to λ^{-1} , therefore we get near $k_0 = k_0' = 0$,

$$A_s = -\frac{4\kappa T \Omega_0}{9M(2\pi u)^2 \lambda'} \left[\frac{C_c^2}{\alpha'} K_{\alpha'}(0, H) + \frac{C_v^2}{\alpha} K_\alpha(0, H) \right], \quad (4.21)$$

where

$$K_\alpha(0, H) = \int_0^\infty dx e^{-x^2} \frac{1}{(1+X^2)^{1/2} \{2X^2 + 2X(1+X^2)^{1/2}\}^{1/2}} \quad (4.22)$$

is a numerical coefficient. By the numerical integration, we get

$$K_{\alpha'}(0, H) = 4.82 \quad \text{for } m_c = 0.05 m \quad \text{and } H = 5 \times 10^4 \text{ Oe.}$$

$$K_{\alpha}(0, H) = 0.338 \quad \text{for } m_v = 0.5 m \quad \text{and } H = 5 \times 10^4 \text{ Oe.}$$

In the absence of a magnetic field, Muto and Ohyama¹⁰⁾ have given the formula for self-energy, which is, for the bottom of *c*-band,

$$\frac{2\kappa T \Omega_0 C_c^2 m_c}{9M(\pi \hbar u)^2} (2w_0), \quad (4.23)$$

where w_0 is the maximum wave number of phonon. The corresponding formula in the present case is

$$\frac{2\kappa T \Omega_0 C_c^2 m_c}{9M(\pi \hbar u)^2} \left(4.82 \frac{1}{\lambda'} \right). \quad (4.24)$$

The ratio of (4.24) to (4.23) is about 1 to 20 for germanium in the presence of the magnetic field of 5×10^4 Oe. Our result expressed by (4.24) is not always valid because of the underlying assumption of restricting the summation over intermediate state within the initial Landau level. A reliable value of the self-energy is derived in the following sub-section.

(B) Approximation of weak magnetic field

Neglecting the magnetic energy and phonon energy compared with the recoil energy of carriers, we can sum up the intermediate states over all Landau levels.

Introducing the integral representation for the delta-function, we obtain

$$f_{ss'}^{s'}(E) = \frac{1}{2\pi \hbar} \int_{-\infty}^{\infty} dt \exp \{ -i(\varepsilon_{s'} + E)t/\hbar \} \sum_{\mu\nu} [(n_{\mu\nu} + 1) \exp(i\omega t) + n_{\mu\nu} \exp(-i\omega t)] \\ \times \{ \sum_{s_1} \beta_{\mu\nu s_1 s}^* \beta_{\mu\nu s_1 s'} \exp(i\varepsilon_{s_1} t/\hbar) \}, \quad (4.25)$$

$$\sum_{s_1} \beta_{\mu\nu s_1 s}^* \beta_{\mu\nu s_1 s'} \exp(i\varepsilon_{s_1} t/\hbar) = \langle s | \beta_{\mu\nu}^* \exp(i\mathcal{H}t/\hbar) \beta_{\nu}^{\mu} | s' \rangle \\ = \frac{2\hbar w}{9NMu} \langle s | [C_c \exp(-i\mathbf{w}\mathbf{r}_c) - C_v \exp(-i\mathbf{w}\mathbf{r}_v)] \exp\{i(\mathcal{H}_c + \mathcal{H}_v)t/\hbar\} \\ \times [C_c \exp(i\mathbf{w}\mathbf{r}_c) - C_v \exp(i\mathbf{w}\mathbf{r}_v)] | s' \rangle \\ = \frac{2\hbar w}{9NMu} \langle s | C_c^2 \exp\{i(\mathcal{H}_v + \mathcal{H}_c^*)t/\hbar\} + C_v^2 \exp\{i(\mathcal{H}_v^* + \mathcal{H}_c)t/\hbar\} \\ - C_c C_v [\exp\{-i\mathbf{w}(\mathbf{r}_v - \mathbf{r}_c)\} \exp\{i(\mathcal{H}_v + \mathcal{H}_c^*)t/\hbar\} \\ + \exp\{i\mathbf{w}(\mathbf{r}_v - \mathbf{r}_c)\} \exp\{i(\mathcal{H}_v^* + \mathcal{H}_c)t/\hbar\}] | s' \rangle, \quad (4.26)$$

where

$$\mathcal{H}_{c,v} = \frac{1}{2m_{c,v}} \left(\mathbf{P} \pm \frac{e}{c} \mathbf{A} \right)^2, \quad \mathcal{H}_{c,v}^* = \frac{1}{2m_{c,v}} \left(\mathbf{P} \pm \frac{e}{c} \mathbf{A} + \hbar \mathbf{w} \right)^2. \quad (4.27)$$

Using the approximation of high temperature for the population of acoustical phonon, we obtain

$$f_{ss'}^{s'}(E) = \frac{\kappa T}{9\pi N M u^2 \hbar} [C_e^2 J_1(E) + C_v^2 J_2(E) - C_e C_v (J_3(E) + J_4(E))], \quad (4.28)$$

where

$$J_j(E) = J_j^{(+)}(E) + J_j^{(-)}(E), \quad (4.29)$$

$$J_1^{(\pm)}(E) = \sum_w \delta_{QQ'} \int_{-\infty}^{\infty} dt \cdot \exp\{i(E_g + E_p - E_{s'} - E \pm \hbar\omega)t/\hbar\} \\ \times \langle P | \exp\{i\mathcal{H}_e^* t/\hbar\} | P' \rangle, \quad (4.30)$$

$$J_2^{(\pm)}(E) = \sum_w \delta_{PP'} \int_{-\infty}^{\infty} dt \cdot \exp\{i(E_g + E_q - E_{s'} - E \pm \hbar\omega)t/\hbar\} \\ \times \langle Q | \exp\{i\mathcal{H}_v^* t/\hbar\} | Q' \rangle,$$

$$J_3^{(\pm)}(E) = \sum_w q_{QQ'}(-w) \int_{-\infty}^{\infty} dt \cdot \exp\{i(E_g + E_{p'} - E_{s'} - E \pm \hbar\omega)t/\hbar\} \\ \times \langle P | \exp(i\mathbf{w} \cdot \mathbf{r}_e) \exp(i\mathcal{H}_e^* t/\hbar) | P' \rangle, \quad (4.31)$$

$$J_4^{(\pm)}(E) = \sum_w q_{PP'}(-w) \int_{-\infty}^{\infty} dt \cdot \exp\{i(E_g + E_{q'} - E_{s'} - E \pm \hbar\omega)t/\hbar\} \\ \times \langle Q | \exp(i\mathbf{w} \cdot \mathbf{r}_v) \exp(i\mathcal{H}_v^* t/\hbar) | Q' \rangle.$$

In order to get the matrix elements of the last factors in the right-hand side of (4.30) and (4.31), we introduce the next operators :

$$u_{x,y} = \lambda' w_{x,y}, \quad O_{x,y} = \lambda'/\hbar \cdot \pi_{x,y}, \quad L_{x,y} = \frac{1}{2\lambda'} x_0, y_0, \\ \pi_{\pm} = \pi_x \pm i\pi_y, \quad L_{\pm} = L_x \pm iL_y, \quad u_{\pm} = u_x \pm iu_y, \\ O^{\pm} = O_x \pm iO_y, \quad x_0 = x - \pi_y/m\omega, \quad y_0 = y + \pi_x/m\omega, \\ \pi = P + \frac{e}{c} A. \quad (4.32)$$

Then, we obtain

$$\exp(i\mathcal{H}_e^* t/\hbar) = \exp\left[-\frac{it}{2m_e \hbar} \left\{ \hbar^2 (k_x + w_x)^2 + \hbar^2 w_{\perp}^2 + \frac{e\hbar}{c} H \right\}\right] \\ \times \exp\left\{ \frac{i\hbar t}{m\lambda^2} (O^+ O^- + u_+ O^- + u_- O^+) \right\}, \quad (4.33)$$

$$\exp\{i\mathbf{w} \cdot \mathbf{r}\} = \exp\{i w_x z\} \exp\{i(u_- L_+ + u_+ L_-) \exp\{-(u_+ O^- - u_- O^+)\}\}. \quad (4.34)$$

We can proceed readily when the following approximation is allowed for the weak magnetic field in the case of a transition from $(0, l, k_0')$ to $(0, l', k_0''')$.

$$\begin{aligned}
 & \langle 0 | \exp \{ i \hbar t / m \lambda^2 \cdot (O^+ O^- + u_+ O^- + u_- O^+) \} | 0 \rangle \delta_{ll'} \\
 & \approx \langle 0 | \exp \{ i \hbar t / m \lambda^2 \cdot (u_+ O^- + u_- O^+) \} | 0 \rangle \delta_{ll'} = \exp \left\{ - \frac{u_+ u_-}{2} (\omega_c t)^2 \right\} \delta_{ll'}.
 \end{aligned}
 \quad (4.35)$$

Then

$$\begin{aligned}
 & \langle 0, l, k_0' | \exp \{ i \mathbf{w} \cdot \mathbf{r}_0 \} \exp \{ i \mathcal{H}_c^* t / \hbar \} | 0, l', k_0''' \rangle \\
 & = \delta_{w_z + k_0''', k_0'} \exp \left\{ \frac{i}{2} \omega_c t \right\} \exp \left\{ \frac{i \hbar t}{2m} (k_0'^2 + w_\perp^2) \right\} \exp \{ -i A u_+ u_- \} \\
 & \quad \times \langle l | \exp \{ i (u_- L_+ + u_+ L_-) \} | l' \rangle \langle 0 | \exp \{ - (u_+ O^- - u_- O^+) \} | 0 \rangle \\
 & \quad \times \langle 0 | \exp \{ i A (u_+ O^- + u_- O^+) \} | 0 \rangle \\
 & = \delta_{w_z + k_0''', k_0'} \exp \left\{ \frac{i}{2} \omega_c t \right\} \exp \left\{ \frac{i \hbar t}{2m} k_0'^2 \right\} \exp \left\{ - \frac{u_+ u_-}{2} \right\} \\
 & \quad \times \exp \left\{ - \frac{A^2}{2} u_+ u_- \right\} \exp \left\{ - \frac{u_+ u_-}{2} \right\} \\
 & \quad \times \{ \text{polynomial in } \lambda' w_\perp \},
 \end{aligned}
 \quad (4.36)$$

where

$$A = i \hbar t / m \lambda^2.$$

The formulae used for the derivation of (4.35) and (4.36) are shown in Appendix I. By the use of the above formulae, (4.30) is calculated and gives

$$\begin{aligned}
 J_1^{(\pm)} & = \delta_{QQ'} \delta_{ll' l'' l'''} \delta_{k_0' k_0'''} \sum_w \int_{-\infty}^{\infty} dt \exp \left[\frac{i \hbar t}{2m_c} \left\{ (k_0' + w_z)^2 - k_0'^2 + w_\perp^2 - \frac{2m_c}{\hbar^2} E \pm \frac{2m_c u}{\hbar} w \right\} \right] \\
 & \quad \times \exp \left\{ - \frac{u_+ u_-}{2} (\omega_c t)^2 \right\} \\
 & = \delta_{QQ'} \delta_{ll' l'' l'''} \delta_{k_0' k_0'''} \frac{2 \sqrt{\pi} m_c \lambda}{\hbar} \sum_w \frac{1}{w_\perp} \\
 & \quad \times \exp \left[- \left(\frac{\lambda}{w_\perp} \right)^2 \left\{ (k_0' + w_\perp)^2 - k_0'^2 + w_\perp^2 - \frac{2m_c}{\hbar^2} E \pm \frac{2m_c u}{\hbar} w \right\}^2 \right] \\
 & = \delta_{QQ'} \delta_{ll' l'' l'''} \delta_{k_0' k_0'''} \frac{2 \sqrt{\pi} m_c \lambda}{\hbar} \frac{V}{(2\pi)^3} \int_0^{2\pi} d\varphi \int_{-\infty}^{\infty} dw_z \int_0^{\infty} dw_\perp \\
 & \quad \times \exp \left[- \frac{\lambda^2}{4} \left\{ w_\perp^2 + 2Z_1^{(\pm)} + \frac{(Z_1^{(\pm)})^2}{w_\perp^2} \right\} \right],
 \end{aligned}
 \quad (4.37)$$

where

$$Z_1^{(\pm)} = (k_0' + w_z)^2 - k_0'^2 - \frac{2m_c}{\hbar^2} E \pm \frac{2m_c u}{\hbar} w. \quad (4.38)$$

$2m_e u v / \hbar$ can be dropped in $Z_1^{(\pm)}$ when $2m_e u \ll \hbar \omega$. This corresponds to the approximation that phonon energy is neglected compared with the recoil energy of carriers, such an assumption is often used in the calculation of carrier mobility in semiconductors.¹¹⁾

Then,

$$\begin{aligned} J_1(E) &= \delta_{qQ'} \delta_{l'l'''} \delta_{k_0' k_0'''} \frac{\sqrt{\pi} \lambda m_e V}{2\pi^2 \hbar} \int_{-\infty}^{\infty} d\omega_z \exp \left\{ -\frac{\lambda^2}{2} Z_1 \right\} \cdot \frac{\sqrt{\pi}}{\lambda} \exp \left\{ -\frac{\lambda^2}{2} |Z_1| \right\} \\ &= \delta_{qQ'} \delta_{l'l'''} \delta_{k_0' k_0'''} \frac{m_e V}{2\pi \hbar} \int_{-\infty}^{\infty} d\omega_z \exp \left\{ -\frac{\lambda^2}{2} (Z_1 + |Z_1|) \right\}. \end{aligned} \quad (4.39)$$

$J_1(0)$ at $k_0' \approx 0$ is immediately obtained (see Appendix III),

$$J_1(0) = \delta_{qQ'} \delta_{l'l'''} \delta_{k_0' k_0'''} \frac{\sqrt{\pi} m_e V}{2\pi \hbar \lambda}. \quad (4.40)$$

Similarly, we have

$$J_2(0) = \delta_{P'P} \delta_{l'l''} \delta_{k_0'' k_0'''} \frac{\sqrt{\pi} m_v V}{2\pi \hbar \lambda}, \quad (4.41)$$

$$\begin{aligned} J_3(E) &\approx \delta_{k_0-k_0', k_0'''-k_0''} \frac{V^{2/3}}{2\pi} \int_0^{\infty} \omega_{\perp} d\omega_{\perp} \exp \left\{ -\lambda^2 \omega_{\perp}^2 \right\} \int_{-\infty}^{\infty} dt \\ &\quad \times \exp \left\{ i \left(E_g + E_{p'} - E_{s'} - E + \frac{\hbar \omega_c}{2} + \frac{1}{\alpha} \frac{k_0^2}{k_0^2} \right) t / \hbar \right\} \exp \left\{ -\frac{\lambda^2 \omega_{\perp}^2}{4} (\omega_c t)^2 \right\} \\ &= \delta_{k_0-k_0', k_0'''-k_0''} \frac{\sqrt{\pi} V^{2/3}}{\pi \lambda \omega_c} \int_0^{\infty} d\omega_{\perp} \exp \left\{ -\lambda^2 \omega_{\perp}^2 - \frac{Z_3^2}{\lambda^2 \omega_c^2} \frac{1}{\omega_{\perp}^2} \right\} \\ &= \delta_{k_0-k_0', k_0'''-k_0''} \frac{V^{2/3}}{2\lambda^2 \omega_c} \exp \left\{ -\frac{2|Z_3|}{\omega_c} \right\}, \end{aligned} \quad (4.42)$$

where

$$Z_3 = \alpha (k_0^2 - k_0'^2) - (E/\hbar). \quad (4.43)$$

$J_4(E)$ is specified by Z_4 which is given by replacing α , k_0 , and k_0' by α' , k_0'' , and k_0''' , respectively, in (4.43).

$J_3(E)$ and $J_4(E)$ are negligibly small compared with $J_1(E)$ and $J_2(E)$ by the order of the ratio $V^{-1/3}$ to λ^{-1} .

Putting (4.40) and (4.41) into (4.28) and taking account of (3.14), $\Gamma_{ss'}^s$ is obtained as

$$\Gamma_s = \frac{2\pi}{\hbar} \frac{2\kappa T \Omega_0 m \sqrt{\pi}}{9M(\hbar u)^2 \lambda} \left(\frac{m_e}{m} C_c^2 + \frac{m_v}{m} C_v^2 \right) \quad (4.44)$$

at $k_0 = k_0' = 0$. When we take the same numerical values of the constants for germanium as in (4.13), we obtain

$$\hbar\Gamma_s \approx 1 \times 10^{-6} \left(\frac{m_c}{m} C_c^2 + \frac{m_v}{m} C_v^2 \right) \text{ eV}, \quad (4.45)$$

and at $k_0 = K$, i. e. at the maximum wave number of photon,

$$\hbar\Gamma_{ss'}^{s'} \approx 2\pi \frac{8\kappa T \Omega_0 m K}{9M(\hbar u)^2} \left(\frac{m_c}{m} \partial_{k_0', k_0'''} C_c^2 + \frac{m_v}{m} \partial_{k_0'', k_0} C_v^2 \right). \quad (4.46)$$

This is larger than the value at $k_0 = k_0' = 0$ by the order of 100, thus in this case the half-value width increases with k_0 and k_0' .

When $H=0$, the above expressions of (4.44) and (4.46) are expected to agree with a transformed formula of the mean collision frequency in the absence of a magnetic field, since the summation over intermediate states includes all Landau levels. For example, we take

$$\frac{1}{\tau} = \frac{4}{9\pi} \frac{\Omega_0 C_c^2 \kappa T m_c}{\hbar^3 u^2 M} \cdot k \quad (4.47)$$

given by Seitz¹¹⁾ for c -band. Transformation by the use of (2.16) gives

$$\Gamma_s = \frac{4}{9\pi} \frac{C_c^2 \kappa T \Omega_0}{\hbar^3 u^2} \frac{m_c}{M} \int k \chi_{00k_0}^2(\mathbf{k}) d\mathbf{k} = \frac{4}{9\pi} \frac{C_c^2 \kappa T \Omega_0}{\hbar^3 u^2} \frac{m_c}{M} \frac{1}{\sqrt{2}\lambda} \int_{2\lambda^2 k_0^2}^{\infty} e^{2\lambda^2 k_0^2} \sqrt{y} e^{-y} dy. \quad (4.48)$$

In particular, at $k_0 = 0$ we have

$$\Gamma_s = \frac{4}{9\pi} \frac{C_c^2 \kappa T m_c \Omega_0}{\hbar^3 u^2 M \lambda} \frac{\sqrt{\pi}}{2\sqrt{2}}, \quad (4.49)$$

which is coincident with the first term in the right-hand side of (4.44) except $\sqrt{2}$. Of course, such an agreement can be easily shown also at $k_0 = K$.

Next, let us consider the self-energy Δ_s near $k_0 = k_0' = 0$. Taking account of (4.39), Appendix III, and putting (4.28) into the first relation of (3.15), we have, for c -band,

$$\begin{aligned} \Delta^{(c)} = & \frac{2\kappa T \Omega_0 m_c C_c^2}{9\pi^2 M (\hbar u)^2} \left[- \int_0^{E_1} \frac{dE}{\sqrt{\alpha' E}} + \frac{\sqrt{\pi}}{2\lambda} \left\{ \int_0^{E_1} \frac{e^{\lambda^2 E/\alpha'}}{E} P\left(\lambda \sqrt{\frac{E}{\alpha'}}\right) dE \right. \right. \\ & \left. \left. - P(\lambda K) \lim_{\delta \rightarrow 0} \left(\int_{-E_1^{(c)}}^{-\delta} + \int_{\delta}^{E_1} \right) \frac{e^{\lambda^2 E/\alpha'}}{E} dE \right\} \right], \quad (4.50) \end{aligned}$$

where $E_1 = \alpha' K^2$ is the energy corresponding to the Brillouin zone, and $E_1^{(c)} = \hbar u w_{max}$ is the maximum phonon energy.

Putting

$$x_1 = \lambda^2 E_1 / \alpha' = (\lambda K)^2, \quad x_1^{(c)} = \lambda^2 \hbar u w_{max} / \alpha' = \hbar u w_{max} / \frac{1}{2} \hbar \omega_c^{(c)}, \quad (4.51)$$

and

$$S_{\alpha'}(x_1, x_1^{(e)}) = 0.980 - P(x_1) \text{Ei}(-x_1^{(e)}) + \frac{1}{\sqrt{x}} \int_1^{x_1} \text{Ei}(x) \frac{e^{-x}}{\sqrt{x}} dx, \quad (4.52)$$

we obtain

$$\mathcal{A}^{(e)} = -\frac{4\kappa T \mathcal{Q}_0 m_c C_c^2}{9\pi^2 M (\hbar u)^2} K \left[1 + \frac{\sqrt{\pi}}{4\lambda K} S_{\alpha'}(x_1, x_1^{(e)}) \right]. \quad (4.53)$$

We can get $\mathcal{A}^{(v)}$ similarly and the self-energy is

$$\begin{aligned} \mathcal{A}_s = \mathcal{A}^{(e)} + \mathcal{A}^{(v)} = & -\frac{4\kappa T m \mathcal{Q}_0 K}{9\pi^2 M (\hbar u)^2} \left[\frac{m_c}{m} C_c^2 \left\{ 1 + \frac{\sqrt{\pi}}{4\lambda K} S_{\alpha'}(x_1, x_1^{(e)}) \right\} \right. \\ & \left. + \frac{m_v}{m} C_v^2 \left\{ 1 + \frac{\sqrt{\pi}}{4\lambda K} S_{\alpha}(x_1, x_1^{(v)}) \right\} \right]. \end{aligned} \quad (4.54)$$

When $S_{\alpha, \alpha'} \ll \lambda K$ is held in the above formula, self-energy \mathcal{A}_s agrees exactly with Muto and Ohyama's result.¹⁰⁾

In the present case, the condition of weak field, $\hbar u w_{mat} \gtrsim \frac{1}{2} \hbar \omega_c$, is held and $S_{\alpha, \alpha'}$ has a small positive value, and the self-energy becomes larger than that in the absence of a magnetic field.

§ 5. Detailed discussions

(i) Half-value width of a sub-peak

Burstein *et al.*¹⁾ inferred a constant half-value width for germanium ($E_g \approx 0.8$ eV) from their observed line shapes which is

$$\hbar \Gamma_s = 1.25 \times 10^{-3} \times E_g = 10^{-3} \text{ eV} \quad (\text{Burstein } et \text{ al.}), \quad (5.1)$$

while our values near $k_0 = k_0' \approx 0$ are

$$\hbar \Gamma_s = 4.0 \times 10^{-4} \text{ eV} \quad (\text{strong field approximation}), \quad (5.2)$$

$$\hbar \Gamma_s = 1.33 \times 10^{-4} \text{ eV} \quad (\text{weak field approximation}). \quad (5.3)$$

Both of them are calculated by using $m_c = 0.05 m$, $m_v = 0.5 m$ and $C_c = C_v = 10$ eV which are not unreasonable values for germanium. Comparing the three values in the above with each other, we find that our calculated values are slightly smaller than that of Burstein *et al.* This small discrepancy seems to come partly from our assumptions used and partly from the additional scattering by the impurities and imperfections in the crystal, the latter of which is especially important because the samples used in the I. M. O.-experiments are very thin. However, it can be said that the results are in good agreement with the observed value.

(ii) Non-diagonal term \mathcal{O}_s

It is necessary to know the non-diagonal term \mathcal{O}_s when we decide whether or not the line shape of the sub-peak is the Lorentzian. In § 2, we showed that the matrix element of the photon perturbation is nearly independent of the carrier's momentum, so that

$$\mathcal{O}_s = \frac{1}{2} \sum_{s' \neq s} \hbar \Gamma_{ss'}^s / (\varepsilon_s - \varepsilon_{s'}) \quad (5.4)$$

is held. Let us estimate \mathcal{O}_s in the both cases which are used in § 4.

(A) Approximation of strong magnetic field

Substituting (4.11) and

$$\varepsilon_s - \varepsilon_{s'} = \alpha(k_0'^2 - k_0''^2) + \alpha'(k_0'^2 - k_0''^2) \quad (5.5)$$

into (5.4), we get

$$\mathcal{O}_s = - \frac{4\kappa T \mathcal{Q}_0 C_v C_v}{9Mu^2 (\hbar u)^2 \lambda'} \delta_{k_0 - k_0'', k_0' - k_0''} \int_{-\infty}^{\infty} dx e^{-x^2 [(A - Bx)^2 - 1]}, \quad (5.6)$$

where

$$x = \lambda'(k_0 - k_0''), \quad A = \frac{2}{\hbar u} (\alpha k_0 + \alpha' k_0') \quad \text{and} \quad B = \frac{\alpha + \alpha'}{\hbar u \lambda'}.$$

\mathcal{O}_s near $k_0 = k_0' = 0$ is readily computed and gives

$$\mathcal{O}_s = -3.7 \times 10^{-2} \quad \text{for } H = 5 \times 10^4 \text{ Oe.} \quad (5.7)$$

When $\mathcal{O}_s \mathcal{Q}_s$ becomes comparable to $\hbar \Gamma_s$ described in (5.2), \mathcal{Q}_s must be of the order of 10^{-2} eV and the corresponding scale of abscissa in the diagram of absorption peak is far apart from the tail. Therefore, we may conclude that the line shape is Lorentzian.

(B) Approximation of weak magnetic field

One of the underlying assumptions in this case is

$$\lambda(k_0'' - k_0) = \lambda w_s \geq 1. \quad (5.8)$$

Hence, the lower limit of integration over w_s must be $1/\lambda$. Putting (4.47) (4.46) into (4.27), we get near $k_0 = k_0' = 0$

$$\mathcal{O}_s = -8.3 \times 10^{-4} \quad \text{for } H = 5 \times 10^4 \text{ Oe.} \quad (5.9)$$

The above value is too small to cause a sizable discrepancy from the Lorentzian.

(iii) Line shape of the first observed peak

It is necessary to know the k_0 dependency of Γ_s when we plot the absorption coefficient versus photon energy. We do so for the first peak which corresponds to the minimum photon energy with use of our results.

Our result of the approximation in the strong field limit, (4.12), can be applied only to clarify the line shape near the absorption edge.

Our attention is thus directed toward the result of the weak field case. Referring to (4.40) ~ (4.43), we may assume that the Γ_s has linear dependency on k_0 and k_0' where these are large, so that the formulae

$$\Gamma_s = \Gamma_0 + \gamma k_0, \quad (5.10)$$

$$\gamma = \frac{4\kappa T \mathcal{Q}_0 m}{9\pi M (\hbar u)^2} \left(\frac{m_v}{m} C_v^2 + \frac{m_c}{m} C_c^2 \right) \quad (5.11)$$

are held.

Putting

$$\hbar\omega_0 = \varepsilon_{00} - \Delta_s + \frac{\hbar^2}{2\mu} k_0^2, \quad (5.12)$$

$$\varepsilon_{00} = E_g + \frac{1}{2} \hbar (\omega_c^{(c)} + \omega_c^{(v)}), \quad (5.13)$$

(3.21) is written as

$$A(\omega) = \text{const} \cdot \frac{1}{\pi} \int_0^\infty d(\hbar\omega_0) \frac{(\hbar\omega_0 - \varepsilon_{00})^{-1/2}}{\hbar\omega_0} \cdot \frac{\hbar\Gamma_s}{(\hbar\omega - \hbar\omega_0)^2 + (\hbar\Gamma_s)^2}. \quad (5.14)$$

In Fig. 3, we plot $A(\omega)$ versus $\hbar\omega$, which corresponds to the observed first peak. The numerical values of the constants are taken for germanium under the field of 5×10^4 Oe.

(iv) η_s and other elements

By the numerical integration we can show that η_s is negligible compared with unity. In section 3, we used the approximation of the weak coupling limit proposed by Toyozawa.³⁾ The main condition is $\Gamma_s \tau_c^s / 2 \ll 1$. This relation is readily proved by any result of the two approximations in the section 4. In the present paper we have neglected the spin effects and band degeneracy which can, however, contribute very slightly to the line shape in so far as the observed first peak is concerned. But when we make the theory more precise we must take into account the effect of optical phonon as well as the scattering by impurities and imperfections in the crystal.

In conclusion, the authors would like to express their thanks to Prof. Y. Uemura and the members* of his laboratory of the University of Tokyo, for their valuable discussions. The authors are indebted to Prof. E. Burstein of the University of Pennsylvania, for his kind information of his group's results before publication.

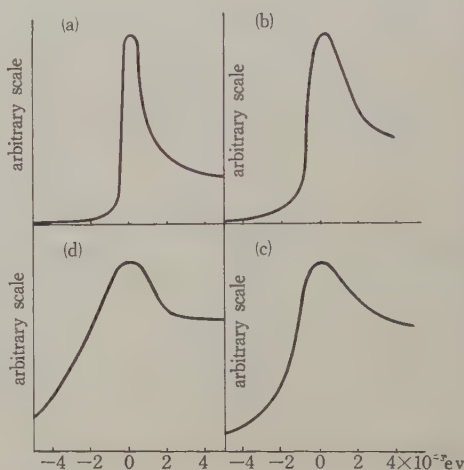


Fig. 3. Line shapes of the first peak of I.M.O.-absorption. (a) Calculated using (5.14), (5.10) and (5.3). (b) Calculated using (5.14), (5.10) and (5.2). (c) Calculated using (5.14) and Burstein *et al.*'s value of (5.1). (d) Observed one by Burstein *et al.* ($H=46.6$ Oe, $H\parallel 110$)

* One of the members, Mr. M. Okazaki (private communication in *Bussei-ron Kenkyu*, 6 of ser. 2 (1959), 248) has developed a theory of I. M. O. by a method different from ours. His result of the scattering probability can be derived by putting $\hbar\omega_0=0$ in (4.1).

Appendix I

The operators O^\pm and L_\pm defined by (4.32) satisfy the relations,

$$\begin{aligned} O^+|n\rangle &= \sqrt{n+1}|n+1\rangle, & O^-|n\rangle &= \sqrt{n}|n-1\rangle, \\ L_+|l\rangle &= \sqrt{l+1}|l+1\rangle, & L_-|l\rangle &= \sqrt{l}|l-1\rangle. \end{aligned} \quad (\text{A} \cdot 1)$$

Then we have the following relations.

$$\langle 0 | e^{-(u_+ o^+ - u_- o^-)} | 0 \rangle = e^{-(u_+ u_-/2)}, \quad (\text{A} \cdot 2)$$

$$\langle 0 | e^{iA(u_+ o^+ + u_- o^-)} | 0 \rangle = e^{-(u_+ u_-/2)}, \quad (\text{A} \cdot 3)$$

$$\langle l | e^{i(u_- L_+ + u_+ L_-)} | l' \rangle = \sqrt{\frac{l!}{l'!}} (iu_-)^l (iu_+)^{l'} \sum_{r=0}^{\min(l, l')} C_r \frac{(-u_+ u_-)^{-r}}{(l-r)!} e^{-(u_+ u_-/2)}. \quad (\text{A} \cdot 4)$$

Appendix II

$$\begin{aligned} \int_{-\infty}^{\infty} dw_z &= 2 \cdot \frac{1}{2\alpha' w_\perp (1+X'^2)^{1/2}} \int_0^{\pi/2} d\theta \left[\frac{\phi}{(1+X'^2)^{1/2} - X' - \cos\theta} \right. \\ &\quad \left. + \frac{1}{(1+X'^2)^{1/2} + X' + \cos\theta} + \frac{1}{(1+X'^2)^{1/2} + X' - \cos\theta} + \frac{1}{(1+X'^2)^{1/2} - X' + \cos\theta} \right] \\ &= \frac{\pi}{\alpha' w_\perp} \cdot \frac{1}{(1+X'^2)^{1/2} \{2X'^2 + 2X'(1+X'^2)^{1/2}\}^{1/2}}, \quad \text{for } X' = \frac{m_e u}{2\hbar w_\perp} \neq 0. \end{aligned} \quad (\text{A} \cdot 5)$$

The divergency of the above integral at $w_\perp = 0$ is removed in the integration over w_\perp by the damping factor of $\exp\{-\lambda'^2 w_\perp^2\}$.

Appendix III

Putting

$$K = 2\pi/\text{lattice constant}, \quad k_E'^2 = k_0'^2 + \frac{E}{\alpha'}, \quad P(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\xi^2} d\xi, \quad (\text{A} \cdot 6)$$

the integrals of (4.39) containing the absolute value of Z_1 are obtained in each range of k_0' as follows:

(i) $k_0' > 0$, $K + k_0'$, $K - k_0' > k_E'$ and $k_0' < 0$, $K + k_0'$, $K - k_0' > k_E'$,

$$\begin{aligned} J_1(E) &= \delta_{QQ'} \delta_{V'VV''} \delta_{k_0'k_0''} \frac{m_e V}{2\pi\hbar} \left[2k_E' + \frac{\sqrt{\pi}}{2\lambda} \exp\{\lambda^2 k_E'^2\} \right. \\ &\quad \left. \times \{P(\lambda(E + k_0')) + P(\lambda(K - k_0')) - 2P(\lambda k_E')\} \right], \end{aligned} \quad (\text{A} \cdot 7)$$

(ii) $k_0' > 0$, $K + k_0' > k_E' > K - k_0'$,

$$J_1(E) = \delta_{QQ'} \delta_{U''''} \delta_{k_0' k_0''''} \frac{m_c V}{2\pi\hbar} \left[K + k_E' - k_0' + \frac{\sqrt{\pi}}{2\lambda} \exp\{\lambda^2 k_E'^2\} \right. \\ \left. \times \{P(\lambda(K + k_0')) - P(\lambda k_E')\} \right], \quad (\text{A} \cdot 8)$$

(iii) $k_0' > 0$, $k_E' > K + k_0'$ and $k_0' < 0$, $k_E' > K - k_0'$,

$$J_1(E) = \delta_{QQ'} \delta_{U''''} \delta_{k_0' k_0''''} \frac{m_c V}{2\pi\hbar} [2K], \quad (\text{A} \cdot 9)$$

(iv) $k_0' < 0$, $K - k_0' > k_E' > K + k_0'$

$$J_1(E) = \delta_{QQ'} \delta_{U''''} \delta_{k_0' k_0''''} \frac{m_c V}{2\pi\hbar} \left[K + k_E' - k_0' + \frac{\sqrt{\pi}}{2\lambda} \exp\{\lambda^2 k_E'^2\} \right. \\ \left. \times \{P(\lambda(K - k_0')) - P(\lambda k_E')\} \right]. \quad (\text{A} \cdot 10)$$

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Contribution from the Three-Pion State to the Axial Vector Coupling Constant in β -Decay

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The effect of the pion cloud to the ratio g_A/g_V in the β -decay is investigated, with the assumption of the conserved current for the vector interaction. The three-pion state is considered as the simplest state which might improve the result of the static theory, which gives $g_A/g_V < 1$. According to the lowest order perturbation calculation, the contribution from this state turns out to be of the positive sign and large. It seems promising to explain the observed ratio $g_A/g_V > 1$ even when the decrease of the bare state probability is taken into account, unless the effect of the suppression of the nucleon pair is too strong.

§ 1. Introduction

The V-A theory of the weak interactions has been widely accepted.^{1),2)} In particular, the idea of the conserved current by Gershtein and Zeldovich,³⁾ Feynman and Gell-Mann¹⁾ has proved to be successful. In their theory it is guaranteed that the observed vector coupling constant g_V in the β -decay is equal to the bare vector coupling constant $g_V^{(0)}$, which can be considered to be identical to that in the μ -e decay, within the electromagnetic correction. This conclusion has been strongly supported by the recent experiments.⁴⁾

On the other hand, some questions still remain in the axial vector coupling constant g_A in the β -decay. Namely, we are tempted to expect that the bare axial vector coupling constant $g_A^{(0)}$ would be equal to $g_V^{(0)}$, since we know that $g_V = g_A$ in the μ -e decay on the one hand,⁵⁾ and the universality of the weak Fermi interaction is well established on the other. If, as stated above, $g_V = g_V^{(0)}$ is assumed in the β -decay, $g_A^{(0)}$ should be equal to g_V . On the other hand, from the general consideration of the static model for the interaction between nucleon and pion, $g_A < g_A^{(0)}$ has been concluded.⁶⁾ Consequently we are led to $g_A < g_V$. This conclusion is not altered by the relativistic perturbation calculation in which also the virtual formation of the nucleon pair is taken into account. Moreover, the magnitude of

g_A is expected to be some ten per cent smaller than that of g_V , from the estimation in the static theory. The experimental value is

$$g_A/g_V = 1.19 > 1,$$

from the decay of the neutron.⁷⁾ No satisfactory explanation has been given for this discrepancy between theory and experiment.* Feynman, Gell-Mann and some authors⁸⁾ have discussed the possibility of the conserved current for the axial vector interaction also. No one has, however, obtained successful results as for the plausibility of the theory itself and its comparison with experiment. The main difficulty comes from the fact that we cannot construct the pion *current* contributing to the axial vector coupling.

An essential point in the discussion by Gershtein and Zeldovich, and Feynman and Gell-Mann, seems to lie in that the β -decay of the pion cloud around the nucleon can compensate the decrease of the strength of the β -decay of the nucleon due to the emission of the pion cloud. Also from the analogy to the electromagnetism, we can understand the possibility of assigning the strength of the β -decay of the pion cloud so as just to cancel the renormalization effect.

It is an interesting suggestion that the pion cloud also should couple into a lepton pair, apart from whether the interaction exists in the form of the current or not. This idea should be taken over to the case of the axial vector interaction, too. For example, we can consider the graphs in Fig. 1, which is resulted from the β -decay of the nucleon created virtually in the pion cloud. From charge conjugation invariance and charge independence of the strong interaction, it is concluded that for the axial vector coupling, a lepton pair should couple only into the odd number of pions. The coupling to a single pion has no effect to the β -decay in which the momentum transfer is almost zero. This is also the consequence of the V-A theory.

Thus the graphs illustrated in Fig. 1 are the lowest order graphs which are the candidate to lead $g_A > g_A^{(0)} (=g_V)$. The contribution from these graphs will hereafter be called the three-pion state contribution. Such a contribution arises from rather complicated structure of the pion cloud, and it has never been considered. However, the importance of the three-pion state contribution has been pointed out recently in connection with the electromagnetic structure of the nucleon, especially the isoscalar part of the charge radius.^{9),10)} The calculation by Hiida¹⁰⁾ et al. has shown that this contribution has a qualitatively desired tendency and

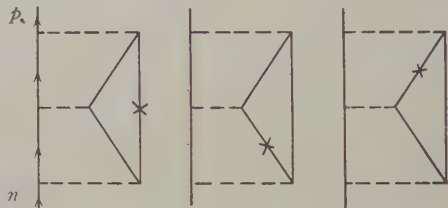


Fig. 1. X on the nucleon line indicates the interaction with a lepton pair.

* C. Iso pointed out a possibility of explaining this ratio by using the strong four-Fermion interaction [Prog. Theor. Phys. **22** (1959), 62], based on Sakata's composite model.

the magnitude is rather large. Such a state may be expected to contribute also to the isoscalar part of the anomalous magnetic moment of the nucleon where the conclusion from the static model is inconsistent with the experiment.¹¹⁾

In § 2, the three-pion state contribution will be calculated by means of the lowest order perturbation. § 3 will be devoted to take account of the conservation of the probability. Results will be summarized in § 4.

§ 2. Three-pion state contribution

The general expression for $g_A/g_A^{(0)}$ is given by

$$g_A^{(0)} \langle p | \mathfrak{F}_\mu(0) | n \rangle = g_A i \bar{u} \gamma_5 \gamma_\mu \tau_+ u, \quad (2.1)$$

where $\mathfrak{F}_\mu(x)$ is the weak interaction source function such that the interaction Hamiltonian $H_w(x)$ is expressed as

$$H_w(x) = i g_A^{(0)} \mathfrak{F}_\mu(x) \bar{\psi}_l(x) \gamma_5 \gamma_\mu (1 + \gamma_5) \psi_\nu(x). \quad (2.2)$$

In (2.1), $|n\rangle$ and $\langle p|$ are the eigenstates of the total Hamiltonian and $\mathfrak{F}_\mu(x)$ is in the Heisenberg representation. In the interaction representation we can write

$$\langle p | \mathfrak{F}_\mu(0) | n \rangle = T \langle p | U(\infty, -\infty) \mathfrak{F}_\mu(0) | n \rangle, \quad (2.3)$$

where $U(\infty, -\infty)$ is the usual S -matrix operator comprised only of the interaction Hamiltonians of the strong interaction.

Assuming the elementary source function

$$\mathfrak{F}_\mu(x) = i Z_2 \bar{\psi}(x) \gamma_5 \gamma_\mu \tau_+ \psi(x), \quad (2.4)$$

we can calculate the total contribution among which the graphs as shown in Fig. 1 are also included. The straightforward calculation of such a graph is, however, exceedingly complicated even in the lowest order perturbation. In the following we apply many simplifying procedures.

First we divide the source function into two parts,

$$\mathfrak{F}_\mu(x) = \mathfrak{F}_\mu^{(\text{static})}(x) + \mathfrak{F}_\mu^{(3\pi)}(x), \quad (2.5)$$

where $\mathfrak{F}_\mu^{(\text{static})}(x)$ is the static nucleon part of the source function given by

$$\left. \begin{aligned} \mathfrak{F}^{(\text{static})}(x) &= Z_2 \chi^* \sigma \tau_+ \chi \rho'(x), \\ \mathfrak{F}_0^{(\text{static})}(x) &= 0, \end{aligned} \right\} \quad (2.6)$$

where $\rho'(x)$ is a form factor, χ is a Pauli spinor. $\mathfrak{F}_\mu^{(3\pi)}(x)$ is the three-pion part of the source function and is related to the S -matrix element of the interaction between three pions and a lepton pair by

$$\begin{aligned} \langle q | S | k_1^i k_2^j k_3^k \rangle &= -i (2\pi)^4 \delta(k_1 + k_2 + k_3 - q) \\ &\times i g_A^{(0)} \langle q | \mathfrak{F}_\mu^{(3\pi)}(0) | k_1^i k_2^j k_3^k \rangle \bar{u}_l \gamma_5 \gamma_\mu (1 + \gamma_5) u_\nu, \end{aligned} \quad (2.7)$$

where the superscripts i, j, k indicate the iso-spin indices.

When the elementary source function is given by (2.4), the introduction of $\mathfrak{F}_\mu^{(3\pi)}$ is only conventional. It is an "effective" source function. We obtain $\mathfrak{F}_\mu^{(3\pi)}$ by calculating the left-hand side of (2.7) in the lowest order perturbation. Three types of graphs in Fig. 2 should be considered.

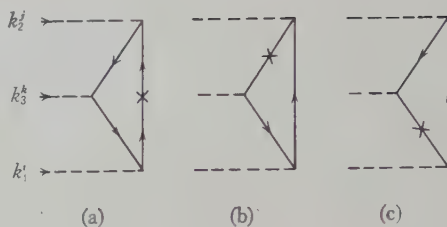


Fig. 2

Corresponding to the graph in Fig. 2 (a) we obtain

$$\begin{aligned} \langle q | \mathfrak{F}_\mu^{(3\pi)} | k_1^i k_2^j k_3^k \rangle_a &= -i(2\pi)^{-4} \frac{1}{\sqrt{8\omega_1\omega_2\omega_3}} \\ &\times G^3 L_\mu(k_1, k_2, k_3; q) Sp(\tau_i \tau_j \tau_k), \end{aligned} \quad (2.8)$$

where

$$\begin{aligned} L_\mu(k_1, k_2, k_3; q) &= \int dk Sp[\gamma_5 \gamma_\mu S_F(k+q) \gamma_5 \\ &\times S_F(k-k_1+q) \gamma_5 S_F(k+k_2) \gamma_5 S_F(k)] \end{aligned} \quad (2.9)$$

is the contribution from the closed loop part. In (2.9) we can omit the argument k_3 , since it is given by k_1, k_2 and q through

$$k_1 + k_2 + k_3 = q.$$

We have adopted the interaction Hamiltonian

$$H(x) = iG\bar{\psi}(x)\gamma_5\tau_3\psi(x)\phi_i(x), \quad (2.10)^*$$

for the pion-nucleon interaction.

For the small values of q ,

$$|q| \ll M,$$

$L_\mu(k_1, k_2; q)$ can be expanded into the power series with respect to q , as

$$\begin{aligned} L_\mu(k_1, k_2; q) &= L_\mu(k_1, k_2; 0) + q_\nu \frac{\partial}{\partial q_\nu} L_\mu(k_1, k_2; q) \Big|_{q=0} \\ &+ \frac{1}{2} q_\lambda q_\nu \frac{\partial^2}{\partial q_\lambda \partial q_\nu} L_\mu(k_1, k_2; q) \Big|_{q=0} + \dots \end{aligned} \quad (2.11)$$

In the first term $L_\mu(k_1, k_2; 0)$, the k -integration is fortunately convergent and we obtain

$$\begin{aligned} L_\mu(k_1, k_2; 0) &= 24\pi^2 M \int_0^1 dx \int_0^x dy \cdot y \\ &\times \left\{ (k_1 + k_2)_\mu \left(-\frac{1}{3\kappa^2} + \frac{M^2 + [(x-y)k_1 - (1-x)k_2]^2}{6\kappa^4} \right) \right\} \end{aligned}$$

* This expression and (2.18) are written in the interaction representation.

$$+ \frac{[(x-y)k_1k_2 - (1-x)k_2^2]k_{1\mu} - [(x-y)k_1^2 - (1-x)k_1k_2]k_{2\mu}}{3\kappa^4} \}, \quad (2.12)$$

where M is the nucleon mass and κ^2 is given by

$$\kappa^2 = M^2 + k_1^2(x-y)(1-x+y) + k_2^2(1-x)x + 2k_1k_2(x-y)(1-x).$$

If we expand (2.12) into a power series with respect to k_1^2/M^2 , k_2^2/M^2 and k_1k_2/M^2 and retain only the first term, the parameter integrals in (2.12) is simplified to be

$$L_\mu(k_1, k_2; 0) = \frac{2\pi^2}{M}(k_1 + k_2)_\mu. \quad (2.13)$$

Since the differentiating with respect to q_ν and setting $q=0$ is equivalent to the appropriate insertion of γ_ν into the nucleon line, the second term in (2.11) is equivalent to the matrix element of the interaction between three pions, a lepton pair, and an "iso-scalar photon." This contribution drops out by including the graph in which the direction of the nucleon line is reversed (generalized Furry's theorem).

The third term in (2.11) can be calculated in the same approximation as in $L_\mu(k_1, k_2; 0)$ and is given by

$$- \frac{\pi^2}{3M} \frac{q^2}{M^2} (k_1 + k_2)_\mu.$$

Thus we obtain the result up to $\sim q^2$,

$$L_\mu(k_1, k_2; q) = \left(1 - \frac{1}{6} \frac{q^2}{M^2}\right) \frac{2\pi^2}{M} (k_1 + k_2)_\mu. \quad (2.14)$$

This shows simply that the nucleon loop part has an extension of the mean square radius $1/M$. As is easily seen, the q -dependence of $L_\mu(k_1, k_2; q)$ is very small. Even in the μ -capture ($q^2 = m_\mu^2$) it is a correction of only -0.2% . In the following we neglect the q -dependence of L_μ completely and write

$$L_\mu(k_1, k_2; q) = L_\mu(k_1, k_2) = \frac{2\pi^2}{M} (k_1 + k_2)_\mu. \quad (2.15)$$

The total contribution from the graphs in Fig. 2 is given by

$$\begin{aligned} \langle q | \mathcal{F}_\mu^{(3\pi)} | k_1^i k_2^j k_3^k \rangle = & -i(2\pi)^{-4} \frac{1}{\sqrt{8\omega_1\omega_2\omega_3}} G^3 \frac{2\pi^2}{M} \\ & \times 2[(k_1 + k_2)_\mu S\hat{p}(\tau_+ \tau_i \tau_k \tau_j) + (k_2 + k_3)_\mu S\hat{p}(\tau_+ \tau_j \tau_i \tau_k) + (k_3 + k_1)_\mu S\hat{p}(\tau_+ \tau_k \tau_j \tau_i)], \end{aligned}$$

where we have multiplied the factor 2 by taking account of the graphs in which the direction of the nucleon loop is reversed. The explicit calculation of the traces yields

$$\langle q | \mathfrak{F}_\mu^{(3\pi)} | k_1^i k_2^j k_3^k \rangle = -i \frac{1}{\sqrt{8\omega_1 \omega_2 \omega_3}} \frac{G^3}{\pi^2} \frac{1}{M} \\ \times (k_{1\mu} \delta_{+i} \delta_{jk} + k_{2\mu} \delta_{+j} \delta_{ki} + k_{3\mu} \delta_{+k} \delta_{ij}), \quad (2.16)$$

where

$$\delta_{+i} = \frac{1}{2} (\delta_{1i} + i\delta_{2i}), \quad \text{etc.}$$

From this we can easily find the form of $\mathfrak{F}_\mu^{(3\pi)}(x)$ as

$$\mathfrak{F}_\mu^{(3\pi)}(x) = -\frac{1}{2\pi^2} G^3 \frac{1}{M} \partial_\mu \phi^*(x) \cdot \phi_i^2(x). \quad (2.17)$$

In this form it is seen that the approximation made in obtaining (2.13) is equivalent to describing the non-local interaction between three pions and a lepton pair by introducing only the first derivative.

We have calculated the nucleon loop part by the lowest order perturbation. On the other hand we know many examples in which the matrix element of the nucleon pair formation is overestimated in the lowest order perturbation calculation of the γ_5 -interaction. It may be the case also in the present calculation. We consider it would, therefore, be more reasonable to multiply (2.16) and (2.17) by the "pair suppression parameter" $\hat{\xi}$, which is conventionally assumed to be $0 < \hat{\xi} < 1$, and to express the final result as a function of $\hat{\xi}$.

Once the effective source function (2.16) or (2.17) (multiplied by $\hat{\xi}$) is obtained, we can calculate the three-pion state contribution $\langle p | \mathfrak{F}_\mu^{(3\pi)} | n \rangle$ with use of the elementary perturbation theory. The lowest order graph is shown in Fig. 3 (a).

As the interaction Hamiltonian for the interaction between the pion and the nucleon in the open polygon, we choose the one in the p -wave static theory,

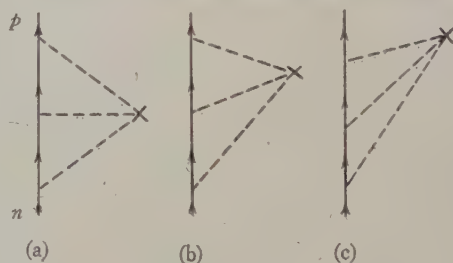


Fig. 3

$$H(x) = \frac{f}{\mu} \chi^* \sigma \tau_i \chi \Delta \phi_i(x) \cdot \rho(x), \quad (2.18) \\ (f = (\mu/2M)G)$$

since this Hamiltonian can be considered to be more suitable than (2.10) unless the nucleon is found in the closed loop.

Corresponding to the graph (a) in Fig. 3, we obtain

$$\langle p | \mathfrak{F}_\mu^{(3\pi)} | n \rangle = -(2\pi)^{-8} \hat{\xi} \left(\frac{f}{\mu} G \right)^3 \frac{1}{\pi^2 M} \int dk_1 \int dk_2 \\ \times \chi^* \left(\hat{k}_2 \tau_j \frac{1}{k_{20} + i\varepsilon} \hat{k}_3 \tau_k \frac{1}{k_{10} - i\varepsilon} \hat{k}_1 \tau_i \right) \chi \\ \times \frac{k_{1\mu} \delta_{+i} \delta_{jk} + k_{2\mu} \delta_{+j} \delta_{ki} + k_{3\mu} \delta_{+k} \delta_{ij}}{(k_1^2 + \mu^2)(k_2^2 + \mu^2)(k_3^2 + \mu^2)}, \quad (2.19)$$

where

$$\hat{\mathbf{k}} = \mathbf{k} \cdot \boldsymbol{\sigma}.$$

In (2.19) we can write

$$\begin{aligned} \tau_j \tau_k \tau_i (k_{1\mu} \delta_{+i} \delta_{jk} + k_{2\mu} \delta_{+j} \delta_{ki} + k_{3\mu} \delta_{+k} \delta_{ij}) &= \tau_+ (3k_{1\mu} + 3k_{2\mu} - k_{3\mu}) \\ &= 4\tau_+ (k_{1\mu} + k_{2\mu} - \tfrac{1}{4} q_\mu), \end{aligned}$$

where we have used

$$k_3 = -k_1 - k_2 + q. \quad (2.20)$$

For the β -decay we may put

$$q = 0, \quad (2.21)$$

and k_3 can be written as

$$k_3 = -k_1 - k_2. \quad (2.22)$$

Substituting into (2.19) we obtain

$$\begin{aligned} \langle p | \tilde{\mathcal{S}}_\mu^{(3\pi)} | n \rangle &= \hat{\epsilon} \frac{8}{\pi^7} \left(\frac{f^2}{4\pi} \right)^3 \left(\frac{M}{\mu} \right)^2 \frac{1}{\mu^4} \int dk_1 \int dk_2 \\ &\times \frac{(k_1 + k_2)_\mu \cdot \chi^* \tau_+ \hat{\mathbf{k}}_2 (\hat{\mathbf{k}}_1 + \hat{\mathbf{k}}_2) \hat{\mathbf{k}}_1 \chi}{(k_{30} + i\epsilon) (k_{10} - i\epsilon) (k_1^2 + \mu^2) (k_2^2 + \mu^2) ((k_1 + k_2)^2 + \mu^2)}. \end{aligned} \quad (2.23)$$

We have omitted the obvious $-i\epsilon$ in the pion propagators.

The k_1 - and k_2 -integrals in (2.23) can be put into the form

$$\begin{aligned} \int dk_1 \int dk_2 (k_1 + k_2) (\hat{\mathbf{k}}_1 K_2^2 + \hat{\mathbf{k}}_2 K_1^2) I(k_1, k_2), \\ \text{for } \mu = 1, 2, 3, \end{aligned} \quad (2.24)$$

$$\begin{aligned} \int dk_1 \int dk_2 (\hat{\mathbf{k}}_1 K_2^2 + \hat{\mathbf{k}}_2 K_1^2) J(k_1, k_2), \\ \text{for } \mu = 0, \end{aligned} \quad (2.25)$$

where

$$K_1 = |\mathbf{k}_1|, \quad K_2 = |\mathbf{k}_2|,$$

and

$$\begin{aligned} (I(k_1, k_2), J(k_1, k_2)) &= \int dk_{10} \int dk_{20} \\ &\times \frac{(1, k_{10} + k_{20})}{(k_{10} - i\epsilon) (k_{20} + i\epsilon) (\omega_1^2 - k_{10}^2) (\omega_2^2 - k_{20}^2) [\omega_3^2 - (k_{10} + k_{20})^2]}. \end{aligned} \quad (2.26)$$

These are easily evaluated to be

$$I(\mathbf{k}_1, \mathbf{k}_2) = \frac{\pi^2}{2} \frac{1}{\omega_1 \omega_2 \omega_3} \left[\frac{1}{\omega_1 \omega_2 \omega_{13}} + \frac{1}{\omega_1 \omega_{13} \omega_{23}} + \text{terms } (1 \leftrightarrow 2) \right], \quad (2.27)$$

and

$$J(\mathbf{k}_1, \mathbf{k}_2) = \pi^2 \frac{\omega_1 - \omega_2}{\omega_1^2 \omega_2^2 \omega_{13} \omega_{23}}, \quad (2.28)$$

where

$$\begin{aligned} \omega_1 &= \sqrt{K_1^2 + \mu^2}, \quad \omega_2 = \sqrt{K_2^2 + \mu^2}, \\ \omega_3 &= \sqrt{(\mathbf{k}_1 + \mathbf{k}_2)^2 + \mu^2} = \sqrt{K_1^2 + K_2^2 + \mu^2 + 2K_1 K_2 x}, \\ \omega_{12} &= \omega_1 + \omega_2, \quad \omega_{13} = \omega_1 + \omega_3, \\ \omega_{123} &= \omega_1 + \omega_2 + \omega_3. \end{aligned}$$

The expression (2.27) is simply the result obtained from the old fashioned perturbation theory. (See the graphs (b) and (c) in Fig. 3 which are pictured in the time order in contrast with (a)). ω_3 contains the direction cosine x of the angle between \mathbf{k}_1 and \mathbf{k}_2 . In this meaning $I(\mathbf{k}_1, \mathbf{k}_2)$ can be considered to be the function of K_1, K_2 , and x .

The expression (2.28) is antisymmetric with respect to ω_1 and ω_2 . Since the rest of (2.25) is symmetric with respect to \mathbf{k}_1 and \mathbf{k}_2 , (2.25) must vanish. Thus

$$\langle p | \mathfrak{F}_0^{(3\pi)} | n \rangle = 0. \quad (2.29)$$

After the angular integrations of \mathbf{k}_1 and \mathbf{k}_2 , (2.24) takes the form

$$\begin{aligned} & \frac{8\pi^2}{3} \sigma \int K_1^2 dK_1 \int K_2^2 dK_2 \\ & \times K_1 K_2 [(K_1^2 + K_2^2) G^{(1)} + 2K_1 K_2 G^{(0)}], \end{aligned} \quad (2.30)$$

where

$$G^{(0,1)}(K_1, K_2) = \int_{-1}^1 dx (1, x) I(K_1, K_2; x), \quad (2.31)$$

where x is the direction cosine of the angle between \mathbf{k}_1 and \mathbf{k}_2 .

Finally, performing the K_1 - and K_2 -integrations, we obtain

$$\begin{aligned} \langle p | \mathfrak{F}^{(3\pi)} | n \rangle &= F^{(3\pi)} \chi^* \sigma_z \chi, \\ F^{(3\pi)} &= C \Phi, \end{aligned} \quad (2.32)$$

where C is a numerical constant given by

$$\begin{aligned} C &= \xi \frac{64}{3} \frac{1}{\pi^3} \left(\frac{f^2}{4\pi} \right)^3 \left(\frac{M}{\mu} \right)^2 \frac{1}{\mu^4} = 0.0154 \xi \frac{1}{\mu^4}. \\ & (f^2/4\pi = 0.08) \end{aligned}$$

Φ is the sum of the integrals given by

$$\Phi = \Phi_1 + \Phi_2 + \Phi_3,$$

$$\Phi_1 = \int_0^K dK_1 \int_0^K dK_2 (K_1^2 + K_2^2) = \frac{2}{3} K^4,$$

$$\begin{aligned} \Phi_2 &= \int_0^K dK_1 \int_0^K dK_2 \frac{K_2}{K_1} (3K_1^2 - K_2^2) \log \frac{\omega_+ + K_1}{\omega_- + K_1} \\ &= \frac{1}{4} K^4 \left[\sum_{n=1}^{\infty} (-1)^{n+1} \left(2 - \frac{1}{2^n} \right) \left(\frac{3}{4} \frac{1}{n(n+1)} - \frac{1}{n^2} \right) \right. \\ &\quad \left. + 2 \sum_{n=\text{odd}} \frac{1}{2^n} \left(\frac{3}{n(n+2)} - \frac{1}{n(n+4)} \right) \right. \\ &\quad \left. + \frac{3}{16} + \frac{9}{8} \log 2 - \frac{1}{2} (\log 2)^2 \right] \\ &= 0.160 K^4, \end{aligned}$$

$$\begin{aligned} \Phi_3 &= - \int_0^K dK_1 \int_0^K dK_2 (K_1 + K_2)^2 \log \frac{\omega_+ + K_1 + K_2}{\omega_- + K_1 + K_2} \\ &= -\frac{1}{2} K^4 \left(\frac{8}{3} \log 2 - \frac{7}{9} \right) = -0.535 K^4, \end{aligned}$$

thus

$$\Phi = 0.292 K^4, \quad (2.33)$$

where

$$\omega_{\pm} = \sqrt{(K_1 \pm K_2)^2 + \mu^2}.$$

In the above estimation we have neglected the pion mass and cutoff K_1 and K_2 by K , though it would be more legitimate to cutoff $K_3 = |\mathbf{k}_1 + \mathbf{k}_2|$ also by K .

Numerical values of

$$F^{(3\pi)} = 448 \tilde{\epsilon} (K/\mu)^4 \times 10^{-5} \quad (2.34)$$

are tabulated in Table I. The fact that the three-pion state contribution (2.34)

Table I. ($\tilde{\epsilon}$ set equal to unity)

K/μ	3	4	5	6
$F^{(3\pi)}$	0.36	1.15	2.80	5.81

is positive and large will be favourable for the understanding of the experimental result which states that $g_A/g_A^{(0)} > 1$.

§ 3. Conservation of the probability

In § 2, we have shown that $F^{(3\pi)}$ is positive and very large. This is, however,

not sufficient for concluding that $g_A/g_A^{(0)}$ is really larger than unity. It is because the probability of the bare nucleon state should have decreased due to the large probability of the emission of the pions and the nucleon pairs around the nucleon. Since the decrease of the bare state probability causes the decrease of the value of $g_A/g_A^{(0)}$, it is necessary to investigate the competition between the increase due to the three-pion state contribution and the decrease due to the decrease of the bare state probability.

An example can be found in the case of the static model.⁶⁾ Let the probability of the state of the one nucleon plus one pion be P_1 , and that of the single nucleon state P_0 . The ratio $F=g_A/g_A^{(0)}$ is given by

$$F=P_0+(1/9)P_1. \quad (3.1)$$

The coefficient of the P_1 , $+(1/9)$ is certainly positive. If, however, we consider the conservation of the probability

$$P_0+P_1=1,$$

then

$$F=(1-P_1)+(1/9)P_1=1-(8/9)P_1, \quad (3.2)$$

which is evidently smaller than unity. Here the decrease of the bare state probability is dominant.

It should be noted that this circumstance can be taken into account also by the perturbation theory. Namely, we calculate the quantity

$$F=Z_2Z_1^{-1}. \quad (3.3)$$

In the static model we obtain

$$\left. \begin{aligned} Z_1^{-1} &= 1 + \frac{1}{6\pi} \frac{f^2}{4\pi} \left(\frac{K}{\mu} \right)^2, \\ Z_2 &= 1 - \frac{3}{2\pi} \frac{f^2}{4\pi} \left(\frac{K}{\mu} \right)^2. \end{aligned} \right\} \quad (3.4)$$

Thus,

$$Z_2Z_1^{-1} = 1 - \frac{1}{2\pi} \frac{f^2}{4\pi} \left(-\frac{1}{3} + 3 \right) \left(\frac{K}{\mu} \right)^2. \quad (3.5)$$

Here also the decrease due to Z_2 , the bare state probability, is dominant over the increase due to Z_1^{-1} .

In the present consideration the similar effects due to the virtual nucleon pair formation will be particularly important. We have, however, no definite method to evaluate them. Assuming that such effects can be taken into account by the appropriate choice of the parameter $\hat{\epsilon}$, we leave it unspecified in the course of the calculation and to see what value of $\hat{\epsilon}$ should be assigned in order to reproduce the experimental result.

If, however, once the effective interaction is specified by (2.16) or (2.17) (multiplied by $\hat{\epsilon}$), the remaining problem essentially is in the domain of the static theory. The evaluation of the decrease of the bare nucleon state probability due to the emission of the pion cloud is contained in the evaluation of

$$\langle p | \hat{\mathfrak{S}}^{(\text{static})} | n \rangle = F^{(\text{static})} \chi^* \sigma \tau_+ \chi. \quad (3.6)$$

This quantity has been evaluated without recourse to the perturbation theory.¹²⁾ If we succeed to evaluate $\langle p | \hat{\mathfrak{S}}_\mu^{(3\pi)} | n \rangle$ in the similar method, e.g., the Chew-Low-Wick method¹³⁾ or dispersion theoretical method, the above estimation of $F^{(\text{static})}$ is already sufficient. But we have calculated the three-pion state contribution only by the perturbation theory. Accordingly it would be more reasonable to calculate the static contribution also by the corresponding perturbation theory.

Since the three-pion state contribution (2.19) is of order f^3 , it will be necessary to calculate

$$F^{(\text{static})} = Z_2 Z_1^{-1} \quad (3.7)$$

at least to order f^4 . By the standard procedure we obtain

$$Z_2 = 1 - B, \quad (3.8)$$

$$B^{(4)} = B_a + B_b = 43.4 (K/\mu)^4 \times 10^{-5}, \quad (3.9)$$

where

$$B_a = \frac{1}{\pi^2} \left(\frac{f^2}{4\pi} \right)^2 \left(\frac{K}{\mu} \right)^4 \left(\frac{3}{4} - \log 2 \right) = 3.69 \left(\frac{K}{\mu} \right)^4 \times 10^{-5}, \quad (3.10)$$

$$B_b = \frac{1}{\pi^2} \left(\frac{f^2}{4\pi} \right)^2 \left(\frac{K}{\mu} \right)^4 9 \left(\log 2 - \frac{5}{8} \right) = 39.7 \left(\frac{K}{\mu} \right)^4 \times 10^{-5}. \quad (3.11)$$

B_a and B_b correspond respectively to the (a) and (b) in Fig. 4. Z_1^{-1} is given by

$$Z_1^{-1} = 1 - L, \quad (3.12)$$

$$-L = -\frac{1}{\pi^2} \left(\frac{f^2}{4\pi} \right)^2 \left(\frac{K}{\mu} \right)^4 \left(\frac{161}{72} - \frac{8}{3} \log 2 \right) = 25.1 \left(\frac{K}{\mu} \right)^4 \times 10^{-5}. \quad (3.13)$$

In these calculations the second order corrections to the nucleon mass and the coupling constant have been subtracted. Also in the fourth order result, the decrease due to Z_2 is dominant.

The above estimation of $F^{(\text{static})}$ is, of course, very unsatisfactory. We, therefore, supplement the discussion with another estimation.

The three-pion state contribution calculated by the lowest order perturbation can also be considered to be the Born approximation term in the dispersion theoretical calculation. The formulation based on the dispersion relation is as follows.¹⁴⁾

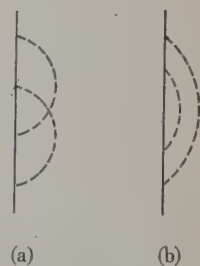


Fig. 4

Consider the quantity

$$F = \langle p | \mathfrak{F}(0) | n \rangle. \quad (3.14)$$

The initial neutron should be considered to be of the variable mass m , different, in general, from the actual mass M , and the difference

$$l = m - M \quad (3.15)$$

is chosen as the dispersion variable. The final result should be evaluated at $l=0$. More specifically, separating out the kinematical part,

$$F(l) = F(l) \chi^* \sigma \tau_+ \chi, \quad (3.16)$$

the ratio $g_A/g_A^{(0)}$ is given by

$$g_A/g_A^{(0)} = F(0), \quad (3.17)$$

from (2.1).

We can assume the dispersion relation of the form,

$$F(0) = 1 - \frac{1}{2\pi i} \int_0^\infty \frac{F^{(abs)}(l)}{l} dl. \quad (3.18)$$

The absorptive part $F^{(abs)}(l)$ is given by

$$F^{(abs)}(l) = F^{(+)}(l) + F^{(-)}(l), \quad (3.19)$$

where

$$F^{(+)}(l) = i \int dx \langle p | \bar{\eta}(x) \mathfrak{F}(0) | 0 \rangle u_n e^{i n x}, \quad (3.20)$$

$$F^{(-)}(l) = -i \int dx \langle p | \mathfrak{F}(0) \bar{\eta}(x) | 0 \rangle u_n e^{i n x}, \quad (3.21)$$

where

$$\eta(x) = (\gamma \cdot \partial + M) \psi(x).$$

The first term 1 in the right-hand side of (3.18) comes from the equal time commutator.

Inserting the complete set, we can write

$$\left. \begin{aligned} F^{(+)}(l) &= i(2\pi)^4 \sum_{s_+} \delta(q - s_+) \langle p | \bar{\eta} | s_+ \rangle \langle s_+ | \mathfrak{F} | 0 \rangle u_n, \\ F^{(-)}(l) &= -i(2\pi)^4 \sum_{s_-} \delta(n - s_-) \langle p | \mathfrak{F} | s_- \rangle \langle s_- | \bar{\eta} | 0 \rangle u_n. \end{aligned} \right\} \quad (3.22)$$

Two kinds of intermediate states are illustrated schematically in Fig. 5.

We fix the square of the momentum transfer as

$$q^2 = 0. \quad (3.23)$$

This leads to

$$s_+^2 = 0,$$

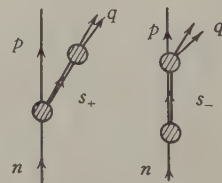


Fig. 5

in order that $F^{(+)}(l)$ does not vanish. There are, however, no states with the strong interaction of vanishing mass. Thus

$$F^{(+)}(l) = 0. \quad (3.24)$$

As s_- -states, we consider one-pion, and two-pion (plus one nucleon*) states and denote as

$$F^{(-)}(l) = F_1^{(-)}(l) + F_2^{(-)}(l), \quad (3.25)$$

where

$$F_1^{(-)}(l) = -i(2\pi)^{-2} \int dN \int dk \delta(N+k-n) \langle p | \mathfrak{F} | Nk^i \rangle \langle Nk^i | \bar{\eta} | 0 \rangle u_n, \quad (3.26)$$

$$F_2^{(-)}(l) = -i(2\pi)^{-5} \frac{1}{2} \int dN \int dk_1 \int dk_2 \delta(N+k_1+k_2-n) \\ \times \langle p | \mathfrak{F} | Nk_1^i k_2^j \rangle \langle Nk_1^i k_2^j | \bar{\eta} | 0 \rangle u_n, \quad (3.27)$$

where N , k^i , etc., represent the nucleon, and the pion of isospin index i in the intermediate states, schematically illustrated in Fig. 6.

The three-pion state contribution arises through the transition from the one-pion state to the two-pion state and the reversed transition as illustrated in Fig. 7 (a), (a'). The transitions between the states with more pions are naturally possible, but the transition from the three-pion state to no-pion state or the reversed transition (Fig. 7(b), (b')) makes no contribution. The situation is similar to that encountered in $F^{(+)}(l)$, as easily seen by replacing the s_+ -state with the three-pion state. Thus we take into account only the one-pion state and the two-pion state as the minimum necessary states.

In the calculation of (3.26) and (3.27), we retain only the Born approximation terms successively in the form of the power series expansion with respect to the *renormalized* coupling constant

f , which appears in the form, for example,**

$$\langle Nk^i | \bar{\eta} | 0 \rangle = i \frac{f}{\mu} \frac{1}{\sqrt{2\omega}} \chi^* \hat{\mathbf{k}} \cdot \boldsymbol{\tau} \chi v(\omega).$$

The lowest order term in $\langle p | \mathfrak{F}^{(3\pi)} | n \rangle$ is of the third order in f and will reproduce the perturbation result obtained in § 2, since this term corresponds to

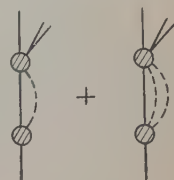


Fig. 6

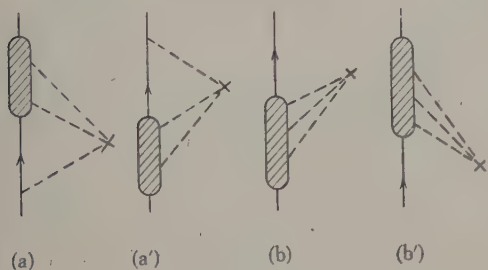


Fig. 7

* In the following this statement will be omitted without confusion.

** $v(\omega)$ is the cutoff factor.

the lowest order irreducible graph. We simply use the result (2.34).

The static part $F^{(\text{static})}(0)$ takes the form

$$F^{(\text{static})}(0) = 1 + A^{(\text{static})} F(0), \quad (3.28)$$

since in the right-hand side of (3.26) and (3.27) there always appears the *actual* decay matrix element $F(0)$.

In order to take account of the one-pion state and the two-pion state, it is necessary to calculate at least to the order f^4 , though it is obviously not sufficient. Giving the details of the calculation in the Appendix, we summarize the result.

To the order f^2 , A in (3.28) can be calculated as follows.

$$A^{(2)} = -\frac{1}{2\pi} \frac{f^2}{4\pi} \left(-\frac{1}{3} + 3 \right) \left(\frac{K}{\mu} \right)^2 \quad (3.29)$$

Substituting this into (3.28) and setting $F(0) = 1$ in the right-hand side, we obtain exactly the same form as the usual perturbation result (3.5).

To the order f^4 , however, we obtain somewhat different results, mainly due to the fact that the procedure of the renormalizations of the coupling constant and the nucleon mass is *ad hoc* in the dispersion relation approach, while it must be performed in the course of the calculation in the usual perturbation method. The dispersion result is given by

$$A^{(4)} = A_v^{(4)} + A_r^{(4)} = -79.6 \left(\frac{K}{\mu} \right)^4 \times 10^{-5},$$

where

$$\left. \begin{aligned} A_v^{(4)} &= \frac{1}{\pi^2} \left(\frac{f^2}{4\pi} \right)^2 \left(\frac{K}{\mu} \right)^4 \frac{4}{9} = 28.8 \left(\frac{K}{\mu} \right)^4 \times 10^{-5}, \\ A_r^{(4)} &= \frac{1}{\pi^2} \left(\frac{f^2}{4\pi} \right)^2 \left(\frac{K}{\mu} \right)^4 \left(\frac{31}{8} - 8 \log 2 \right) = -108.4 \left(\frac{K}{\mu} \right)^4 \times 10^{-5}. \end{aligned} \right\} \quad (3.30)$$

A_v and A_r corresponds to the graph (a) and (b) in Fig. 8, or to the vertex part ($-L$) and the wave function renormalization part ($-B$), respectively. We see that the $A_r^{(4)}$ is, in its absolute value, much larger than the corresponding $B^{(4)}$. It seems, therefore, necessary to compare them in some detail.

"Graphs" corresponding to $A_r^{(4)}$ are listed in Fig. 9, together with the magnitude of each contribution.

The graph (a) in Fig. 9 is an "improper graph," which does not appear in the usual method of perturbation, has appeared as a part of the graph in Fig. 10 (a). Namely, by the substitution indicated in Fig. 10 (b), the graph in Fig. 9 (a) has been obtained together with the graph in Fig. 9 (b). The contributions from the graph (c) and (d) in Fig. 9 agree exactly with the results from the perturbation theory (3.10) and (3.11).

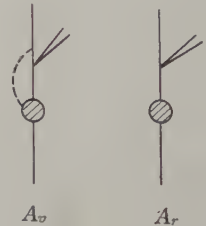


Fig. 8

In the following we shall tentatively use the dispersion theoretical estimation for the static contribution instead of the usual perturbation result, simply because the former is more "severe" than the latter.

§ 4. Results

For the convenience we modify the expression (2.34) for the three-pion state contribution as follows:

$$F^{(3\pi)}(0) = A^{(3\pi)} F(0), \quad (4.1)$$

where

$$A^{(3\pi)} = 448 \hat{\xi} (K/\mu)^4 \times 10^{-5}. \quad (4.2)$$

This is based on the assumption that the strength of the weak axial vector interaction of the nucleon in the closed loop part is equal to that observed in the static case.

Combining (3.28) and (4.1), we obtain

$$F(0) = 1 + AF(0), \quad (4.3)$$

where

$$A = A^{(\text{static})} + A^{(3\pi)}. \quad (4.4)$$

From (4.3), $F(0)$ is given by

$$F(0) = \frac{1}{1-A}. \quad (4.5)$$

Numerical results are tabulated in Table II for various values of K , the cut-off momentum, and the "pair suppression parameter" $\hat{\xi}$.

The dependence of the results on the cutoff is too strong to allow the direct comparison between these results and the observed value. But it is lucky that $A^{(4)}$ and $A^{(3\pi)}$ are of the same dependence on the cutoff ($\sim K^4$). Our main concern, therefore, should be confined to the rather qualitative consideration, whether $F(0)$ is larger than unity or not, for the appropriate region of K .

At $\hat{\xi}=1$, F is larger than unity for $K/\mu \gtrsim 3$. For $K/\mu \gtrsim 4$, the present approximation of the power series expansion with respect to f times K fails and no definite conclusion will be expected. At $\hat{\xi}=1/2$, F is larger than unity for $K/\mu \gtrsim 5$. At $\hat{\xi}=1/3$, F is always smaller than unity for all values of $K/\mu < 6$. Even in

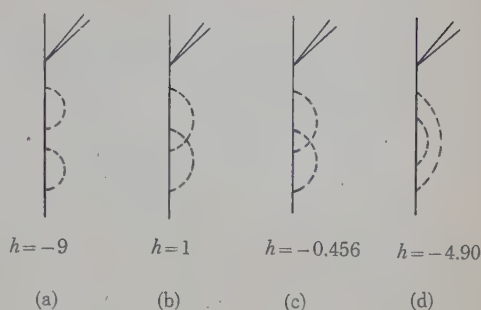


Fig. 9. The values of h represent the relative magnitudes of each contribution. (See the caption to Table III)

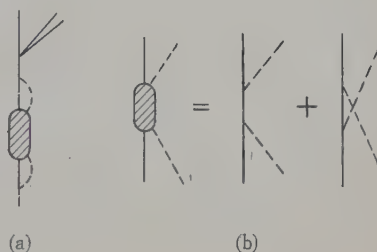


Fig. 10

Table II

K/μ	3	4	5	6
$A^{(2)}$	-0.31	-0.54	-0.85	-1.22
$A^{(4)}$	-0.06	-0.20	-0.50	-1.03
$A^{(\text{static})}$	-0.37	-0.75	-1.35	-2.26
$A^{(3\pi)}$	$+0.36\xi$	$+1.15\xi$	$+2.80\xi$	$+5.81\xi$
$A \begin{cases} \xi=1 \\ \xi=1/2 \\ \xi=1/3 \end{cases}$	$\begin{cases} -0.01 \\ -0.19 \\ -0.25 \end{cases}$	$\begin{cases} +0.40 \\ -0.17 \\ -0.37 \end{cases}$	$\begin{cases} +1.45 \\ +0.05 \\ -0.42 \end{cases}$	$\begin{cases} +3.55 \\ +0.65 \\ -0.32 \end{cases}$
$F = \frac{1}{1-A} \begin{cases} \xi=1 \\ \xi=1/2 \\ \xi=1/3 \end{cases}$	$\begin{cases} 0.99 \\ 0.84 \\ 0.80 \end{cases}$	$\begin{cases} 1.66 \\ 0.85 \\ 0.73 \end{cases}$	$\begin{cases} - \\ 1.05 \\ 0.70 \end{cases}$	$\begin{cases} - \\ 2.83 \\ 0.76 \end{cases}$

this case, however, the result is *better* (A is larger) than that of the second order perturbation.

The condition that the effect of the three-pion state improves the second order perturbation result is

$$A^{(3\pi)} + A^{(4)} > 0,$$

or

$$\xi > 79.6/448 = 0.18. \quad (4.6)$$

We can summarize as follows. The three-pion state contribution can be expected to be large enough to reproduce the experimental result $g_A/g_A^{(0)} > 1$, even if the decrease of the bare nucleon state probability is taken into account, unless the formation of the nucleon pair is suppressed too strongly. The required lower limit of ξ will be $\sim 1/2$. This value can be lowered to $\sim 1/3$ if the $F^{(\text{static})}$ is estimated by the usual fourth order perturbation.* Even when the three-pion state contribution is insufficient to reproduce the observed result completely, it is still of the desired direction if $\xi \gtrsim 0.18$ from the dispersion theoretical estimation, or $\xi \gtrsim 0.04$ from the usual perturbation estimation.

§ 5. Concluding remarks

In the present calculation of the three-pion state contribution the treatment of the closed loop part is obviously most unsatisfactory. On the other hand, more refined treatment can be expected for the interaction between pions and the nucleon in the open polygon.

If we confine ourselves only to the p -wave pions, the Chew-Low-Wick method can be applied to the expression,

* In this case $|A^{(2)}| \gg |A^{(4)}|$ always for $K/\mu \lesssim 6$.

$$\begin{aligned}
\langle p | \int \mathfrak{F}^{(3\pi)}(x) d\mathbf{x} | n \rangle = & -\frac{G^3}{2\pi^3} \frac{1}{M} \frac{1}{(2\pi)^9} \int \frac{d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3}{\sqrt{8\omega_1 \omega_2 \omega_3}} \\
& \times i\mathbf{k}_1 \langle p | \left\{ a_{k1}^- a_{k2}^i a_{k3}^i \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) + a_{k1}^- a_{k2}^i a_{k3}^{i*} \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3) \right. \\
& + a_{k1}^- a_{k2}^{i*} a_{k3}^i \delta(\mathbf{k}_1 - \mathbf{k}_2 + \mathbf{k}_3) + a_{k1}^- a_{k2}^{i*} a_{k3}^{i*} \delta(\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) \\
& \left. - \left(\text{terms obtained by } a_{k1}^- \rightarrow a_{k1}^{+*}, \text{ and } \right. \right\} | n \rangle, \\
& \left. \mathbf{k}_1 \rightarrow -\mathbf{k}_1 \text{ in the } \delta\text{-functions.} \right)
\end{aligned}$$

where a_k^i is an annihilation operator of the pion of the momentum k and the isospin index i , and

$$a_{\bar{k}} = (a_k^1 + i a_k^2) / \sqrt{2}.$$

Also the dispersion relation approach developed in § 3 can be applied to the three-pion state contribution. In this case we can include also the s -wave pions by using the experimental s -wave phase shifts for the pion-nucleon scattering.

A troublesome problem may arise whether the higher order contributions, e.g. the five-, seven-, ... pion state contributions, can be neglected or not. If this is really the case, it would be almost meaningless to consider the lowest order configuration alone. On the other hand we know many examples that the lowest order perturbation yields a good fit to the experiment and we expect that the present discussion serves as a useful first step.

Acknowledgement

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Appendix Dispersion relation approach to the static part

At first we calculate $F_1^{(-)}(l)$, (3.26). The first factor in (3.26) can be written as

$$\langle p | \mathfrak{F} | N k^i \rangle \equiv W_1(k^i) = W_1^{(c)} - i(1/\sqrt{2}\omega) \int dx \langle p | T(J_i(x) \mathfrak{F}(0)) | N \rangle e^{ikx}, \quad (\text{A.1})$$

where

$$J_i(x) = (\square - \mu^2) \phi_i(x). \quad (\text{A.2})$$

$W^{(c)}$ comes from the equal time commutator and does not depend on k . Regarding W as the function of $\mathcal{A}^2 = (N-p)^2 = (k-q)^2$ and ω , we assume the dispersion relation of the form

$$W_1(\mathcal{A}^2, \omega) = W_1^{(c)}(\mathcal{A}^2) - \frac{1}{2\pi i} \int \frac{W_1^{(+)}(\mathcal{A}^2, \omega') + W_1^{(-)}(\mathcal{A}^2, \omega')}{\omega' - \omega} d\omega'. \quad (\text{A.3})$$

Intermediate states contributing to $W^{(+)}$ and $W^{(-)}$ are illustrated in Fig. 11.

If we choose the one-nucleon state as s_+ - and s_- -states, we obtain

$$\left. \begin{aligned} W_{10}^{(+)}(\Delta^2, \omega^i) &= -2\pi i (1/\sqrt{2\omega}) i(f/\mu) \delta(\omega) \times F(0) \chi^* \hat{\mathbf{k}} \sigma \tau_i \tau_+ \chi v(\omega), \\ W_{10}^{(-)}(\Delta^2, \omega^i) &= 2\pi i (1/\sqrt{2\omega}) i(f/\mu) \delta(\omega) \times F(0) \chi^* \sigma \hat{\mathbf{k}} \tau_+ \tau_i \chi v(\omega), \end{aligned} \right\} \quad (\text{A} \cdot 4)^*$$

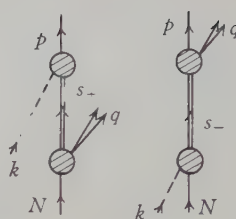


Fig. 11

where we have neglected μ/M . We have used

$$\left. \begin{aligned} \langle p | \mathfrak{F} | N \rangle &= F(0) \chi^* \sigma \tau_+ \chi, \\ \langle N k' | \bar{\eta} | 0 \rangle u_n &= (1/\sqrt{2\omega}) i(f/\mu) \chi^* \hat{\mathbf{k}} \tau_i \chi v(\omega), \end{aligned} \right\} \quad (\text{A} \cdot 5)$$

where $v(\omega)$ is the cutoff factor. Substituting (A·4) into (A·3), we obtain

$$W_{10}(\Delta^2, \omega^i) = i(1/\sqrt{2\omega}) (f/\mu) F(0) (1/\omega) v(\omega) \chi^* (-\hat{\mathbf{k}} \sigma \tau_i \tau_+ + \sigma \hat{\mathbf{k}} \tau_+ \tau_i) \chi. \quad (\text{A} \cdot 6)$$

If we choose the one pion and one nucleon state as s_{\pm} -states, we obtain

$$W_{11}^{(+)} = -i(2\pi)^{-2} (1/\sqrt{2\omega}) \int dN' \int d\mathbf{k}' \delta(N' + k' - p + k) \langle p | J_i | N' k'^j \rangle \langle N' k'^j | \mathfrak{F} | N \rangle, \quad (\text{A} \cdot 7)$$

$$W_{11}^{(-)} = i(2\pi)^{-2} (1/\sqrt{2\omega}) \int dN'' \int d\mathbf{k}'' \delta(N'' + k'' - N - k) \langle p | \mathfrak{F} | N'' k''^j \rangle \langle N'' k''^j | J_i | N \rangle. \quad (\text{A} \cdot 8)$$

These are illustrated in Fig. 12. Since we need $F(0)$ up to the order $\sim f^4$, we approximate as

$$\begin{aligned} \langle N' k'^j | \mathfrak{F} | N \rangle, \quad \langle p | \mathfrak{F} | N'' k''^j \rangle &\sim f, \\ \langle p | J_i | N' k'^j \rangle, \quad \langle N'' k''^j | J_i | N \rangle &\sim f^2. \end{aligned}$$

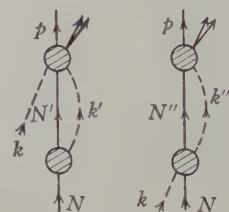


Fig. 12

From (A·1) and (A·6), we obtain

$$\begin{aligned} \langle N' k'^j | \mathfrak{F} | N \rangle &= -i(f/\mu) F(0) (1/\sqrt{2\omega'}) (1/\omega') v(\omega') \\ &\quad \times \chi^* (\hat{\mathbf{k}}' \sigma \tau_j \tau_+ - \sigma \hat{\mathbf{k}}' \tau_+ \tau_j) \chi, \\ \langle p | \mathfrak{F} | N'' k''^j \rangle &= -i(f/\mu) F(0) (1/\sqrt{2\omega''}) (1/\omega'') v(\omega'') \chi^* (\hat{\mathbf{k}}'' \sigma \tau_j \tau_+ - \sigma \hat{\mathbf{k}}'' \tau_+ \tau_j) \chi, \end{aligned} \quad (\text{A} \cdot 9)$$

and the matrix elements for pion-nucleon “scattering” are

* We omit the term $W^{(0)}$ hereafter, because this term contributes only to the three-, five-, ... pion states contributions.

$$\begin{aligned}
 \langle p | J_i | N' k'^j \rangle &= - (f/\mu)^2 (1/\sqrt{2\omega'}) v(\omega) v(\omega') \\
 &\quad \times \chi^* \left(\frac{1}{\omega} \hat{k}' \hat{k} \tau_j \tau_i + \frac{1}{\omega'} \hat{k} \hat{k}' \tau_i \tau_j \right) \chi, \\
 \langle N' k''^j | J_i | N \rangle &= (f/\mu)^2 (1/\sqrt{2\omega''}) v(\omega) v(\omega'') \\
 &\quad \times \chi^* \left(\frac{1}{\omega} \hat{k}'' \hat{k} \tau_j \tau_i - \frac{1}{\omega''} \hat{k} \hat{k}'' \tau_i \tau_j \right) \chi.
 \end{aligned} \tag{A.10}$$

Substituting (A.9) and (A.10) into (A.7) and (A.8), and performing the phase volume integrations (N' , k' -integrations), we obtain

$$\begin{aligned}
 W_{11(+)} &= i(2\pi)^{-2} (1/\sqrt{2\omega}) (f/\mu)^3 F(0) [K + \omega \log(\omega/K + \omega)] \\
 &\quad \times \chi^* \left[\frac{1}{3} \hat{k} \sigma \tau_i \tau_+ - \frac{1}{3} (2\hat{\sigma} \hat{k} + \hat{k} \sigma) (2\tau_+ \tau_i + \tau_i \tau_+) \right. \\
 &\quad \left. - 3\hat{k} \sigma \tau_i \tau_+ + \frac{1}{3} \hat{k} \sigma \tau_j \tau_+ \right] \chi, \\
 W_{11(-)} &= -i(2\pi)^{-2} (1/\sqrt{2\omega}) (f/\mu)^3 F(0) [K + \omega \log(K - \omega/\omega)] \\
 &\quad \times \chi^* \left[-\frac{1}{3} \hat{\sigma} \hat{k} \tau_+ \tau_i + \frac{1}{3} (2\hat{k} \sigma + \hat{\sigma} \hat{k}) (2\tau_i \tau_+ + \tau_+ \tau_i) \right. \\
 &\quad \left. + 3\hat{\sigma} \hat{k} \tau_+ \tau_i - \frac{1}{3} \hat{\sigma} \hat{k} \tau_+ \tau_j \right] \chi.
 \end{aligned} \tag{A.11}$$

In the above calculation we have neglected the pion mass and ω/M . This approximation will apply throughout this Appendix.

Substituting (A.4), (A.11) into (3.26), we obtain

$$F_1^{(-)}(l) = F_{10}^{(-)}(l) + F_{11(+)}^{(-)}(l) + F_{11(-)}^{(-)}(l),$$

where

$$\begin{aligned}
 F_{10}^{(-)}(l) &= (i/2\pi) (f/\mu)^2 F(0) l^2 v(l) \left(-\frac{1}{3} + 3\right), \\
 F_{11(+)}^{(-)}(l) &= \frac{1}{2} i(2\pi)^{-4} (f/\mu)^4 F(0) \int dN \int d\mathbf{k} \delta(N + k - n) \\
 &\quad \times (k^2/\omega) v(\omega) [K + \omega \log(\omega/K + \omega)] \left(\frac{1}{9} - \frac{25}{9} - 1 + \frac{1}{9}\right), \\
 F_{11(-)}^{(-)}(l) &= -\frac{1}{2} i(2\pi)^{-4} (f/\mu)^4 F(0) \int dN \int d\mathbf{k} \delta(N + k - n) \\
 &\quad \times (k^2/\omega) v(\omega) [K + \omega \log(k - \omega/\omega)] \left(-1 + \frac{1}{9} + 9 - 1\right),
 \end{aligned} \tag{A.12}$$

where

$$l = m - M = \omega, \quad (m = n_0) \tag{A.13}$$

in the present approximation.

Performing the phase volume integrations and substituting into (3.18), we obtain

$$F_{10}(0) = - (1/2\pi) (f^2/4\pi) F(0) (K/\mu)^2 \left(-\frac{1}{3} + 3\right), \tag{A.14}$$

$$\begin{aligned}
 F_{11(+)}(0) &= (1/8\pi^2) (f^2/4\pi)^2 F(0) (K/\mu)^4 \left(-\frac{1}{9} + \frac{25}{9} + 1 - \frac{1}{9}\right), \\
 F_{11(-)}(0) &= (1/8\pi^2) (f^2/4\pi)^2 F(0) (K/\mu)^4 \left(1 - \frac{1}{9} - 9 + 1\right).
 \end{aligned} \tag{A.15}$$

Next we enter into the calculation of $F_2^{(-)}(l)$, (3.27). In (3.27), we can write

$$\begin{aligned} \langle p | \mathfrak{F} | N k_1^i k_2^j \rangle &\equiv W_2(k_1^i, k_2^j) \\ &= -i(1/\sqrt{2\omega_1}) \int dx \langle p | T(J_i(x) \mathfrak{F}(0)) | N k_2^j \rangle e^{ik_1 x}, \end{aligned} \quad (\text{A} \cdot 16)$$

where we have again omitted the equal time commutator.

Assuming the similar dispersion relation as (A.3), we obtain up to the order $\sim f^2$.

$$\begin{aligned} W_2(k_1^i, k_2^j) &= -\frac{2M}{\sqrt{2\omega_1}} \left[\frac{1}{(p-k)^2 + M^2} \langle p | J_i | N' \rangle \langle N' | \mathfrak{F} | N k_2^j \rangle \right. \\ &\quad \left. + \frac{1}{(N+k_1+k_2)^2 + M^2} \langle p | \mathfrak{F} | N'' \rangle \langle N'' | J_i | N k_2^j \rangle \right] \end{aligned} \quad (\text{A} \cdot 17)$$

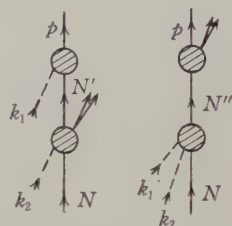


Fig. 13

These are illustrated in Fig. 13.

Using (A.9), (A.10) and similar expressions, we obtain

$$\begin{aligned} W_2(k_1^i, k_2^j) &= (1/\sqrt{4\omega_1\omega_2}) (f/\mu)^2 F(0) v(\omega_1) v(\omega_2) \\ &\quad \times \chi^* \left[\frac{1}{\omega_1\omega_2} (-\hat{k}_1 \hat{k}_2 \sigma \tau_i \tau_j \tau_+ + \hat{k}_1 \sigma \hat{k}_2 \tau_i \tau_+ \tau_j) \right. \\ &\quad \left. - \frac{1}{\omega_1 + \omega_2} \left(\frac{1}{\omega_1} \sigma \hat{k}_2 \hat{k}_1 \tau_+ \tau_j \tau_i + \frac{1}{\omega_2} \sigma \hat{k}_1 \hat{k}_2 \tau_+ \tau_i \tau_j \right) \right] \chi. \end{aligned} \quad (\text{A} \cdot 18)$$

Also the analytic continuation of (A.10) becomes

$$\begin{aligned} \langle N k_1^i k_2^j | \bar{\eta} | 0 \rangle &= (1/\sqrt{4\omega_1\omega_2}) (f/\mu)^2 v(\omega_1) v(\omega_2) \\ &\quad \times \chi^* \left(\frac{1}{\omega_2} \hat{k}_2 \hat{k}_1 \tau_j \tau_i + \frac{1}{\omega_2} \hat{k}_1 \hat{k}_2 \tau_i \tau_j \right) \chi. \end{aligned} \quad (\text{A} \cdot 19)$$

Substituting (A.18) and (A.19) into (3.27), we obtain

$$\begin{aligned} F_2^{(-)}(l) &= -i(2\pi)^{-5} \frac{1}{2} (f/\mu)^4 F(0) \int dN \int d\mathbf{k}_1 \int d\mathbf{k}_2 \delta(N+k_1+k_2-n) \\ &\quad \times \frac{\mathbf{k}_1^2 \mathbf{k}_2^2}{4\omega_1\omega_2} \left[\frac{1}{\omega_1\omega_2} \left(-\frac{1}{9} \frac{1}{\omega_2} - \frac{25}{9} \frac{1}{\omega_1} + \frac{1}{\omega_2} + \frac{1}{9} \frac{1}{\omega_1} \right) \right. \\ &\quad \left. - \frac{1}{\omega_1 + \omega_2} \left(\frac{2}{\omega_1\omega_2} + \frac{9}{\omega_1^2} + \frac{9}{\omega_2^2} \right) \right], \end{aligned} \quad (\text{A} \cdot 20)$$

where

$$l = \omega_1 + \omega_2 \quad (\text{A} \cdot 21)$$

from the energy conservation law. After the phase volume integrations (N , \mathbf{k}_1 , \mathbf{k}_2 -integrations), and substituting into (3.18), we obtain

Table III. Magnitudes of each contribution in (A·14), (A·15), and (A·22) are tabulated together with the corresponding graphs. The coefficient h is defined by

$$A = \begin{cases} h(1/2\pi)(f^2/4\pi)(K/\mu)^2, & \text{for } F_{10}, \\ h(1/8\pi^2)(f^2/4\pi)^2(K/\mu)^4, & \text{for } F_{11(\pm)}, F_2, \end{cases}$$

for each contribution.

Classification	Graph	h	Classification	Graph	h
F_{10}		1/3	F_2		-1/18
		-3			-25/18
$F_{11(+)}$		-1/9			1/2
		25/9			1/18
		1			
		-1/9			
$F_{11(-)}$		1			$-8(3/4 - \log 2) = -0.456$
		-1/9			
		-9			$-72(\log 2 - 5/8) = -4.90$
		1			

$$F_2(0) = (1/8\pi^2)(f^2/4\pi)^2 F(0)(K/\mu)^4 \left[\frac{1}{2} \left(-\frac{1}{9} - \frac{25}{9} + 1 + \frac{1}{9} \right) - 8 \left(\frac{3}{4} - \log 2 \right) - 72 \left(\log 2 - \frac{5}{8} \right) \right]. \quad (\text{A} \cdot 22)$$

In Table III, the results (A·14), (A·15) and (A·22) are summarized with the corresponding "dispersion graphs."

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On the Redundant Solutions of the Bethe-Salpeter Equation

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The abnormal solutions of the Bethe-Salpeter equation are examined. It is shown that they are not permissible in quantum field theory. A special example of the soluble Bethe-Salpeter equation is presented to show that it actually has many redundant solutions.

§ 1. Introduction

Wick¹⁾ and Cutkosky²⁾ have introduced an example in which the Bethe-Salpeter equation leads us to a set of abnormal solutions. The abnormal solutions are supposed to appear closely bound up with the relative time which does not show itself in the ordinary formulation of the nonrelativistic two-body problems. Green and Biswas³⁾⁴⁾⁵⁾ have once discussed such standpoint and remarked that the strangeness might be the new quantum number due to the relative time. This statement would be very interesting to us, if we could show that the abnormal solution would describe two-body states permissible in quantum field theory. It has, however, never been proved that any solution of the Bethe-Salpeter equation can be an eigenstate of the total Hamiltonian, although it has been shown⁶⁾ that all the eigenstates of the Hamiltonian are solutions of the Bethe-Salpeter equation. Therefore it seems to us that under certain circumstances the Bethe-Salpeter equation has physically meaningless solutions. Scarf and Umezawa⁷⁾ examined the abnormal solutions in Wick-Cutkosky's example with use of the Sakata-Taketani amplitude⁸⁾. They concluded that the abnormal solutions cannot describe any physical state, because all of them degenerate and have singularities at $\lambda^2 = \frac{1}{4}$ (λ ; a coupling constant) and, therefore, the two-body propagator cannot be well normalized at this point. As shown in the example in § 3 of the present paper, however, such degeneration is not a general feature of abnormal solutions: we need to find another physical reason in order to exclude the abnormal solutions.

In § 2, we give a reason for it which will suggest that the abnormal solutions conflict with general principles of quantum field theory.

In § 3, it is noted that the relative time plays an essential role even in non-relativistic two-body problems as far as use is made of the Bethe-Salpeter equation. We shall present an example of the nonrelativistic Bethe-Salpeter equation which has some unphysical solutions due to the relative time.

§ 2. General survey of the abnormal solutions

Let us consider an interaction between two fields, ϕ_a (mass; m_a) and ϕ_b (mass; m_b), and a field ϕ_c . The interaction Hamiltonian density is

$$H'(x) = g \bar{\phi}_{a,i}(x) O_{ij}^m \phi_{b,j}(x) \phi_{c,m}(x) + h. c., \quad (2.1)^*$$

where indices i, j and m indicate the components of the fields ϕ_a , ϕ_b and ϕ_c respectively and $\bar{\phi}$ is a conjugate field of ϕ . The Bethe-Salpeter equation for the particles a and b is of the following form in the lowest ladder approximation:

$$\begin{aligned} \varphi_{ij}(x_a, x_b) = & -\delta g^2 \int dx_a' dx_b' D_a^{ii}(x_a - x_b') D_b^{jj}(x_b - x_a') \\ & \times O_{ij'}^{m\dagger} \Delta_c^{mm}(x_b' - x_a') O_{j'j}^m \varphi_{ij'}(x_a', x_b'). \end{aligned} \quad (2.2)$$

Here D_a , D_b and Δ_c are the propagators of the fields ϕ_a , ϕ_b and ϕ_c respectively. The constant δ is defined as follows:

$$T\{\phi_c(x) \phi_b(x) \phi_c^*(x') \phi_a(x')\} = \delta T\{\phi_c(x) \phi_c^*(x') \phi_a(x') \phi_b(x)\}. \quad (2.3)$$

The symbol $O^{m\dagger}$ stands for the hermitian conjugate of the matrix O^m . In the following we shall omit the indices representing field components. Introducing the center of gravity coordinates X_μ , the relative coordinates x_μ and a function \mathcal{D} as follows:

$$\begin{aligned} X_\mu = & \frac{1}{(m_a + m_b)} (m_a x_{a\mu} + m_b x_{b\mu}), \quad x_\mu = x_{b\mu} - x_{a\mu}, \\ \mathcal{D}(X_\mu - X_\mu', x_\mu, x_\mu') = & D_a(x_a - x_b') D_b(x_b - x_a'), \end{aligned} \quad (2.4)$$

one finds

$$\varphi(X_\mu, x_\mu) = -\delta g^2 \int dX' dx' \mathcal{D}(X_\mu - X_\mu', x_\mu, x_\mu') O^\dagger \Delta_c(x_\mu') O \varphi(X_\mu', x_\mu'). \quad (2.5)$$

Let us now use the reference system of the center of gravity, where $\varphi(X_\mu, x_\mu)$ can be written as $\tilde{\varphi}(X_0, x_\mu)$.** In the approximation of neglecting the retardation effects in Δ_c , Eq. (2.5) turns out to be the following relation:

$$\tilde{\varphi}(X_0, \mathbf{x}) = -\delta g^2 \int dX_0' d\mathbf{x}' \mathcal{D}(X_0 - X_0', \mathbf{x}, \mathbf{x}') O^\dagger \tilde{\Delta}(\mathbf{x}') O \tilde{\varphi}(X_0', \mathbf{x}'). \quad (2.6)***$$

Here

$$\tilde{\varphi}(X_0, \mathbf{x}) = \tilde{\varphi}(X_0, x_\mu)_{x_0=0}, \quad \mathcal{D}(X_0, \mathbf{x}, \mathbf{x}') = \int d\mathbf{X} \mathcal{D}(X_\mu, x_\mu, x_\mu')_{x_0=x_0'=0} \quad (2.7)$$

and

$$\tilde{\Delta}(\mathbf{x}) = \frac{1}{(2\pi)^3} \int d\mathbf{k} \Delta_c(k_\mu)_{k_0=0} e^{i\mathbf{k}\cdot\mathbf{x}}.$$

* Double indices mean the summation for them.

** The index "0" means the fourth component of a four-vector.

*** If ϕ_a or ϕ_b is a Bose field, we must use the Sakata-Taketani amplitude instead of the usual Bethe-Salpeter one as was discussed by Scarf and Umezawa.⁷⁾

It must be noted that Eq. (2.6) has no relative time in the amplitude $\tilde{\varphi}$.

In general, the Bethe-Salpeter equation (2.5) has two kinds of solution, which we call the normal and abnormal solution. The former has a corresponding solution in Eq. (2.6) and the latter does not. Thus the appearance of the abnormal solutions is due to the relative time in the Bethe-Salpeter amplitude.

On the other hand, we have the following equation in the scattering problem of the particles a and b :

$$\begin{aligned} \chi(X_\mu, x_\mu) = & \chi^{in}(X_\mu, x_\mu) - \delta g^2 \int dX' dx' \mathcal{D}(X_\mu - X'_\mu, x_\mu, x'_\mu) \\ & \times O^\dagger \mathcal{A}_e(x'_\mu) O \chi(X'_\mu, x'_\mu). \end{aligned} \quad (2.8)$$

Here χ^{in} is the incident wave of a form

$$\chi^{in}(X_\mu, x_\mu) \sim e^{i(K_\mu X_\mu + k_\mu x_\mu)}, \quad (2.9)$$

where K_μ and k_μ are the total and relative energy-momentum respectively:

$$K_\mu = (k_{a\mu} + k_{b\mu}), \quad k_\mu = \frac{1}{m_a + m_b} (m_a k_{b\mu} - m_b k_{a\mu}). \quad (2.10)$$

Since the relative momentum k_μ is a space-like vector,* we can introduce the Lorentz transformation $k_\mu \rightarrow (k', 0)$, under which incident wave $\exp i(K_\mu X_\mu + k_\mu x_\mu)$ is transformed to $\exp i(K'_\mu X'_\mu + k' x')$. In this way the relative time in the incident wave disappears. Since any solution of (2.8) is determined uniquely by the incident wave χ^{in} , disappearance of the relative time in the incident wave indicates that the relative time plays no important role in the scattering problem. Indeed, it can be shown that there is no abnormal solution at all in scattering problems. To see this, we consider the non-retardation approximation of Eq. (2.8):

$$\tilde{\chi}(X_0, \mathbf{x}) = \tilde{\chi}^{in}(X_0, \mathbf{x}) - \delta g^2 \int dX'_0 d\mathbf{x}' \mathcal{D}(X_0 - X'_0, \mathbf{x}, \mathbf{x}') O^\dagger \tilde{\mathcal{A}}(\mathbf{x}') O \tilde{\chi}(X'_0, \mathbf{x}'). \quad (2.11)$$

In this approximation, any solution χ having $\chi^{in} \sim \exp i(-K_0 X_0 + k_\mu x_\mu)$ changes into $\tilde{\chi}$, whose incident wave is $\tilde{\chi}^{in} \sim \exp i(-K_0 X_0 + k \mathbf{x})$. Thus, we see that there exists one to one correspondence between solutions of Eqs. (2.8) and (2.11). We shall write this correspondence in a following form

$$\chi = \hat{\xi}(\tilde{\chi}) \quad \text{and} \quad \tilde{\chi} = \hat{\xi}^{-1}(\chi), \quad (2.12)$$

where $\hat{\xi}$ is a linear function having its inverse. (2.12) indicates that no abnormal solution appears in the scattering problem. This is a situation different from that

* Here it is assumed that both m_a and m_b are larger than zero. If one of them, for example, m_a , is zero and the other is not, the relative energy-momentum k_μ lies on the light cone. In this case, however, defining the center of gravity and relative coordinates as $X_\mu = \alpha x_{a\mu} + (1-\alpha) x_{b\mu}$ and $x_\mu = x_{b\mu} - x_{a\mu}$ with $0 < \alpha < -2k_{a\mu} k_{b\mu} / (m_b^2 - 2k_{a\mu} k_{b\mu})$, we see k_μ to be space-like, although this definition is based on the ambiguous concept of the center of gravity in the relativistic theory.

in the bound state problem. Further we remark that the iteration method applied to (2·8) leads us to the Feynman diagrams of the scattering problem. This means that χ (and also $\tilde{\chi}$) is an eigenstate of the total Hamiltonian. We thus see that the scattering wave solution of the Bethe-Salpeter equation is normal.

We shall now show that the abnormal solutions in the bound state problem are not permissible in quantum field theory. As an example we shall consider the case shown in Fig. 1, where an abnormal solution is indicated by a dotted line. There is assumed that the system is specified by the Hamiltonian $H=H^0+\eta H'$, where H^0 is the sum of the free Hamiltonians of the fields ϕ_a and ϕ_b . Choosing a small positive number η with a same dimension as the coupling constant g , we have the abnormal solution for $g=G+\eta$ and we have not for $g=G-\eta$. Let us now introduce the time dependent Hamiltonian as follows:

$$H(t)=H^0+g(t)H' \quad (2\cdot14)$$

where

$$g(t)=G-\eta \quad \text{for } t>T$$

and

$$g(t)=G+\eta \quad \text{for } t<-T \quad (2\cdot15)$$

with large T . Here $g(t)$ varies very slowly from $G+\eta$ to $G-\eta$ in a time interval from $-T$ to T . If the abnormal solution would have any physical reality, it would describe a bound state $|B\rangle$, which is an eigenstate of the Hamiltonian $H^{(+)}=H^0-(G+\eta)H'$. Suppose that we have the state $|B\rangle$ at a time $t_1<-T$. Then the state changes into

$$|B'\rangle = P \int_{t_1}^{t_2} dt e^{-iH(t)} |B\rangle = \sum_n |n\rangle \langle n| P \int_{t_1}^{t_2} dt e^{-iH(t)} |B\rangle \quad (2\cdot16)$$

at a time $t_2>T$. Here $|n\rangle$'s are the eigenstates of the Hamiltonian $H^{(-)}=H^0+(G-\eta)H'$. Since $g(t)$ varies very slowly, the expectation value of $dH(t)/dt$ may be very small. Then mean values of the energies of states $|n\rangle$ effective in the summation $\sum_n |n\rangle \langle n| B'\rangle$ are close to (m_a+m_b) , because the binding energy of the abnormal state concerned here is nearly zero. Owing to the largeness of the binding energies of the normal bound states, they cannot contribute to the summation. Accordingly $|n\rangle$ must be the low energy scattering states given by $H^{(-)}$, for $H^{(-)}$ has no bound state in the vicinity of the energy value (m_a+m_b) . In other words, through the long time interval the scattering amplitude χ_n appears gradually, while the abnormal Bethe-Salpeter amplitude diminishes. Then, applying

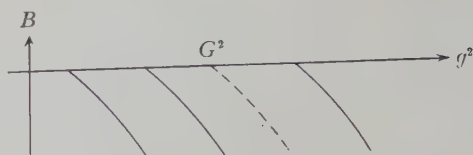


Fig. 1. The dotted line indicates the abnormal solution. The abscissa and ordinate represent the square of the coupling constant and the binding energy $B=E-(m_a+m_b)$ respectively.

(2.12) to writing $\Sigma\chi_n$ in terms of $\Sigma\tilde{\chi}_n$, we see that this means the gradual appearance of $\Sigma\tilde{\chi}_n$. It should, however, be noted here that, according to the non-retardation formalism (2.11), such appearance of $\Sigma\tilde{\chi}_n$ is impossible, since in this formalism the norm of $\Sigma\tilde{\chi}_n$ cannot change from zero to finite (or from finite to zero) through the long time interval. This is because, as is seen in Eq. (2.11), when the coupling constant varies very slowly, every $\tilde{\chi}_n$ makes transition only into the low energy scattering state, but not into any bound state. Hence, the normal state as a whole cannot be created or destructed. Thus we see that the state $|B\rangle$ disappears without inducing any scattering state; and, in other words, its norm is not conserved. This is inconsistent with the law of conservation of probability in quantum field theory.

§ 3. An example of the Bethe-Salpeter equation

To see the nature of the abnormal solutions and the relative time, we shall consider the static model for the neutral scalar meson with the scalar coupling. The Hamiltonian is of the following form:

$$H = \int dx \{ m\psi^* \psi + \frac{1}{2}(\pi^2 + (\nabla\phi)^2 + \mu^2\phi^2) + g\psi^* \phi \psi - \delta m\psi^* \psi \}, \quad (3.1)$$

where ψ and ϕ stand for the field variables of a fermion (mass; m) and a scalar meson (mass; μ) respectively and π is the conjugate momentum of ϕ . As is well known, the eigenvalue problem of the Hamiltonian (3.1) is exactly soluble: in the two-body case, energy eigenvalue is given by

$$E = 2m - \frac{g^2}{4\pi} \frac{e^{-\mu r}}{r}. \quad (3.2)$$

Here r is the spacial distance between two fixed fermions.

Now, in the lowest ladder approximation, the Bethe-Salpeter equation is written as follows.

$$\left(i\frac{\partial}{\partial t_1} - m\right)\left(i\frac{\partial}{\partial t_2} - m\right)\varphi(x_1, x_2) = g^2 \mathcal{A}_F(x_1 - x_2)\varphi(x_1, x_2). \quad (3.3)$$

Comparing the solutions of Eq. (3.3) with the true bound state given by (3.2), we shall examine whether (3.3) has any redundant solution or not. Denoting the respective time-coordinate of two fermions by t_1 and t_2 and introducing T and t by the relations

$$T = \frac{t_1 + t_2}{2} \quad \text{and} \quad t = t_1 - t_2, \quad (3.4)$$

we obtain

$$\left\{\left(\frac{i}{2}\frac{\partial}{\partial T} - m\right)^2 + \frac{\partial^2}{\partial t^2}\right\}\varphi = g^2 \mathcal{A}_F(x_1 - x_2)\varphi. \quad (3.5)$$

This leads us to

$$\left\{ \left(-\frac{1}{2} (E-2m) \right)^2 + \frac{\partial^2}{\partial t^2} \right\} f(t, r) = g^2 J_F(x_1 - x_2) f(t, r), \quad (3.6)$$

where

$$\varphi(x_1, x_2) = e^{-iET} f(t, r). \quad (3.7)$$

Here E is the total energy of the system. When we perform the analytic continuation¹⁾ of the function $f(t, r)$, in the complex plane of the relative time t , we get

$$\left(-\frac{\partial^2}{\partial x^2} + \lambda^2 U(x, r) \right) f = -\varepsilon^2 f, \quad (3.8)$$

where

$$\lambda^2 = \left(\frac{g}{2\pi} \right)^2, \quad \varepsilon = \frac{1}{2} (E-2m), \quad x = -it \quad (3.9)$$

and

$$U(x, r) = \frac{-1}{(2\pi)^2} \int dk dk' \frac{e^{i(kx' - kx)}}{k^2 + k'^2 + \mu^2} = \frac{\mu}{\sqrt{r^2 + x^2}} K_0'(\mu \sqrt{r^2 + x^2}). \quad (3.10)$$

Here $K_0'(z)$ is a derivative of the modified Bessel function of the 2nd kind. We have assumed that $E < 2m$. A change of the variable, i.e. $x \rightarrow rx$, leads us to

$$\left(-\frac{\partial^2}{\partial x^2} + \lambda^2 V(x, \mu r) \right) f = -\alpha^2 f, \quad (3.11)$$

where

$$V(x, \mu r) = \frac{\mu r}{\sqrt{1+x^2}} K_0'(\mu r \sqrt{1+x^2}) \quad (3.12)$$

and

$$\alpha = r\varepsilon. \quad (3.13)$$

As is easily seen from (3.11), the eigenvalue ε has a form

$$\varepsilon = \alpha(\mu r)/r. \quad (3.14)$$

Eq. (3.11) is the equation for the one-dimensional energy eigenvalue problem with the potential $\lambda^2 V(x, \mu r)$.

In the case of $\mu \neq 0$, the form of $V(x, \mu r)$ is pictured in Fig. 2; for a large value of $|x|$, it is proportional to $|x|^{-3/2} \exp(-\mu r|x|)$. We can easily show that the short range

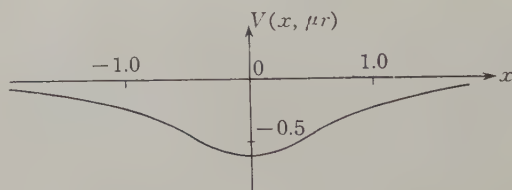


Fig. 2. The case $\mu r=1$

potential of this type has a finite number of discrete levels for any value of λ^2 . When $\mu=0$, (3.14) is just of the Coulomb type. In this case, $V(x, \mu r)$ becomes

$$\lim_{\mu \rightarrow 0} V(x, \mu r) = -\frac{1}{1+x^2}. \quad (3.15)$$

We shall show in the Appendix that this long range potential gives us a degeneration of an infinite number of solutions at $\lambda^2 = \frac{1}{4}$.* (See Fig. 4)

To study the properties of the solutions in some more detail, we investigate the eigenvalue satisfying the condition

$$\lim_{|\alpha| \rightarrow 0} \frac{\lambda^2}{|\alpha|} = A^2, \quad (3.16)$$

where A^2 is non-zero and finite. Introducing $x' = x|\alpha|$, we obtain

$$\left\{ -\frac{\partial^2}{\partial x'^2} + \frac{\lambda^2}{|\alpha|} \frac{1}{|\alpha|} V\left(\frac{x'}{|\alpha|}, \mu r\right) \right\} f = -f. \quad (3.17)$$

When $|\alpha|$ is small, (3.17) can be reduced to the form

$$\left(\frac{\partial^2}{\partial x'^2} + CA^2 \delta(x') \right) f = f, \quad (3.18)$$

where

$$C = \frac{-1}{|\alpha|} \int_{-\infty}^{\infty} dx V\left(\frac{x}{|\alpha|}, \mu r\right) = \pi e^{-\mu r}. \quad (3.19)$$

Then, denoting the Fourier component of $f(x', r)$ by

$$F(p) = \int_{-\infty}^{\infty} f(x', r) e^{-ipx'} dx', \quad (3.20)$$

one finds

$$F(p) = CA^2 \frac{f(0, r)}{1 + p^2}. \quad (3.21)$$

We thus obtain

$$f(x', r) = \frac{CA^2}{2} f(0, r) e^{-|x'|}. \quad (3.22)$$

Now let a value of x' be zero in Eq. (3.22). Taking account of Eqs. (3.13), (3.16) and (3.19), we obtain

$$2|\varepsilon| = \frac{g^2}{4\pi} \frac{e^{-\mu r}}{r}. \quad (3.23)$$

Here ε must be negative, for we have assumed $E < 2m$ in the analytic continuation of the function f . The eigenvalue (3.23) is nothing but the eigenvalue (3.2) obtained by solving $H|\rangle = E|\rangle$. Although Eq. (3.11) has many other eigenvalues shown in Figs. 3 and 4, all of them are physically meaningless, because (3.2) is

* Although the degeneration of this type also occurs in Wick-Cutkosky's solutions,^{1), 2)} it does not represent any feature of the abnormal solutions. This can be seen from the case $\mu \neq 0$ in our example. The appearance of this degeneration seems due to $\mu = 0$.

the only eigenvalue of the Hamiltonian (3.1) in the two-body case. Thus the relative time in Eq. (3.3) actually gives rise to many redundant solutions.*

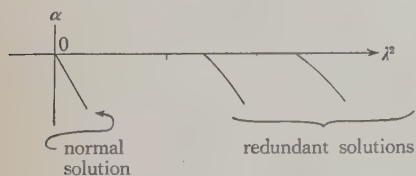


Fig. 3. The case $\mu \neq 0$

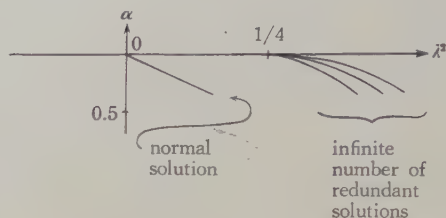


Fig. 4. The case $\mu = 0$

Finally we discuss the non-retardation formalism of Eq. (3.3). Neglecting the retardation effect in \mathcal{A}_F , one finds

$$\left(-\frac{\partial^2}{\partial x^2} + \varepsilon^2\right)f = \frac{g^2}{(2\pi)^4} \int dk dp dk \frac{e^{ikr}}{k^2 + \mu^2} \frac{e^{ipx}}{2\pi} F(p-k). \quad (3.24)$$

As can be seen easily, this is just equal to Eq. (3.18), whose eigenvalue has been found to be given by Eq. (3.23). Thus in the non-retardation formalism, we get the eigenvalue (3.2) only.

In this section we have compared the solutions of the Bethe-Salpeter equation (3.3) with the true energy eigenvalue given by (3.2) to show that the relative time gives rise to the redundant solutions which must be excluded. It is remarked, however, that even if we did not know the eigenvalue of the Hamiltonian, we could exclude all of abnormal solutions by means of the method discussed in § 2. In other words, we should omit all the solutions which do not appear in the non-retardation formalism.

§ 4. Concluding remarks

We have examined the solutions of the Bethe-Salpeter equation and suggested that its abnormal solution is not permissible in quantum field theory. This statement was further proved in § 3 in a special example of the Bethe-Salpeter equation.

In the static model, the binding energy is nothing but the potential between two particles. Therefore it is supposed that the bound states in the normal and abnormal potentials obtained from the static Bethe-Salpeter equation correspond respectively to the normal and abnormal solutions of the covariant one. Indeed, it will be considered that Wick-Cutkosky's solutions^{1),2)} correspond to the bound states in the potential which is the solution of Eq. (A.1). Appearance of the degeneration at $\lambda^2 = \frac{1}{4}$ in both Wick-Cutkosky's and our case seems to be understood in this way.

* If $\lambda^2 < 1/4$, there is no redundant solution for any value of μ .

Although our discussion is based on the lowest ladder approximation, it may be possible to generalize it to the case including the higher order corrections. Nevertheless, it has not yet been clear whether the complete Bethe-Salpeter equation without any approximations has actually the redundant solutions or not; further investigations on this point are desired.

Appendix

In the case of $\mu=0$, Eq. (3.11) becomes

$$\left(\frac{d^2}{dx^2} + \frac{\lambda^2}{1+x^2}\right)f = \alpha^2 f. \quad (\text{A} \cdot 1)$$

With a slight change of variables, we can rewrite (A.1) as follows:

$$\left(\frac{d^2}{dx^2} + \frac{\lambda^2}{\beta^2 + x^2}\right)f = -\frac{1}{4}f \quad (\text{A} \cdot 2)$$

where

$$\beta = 2|\alpha|. \quad (\text{A} \cdot 3)$$

Now we shall examine the eigenvalues λ^2 satisfying the condition $\lim_{\beta \rightarrow 0} \lambda^2/\beta \rightarrow \infty$ and show that all of them converge to $\lambda^2 \rightarrow \frac{1}{4}$ when $\beta \rightarrow 0$.

As the solution f is an even or odd function of x , it is sufficient to consider the case in which x lies in the interval $(0, \infty)$. Assuming

$$\beta \ll 1, \quad (\text{A} \cdot 4)$$

we divide the interval $(0, \infty)$ into two parts,

$$0 < x < \gamma \quad \text{and} \quad \gamma < x, \quad (\text{A} \cdot 5)$$

where we choose γ in order to satisfy

$$\beta \ll \gamma \ll 1. \quad (\text{A} \cdot 6)$$

In the first interval, neglecting the term of order β , we obtain the following equation,

$$\left(\frac{d^2}{dy^2} + \frac{\lambda^2}{1+y^2}\right)f = 0, \quad y = \frac{x}{\beta}. \quad (\text{A} \cdot 7)$$

Eq. (A.7) is of the hypergeometric type. Its solutions are

$$f^{\text{odd}} = f^{(+)} - f^{(-)} \quad (\text{A} \cdot 8)$$

and

$$f^{\text{even}} = f^{(+)} + f^{(-)}, \quad (\text{A} \cdot 9)$$

where

$$f^{(\pm)}(y) = (1+y^2)F\left(\frac{3}{2} + \rho, \frac{3}{2} - \rho; 2; \frac{1}{2}(1 \pm iy)\right), \quad (\text{A} \cdot 10)$$

$$\rho = (\tfrac{1}{4} - \lambda^2)^{1/2}.$$

In the second interval, again neglecting the term of order β , we write

$$-\frac{d^2}{dx^2}f + \left(-\frac{1}{4} + \frac{\frac{1}{4} - \rho^2}{x^2}\right)f = 0. \quad (\text{A} \cdot 11)$$

This is the confluent hypergeometric equation whose solution (with the boundary condition $f(\infty) \rightarrow 0$) is given by the Whittaker function

$$W_{0\rho}(x) = \frac{\Gamma(-2\rho)}{\Gamma(\frac{1}{2} - \rho)} M_{0,\rho}(x) + \frac{\Gamma(2\rho)}{\Gamma(\frac{1}{2} + \rho)} M_{0,-\rho}(x). \quad (\text{A} \cdot 12)^*$$

Now we shall investigate the eigenvalues λ^2 corresponding to the solutions f^{odd} . First, we will show that if $0 < \lambda^2 < \frac{1}{4}$, the solutions $f_{1\text{st}}$ and $f_{2\text{nd}}$ (defined by (A·8) and (A·12) respectively) cannot join smoothly at $x = \gamma$. This means that there is no eigenvalue satisfying $0 < \lambda^2 < \frac{1}{4}$. In fact, since $\beta \ll \gamma \ll 1$, y is large enough at $x = \gamma$, we may evaluate $f_{1\text{st}}(y)$ by means of the asymptotic formula of the hypergeometric functions. Then one finds, neglecting a normalization factor,

$$f_{1\text{st}}(y) \sim y^{\rho+1/2}(1 + \dots) + A(\rho)y^{-\rho+1/2}(1 + \dots), \quad (\text{A} \cdot 13)$$

$$A(\rho) = 2^{2\rho} \tan(\tfrac{1}{4}\pi - \tfrac{1}{2}\pi\rho) \frac{\Gamma(2\rho)}{\Gamma(-2\rho)}. \quad (\text{A} \cdot 14)$$

Using $x = \beta y$, we rewrite this as follows,

$$f_{1\text{st}}(x) \sim x^{\rho+1/2}(1 + \dots) + A(\rho)\beta^{2\rho}x^{-\rho+1/2}(1 + \dots). \quad (\text{A} \cdot 15)$$

Similarly, $f_{2\text{nd}}(x)$ may be evaluated for small value of x by means of the power series expansion of the functions $M_{0,\pm\rho}(x)$. Again neglecting a normalization factor, one finds

$$f_{2\text{nd}}(x) \sim x^{\rho+1/2}(1 + \dots) + B(\rho)x^{-\rho+1/2}(1 + \dots), \quad (\text{A} \cdot 16)$$

$$B(\rho) = 2^{4\rho} \frac{\Gamma(\rho)}{\Gamma(-\rho)}. \quad (\text{A} \cdot 17)$$

As λ^2 varies from 0 to $\frac{1}{4}$, $A(\rho)$ and $B(\rho)$ vary monotonously from 0 to -1 and from -2 to -1 respectively. Since $A(\rho)\beta^{2\rho}$ stays quite close to 0, $f_{1\text{st}}$ and $f_{2\text{nd}}$ cannot join smoothly at $x = \gamma$. Hence there is no eigenvalue satisfying $0 < \lambda^2 < \frac{1}{4}$.

Let us now consider the case of $\lambda^2 > \frac{1}{4}$. We see that ρ is a pure imaginary, say $\rho = i\sigma$, $\sigma = (\lambda^2 - \frac{1}{4})^{1/2}$. One then finds that

$$f_{1\text{st}} \sim \sqrt{x} \sin\{\sigma \log(x/\beta) - \sigma \log 2 + \pi/4 + \tan^{-1}e^{\pi\sigma} + \theta(2\sigma)\} \quad (\text{A} \cdot 18)$$

and

$$f_{2\text{nd}} \sim \sqrt{x} \sin\{\sigma \log x - 2\sigma \log 2 + \pi/2 + \theta(\sigma)\}, \quad (\text{A} \cdot 19)$$

where $\theta(\sigma)$ is defined by

* Functions $M_{0,\pm\rho}(x)$ are defined in "Whittaker and Watson: *A Course of Modern Analysis*, p. 339."

$$\Gamma(-i\sigma) = |\Gamma(-i\sigma)| e^{i\theta(\sigma)}. \quad (\text{A} \cdot 20)$$

(A·18) and (A·19) can be joined smoothly at $x=r$, if

$$\sigma \log \frac{\beta}{2} + \frac{\pi}{4} - \text{Tan}^{-1} e^{\pi\sigma} + \theta(\sigma) - \theta(2\sigma) = n\pi. \quad (\text{A} \cdot 21)$$

To solve Eq. (A·21), we examine the analytic properties of a function

$$\mathcal{J}(z) = \frac{\Gamma(-z)}{\Gamma(-2z)}. \quad (\text{A} \cdot 22)$$

Since $\Gamma(z)$ is regular except at $z=0, -1, -2, \dots$ and has no zero on the imaginary axis of z as seen from the formula

$$|\Gamma(i\sigma)| = \sqrt{\frac{\pi}{\sigma \sinh \sigma}}, \quad (\text{A} \cdot 23)$$

$\log \mathcal{J}(z)$ is regular at $z=i\sigma$ and therefore $\theta(\sigma) - \theta(2\sigma) (= \text{Im} \log \mathcal{J}(i\sigma))$ is finite for any real value of σ . Then one finds that the roots σ of Eq. (A·21) are small enough, unless $\sigma \log \beta/2$ is very large and the equation cannot be satisfied. Writing for small σ

$$\text{Tan}^{-1} e^{\pi\sigma} = \frac{\pi}{4} + \frac{\pi}{2} \sigma + O(\sigma^2)$$

and

$$\theta(\sigma) - \theta(2\sigma) = a\sigma + O(\sigma^2), \quad (\text{A} \cdot 24)$$

we obtain from Eq. (A·21)

$$\begin{aligned} \sigma &= \frac{n\pi}{(\log(\beta/2) - \pi/2 + a)}, \\ \lambda^2 &= \frac{1}{4} + \frac{(n\pi)^2}{(\log r|\varepsilon| - \pi/2 + a)^2}. \end{aligned} \quad (\text{A} \cdot 25)$$

In the same way we can see that the eigenvalues of the functions f^{even} are given as follows,

$$\lambda^2 = \frac{1}{4} + \frac{(n\pi)^2}{(\log r|\varepsilon| + \pi/2 + a)^2}. \quad (\text{A} \cdot 26)$$

Eqs. (A·25) and (A·26) give an infinite number of eigenvalues which tend to $\lambda^2 = \frac{1}{4}$ when $\varepsilon \rightarrow 0$.

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A Note on the Ordinary and Anomalous Thresholds in Perturbation Theory[†]

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Some further investigations are made for the singular point of the matrix element corresponding to the general Feynman graph.

In a previous work¹⁾ we investigated the thresholds of the matrix element corresponding to the general Feynman graph by using the explicit expression for the denominator function $V(x)$.²⁾ Recently, a very similar investigation has been made independently by Landau.³⁾ He found an elegant method for obtaining the zero extremum determined by

$$\begin{aligned} \partial V / \partial x_i &= 0 & \text{for } i \in B \equiv \{1, 2, \dots, r\}, \\ x_j &= 0 & \text{for } j \in A \equiv \{r+1, \dots, \alpha\}. \end{aligned} \quad (1)$$

But since he did not use the explicit expression for $V(x)$, the behaviors in the neighborhood of the zero extremum, which were investigated in detail for the ordinary threshold in NIII, were not treated by him.* So, applying Landau's idea to the explicit form of $V(x)$, we can present simpler proofs for the results in NIII and some further extensions.

When $x_j=0$ ($j \in A$), from Theorem 2 in NIII it suffices to consider the reduced Feynman graph B^* which is obtained by *shrinking* each line of A to one point. Each propagator contains a constant momentum q_i , which is not uniquely determined by external momenta k_l since any arbitrary constant momentum can additionally flow along each independent closed circuit. We can therefore choose q_i so that

$$\sum_i \pm x_{c_i} q_{c_i} = 0 \quad (2)$$

for independent circuits C , where the double sign corresponds to the relative direction of q_{c_i} on C . Then (2) holds also for any other closed circuit because it is composed as a linear combination from the independent circuits. (2) is just Landau's

[†] Notations are the same as in reference 1).

* Therefore Landau's argument is insufficient since the condition (1) alone cannot exclude the saddle point which does not correspond to any singularity.

condition. It should be noticed that this is independent of the extremum condition. It is of interest that the momentum conservation at each vertex, which is written in such a form as

$$(\sum \pm q_i) + (\sum \pm k_i) = 0, \quad (3)$$

and Eq. (2), respectively, have a strong resemblance to Kirchhoff's first and second laws for a network of electric current if we regard q_i as a current and x_i as a resistance.⁴⁾

For the choice of (2) we have simply

$$V = \sum_i x_i (m_i^2 + q_i^2), \quad (4)$$

$$Y_i \equiv (1/2x_i) (\partial V / \partial q_i) = q_i, \quad (5)$$

$$\partial V / \partial x_i \equiv m_i^2 + Y_i^2 = m_i^2 + q_i^2. \quad (6)$$

Therefore from (1) and (6) the condition for the zero extremum is

$$m_i^2 + q_i^2 = 0. \quad (7)$$

The simultaneous equations (2), (3) and (7) determine the zero extremum under the important restrictions $x_i > 0$ and $\sum x_i = 1$. When B^* is an intermediate-state set, Theorem 7 in NIII is easily obtained by solving these equations. If the number of the conditions imposed by (3) and (7) is more than the freedoms of q_i 's, no solution can generally be found for such a graph B^* .

The expression for Y_i in terms of k_i was given in NI(3.15). Under the present choice of (2), as q_i is equal to Y_i as in (5), NI(3.15) represents just the solution of Eqs. (2) and (3) involving the parameters x_i .

Now, let us denote the value of x_i corresponding to the zero extremum by x_i^0 , and consider its neighborhood

$$x_i = x_i^0 + \varepsilon y_i, \quad (\varepsilon: \text{infinitesimal}) \quad (8)$$

$$\text{with} \quad \sum_{i=1}^r y_i = 0. \quad (9)$$

(2) is now rewritten as

$$\sum_i \pm x_{\alpha_i}^0 q_{\alpha_i} = 0 \quad (10)$$

and thus q_i is fixed. We then have

$$V(x) = -\varepsilon^2 (1/U) \sum_c U_c (\sum_i \pm y_{\alpha_i} q_{\alpha_i})^2, \quad (11)$$

where U and U_c may be regarded as (positive) constants if we are interested only in the lowest order terms with respect to ε . If in the reduced graph B^* two or more external lines start from the same vertex (this situation often occurs owing to the reduction of the original graph), then we denote the sum of them by k_i' . Then in B^* q_i is of course expressible by a linear combination of k_i' 's.

If these momenta k'_i have Euclidean character,* i.e., if it always holds that

$$-(\sum_i a_i k'_i)^2 \geq 0 \quad (12)$$

for arbitrary real numbers a_i , then we have $V(x) \geq 0$ from (11). Namely, *the point $x_i = x_i^0$ surely gives the minimum of $V(x)$ ("threshold minimum") if the effective external momenta of the reduced graph have Euclidean character.* This statement is an extension of Theorem 8 in NIII. An interesting problem is its inverse statement: "Can the threshold minimum appear when k'_i 's have *not* Euclidean character?". This is an open question, but for the lowest order graphs (i.e., the graphs consisting of only one closed circuit) we can easily show that Euclidean character is the necessary (and sufficient) condition for the threshold minimum, because without changing the expression (11) we can replace y_i by y'_i which is an arbitrary real number free from the restriction (9), as is seen by putting

$$y_i = y'_i - x_i^0 \sum_{k=1}^r y_k'. \quad (13)$$

Finally, we discuss the variation of $x_j \in A$:

$$x_j = \varepsilon' y_j \geq 0 \quad (\varepsilon' = O(\varepsilon)) \quad (14)$$

with

$$\varepsilon \sum_{i=1}^r y_i + \varepsilon' \sum_{j=r+1}^{\alpha} y_j = 0$$

instead of (9). We then have

$$V(x) = \varepsilon' V_A(y_j) + \varepsilon^2\text{-term} \quad (11), \quad (15)$$

where V_A is the V of the graph A (which is obtained from the original graph by *opening* each line of B) in which the new external momenta q_i are determined as the solution of the simultaneous equations (2), (3) and (7). Therefore it will in general be rather difficult to write down the explicit form of $V_A(y_j)$ except for the ordinary threshold, for which the formula (15) was already applied to the calculation of the nucleon structure.⁵⁾

The author wishes to express his sincere thanks to Prof. H. Yukawa for informing the author of the Landau's work.

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* This involves the case in which all k'_i 's are time-like and parallel (ordinary threshold). The other cases can occur only in the unphysical region.

Coupling Types and Strengths of the Y - N - K Interactions

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Analyses are made on K - N scattering with use of the dispersion relations for the charge exchange scattering $K^+n \rightarrow K^0p$ and for the ordinary elastic scattering $K^+p \rightarrow K^+p$. It is shown that (ΛNK) and (ΣNK) interactions are both pseudoscalar types or pseudoscalar and scalar types, respectively. As for the interaction strengths, it is preferable that $g_\Lambda^2 \approx 5$ and $g_\Sigma^2 = 0$ with the existing experimental data.

§ 1. Introduction

A number of analyses on the K - N scattering¹⁾⁻⁷⁾ have recently been made with use of the dispersion relations. Though definite conclusions have not been deduced because of the poor experimental data, the results obtained are as follows, provided that the $K^+ - p$ scattering potential is repulsive,

1) Λ and Σ have the same parities and $(Y NK)$ interactions are of both ps -types.¹⁾⁻⁶⁾

2) the $K^- - p$ scattering potential is attractive.⁵⁾

But the possibility that Λ and Σ have opposite parities is not excluded.⁷⁾

As is known, in the dispersion relations for the K^+p and K^-p elastic scatterings which have been used by many authors, the bound state contributions from Λ and Σ appear always in the same signs. On the other hand, in the dispersion relations for the charge exchange scattering $K^+n \rightarrow K^0p$ and $K^-p \rightarrow \bar{K}^0n$, the bound state contributions have opposite signs for Λ and Σ . As seen in § 2, this charge exchange scattering dispersion integrals are more convergent than the elastic ones. We, therefore, can decide each interaction type and coupling constant of (ΛNK) and (ΣNK) by making use of two dispersion relations.

The dispersion relation used by Matthews-Salam²⁾ is not good in the convergency of cross section integral, while Igi improved on the convergency by using the subtracted dispersion relation which had a pole at zero kinetic energy. The results, therefore, depend on the way of extrapolation of the cross section into the unphysical region, especially on the slope at zero energy. It will be significant to recalculate by using the subtracted dispersion relation at non-zero energy.

In § 2 the charge exchange dispersion relations are derived and the numerical analyses are done in § 3. The consequences are as follows. Of four possible

cases of interaction types for (ΛNK) and (ΣNK),

- (A) Λ : ps and Σ : ps,
- (B) Λ : ps and Σ : s,
- (C) Λ : s and Σ : ps,
- (D) Λ : s and Σ : s,

the case (C) and (D) are excluded. The cases (A) and (B) are fitted and as for the coupling constants $g_A^2 \approx 5$ and $g_\Sigma^2 = 0$ in both cases.

§ 2. Dispersion relation for charge exchange scattering

Prior to numerical analyses, the forward dispersion equations for the charge exchange scattering and the kinematical relations will be derived.

For the iso-spin $I=0$ and 1 states, we have*

$$D_0^\pm(\omega) = \frac{P}{\pi} \int_0^\infty \left[\frac{A_0^\pm(\omega')}{\omega' - \omega} + \frac{-A_0^\mp(\omega') + 3A_1^\mp(\omega')}{2(\omega' + \omega)} \right] d\omega', \quad (1 \cdot a)$$

$$D_1^\pm(\omega) = \frac{P}{\pi} \int_0^\infty \left[\frac{A_1^\pm(\omega')}{\omega' - \omega} + \frac{A_0^\mp(\omega') + A_1^\mp(\omega')}{2(\omega' + \omega)} \right] d\omega', \quad (1 \cdot b)$$

where D_I^\pm and A_I^\pm are the real and imaginary parts of the forward $K-N$ and $\bar{K}-N$ scattering with the iso-spin state I ($M_I^\pm = D_I^\pm + iA_I^\pm$). P stands for the principal value. Denoting the causal amplitudes for $K^+n \rightarrow K^0p$ and $K^-p \rightarrow \bar{K}^0n$ scattering by M_{ee}^+ and M_{ee}^- respectively, we can get a relation:

$$M_{ee}^\pm = -\frac{1}{2}(M_0^\pm - M_1^\pm). \quad (2)$$

Assuming that parities of charged K and neutral K are the same ones, we obtain from Eqs. (1·a) and (1·b)

$$D_{ee}^\pm(\omega) = +P_\pm^{(\Lambda)}(\omega) - P_\pm^{(\Sigma)}(\omega) + \frac{P}{\pi} \int_K^\infty d\omega' \left[\frac{A_{ee}^\pm(\omega')}{\omega' - \omega} - \frac{A_{ee}^\mp(\omega')}{\omega' + \omega} \right] - \frac{1}{\pi} \int_a^K d\omega' \frac{A_{ee}^\mp(\omega')}{\omega' \pm \omega}, \quad (3 \cdot a)$$

with

$$\begin{aligned} P_\pm^{(\gamma)} &= \frac{g_\gamma^2}{\omega_\gamma \pm \omega} \frac{(Y+N)^2 - K^2}{4NY} && \text{for } s\text{-coupling,} \\ &= -\frac{g_\gamma^2}{\omega_\gamma \pm \omega} \frac{K^2 - (Y-N)^2}{4NY} && \text{for } ps\text{-coupling,} \end{aligned} \quad (4)**$$

* The natural unit $\hbar=c=1$ will be used throughout this paper.

** The letter indicating a particle is used also as a symbol of its mass value.

$$\omega_Y = (Y^2 - N^2 - K^2)/2N \quad \text{and} \quad a = [(A + \pi)^2 - N^2 - K^2]/2N.$$

Eq. (3·a) can be rewritten into

$$D_{ex}^{\pm}(\omega) = +P_{\pm}^{(\Lambda)}(\omega) - P_{\pm}^{(\Sigma)}(\omega) - \frac{P}{8\pi^2} \int_{\kappa}^{\infty} k' d\omega' \left[\frac{\sigma_0^{+}(\omega') - \sigma_1^{+}(\omega')}{\omega' \mp \omega} - \frac{\sigma_0^{-}(\omega') - \sigma_1^{-}(\omega')}{\omega' \pm \omega} \right] + \frac{1}{2\pi} \int_a^{\kappa} d\omega' \frac{A_0^{-}(\omega') - A_1^{-}(\omega')}{\omega' \pm \omega}. \quad (3 \cdot b)$$

Here all σ_f^{\pm} 's denote the total cross section. It will be noticed that in Eqs. (3·a) and (3·b) bound state contributions $P_{\pm}^{(\Lambda)}$ and $P_{\pm}^{(\Sigma)}$ have opposite signs.

Since we have no detailed knowledge of A_{ec}^{\pm} at present, we analyse the K - N scattering with use of Eq. (3·b) in § 3. It seems that the charge exchange dispersion relations have more rapid convergency than the ordinary elastic scattering ones, so that it is unnecessary to perform any subtractions, provided that in high energy region $\sigma_n^{\pm} \simeq \sigma_p^{\pm}$.

The charge independent theory gives

$$\sigma_0^{\pm} - \sigma_1^{\pm} = \pm 2(\sigma_n^{\pm} - \sigma_p^{\pm}). \quad (5)$$

As far as the s -wave amplitude is predominant in low energies we can get the expression

$$D_{ex}^{\pm} = \pm \sqrt{\frac{W^2}{N^2} \frac{\sigma_{ex}^{\pm}}{4\pi} - \left\{ \frac{k}{4\pi} (\sigma_n^{\pm} - \sigma_p^{\pm}) \right\}^2}, \quad (6)$$

where W is the total energy in the center of mass system.

§ 3. Calculation and results

In the elastic dispersion relations which have been used by Igi, the subtraction is performed at the zero kinetic energy in order that the integration of the cross section converges rapidly and then the dispersion integral has a pole at zero kinetic energy. The obtained results, therefore, depend on the way of extrapolation of the cross section into the unphysical region.

At first, the recalculations will be made by using the dispersion equation subtracted at non-zero energy. The elastic dispersion relation subtracted at the energy ω_0 can be written as

$$\begin{aligned} D_p^{+}(\omega) - \frac{1}{2} \left(1 + \frac{\omega}{\omega_0} \right) D_p^{+}(\omega_0) - \frac{1}{2} \left(1 - \frac{\omega}{\omega_0} \right) D_p^{-}(\omega_0) \\ = P^{(\Lambda)}(\omega, \omega_0) + P^{(\Sigma)}(\omega, \omega_0) + \frac{\omega^2 - \omega_0^2}{4\pi^2} P \int_{\kappa}^{\omega_{\max}} d\omega' \frac{k'}{\omega'^2 - \omega_0^2} \\ \times \left[\frac{\sigma_p^{+}(\omega')}{\omega' - \omega} + \frac{\sigma_p^{-}(\omega')}{\omega' + \omega} \right] + \frac{\omega^2 - \omega_0^2}{\pi} \int_a^{\kappa} \frac{d\omega'}{\omega'^2 - \omega_0^2} \frac{A_p^{-}(\omega')}{\omega' + \omega} \end{aligned} \quad (7)$$

with

$$\begin{aligned}
 P^{(r)}(\omega, \omega_0) &= - \frac{(\omega^2 - \omega_0^2) g_Y^2}{(\omega_0^2 - \omega_Y^2)(\omega_Y + \omega)} \frac{(Y+N)^2 - K^2}{4NY} \quad \text{for } s, \\
 &= \frac{(\omega^2 - \omega_0^2) g_Y^2}{(\omega_0^2 - \omega_Y^2)(\omega_Y + \omega)} \frac{K^2 - (Y-N)^2}{4NY} \quad \text{for } ps.
 \end{aligned} \quad (8)$$

Here we put $\omega = 1.2K$ (100 Mev) and $\omega_0 = 1.1K$ (50 Mev). The contribution from the unphysical region is unknown but will be small compared with the contribution from the physical region if A_p^- does not behave singularly in the unphysical region. However, in order to compare with Igi's result, we shall take into account this contribution making use of the curve for A_p^- assumed by him (Fig. 1).*

Using the data which have been reported at CERN⁸⁾ and Kiev Conferences⁹⁾ and assuming that the $K^+ - p$ scattering potential is repulsive and the $K^- - p$ scattering potential attractive, we get D_p^+ (1.1K) = -1.21, D_p^+ (1.2K) = -1.22 and $D_p^-(1.1K) = 1.89$, where we have used the relation

$$D_p^\pm(\omega) = \pm \sqrt{\frac{W^2}{N^2} \frac{\sigma_p^\pm(\text{elastic})}{4\pi} - \left(\frac{k\sigma_p^\pm(\text{total})}{4\pi} \right)^2}.$$

The first integral term in Eq. (7) is 0.099, 0.116 and 0.124 for the cut-off energy $\omega_{\max} = 3K, 4K$ and $5K$, respectively. Substituting those values and Eq. (8) into Eq. (7), we obtain

$$P^{(\Lambda)} + P^{(\Sigma)} = \begin{cases} 0.04 + 0.01 \\ 0.03 + 0.01 \\ 0.02 + 0.01 \end{cases} \quad \text{for } \omega_{\max} = \begin{cases} 3K \\ 4K \\ 5K, \end{cases} \quad (9)$$

where the second term denotes the contribution from the unphysical region and

$$P^{(\Lambda)} + P^{(\Sigma)} = \begin{cases} 0.008 g_\Lambda^2 + 0.006 g_\Sigma^2 \\ 0.008 g_\Lambda^2 - 0.13 g_\Sigma^2 \\ -0.15 g_\Lambda^2 + 0.006 g_\Sigma^2 \\ -0.15 g_\Lambda^2 - 0.13 g_\Sigma^2 \end{cases} \quad \text{for } \begin{cases} \text{case (A)} \\ \text{case (B)} \\ \text{case (C)} \\ \text{case (D)}. \end{cases} \quad (10)$$

From Eq. (9) the case (D) is excluded.

* $A_p^-(\omega) = (1/4\pi) \cdot k\sigma_p^-$ does not in general behave like constant near $\omega = K$.

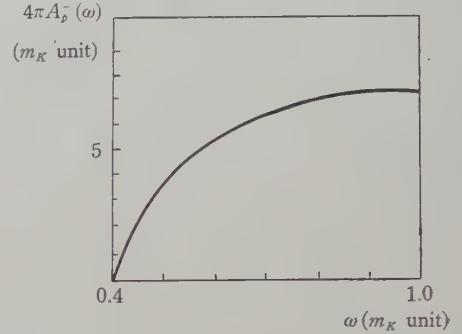
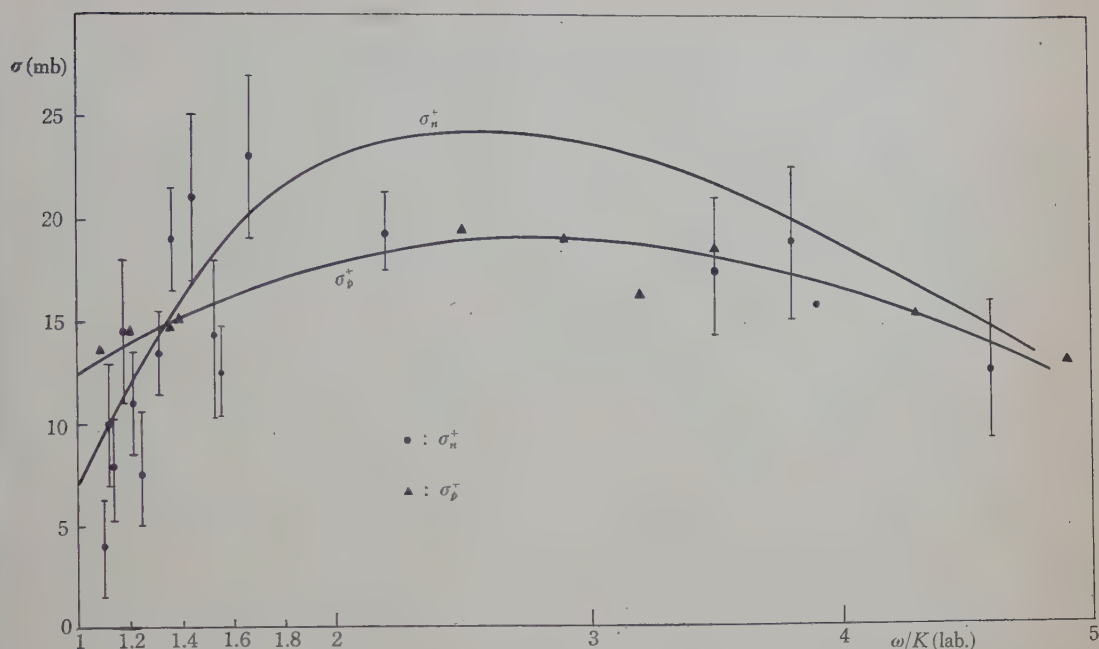


Fig. 1. Curve for A_p^- in the unphysical region

Fig. 2. Cross sections σ_p^+ and σ_n^+ vs. energies⁸⁾

Next, let us analyse the charge exchange scattering making use of the dispersion relation (3·b). We take $\omega=1.1K$ and $1.2K$. From the experimental data, the cross section of $K^-n \rightarrow K^0p$ scattering is about 2mb at 50Mev and 3mb at 100Mev in laboratory system and $D_1^- < 0$ and $|D_0^-| < |D_1^+|$ in low energies, so that $D_{ex}^+ = -1/2 \cdot (D_0^+ - D_1^+) < 0$. The cross sections σ_p^+ and σ_n^+ vs. the energy are shown in Fig. 2.* In this case the unphysical region integral is negligibly small. From Eq. (6), we get $D_{ex}^+ (1.1K) \approx -0.49$ and $D_{ex}^+ (1.2K) \approx -0.61$. Substituting these values into Eq. (3·b), we get, for $\omega=1.1K$

$$\begin{pmatrix} 0.80 \\ -0.043 \end{pmatrix} g_{\Lambda}^2 - \begin{pmatrix} 0.70 \\ -0.031 \end{pmatrix} g_{\Sigma}^2 = -0.49 + I_+ - I_-, \quad (11 \cdot a)$$

and for $\omega=1.2K$

$$\begin{pmatrix} 0.74 \\ -0.040 \end{pmatrix} g_{\Lambda}^2 - \begin{pmatrix} 0.66 \\ -0.028 \end{pmatrix} g_{\Sigma}^2 = -0.61 + I_+ - I_-. \quad (11 \cdot b)$$

Here the upper and lower values in the bracket denote the scalar and pseudo-scalar coupling types respectively and

* Though the data at CERN Conference give a small cross section for $K^+n \rightarrow K^+n$ and $K^+n \rightarrow K^0p$ scattering, we get $\sigma_n^+ = \sigma_{el}^+ + \sigma_{ex}^+ \approx \sigma_p^+/2$ from the charge independent theory. Thus $\sigma_n^+ \approx 6\text{mb}$ when the data σ_p^+ are true at zero energy.

$$I_+ \equiv \frac{1}{4\pi^2} \int_K^{\omega_{\max}} k' d\omega' \frac{\sigma_n^+ - \sigma_p^+}{\omega' - \omega} \quad \text{and} \quad I_- \equiv \frac{1}{4\pi^2} \int_K^{\omega_{\max}} k' d\omega' \frac{\sigma_p^- - \sigma_n^-}{\omega' + \omega}.$$

The numerical values of I_+ are shown in Table I. As for I_- , we have hardly the data on σ_n^- , so we cannot estimate I_- . But it is expected that I_- is positive and small because σ_n^- (elastic) $\approx \sigma_p^-$ (elastic) in low energy and σ_n^- (absorption) $< \sigma_p^-$ (absorption).⁸⁾ Although in the following calculation I_- is put to be zero, the qualitative consequences are not altered. The magnitude of coupling constants obtained in this way shows the lower limit.

Table I. The values of I_+

ω \diagdown ω_{\max}	3K	4K	5K
1.1K	0.22	0.25	0.29
1.2K	0.30	0.35	0.39

As seen from Eqs. (11) and Table I, the case (C), scalar type for (ΛNK) interaction and pseudoscalar type for (ΣNK) interaction, is clearly rejected. Thus the remaining possibilities are cases (A) and (B). In case (A): both pseudoscalar types for (ΛNK) and (ΣNK) interactions, we get from Eqs (9), (11·a) and (11·b)

$$1.4g_\Lambda^2 + g_\Sigma^2 \approx \begin{pmatrix} 8 \\ 6.5 \\ 5 \end{pmatrix} \quad \text{for} \quad \omega_{\max} = \begin{pmatrix} 3K \\ 4K \\ 5K \end{pmatrix}, \quad (12\cdot a)$$

and at $\omega = 1.1K$

$$1.4g_\Lambda^2 - g_\Sigma^2 \gtrsim \begin{pmatrix} 9 \\ 8 \\ 7 \end{pmatrix} \quad \text{for} \quad \omega_{\max} = \begin{pmatrix} 3K \\ 4K \\ 5K \end{pmatrix}. \quad (12\cdot b)$$

It seems that Eqs. (12·a) and (12·b) are inconsistent, but the numerical values of the right-hand sides in Eqs. (12·a) and (12·b) are not exact because of the poor experimental data, so we may conclude:*

$$g_\Sigma^2 \approx 0, \quad g_\Lambda^2 \approx 5. \quad (13)$$

On the other hand, in case (B)

$$1.4g_\Lambda^2 - 21g_\Sigma^2 \approx \begin{pmatrix} 8 \\ 6.5 \\ 5 \end{pmatrix} \quad \text{for} \quad \omega_{\max} = \begin{pmatrix} 3K \\ 4K \\ 5K \end{pmatrix}, \quad (14\cdot a)$$

and at $\omega = 1.1K$

* Only the value of g_Λ^2 is slightly changed when Eq. (11·b) at $\omega = 1.2K$ is used.

$$1.4g_{\Lambda}^2 + 22g_{\Sigma}^2 \approx \begin{pmatrix} 9 \\ 8 \\ 7 \end{pmatrix} \quad \text{for} \quad \omega_{\max} = \begin{pmatrix} 3K \\ 4K \\ 5K \end{pmatrix}. \quad (14 \cdot b)$$

Only when $g_{\Sigma}^2 \approx 0$, Eqs. (14·a) and (14·b) are consistent. Then, the result on the coupling strengths is same as in case (A).

To get case (A) Igi used the data in which the $K^+ - p$ scattering cross section was decreased more rapidly than the present experimental ones in low energy region ($\sigma_p^+ \approx 6\text{mb}$ at zero kinetic energy). In our calculation such treatment is not needed. This difference seems to be due to the ambiguity in the way of extrapolation into the unphysical region, namely the contribution from the principal integral at $\omega = K$. As the absence of (ΣNK) interaction does not contradict with the existing experimental data, this conclusion is not queer. However, our estimation contains a little ambiguity due to the poor experimental data. Furthermore, it will be doubtful to use Eq. (6) under the assumption that the s -wave is predominant in the charge exchange scattering near $\omega = 1.2K$. At $\omega = 1.1K$ this assumption does not seem to be unreasonable. The values of the coupling constants will vary slightly with the exact data, but the qualitative consequences will not be altered.

There is a question about the unphysical region. However, the contribution from the unphysical region to the dispersion integral is small, if we assume that $A^-(\omega)$ does not behave singularly.

We may also obtain each coupling constant g_{Λ}^2 and g_{Σ}^2 from the processes $\gamma + N \rightarrow K^+ + Y$, making use of the poleology which was shown in case of the $\gamma - \pi$ production by Taylor et al.⁹⁾ On the other hand, the dispersion equation for the $K^- n$ scattering has the bound state corresponding to Σ alone, so the coupling type and strength of (ΣNK) will be obtained from it. It will, therefore, be interesting to check our result, $g_{\Sigma}^2 = 0$ and $g_{\Lambda}^2 \approx 5$.

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Propagation of Shock Waves in Inhomogeneous Gases. I

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Chisnell's method of shock propagation is generalized for the case of inhomogeneous gravitating gases. The relation between shock strength and initial pressure is derived to the first approximation, using polytropic index as parameter. The strength of a shock wave, which is generated near center and progresses outwards, increases rapidly, being proportional to some inverse power of pressure near the surface. Applying the results to the Eddington model, some speculation concerning the origin of nova explosion is made.

§ 1. Introduction

Although many papers have appeared which treat the problem of propagation of shock wave or detonation in fluids, especially in ideal gases, most of them are concerned with homogeneous medium. As for the case of inhomogeneous one for which physical quantities (density, pressure, etc.) are functions of position, for example, gas under a gravitation field, it would be said that almost nothing has been investigated. This case could also be studied in principle, and the variation of shock strength with position would be calculated by the method of characteristics.¹⁾ However, this is extremely cumbersome, and it might be difficult to treat without an electronic computer. Then, it becomes a requisite problem to find a method which can give the overall picture of the shock propagation even approximately.

The following two methods may be considered for the approximate solution: The method of similarity solution and that of infinitesimal discontinuity. The first has been used by many authors for various problems: Taylor²⁾ derived the solution of blast waves, and soviet hydrodynamists³⁾ are performing extensive studies along this method.* Though the solutions by this method are exact, this method suggests only the possibility of existence of such solutions. It does not seem clear what approximation they adopt for the given problems.**

In the second method, the inhomogeneous fluid is divided into layers by infinitesimal contact discontinuity surfaces, and the incident shock wave is reflected by and refracts (penetrates) through these surfaces. Using this method, Chisnell⁴⁾

* These are mainly concerned with solutions for homogeneous gases.

** For the existence of such solutions, certain similarity relations should be imposed among physical quantities, which are not satisfied in general problems.

discussed the propagation of shock wave through one-dimensional gas, having a finite layer in which density increases but pressure is constant.* In his first approximation, where double reflection of reflected waves is neglected, the strength of penetrating shock wave happens to depend only on density itself, but not on its spatial distribution, while, in the second approximation, which takes into account double reflection, the latter dependency does appear. Adopting a special density distribution, Chisnell showed that the second approximation is sufficiently correct, i.e. it substantially coincides with the case at a distance, when the whole increment of density occurs on a sheet (finite contact discontinuity). Moreover, it was shown that the first approximation has a certain discrepancy at high density. Though the calculation of the second approximation becomes considerably complicated, the method is very intuitive, and it seems suitable to get a general survey of various problems even with the first approximation only. As stated above, Chisnell treated only the case of constant pressure (ordinary contact discontinuity). Since, however, pressure also changes in general inhomogeneous gases, it is necessary to extend this method to general cases.

Now it becomes more and more important in astrophysics to treat the problem of shock relations and shock propagation in stellar interior and interstellar matter. The main subject of shock relations is to consider the effects of radiation and magnetic field, and many studies are being performed along these lines.⁵⁾ As for the propagation, there is a series of studies by Kopal⁶⁾, Sedov⁷⁾ and others in connection with nova and supernova explosions. However, all of them used the similarity solutions, and merely show the possible existence of such solution, as mentioned above. Kopal⁶⁾, for instance, sought the shock wave propagating with constant Mach number, but it cannot be considered generally that the shock wave generated in stellar interior propagates with constant strength. What we require is to make the question clear: With what strength the shock wave internally generated propagates and arrives at stellar surface? When this problem is answered, it may be possible to connect the internal nuclear reactions with surface phenomena of nova and supernova. Hayakawa et al.⁸⁾ estimated the initial expansion velocity of supernova with the assumption that the shock wave was generated with the strength determined by the ratio of surface and center pressures. However, such estimation cannot be regarded as well grounded.

Thus, it may be said that the problem of shock propagation in inhomogeneous gases is yet almost untouched in hydrodynamics as well as in astrophysics. The aim of this paper is to generalize Chisnell's method to get a formula for the first approximation in the case of general inhomogeneous gases, and then to find the way to application or estimation in various astrophysical problems. In § 2, Chisnell's method is briefly described to the first approximation, and this is generalized in § 3. The calculated results for several polytropes are compared

* In the following we call this method "Chisnell's method".

in § 4, and using these results, some astrophysical application or speculation is made in § 5, especially concerning the Eddington model.

Of course, there are many limitations, when we apply the results here derived to astrophysical problems. First, we take only the first approximation, which neglects double reflection and reflection from the base (center). To what extent this is correct is determined only by taking second approximation in each case. Secondly, the formula is exactly applicable only to one-dimensional cases because we did not take into account the damping of spherical wave. Further, the method treats stationarily generated shock wave, and strictly speaking, it is not applicable to shock pulse, i.e. the case of instantaneous explosion. Lastly, we deal here only with ideal gases. In order to be applied to stellar interior or interstellar matter, the effects of radiation and magnetic field must be considered.

We have a program of investigating these subjects to be carried out in near future. Nevertheless, it seems worth while to publish the present results to get more justifiable estimation in astrophysical problems, because at present we have only poor knowledge about the shock propagation.

§ 2. Chisnells' method

As stated in § 1, Chisnell divided the layer of variable density and constant pressure into layers with infinitesimal density jumps, and considered the reflection and penetration of a shock wave at these jump positions. In Fig. 1, on the dotted line between states 1 and 5 exists the initial infinitesimal density jump $d\rho$, i.e.

$$\rho_5 = \rho_1 + d\rho. \quad (2.1)$$

An incident shock wave in the gas of state 1* generates at the jump position $\langle 1, 5 \rangle$ a reflected wave $\langle 2, 3 \rangle$,** penetrated shock wave $\langle 5, 4 \rangle$ and discontinuity surface $\langle 3, 4 \rangle$.

On both sides of the contact discontinuity, fluid velocities u and pressures p are equal. Thus the conditions

$$u_1 = u_5, *** \quad u_3 = u_4 \quad (2.2)$$

and
$$p_1 = p_5, \quad p_3 = p_4 \quad (2.3)$$

are imposed.⁹⁾ Next, when we represent the strength of the shock wave (compression or rarefaction wave) by the ratio of the pressures before and behind the

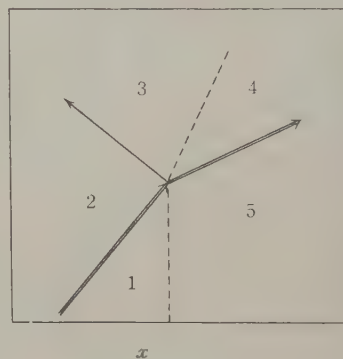


Fig. 1. Reflection and penetration of a shock wave by a contact discontinuity.

* Behind the shock wave, the state of gas changes from 1 to 2. In this meaning, we denote this shock wave as $\langle 1, 2 \rangle$. For the reflected wave and discontinuity line we use the same notation. We can also represent the waves by their strength, i. e. z_{12} . See the following.

** This is a compression or rarefaction wave.

*** $u_1 = u_5 = 0$, when the gas is initially at rest.

wave, the difference of strength between incident shock $z_{12}=p_2/p_1$ and penetrated shock $z_{54}=p_4/p_5$ must be infinitesimal,

$$z_{54}=z_{12}+dz. \quad (2.4)$$

Therefore, the strength of reflected wave z_{23} is nearly unity.*

Physical quantities before and behind a shock wave are connected by Rankine-Hugoniot relations, which are represented for ideal gases (e.g. for $\langle 1, 2 \rangle$) by

$$\rho_2 = \rho_1 \frac{\lambda^2 + z_{12}}{1 + \lambda^2 z_{12}}, \quad (2.5)$$

$$u_2 = u_1 \pm \phi(z_{12}, p_1, \rho_1), \quad (2.6)$$

$$U = u_1 \pm \left\{ \frac{p_1(\lambda^2 + z_{12})}{\rho_1(1 - \lambda^2)} \right\}^{1/2}, \quad (2.7)$$

where

$$\phi(z_{12}, p_1, \rho_1) = (z_{12} - 1) \left\{ \frac{p_1(1 - \lambda^2)}{\rho_1(\lambda^2 + z_{12})} \right\}^{1/2}, \quad (2.8)$$

$$\lambda^2 = \frac{\gamma - 1}{\gamma + 1}. \quad (2.9)$$

U is the propagation velocity of shock and γ the ratio of specific heats. The plus sign in (2.6) and (2.7) applies to forward facing waves. Though reflected waves are in general not shock waves, the above relations also hold as far as the second order in $z-1$.

Now, from the conditions (2.2) and (2.3), we get

$$\phi(z_{12}, p_1, \rho_1) - \phi(z_{23}, p_2, \rho_2) = \phi(z_{54}, p_5, \rho_5), \quad (2.10)$$

$$z_{12} z_{23} = z_{54}. \quad (2.11)$$

When we introduce (2.1), (2.4) and (2.9) into these formulae, and take the first order terms in $d\rho$ and dz , we obtain

$$\frac{1}{\rho_1} \frac{d\rho}{dz} = \frac{2}{z_{12} - 1} - \frac{1}{\lambda^2 + z_{12}} + \frac{2}{z_{12} - 1} \left\{ \frac{1 + \lambda^2 z_{12}}{z_{12}(1 + \lambda^2)} \right\}^{1/2}, \quad (2.12)$$

or by integration

$$\rho(z) = A f(z),^{**} \quad (2.13)$$

$$f(z) = \frac{(z-1)^2}{(\lambda^2 + z)} \left\{ \sqrt{\lambda^2 + \frac{1}{z}} + \lambda \right\}^{2\lambda/\sqrt{1-\lambda^2}} \left\{ \frac{\sqrt{\lambda^2 + 1} - \sqrt{\lambda^2 + \frac{1}{z}}}{\sqrt{\lambda^2 + 1} + \sqrt{\lambda^2 + \frac{1}{z}}} \right\}^2. \quad (2.14)$$

Here A is an integration constant to be determined by initial strength. From (2.13) and (2.14) we know the strength of shock at a given density.

* $z=1$ corresponds to the strength zero.

** ρ and z are taken instead of ρ_1 and z_{12} .

§ 3. Generalization of Chisnell's method

When we consider a gas at rest under the action of gravity, where pressure as well as density changes, we must extend the condition for contact discontinuity in § 2.

In the first place, in order that the condition of contact of gas is to be satisfied, the velocities on both sides of discontinuity surface must be equal. Thus the condition (2.2) generally remains appropriate. The question then is how to generalize the condition (2.3) to the case of non-uniform pressure.

We now consider that the gas is in equilibrium under an arbitrary mass force $f(x)$:*

$$\frac{1}{\rho} \frac{dp}{dx} + f(x) = 0.$$

Then, the pressure difference Δp across the thickness Δx (Fig. 2) becomes as follows :

$$\Delta p = -\rho f(x) \Delta x. \quad (3.1)$$

Introducing the mass included in this thickness

$$\Delta m = \rho \Delta x,$$

we get

$$p_5 - p_1 = \Delta p = -f(x) \Delta m. \quad (3.2)$$

On the other hand, the layer after the penetration of shock includes the same mass Δm as before, and the position of the layer does not change at the instant of penetration, and moreover, there exists no acceleration of the layer because of the relation $u_3 = u_4$. Thus the right-hand side of (3.2) is also equal to $p_4 - p_3$:

$$p_4 - p_3 = -f(x) \Delta m.$$

As the result, the following relation holds instead of (2.3) :

$$p_5 - p_1 = p_4 - p_3 (= \Delta p).^{**} \quad (3.3)$$

Then we get the general formula

* This force may be a given external gravity or self-gravity; only one condition is required that it is the function of position x .

** This condition is not limited to one-dimensional cases. For instance, in the case of spherical symmetry, for which $\Delta m = \rho 4\pi r^2 \Delta r$, the formula (3.2) can be put as

$$\Delta p = -\frac{f(r)}{4\pi r^2} \cdot \Delta m = -\varphi(r) \Delta m,$$

and then the same arguments as the above follow.

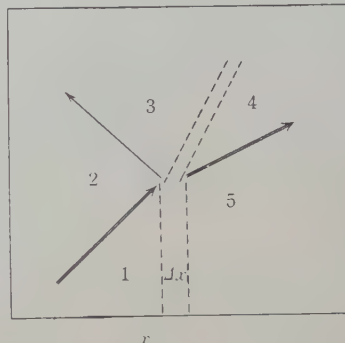


Fig. 2. Infinitesimal layer Δx including mass Δm .

$$z_{12} z_{23} = (1 + 4p/p_1) z_{34} - 4p/p_1, \quad (3.4)$$

corresponding to (2.11).^{*} When the initial distribution is of the polytrope k , i.e.

$$p \propto \rho^k, \quad (3.5)$$

we have the relation

$$4p/p_1 = k \Delta \rho / \rho_1. \quad (3.6)$$

Introducing (3.4) and (3.6) into (2.10), and taking only the first order terms in $d\rho$ and dz , we obtain after all

$$\frac{dp}{p dz} = \frac{k}{\rho} \frac{d\rho}{dz} = \frac{1}{\lambda^2 + z} - \frac{2}{z-1} - \frac{2}{z-1} \left\{ \frac{1 + \lambda^2 z}{z(1 + \lambda^2)} \right\}^{1/2} \cdot \quad (3.7)$$

$$2 \left\{ \frac{1 + \lambda^2 z}{z(1 + \lambda^2)} \right\}^{1/2} + \frac{k-1}{k}$$

Chisnell's case (2.12) corresponds to $k=0$ (isobar), which can be easily seen by putting $k=0$ in (3.7).

Integrating (3.7), we can get the relation between ρ and z . It is remarkable that this relation does not depend on the character of gravitation field $f(x)$, but on the initial distribution, that is, on the polytropic index k , as parameter.

The differential equation (3.7) cannot in general be integrated in an explicit form, except for isothermal case $k=1$, which gives

$$\rho(z) = A g(z), \quad (3.8)$$

$$g(z) = \frac{1}{(z-1)} \left\{ \frac{\sqrt{\frac{z}{1+\lambda^2 z}} + \frac{1}{\lambda}}{\sqrt{\frac{z}{1+\lambda^2 z}} - \frac{1}{\lambda}} \right\}^{\frac{2\sqrt{1+\lambda^2}}{\lambda}} \left\{ \frac{\sqrt{\frac{(1+\lambda^2)z}{1+\lambda^2 z}} + 1}{\sqrt{\frac{(1+\lambda^2)z}{1+\lambda^2 z}} - 1} \right\}^{\frac{4}{(1+\lambda^2)^{3/2}}}$$

$$\times \exp \left\{ \frac{\lambda}{\sqrt{1-\lambda^2}} \tan^{-1} \left[\frac{1}{\lambda} \sqrt{\frac{(1-\lambda^2)z}{1+\lambda^2 z}} \right] \right\}. \quad (3.9)$$

Here A is again an integration constant given by the initial strength.

§ 4. Results of integration

Here we describe and compare the integrated results for the following cases: $k=1$ (isothermal), $4/3$ (corresponding to the Eddington model) and $k=5/3$ (adiabatic). As for γ we take the value $5/3$ of one atomic gas corresponding to stellar interior. In Table I and Fig. 3, we show the relations between shock strength z and initial pressure p , where the integration constant is so determined that at pressure p_0 the incident shock strength z_0 is 2.2.

^{*} When we put $4p \rightarrow 0$, we obtain the formula (2.3) again.

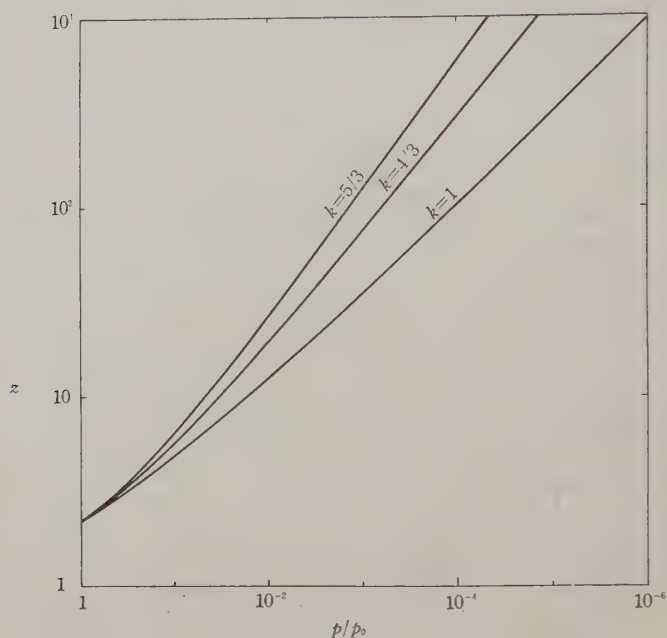
Fig. 3. Shock strength z versus initial pressure p/p_0 .

Table I.

z	$\log(p/p_0)$		
	$k=1$	$k=5/3$	$k=4/3$
2.2	0	0	0
3.0	-0.446	-0.349	-0.380
4.0	0.801	0.621	0.678
5.0	1.054	0.812	0.888
7.0	1.413	1.079	1.184
9.0	1.670	1.267	1.393
10.0	1.775	1.343	1.478
30.0	2.846	2.108	2.335
50.0	3.325	2.444	2.713
70.0	3.638	2.663	2.959
100.	3.968	2.870	3.193
300.	4.995	3.582	3.998
500.	5.446	3.909	4.367
700.	5.778	4.123	4.609
1000.	6.072	4.328	4.841

When the relation of other incident strength z_0' is desired, it can be obtained by the parallel shift of the curve z_0 as far as z becomes z_0' at $p/p_0=1$, (see Fig. 3).

This is because of the following situation: We write the formula (3.7) in the form

$$d \ln p / dz = \varphi(z; k), \quad (4.1)$$

where φ is the right-hand member of (3.7). Integrating and determining the constant so that $z=z_0$ at p_0 , we get

$$\ln(p/p_0) = \psi(z; k) - \psi(z_0; k), \quad (4.2)$$

where

$$\psi(z; k) = \int_a^z \varphi(z; k) dz \quad (4.3)$$

(a is arbitrary).

From Fig. 3, we see the strength increases rapidly as pressure decreases, but the degree of increment differs among different k 's; it is the least for isothermal distribution and the larger for larger k . For $z \gg 1$, we get from (4.1)

$$z \propto (p/p_0)^{-a(\gamma, k)}, \quad (4.4)$$

where

$$a(\gamma, k) = \frac{2 \left\{ \frac{\lambda^2}{1+\lambda^2} \right\}^{1/2} + \frac{k-1}{k}}{1 + 2 \left\{ \frac{\lambda^2}{1+\lambda^2} \right\}^{1/2}}. \quad (4.5)$$

In our case $\gamma=5/3$, hence $\lambda^2=1/4$, the following values are obtained:

$a=0.472$	for $k=1$
0.604	4/3
0.683	5/3.

Conversely, when the strength of generated shock wave is infinitesimal, i.e. z nearly equals to one, it follows that

$$z-1 \propto p^{-b(k)}, \quad (4.6)$$

where

$$b(k) = (3k-1)/4k. \quad (4.7)$$

This means that $z=1$ is attained only at infinite pressure. In other words, sound waves generated in stellar interior, where pressure is enormously large but finite, cannot grow up to shock waves.

As the result of increasing strength from center to surface, density, temperature and fluid velocity after the shock also increase considerably. These results will be discussed in relation to the Eddington model in the next section.

§ 5. Application to astrophysics

Here, using Eddington's standard model, we discuss more closely the propagation of shock wave which has been generated in some way in stellar interior. As is well known, this corresponds to the polytrope of $k=4/3$ and is the solution of Lane-Emden equation of index $3^{10)}$

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) = -\theta^3, \quad (5.1)$$

with the boundary condition $\theta=1$, $d\theta/d\xi=0$ at $\xi=0$. Temperature T , pressure p and density ρ are expressed by non-dimensional quantity θ as

$$T=T_c\theta, \quad p=p_c\theta^4 \quad \text{and} \quad \rho=\rho_c\theta^{3*}. \quad (5.2)$$

ξ relates with radius r by

$$r=\alpha\xi, \quad \alpha=\left(\frac{K}{\pi G} \rho_c^{-2/3}\right)^{1/2}. \quad (5.3)$$

K and G have the usual meaning. The solution is known already.¹¹⁾

Now we consider a shock wave of strength z_0 being generated at $r=r_0$ of the standard model, and propagating outwards. Then we can get the strength $z(r_1)$, when the shock arrives at an arbitrary point r_1 , by the following procedure: Determine $p_1(r_1)/p_0(r_0)$ from the standard model solution, and take the corresponding value $z_1(p_1)$ from Table I of the foregoing section. $z_1(p_1)$ being thus determined, all the physical quantities behind the shock wave can also be gained by using (2.4)~(2.6). For example, the ratio of temperatures behind and before the shock wave is

$$T_2/T_1=z(1+\lambda^2 z)/(\lambda^2+z) \equiv \tau. \quad (5.4)$$

Taking the temperature of the standard model at r_1 as T_1 , it follows

$$(T/T_c) \text{ behind shock} = (T/T_c) \text{ standard model} \times \tau. \quad (5.5)$$

In Fig. 4, the values of the physical quantities are shown for the following three cases.

- i) $r_0=0.145$, $q_0=0.125$, $z_0=2.2$
- ii) $r_0=0.145$, $q_0=0.125$, $z_0=3.0$
- iii) $r_0=0.290$, $q_0=0.518$, $z_0=2.2$.**

Assuming $T_c=3.8 \times 10^8$ °K and $\rho_c=7.7 \times 10^4$ gr/cm³ (these values correspond to the star of total mass $\sim 1M_\odot$, radius $\sim 0.1R_\odot$ and mean molecular weight $\mu=2$), physical quantities behind the shock at $r_1=0.8700$ and $q_1=0.9997$ for the case i) are as follows:

* Suffix c represents the value at center.

** q_0 is the mass ratio included in the radius r_0 .

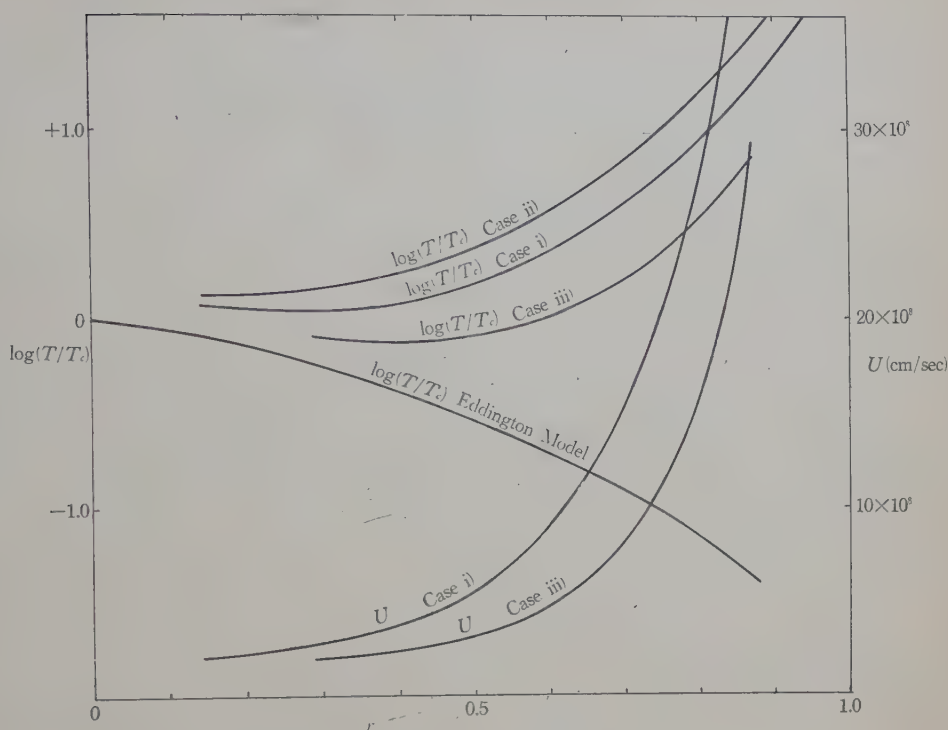


Fig. 4. Physical quantities behind shock wave in standard model.

$$T = 65 \times 10^8 \text{ }^\circ\text{K}$$

$$\rho = 25 \text{ gr/cm}^3$$

$$u = 3.9 \times 10^9 \text{ cm/sec.}$$

The propagation velocity of the shock increases from $U_0 = 2.1 \times 10^8 \text{ cm/sec}$ at r_0 to $U_1 = 5.1 \times 10^9 \text{ cm/sec}$ at r_1 and the time required for the propagation is *about five sec*. Thus we see that the temperature and the expansion velocity near the surface become fairly large. The limitations noted in § 1 will change the values appreciably: Double reflection will strengthen, spherical damping and radiation loss (especially near surface) will weaken the shock.

Nevertheless, these results might be applied to the stellar explosion phenomenon as the first approximation. If some core instability induces the shock waves in stellar interior, they are exaggerated more and more as they progress outwards, and at last, it could be possible that instantaneous nuclear reaction could take part in near the surface. From these considerations it would be speculated that nova explosion, which has hitherto been considered as occurring near the surface, might be triggered by shock waves generated in interior. As for supernova explosion, it is possible that nuclear detonation would proceed through envelope because of the high temperature behind shock wave. This will cause the explosion more violently.

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On a New Approach to Cosmology. II*—The Problem of Local Gravitation—*

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As a sequel to the previous paper, an attempt is made to develop a general method for attacking at the problem of local gravitational field due to such a large scale aggregation of matter that the effect of the cosmic expansion cannot be ignored. The formalism of this paper will provide us with a basis for treating the dynamical motion of galaxies within the Supergalaxy, together with the reexamination of the velocity-distance relation of galaxies.

§ 1. Introduction

In the previous paper^{1,*)} we proposed a cosmological theory based on a scalar field ϕ , and discussed a method for constructing the smoothed-out model of the universe and several problems concerning Hubble's velocity-distance relation of the extra-galaxies. We also pointed out in I that the ϕ -field would describe the combined effect of cosmic expansion and Newtonian gravitation due to the local aggregation of matter such as galaxies. This expectation must, however, be partly given up after a detailed examination, because, if we make the field equation for ϕ (cf. § 2) correspond to Laplace's equation in the Newtonian theory of gravitation, we meet with a difficulty in stipulating the equation corresponding to Poisson's equation.

In order to avoid this difficulty, we cannot help appealing to the general theory of relativity as concerning the problem of local gravitation, though the method of attacking developed in I for problems of the universe as a whole would remain true. Then the cosmological background space-time stipulated by ϕ -field must play a significant role as a kind of the boundary condition for the metric specifying the local gravitational field in question, or as anything beyond that. To make such a formalism possible the smoothed-out model-universe itself must be, as a matter of course, a solution of the field equations of the general theory of relativity just as the Minkowski space-time is. This formalism can also be supported from another side. Namely, as was stated in I, the limit of applicability of general relativity was stipulated by the following inequality relation:

$$GM/c^2 L \approx G\rho t^2 \ll 1, \quad (1.1)$$

* In the following we shall refer to this as I.

where G is Newton's gravitation constant, M and L ($M \approx \rho L^3$, $L \approx ct$) are the mass and linear dimension characteristic of the dynamical system respectively. However, this inequality relation is always satisfied by any real dynamical system, except the universe as a whole.

Hitherto the large scale aggregation of matter in the universe has been usually meant for galaxies or clusters of them. And moreover it is known that systematic motion due to the cosmic expansion can be ignored for such dynamical systems. Accordingly, in the current treatment²⁾ of local aggregations of matter in the universe, they are regarded as the assembly of point masses embedded in the expanding universe. We cannot, however, apply such a point-mass model to the Supergalaxy³⁾ (or its analogues) whose constituent members are galaxies and clusters of them. Because this is such a large-scale system that the systematic recessional motion of constituent members due to the cosmic expansion cannot be ignored. Our object, "local aggregation of matter", is in short meant for such a large system.

The purpose of this paper is to treat the problem of local gravitation along the line of thought stated above. In § 2 we shall examine the applicability of the field equation for ψ to the present problem, after summarizing the essential points of I. The kinematical and dynamical character of the Supergalaxy will be discussed in § 3. In §§ 4-6 the plan stated in § 1 will be realized in a concrete form. In § 7 we shall discuss the properties to be satisfied by the model Supergalaxy with spherical symmetry, though such a model is of an over-simplification. We shall investigate in § 8 the motion of a test particle in the local exterior gravitational field due to the model Supergalaxy. The last three sections will give us some prospect of the theory concerning the rotational and expansive motion of the Supergalaxy which will be developed in future.

§ 2. Applicability of the field equation for ψ to the present problem

(i) *Summary of I* As was stated in I, the relation of our theory to the general theory of relativity can be expressed schematically as follows:

(General theory of relativity) (Our theory)

(Metric) :

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \xrightarrow{\text{smoothing-out}} ds^2 = \psi^2 [c^2 dt^2 - dr^2 - A^2(r) (d\theta^2 + \sin^2 \theta d\varphi^2)]$$

(Field equation) :

$$R_{\mu\nu} - (1/2) g_{\mu\nu} R = -\kappa T_{\mu\nu} \xrightarrow{\text{contraction}} R = \kappa T,$$

with usual notations; in this paper we shall consider for simplicity the case $A(r) = r$ among three possible forms of $A(r)$. In the above scheme, κT is sti-

pulated in such a way that it satisfies the so-called Mach's relation materially specifying the universe as a whole, i.e.

$$(4\pi/3)GT = \gamma h^2, \quad (\gamma \sim 1), \quad (2.1)$$

where h is Hubble's constant. Then, for the case $A(r) = r$, we have

$$\square \phi = 0, \quad \begin{cases} \phi = \ln \psi, & \text{for } \gamma = 1 \\ \phi = \psi^{1-\gamma}, & \text{for } \gamma \neq 1, \end{cases} \quad (2.2)$$

where \square is the d'Alembertian operator with respect to the representation space-time defined by the metric $ds_0^2 = \phi^{-2} ds^2$.

Moreover, we have stipulated the smoothed-out universe by the condition $\phi = \phi_0(t)$, say, so that from (2.2) we obtain

$$\phi_0 = \text{a linear function of } t.$$

Then we can transform the metric ds^2 into the standard form,

$$ds^2 = c^2 d\tau^2 - \Phi^2(\tau) (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2), \quad (2.3)$$

in terms of the relations:

$$d\tau = \phi(t) |dt|; \quad \Phi(\tau) = \phi\{t(\tau)\}, \quad (2.4)$$

where τ is the cosmic time. The explicit form of $\Phi(\tau)$ is

$$\left. \begin{aligned} \Phi(\tau) &= (\tau/\tau_0)^{1/(2-\gamma)}, \quad h = 1/(2-\gamma)\tau, & q &= \gamma - 1 & : (\gamma \neq 2) \\ \Phi(\tau) &= \exp(\tau/\tau_0 - 1), \quad h = 1/\tau_0, & & & : (\gamma = 2) \end{aligned} \right\}, \quad (2.5)$$

where $h \equiv (1/\phi) (d\phi/d\tau)$, $qh^2 \equiv (1/\phi) (d^2\phi/d\tau^2)$, and τ_0 is an integration constant specifying, without loss of generality, the present value of the cosmic time.

As is easily seen, the introduction of a new scalar field ϕ instead of the original one ψ is of great advantage for us, because the non-linear field equation for ϕ with source term specified by (2.1) can be reduced to a d'Alembertian equation without source, (2.2). Therefore, as was stated in I, the exterior field due to an aggregation of matter embedded in the expanding universe must be expressed by the solution of a type $\phi = \phi_0 + \phi_1(t, r, \theta, \varphi)$, where ϕ_1/ϕ_0 expresses the Newtonian gravitation or its analogue. But this interpretation offers a difficulty, as will be shown in the following, when we attempt to consider the fitting of this solution to the inner solution for the local aggregation of matter.

(ii) *The problem of local gravitation* Now let T_1 be the deviation of the spur, T , of the energy-momentum tensor for a local aggregation of matter from the value (2.1) for the smoothed-out universe. Then we can express the difference between T and T_1 as

$$(4\pi/3) (T - T_1) = \gamma h^2. \quad (2.6)$$

Therefore we get

$$\square\phi = (4\pi G/3)T_1 \times \begin{cases} \exp(2\phi), & \text{for } \gamma=1 \\ (1-\gamma)\phi^{(3-\gamma)/(1-\gamma)}, & \text{for } \gamma \neq 1 \end{cases}, \quad (2.7)$$

instead of (2.2). Now we shall characterize the weak field by the condition $|\phi_1/\phi_0| \ll 1$, where ϕ_0 is a linear function of t corresponding to $\phi(\tau)$ in (2.5). Then we can obtain after some calculation from (2.7) with (2.4)

$$\square\{U\phi^{1-\gamma}\} \approx -(4\pi G/3)T_1\phi^{3-\gamma}, \quad (2.8)$$

with

$$ds^2 = (1 - U/c^2)^2 [c^2 d\tau^2 - \phi^2(\tau) (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2)], \quad (2.9)$$

where

$$U/c^2 = 1 - \exp(\phi_1) \quad (\text{for } \gamma=1), \quad 1 - (1 + \phi_1/\phi_0)^{1/(1-\gamma)} \quad (\text{for } \gamma \neq 1).$$

If we further assume that the retardation effect can be ignored, Eq. (2.8) is reduced to

$$\phi^{-2}\Delta U \approx (4\pi G/3)T_1, \quad (2.10)$$

where Δ denotes the ordinary Laplacian operator. As is easily seen, for any given positive function T_1 , the gradient of U , where U is the solution of (2.10) fitting to the exterior solution at the boundary, becomes positive, i.e. the gravitational force due to a local aggregation of matter is repulsive.*

Then from what reason does such a difficulty originate? It could be attributed to the following situation: The coupling constant κ in our field equation " $R=\kappa T$ " is determined from Einstein's field equations in general relativity. Then the contraction procedure practically has the same effect** as altering the sign and numerical value of κ such that $\kappa \rightarrow -\kappa/3$, so far as the function of κ is concerned. Because the desired equation should be of the form

$$\phi^{-2}\Delta U \approx -4\pi GT_1, \quad (2.11)$$

in place of (2.10). Hence, as was stated in § 1, we must take another method of approach.

§ 3. Characteristic features of the Supergalaxy

For simplicity's sake, we regard the Supergalaxy as a spherically symmetric dynamical system. Then the strength of its gravitational field at the outer region can be specified numerically by the following dimensionless number:

$$\eta \equiv GM/c^2 L, \quad (3.1)$$

* In Gürsey's paper⁴⁾ no account is taken of this point.

** Thus we can see that the contraction procedure plays a role similar to the insertion of the cosmic term in Einstein's original field equations, so that, in our formalism, the tendency of matter to the systematic outward motion is elucidated without appealing to such an ad hoc term.

where M is the mass and L is the radius. If we put

$$M = (4\pi/3)\rho L^3, \quad L = ct, \quad (3.2)$$

(3.1) is rewritten as

$$\gamma = (4\pi/3)G\rho t^2. \quad (3.3)$$

Now we put

$$GM/L^3 = \zeta h^2 L \quad (3.4)$$

in order to estimate the effect of the cosmic expansion at the outer region in question, where ζ is a certain dimensionless parameter. The above expression can also be written as

$$GM/L = \zeta v_{ex}^2, \quad (3.5)$$

where v_{ex} is the expansion velocity specified by Hubble's velocity-distance relation, i.e.

$$v_{ex} = hL. \quad (3.6)$$

Further we obtain from (3.4) and (3.2)

$$(4\pi/3)G\rho = \zeta h^2. \quad (3.7)$$

Since (3.7) has a form similar to Mach's relation (2.1), we may find a restriction on the admissible range of ζ from these two expressions. But, as will be treated in § 4, the expression for the mean density ρ_0 of the smoothed-out universe corresponding to (2.1) is

$$(4\pi/3)G\rho_0 = (1/2)h^2. \quad (3.8)$$

Eliminating h^2 from the above two equations, we obtain

$$2\zeta = \rho/\rho_0 \gtrsim 1, \quad (3.9)$$

since, by virtue of the definition of ρ , $\rho/\rho_0 \gtrsim 1$. Accordingly, from (3.5), we have

$$2\gamma c^2 \gtrsim v_{ex}^2, \quad (3.10)$$

which means that, in the terminology of Newtonian dynamics, the kinetic energy of the cosmic expansion is less than the potential of the Supergalaxy at its outer region.

Now we shall check the numerical values. According to de Vaucouleurs,³⁾ we have

$$\dot{M} = 10^{15} \odot, \quad L \approx 1.5 R_1, \quad R_1 \varepsilon_1 = 1100 \text{ km/sec}, \quad (3.11)$$

where \odot is the solar mass, R_1 the distance of our Galaxy from the center (near the Virgo cluster) of the Supergalaxy, and ε_1 is a parameter specifying the rate of the expansion of the Supergalaxy such that

$$\varepsilon_1 = 0.632 h. \quad (3.12)$$

From the above two expressions and (3.6), we obtain

$$v_{ex} \approx 2.6 \times 10^3 \text{ km/sec.} \quad (3.13)$$

On the other hand, according to Sandage⁵⁾ the value of h^{-1} is $(9.9 \sim 20.0) \times 10^9$ years. Accordingly we shall adopt the following value tentatively

$$h^{-1} = 10^{10} \text{ years} = 3 \times 10^{17} \text{ sec.} \quad (3.14)$$

Then the expression (3.12) enables us to obtain

$$\varepsilon_1 = 0.21 \times 10^{-17} \text{ sec}^{-1}. \quad (3.15)$$

Inserting (3.15) into (3.11), we obtain

$$L \approx 7.8 \times 10^{25} \text{ cm,} \quad (3.16)$$

which corresponds to $R_1 \approx 16$ Mpc in place of 10 Mpc due to de Vaucouleurs. This is because de Vaucouleurs' value corresponds to $h^{-1} = 6.2 \times 10^9$ years.

We have, however, from (3.1) and (3.10)

$$M \gtrsim v_{ex}^2 L / 2G. \quad (3.17)$$

Inserting (3.13) and (3.16) into the right-hand side of the above inequality relation, we obtain

$$M \gtrsim 2.0 \times 10^{16} \odot, \quad (3.18)$$

which is twenty times as large as the value given by (3.11). Such a discrepancy is mainly due to the simplified assumption of spherical symmetry, despite of the existence of the flattening of the order of 1/5 in the real Supergalaxy. After this correction is made, M becomes instead of (3.18)

$$M \gtrsim 4 \times 10^{15} \odot, \quad (3.19)^*$$

where the explanation of the remaining factor 4 will be attributed to the value of h and to de Vaucouleurs' erroneous assumption that the dynamics of the Supergalaxy is the same as that of Magellanic Clouds.³⁾ Further, inserting (3.16) and (3.19) into (3.1), we obtain

$$\eta \gtrsim 7.6 \times 10^{-6} \leq 1, \quad (3.20)$$

which shows that the local gravitational field due to the Supergalaxy can be treated by the general theory of relativity (cf. (1.1)), though the effect of the cosmic expansion operates considerably for such a large-scale system.

§ 4. Einstein's field equations for the non-uniform expanding universe

The metric (2.3) of the smoothed-out universe in our cosmology is the same

* The inequality relation (3.10), however, is broken down for this value of M together with (3.13) and (3.14). In order to avoid this discrepancy, we must examine in more detail an appropriateness of various numerical values adopted above.

as that in the general relativistic cosmology, except the procedure for stipulating the scale factor $\Phi(t)$.* However, in order to make our model universe play the part of the boundary condition in the general relativistic treatment, which will be developed in the following, for the local gravitational field due to the Supergalaxy, say, in the expanding universe, our model itself must satisfy the field equations of general relativity. According to our investigation,** sufficient and almost necessary conditions to such a fulfilment are as follows:

$$\left. \begin{aligned} \kappa c^2 \rho_0 &= 3h^2, \\ \kappa p_0 &= -(1+2q)h^2 = (1-2r)h^2, \end{aligned} \right\} \quad (4.1)$$

where ρ_0 and p_0 are the smoothed-out density and stress respectively in the sense of general relativity, where h is given by (2.5).

Now we shall consider the local gravitational field in the expanding universe as mentioned above. According to McVittie,²⁾ we assume that the metric for such a non-uniform expanding universe can be written as

$$ds^2 = Ddt^2 - A_1 dx^2 - A_2 dy^2 - A_3 dz^2, \quad (4.2)$$

with

$$\left. \begin{aligned} D &= c^2(1 - \kappa\psi), \\ A_i &= \Phi^2(t)[1 + \kappa\{\psi + 2\phi_i/c^2\}], \quad (i=1, 2, 3) \end{aligned} \right\} \quad (4.3)$$

where ψ^{***} and ϕ_i are functions of (t, x, y, z) in general. Einstein's field equations for the above metric can be written down with Dingle's formulae.

Taking account of (3.20), we can ignore the terms of the order of κ^2 compared with 1, because $\kappa\psi$ is of the same physical meaning as that of η in § 3. Then Einstein's field equations are reduced as follows:

$$\left. \begin{aligned} \kappa c^2 T_4^4 &= \kappa c^2 \rho_0 + \kappa [2h(\dot{\psi}_1 + \dot{\psi}_2 + \dot{\psi}_3)/c^2 - (c/\Phi)^2 \{ \Delta\psi + (1/c^2) \sum_{l,m,n} (\partial^2 \psi_l / \partial x_m^2) \\ &\quad + \partial^2 \psi_l / \partial x_n^2 \} \}], \\ \kappa c^2 T_i^i &= -\kappa p_0 + \kappa [K(\psi) - \Phi^{-2} (\partial^2 \psi_m / \partial x_n^2 + \partial^2 \psi_n / \partial x_m^2) \\ &\quad + \{ (\ddot{\psi}_m + \ddot{\psi}_n) + 3h(\dot{\psi}_m + \dot{\psi}_n) \} / c^2], \\ \kappa c^2 T_4^i &= \kappa (c/\Phi)^2 [(\dot{\psi} + h\psi)_{,i} + (\dot{\psi}_m + \dot{\psi}_n)_{,i} / c], \\ \kappa c^2 T_m^l &= \kappa c^2 T_l^m = (\kappa/\Phi^2) (\partial^2 \psi_n / \partial x_l \partial x_m), \end{aligned} \right\} \quad (4.4)$$

with

$$\left. \begin{aligned} h &\equiv \dot{\Phi}/\Phi, \quad qh^2 \equiv \ddot{\Phi}/\Phi, \\ K(\psi) &\equiv \ddot{\psi} + 4h\dot{\psi} + (1+2q)h^2\psi, \end{aligned} \right\} \quad (4.5)$$

* In the following we shall designate the cosmic time as t .

** This problem will be treated in another paper.

*** This ψ is different from the same symbol ψ in § 2.

where lmn stands for any cyclic permutation of 123 and $\dot{\phi} = \partial\phi/\partial t$, $\phi_{,i} = \partial\phi/\partial x_i$, etc. In the above derivation, we have used the expressions in (4.1), on the assumption that $|\dot{\phi}| \sim \hbar\phi$.

Now we shall further take the so-called $c \rightarrow \infty$ approximation. Because, as was shown in (3.13) of § 3, the velocity appearing in the dynamical system in question does not exceed the order of 10^3 km/sec. Then from (4.4) we obtain

$$\left. \begin{aligned} T_4^4 &= \rho_0 - \Phi^{-2} \Delta\phi, \\ c^2 T_i^i &= -p_0 + K(\phi) - \Phi^{-2} (\partial^2 \phi_m / \partial x_n^2 + \partial^2 \phi_n / \partial x_m^2), \\ T_4^i &= \Phi^{-2} (\dot{\phi} + \hbar\phi)_{,i}, \\ T_m^i &= T_l^m = \Phi^{-2} (\partial^2 \phi_n / \partial x_l \partial x_m). \end{aligned} \right\} \quad (4.6)$$

Now we shall assume that the energy-momentum tensor T_μ^ν for the local aggregation of matter in question can be denoted by that of a perfect fluid, i.e.

$$T_\mu^\nu = (\rho + p/c^2) v_\mu v^\nu - (p/c^2) \delta_\mu^\nu, \quad (4.7)$$

with usual notations. After the above two kinds of approximation are applied to (4.7) with (4.2), we have

$$\left. \begin{aligned} T_4^4 &= \rho, \quad T_4^i = \rho U_i / \Phi, \\ c^2 T_j^i &= - (p \delta_j^i + \rho U_i U_j), \end{aligned} \right\} \quad (i, j = 1, 2, 3) \quad (4.8)$$

where U_i is the physical component of the fluidal velocity with respect to the ensemble of coordinate directions, i.e. $U_i = \Phi v^i / v^4$. Inserting (4.8) into (4.6), we obtain

$$\Phi^{-2} \Delta\phi = -\tilde{\rho}, \quad (4.9)$$

$$\tilde{p} + K(\phi) + \rho U_i^2 = \Phi^{-2} (\partial^2 \phi_m / \partial x_n^2 + \partial^2 \phi_n / \partial x_m^2), \quad (4.10)$$

$$\rho U_i = \Phi^{-1} (\dot{\phi} + \hbar\phi)_{,i}, \quad (4.11)$$

$$\rho U_l U_m = -\Phi^{-2} (\partial^2 \phi_n / \partial x_l \partial x_m), \quad (4.12)$$

with

$$\tilde{\rho} \equiv \rho - \rho_0; \quad \tilde{p} \equiv p - p_0. \quad (4.13)$$

The above system of equations is the fundamental equation of the local gravitational field in question. In the estimation of order of magnitudes, the operation $\Phi^{-2} \Delta$ can be regarded as the division by L^2 , where L is the quantity which has appeared in § 3. Accordingly, if each term in (4.9)–(4.12) is of the same order, the following relations concerning the order of magnitude can be derived:

$$\left. \begin{aligned} \tilde{p} &\sim \rho U_i^2 \sim \hbar^2 \phi, \quad |U_i| \sim v_{ec}, \\ |\phi_{,m}| &\sim v_{ec}^2 \phi, \end{aligned} \right\} \quad (4.14)$$

which are consistent with the order of magnitude estimations in § 3. From this

we can see that each term in (4.9)–(4.14) could become comparable.

Now it must be remarked that (1) the source term of the generalized version of Poisson's equation (4.9) is not ρ itself but the difference between ρ and ρ_0 , the density of the smoothed-out universe, and that (2) the stress p does not appear alone but as the form \tilde{p} only. From the above two facts, we see that $\tilde{\rho}$ (but not ρ) is the effective density contributing to ψ and that any inpropriety does not appear even for the model such that $p_0 < 0$, since the pressure gradient $\partial p / \partial x_i$ is equal to $\partial \tilde{p} / \partial x_i$ irrespective of the sign of p_0 . Another important point to be stressed is that, in our treatment, there is no such assumption about the behavior of the fluid that it is at rest with respect to the coordinate system used now, contrary to McVittie's² point-model of a galaxy embedded in the expanding-universe.

§ 5. Consistency of our fundamental equations and hydrodynamical equations for ρ , p and U_i

Though it is easily seen that the expression (4.6) automatically satisfies the equations $T_{\mu\nu} = 0$ under the approximation considered above, it does not assure that the consistency of (4.9)–(4.12) is satisfied automatically; so we have to examine the consistency relations for them.

Eliminating ρ and U_i from (4.9), (4.11) and (4.12), we obtain

$$\partial^2 \phi_n / \partial x_i \partial x_m = (\Phi \psi)^{\cdot, i} (\Phi \psi)^{\cdot, m} / (\Delta \psi - \rho_0 \Phi^2). \quad (5.1)$$

In a similar manner we have

$$\begin{aligned} [\tilde{p} + K(\psi)] \Phi^2 &= \frac{[(\Phi \psi)^{\cdot, 1}]^2}{\Delta \psi - \rho_0 \Phi^2} + \left(\frac{\partial^2 \psi_2}{\partial x_3^2} + \frac{\partial^2 \psi_3}{\partial x_2^2} \right) \\ &= \frac{[(\Phi \psi)^{\cdot, 2}]^2}{\Delta \psi - \rho_0 \Phi^2} + \left(\frac{\partial^2 \psi_3}{\partial x_1^2} + \frac{\partial^2 \psi_1}{\partial x_2^2} \right) \\ &= \frac{[(\Phi \psi)^{\cdot, 3}]^2}{\Delta \psi - \rho_0 \Phi^2} + \left(\frac{\partial^2 \psi_1}{\partial x_2^2} + \frac{\partial^2 \psi_2}{\partial x_1^2} \right). \end{aligned} \quad (5.2)$$

These are nothing but the consistency relations to be satisfied. Eliminating ψ_i from (5.1) and (5.2), we obtain

$$\frac{\partial}{\partial x_i} [\{\tilde{p} + K(\psi)\} \Phi^2] = \sum_{i=1}^3 \frac{\partial}{\partial x_i} \left[\frac{(\Phi \psi)^{\cdot, i} (\Phi \psi)^{\cdot, i}}{\Delta \psi - \rho_0 \Phi^2} \right], \quad (5.3)$$

so that \tilde{p} can be determined in terms of ψ .

Moreover, from (4.11), we obtain

$$(\rho U_i)_{,m} - (\rho U_m)_{,i} = 0, \quad (5.4)$$

which is the same restriction on the possible types of motion as McVittie's.² Using the three-dimensional vector notation, we can rewrite the above as

$$\text{curl } \vec{U} = [\nabla (\ln \rho) \times \vec{U}], \quad (5.5)$$

which shows that the velocity field \vec{U} is curl-free when and only when it is parallel to the direction of $\nabla\rho$.

Hydrodynamical equations for ρ , p and \vec{U}

Our fundamental equations (4.9)–(4.12) are reduced, on the condition that $\phi \rightarrow 1$ ($h \rightarrow 0$) i.e. when the cosmological background space-time is tend to Minkowski's space-time, to McVittie's expressions (6.202)–(6.205) of § 6.2 in his book.²⁾ Therefore we can expect to derive quasi-Newtonian equations of hydrodynamical motion in the expanding universe from our fundamental equations according to his procedure. For this purpose, if we eliminate ζ' and ϕ_t from these equations as far as possible, the results are

$$\partial\rho/\partial t + \phi^{-1}\nabla \cdot (\rho\vec{U}) + h\{3\rho - (1+2q)\rho_0\} = 0, \quad (5.6)$$

$$D\vec{U}/Dt + (\phi^{-1}/\rho)\{\nabla\rho - (1-q)h^2\nabla\phi\} + h\{1 + (1+2q)(\rho_0/\rho)\}\vec{U} = 0, \quad (5.7)$$

where $D/Dt \equiv \partial/\partial t + \phi^{-1}(\vec{U} \cdot \nabla)$. It is easily seen that these equations correspond to the equations of continuity and of momentum conservation respectively. But we must remark the following facts: (1) In these equations there appears not only the density ρ but the density ρ_0 of the cosmological background, except in the case $q = -0.5$ corresponding to the Einstein-de Sitter model-universe (cf. I), (2) the gradient term of ϕ which did not appear in McVittie's theory appears in the coupled form with the term $(1-q)h^2$ which specifies the cosmological background, though such a coupling vanishes when $q = 1$ (this case corresponds to the steady-state model-universe (cf. I)).

Now it must be pointed out that the main purpose of McVittie's investigation lies in showing the effectiveness of an approximation of Einstein's field equations for solving the Newtonian hydrodynamical equations, while the main purpose of this sub-section is to provide for a definition (to be discussed in the next section) of the boundary for a local aggregation of matter.

§ 6. Definition of boundary for the local aggregation of matter

Now let Ω be the domain of coordinate volume in which a local aggregation of matter is distributed and whose boundary is specified by the symbol Σ , and \mathcal{V} be the coordinate volume enclosed by Σ . Then we can define the following three kinds of mass corresponding to the three kinds of density which have appeared in (4.13):

$$\left. \begin{aligned} \mathfrak{M} &= \int_{\Omega} \rho d^3x, & \mathfrak{M}_0 &= \int_{\Omega} \rho_0 d^3x = \rho_0 \mathcal{V}, & \Phi^3, \\ \widetilde{\mathfrak{M}} &= \int_{\Omega} \tilde{\rho} d^3x = \mathfrak{M} - \mathfrak{M}_0, \end{aligned} \right\} \quad (6.1)$$

in which \mathfrak{M} is the true mass of the local aggregation of matter in question and

$\tilde{\mathcal{M}}$ is the effective mass contributing to the main part, $\kappa\phi$, of the local gravitational field.

It must be remarked, however, that the boundary Σ of \mathcal{Q} changes in general depending on the cosmic time t . Namely, Σ is independent of t only when the system of coordinates is co-moving, i.e. $\vec{U}=0$ on Σ . But in general \vec{U} does not vanish. That is, there must exist an intimate connection between the t -dependency of Σ and the behavior of the velocity field \vec{U} on Σ . In order to make this circumstance clear, let the boundary $\Sigma=\Sigma(t)$ at the epoch t be designated as

$$F(\vec{r}, t)=0, \quad (6.2)$$

where \vec{r} is the position vector of a point on $\Sigma(t)$ relative to the origin of the coordinate system. Then the boundary $\Sigma(t+\delta t)$ at the epoch $(t+\delta t)$ is given by

$$F(\vec{r}+\vec{n}u_n\delta t, t+\delta t)=0, \quad (6.3)$$

where u_n is the normal component of the displacement coordinate velocity of the point on Σ , and \vec{n} is the unit normal specified by

$$\vec{n}=\nabla F/|\nabla F|. \quad (6.4)$$

From (6.2) and (6.3), we have

$$\partial F/\partial t+u_n(\vec{n}\cdot\nabla F)=0. \quad (6.5)$$

Inserting (6.4) into the above equation, we obtain

$$u_n=-(\partial F/\partial t)/|\nabla F|. \quad (6.6)$$

Now we shall integrate the equation of continuity (5.6) with respect to d^3x over \mathcal{Q} . Then after some reductions, we have

$$\dot{\mathcal{M}}=(1+2q)h\mathcal{M}_0+\Phi^2\int_{\Sigma}\rho\{u_n-(U_n/\Phi)\}dS, \quad (6.7)$$

where U_n is the normal component of \vec{U} , dS is the surface element of Σ . In the above derivation the term $\Phi^2\int_{\Sigma}\rho u_n dS$ manifests the effect due to the t -dependency of Σ (as shown in Fig. 1). (6.7) with (6.6) is nothing but the relation in question between the t -dependency of Σ and the velocity \vec{U} on Σ .

Next we shall consider the differential equation (4.9). It is the Poisson equation, so that its solution vanishing at infinity is given by

$$\begin{aligned} \phi &= (\Phi^2/4\pi) \\ &\times \int_{\mathcal{Q}} \{\bar{\rho}(\vec{r}', t)/|\vec{r}-\vec{r}'|\} d^3x'. \end{aligned} \quad (6.8)$$

As a matter of course, the above ex-

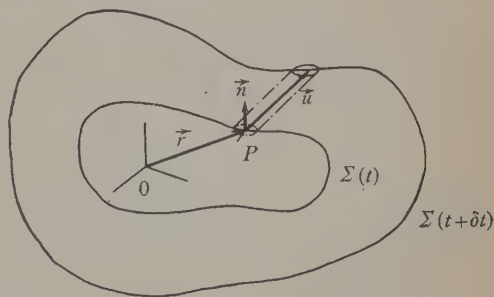


Fig. 1

pression has an intimate connection with (6.7), because (4.9) is used in the derivation of (5.6).

The definition of Σ' specified by (6.2) is not so simple because of the condition (6.7) with (6.6) and its relation with (6.8). According to the analogy from the usual Newtonian hydrodynamics, it seems plausible that the following three conditions would be satisfied simultaneously:

$$(i) \quad u_n = U_n / \Phi \quad \text{on } \Sigma', \quad (6.9)$$

$$(ii) \quad \tilde{\rho} = 0 \quad \text{on } \Sigma', \quad (6.10)$$

$$(iii) \quad \dot{\mathcal{M}} = 0 \quad \text{for } \mathcal{Q}. \quad (6.11)$$

Assuming (i), however, we have from (6.7)

$$\dot{\mathcal{M}} = (1 + 2q)h\mathcal{M}_0, \quad (6.12)^*$$

so that the condition (iii) cannot be satisfied, except in the case $q = -0.5$. In the same way, assuming (ii), we have from (6.7)

$$\dot{\mathcal{M}}/\mathcal{M}_0 = (1 + 2q)h + (1/\mathcal{V}) \int_{\Sigma} dS \{u_n - (U_n/\Phi)\}, \quad (6.13)$$

if we use the definition of \mathcal{M}_0 . This formula satisfies (i) and (iii) at the same time only when $q = -0.5$.

Thus we can see that such a boundary Σ' as satisfying the conditions (i)–(iii) is possible only when $q = -0.5$. We have, however, no positive observational facts supporting the case $q = -0.5$. In these respects, we should like to define Σ' with (ii) alone. Because this is the only definition that the equation for ψ of the exterior region for the local aggregation of matter reduces to the Laplacian equation, i.e.

$$\Delta\psi = 0, \quad (6.14)$$

in a manner similar to the Newtonian theory of gravitation.

§ 7. The gravitational field due to the spherically symmetric aggregation of matter

In the foregoing three sections, we have derived the fundamental equations describing the gravitational field due to the local aggregation of matter in the expanding universe and discussed several important problems to be considered. In this section we shall apply the formalism developed above to the case in which the distribution of matter is spherically symmetric.

Now if we take into consideration the metric (4.2) with (4.3), for the spherically symmetric field, the following conditions, i.e.

* It is necessary to investigate whether this equation relates to the continuous creation of matter⁹⁾ or not.

$$\psi = \psi(t, r); \quad \psi_i = \psi_0(t, r), \text{ say, } (i=1, 2, 3) \quad (7.1)$$

must be imposed, where $r = (x^2 + y^2 + z^2)^{1/2}$, because otherwise r does not necessarily mean the polar coordinate as shown in Appendix I. Then, from (6.8) with $\tilde{\rho} = \tilde{\rho}(t, r)$, we have

$$\kappa\psi = (8\pi G\Phi^2/c^2) \left[(1/r) \int_0^r \tilde{\rho}(t, r') r'^2 dr' + \int_r^{a(t)} \tilde{\rho} r' dr' \right], \quad (7.2)$$

where a is the radial coordinate of the spherical boundary Σ , and $\kappa = 8\pi G/c^2$ as usual. Especially for an exterior point specified by $r \geq a$, (7.2) is reduced to

$$\kappa\psi_e = (8\pi G\Phi^2/c^2 r) \int_0^a \tilde{\rho}(t, r') r'^2 dr',$$

or

$$\kappa\psi_e = 2G\tilde{\mathfrak{M}}/c^2 l, \quad (l \equiv \Phi \cdot r), \quad (7.3)$$

if we take into consideration the definition of $\tilde{\mathfrak{M}}$, where l is the proper radial distance for the smoothed-out universe. We may say that (7.3) expresses an exterior field of Schwarzschild's type due to the effective gravitating mass $\tilde{\mathfrak{M}}(t)$.

Next we shall consider the velocity field U_i . On account of the spherical symmetry of ψ and ρ , (4.11) can be reduced to

$$U_i = U(t, r) x_i / r, \quad (7.4)$$

with

$$U(t, r) = (1/\rho\Phi) (\partial/\partial r) (\dot{\psi} + h\psi), \quad (7.5)$$

where ψ is given by (7.2). From (7.4) we have $\text{curl } \vec{U} = 0$, so that \vec{U} is parallel to the direction of $\nabla\rho$ (cf. (5.5)), i.e. the motion is purely radial and its velocity is given by (7.5). Also, F in (6.2) is given by

$$F = r^2 - a^2 \quad \text{for the spherical boundary } \Sigma.$$

Accordingly,

$$u_n = \dot{a} \quad \text{on } \Sigma. \quad (7.6)$$

Inserting (7.6) and $U_n = U$ specified by (7.5) into (6.13), we obtain

$$\dot{\mathfrak{M}}/\mathfrak{M}_0 = (1 + 2q)h + 3[(\dot{a}/a) - (U(t, a)/L)] = \dot{\mathfrak{M}}_0/\mathfrak{M}_0 - 3U(t, a)/L, \quad (7.7)$$

where $L \equiv \Phi \cdot a$. From (6.1) and (7.7), we have

$$\dot{\tilde{\mathfrak{M}}} = -3U(t, a)\mathfrak{M}_0/L = -4\pi\rho_0 L^2 U(t, a), \quad (7.8)$$

which means that the temporal change of $\tilde{\mathfrak{M}}$ is due to the net flow of matter passing through the boundary.

Now let us make an order estimation for $\tilde{\mathfrak{M}}$ by means of (7.8). The symbol L in (7.8) has the same physical meaning as that in § 3, and further we may

assume that $|U(t, a)| \lesssim v_{ex}$, since $|U(t, r)|$ is at most of the order of v_{ex} (cf. (4.14)). Moreover, from (4.1), we have $4\pi\rho_0 = 3h^2/2G$. Accordingly, (7.8) enables us to obtain

$$|4\pi\rho_0 L^2 U(t, a)| \lesssim 3v_{ex}^2/2G, \quad (7.9)$$

if we use the expression (3.5). Inserting the value of v_{ex} given by (3.13) into (7.9), we obtain

$$|\dot{\mathcal{M}}| \lesssim 0.2 \odot \text{ per sec}, \quad (7.10)$$

which seems to be plausible, in spite of such a rough estimation as the above.

Inserting (7.2) into (7.5), we have

$$U = -(\Phi/\rho r^2) \int_0^r (\partial\tilde{\rho}/\partial t + 3h\tilde{\rho}) r'^2 dr'. \quad (7.11)$$

Especially for $r=a$

$$U(t, a) = -(\Phi/\rho_0 a^2) \int_0^a (\partial\tilde{\rho}/\partial t + 3h\tilde{\rho}) r'^2 dr'. \quad (7.11')$$

The insertion of the above expression into (7.8) enables us to deduce

$$\dot{\mathcal{M}} = 4\pi \int_0^a \{(\partial/\partial t)(\tilde{\rho}\Phi^2)\} r'^2 dr', \quad (7.12)$$

from which we see that $\dot{\mathcal{M}} = \text{const.}$ when and only when $\partial(\tilde{\rho}\Phi^2)/\partial t = 0$, i.e. the system of coordinates employed is co-moving with the local aggregation of matter.

Now inserting (7.4) and the second expression of (7.1) into (4.10) and (4.12), we obtain the following conditions:

$$\{\phi_0'' - \phi_0'/r = -\rho U^2 \Phi^2, \quad (7.13)$$

$$\{\tilde{p} = \Phi^{-2}(\phi_0'' + \phi_0'/r) - K(\phi), \quad (7.14)$$

where $K(\phi)$ is given by (4.5) with (7.2), and the dash denotes differentiation with respect to r . These two equations are nothing but the consistency relations mentioned in § 5 when the distribution of matter is spherically symmetric and the fluidal motion is curl-free. Integrating (7.13) with respect to r , we have

$$\phi_0'/r = [\phi_0'/r]_{r=0} - \Phi^2 \int_0^r \{\rho U^2(t, r)/r\} dr, \quad (7.15)$$

where U is given by (7.9) and $\rho = \tilde{\rho} + \rho_0$. Next we have, from (7.13), (7.14) and (7.15),

$$\tilde{p}(r) = 2\Phi^{-2}[\phi_0'/r]_{r=0} - \int_0^r (1/r^2) \{(\partial/\partial r)(r^2 \rho U^2)\} dr - K(\phi), \quad (7.16)$$

since $U=0$ at $r=0$ (cf. (7.11)). Especially when $r=a$, the above expression is reduced to

$$\tilde{\rho} = 2\Phi^{-2}[\phi_0'/r]_{r=0} - \int_0^a (1/r^2) \{(\partial/\partial r)(r^2 \rho U^2)\} dr - [K(\phi)]_{r=a}, \quad (7.17)$$

From (7.2), however, we have

$$\left. \begin{aligned} 4\pi L\dot{\phi}(a) &= \ddot{\mathfrak{M}}, & 4\pi L\dot{\phi}(a) &= \ddot{\mathfrak{M}} - h\dot{\mathfrak{M}}, \\ 4\pi L\ddot{\phi}(a) &= \ddot{\mathfrak{M}} - 2h\dot{\mathfrak{M}} + (2-q)h^2\mathfrak{M}, \end{aligned} \right\} \quad (7.18)$$

where $\dot{\phi}(a) = [\dot{\phi}]_{r=a}$, for example. Inserting these into (4.5), we obtain

$$[K(\phi)]_{r=a} = (1/4\pi L)[\ddot{\mathfrak{M}} + 2h\dot{\mathfrak{M}} + (q-1)h^2\mathfrak{M}]. \quad (7.19)$$

Therefore, from (7.16), (7.17) with (7.19), we have

$$\tilde{p}(r) = \tilde{p}(a) + \int_r^a (1/r) \{(\partial/\partial r)(r^2 \rho U^2)\} dr + J(\mathfrak{M})/4\pi L - K(\phi), \quad (7.20)$$

where

$$J(\mathfrak{M}) \equiv \ddot{\mathfrak{M}} + 2h\dot{\mathfrak{M}} + (q-1)h^2\mathfrak{M}. \quad (7.21)$$

Moreover, from (7.16) and (7.19), we have

$$[\phi_0'/r]_{r=0} = (\Phi^2/2)[\tilde{p}(a) + \int_0^a (1/r^2) \{(\partial/\partial r)(r^2 \rho U^2)\} dr + J(\mathfrak{M})/4\pi L], \quad (7.22)$$

which should be inserted into the right-hand side of (7.15). Integrating (7.15) further with respect to r , we obtain

$$\begin{aligned} \phi_0(r) &= \{(\Phi r)^2/2\} [\tilde{p}(a) + \int_0^a (1/r^2) \{(\partial/\partial r)(r^2 \rho U^2)\} dr + J(\mathfrak{M})/4\pi L] \\ &\quad - \Phi^2 \int_0^r r dr \left[\int_0^r \rho \{U^2(t, r')/r'\} dr' \right], \end{aligned} \quad (7.23)$$

where we have assumed without loss of generality that $\phi_0=0$ at $r=0$.

In the above our fundamental equations are formally solved for the local gravitational field due to a local aggregation of matter whose hydrodynamical motion is curl-free. From our derivation it is easily seen that a variety of possibility remains for the effective density $\tilde{\rho}$. It must be remarked, however, that the following requirement is necessary for $\tilde{\rho}$: the effective stress \tilde{p} given by (7.20) must be made non-negative, because we made \tilde{p} correspond to the pressure in the Newtonian hydrodynamics.

§ 8 Motion of a test particle in the exterior field due to the local aggregation of matter with spherical symmetry

As was shown by (7.3), the main part ψ of the exterior field due to the local aggregation of matter with spherical symmetry takes Schwarzschild's type independently of the functional form of $\tilde{\rho}$. Under the approximation of neglecting the term ψ_i/c^2 compared with ψ , the metric (4.2) with (4.3) and (7.3) is written as

$$ds^2 = e^\nu c^2 dt^2 - e^\mu (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2), \quad (8.1)$$

with

$$\left. \begin{aligned} e^\nu &= 1 - 2G\tilde{\mathcal{M}}/c^2 l, \\ e^\mu &= \Phi^2(t) [1 + 2G\tilde{\mathcal{M}}/c^2 l], \end{aligned} \right\} \quad (l \equiv \Phi \cdot r) \quad (8.2)$$

where $\tilde{\mathcal{M}} = \tilde{\mathcal{M}}(t)$ in general.

Because the motion of a test particle must be described with the geodesic equation in our formalism, the equations of motion of the test particle moving in the plane $\theta = \pi/2$ can be written as follows:

$$(d^2 r/ds^2) + (d\mu/ds)(dr/ds) - r(d\varphi/ds)^2 + (\nu' - \mu')(e^{\nu-\mu}/2)(cdt/ds)^2 + (\mu'/2)e^{-\mu} = 0, \quad (8.3)$$

$$r^2 e^\mu (d\varphi/ds) = H/c, \quad (8.4)$$

$$(cdt/ds)^2 = e^{-\nu} + e^{\mu-\nu} \{ (dr/ds)^2 + r^2 (d\varphi/ds)^2 \}, \quad (8.5)$$

where $\mu' = \partial\mu/\partial r$, $\nu' = \partial\nu/\partial r$, and H is an integration constant. From (8.5) with (8.2) we have

$$cdt/ds = [1 - (2G\tilde{\mathcal{M}}/c^2 l) - (v/c)^2 \{1 + (2G\tilde{\mathcal{M}}/c^2 l)\}]^{-1/2}, \quad (8.6)$$

where $v^2 \equiv \dot{r}^2 + r^2 \dot{\varphi}^2$ in which $\dot{r} = dr/dt$, $\dot{\varphi} = d\varphi/dt$.

Now let us make a quasi-Newtonian approximation stipulated by the following conditions:

- (1) Deviations of e^ν and $\Phi^{-2}e^\mu$ from unity are ignored, after the necessary calculations are made by inserting (8.3) and (8.4) into (8.2).
- (2) The velocity v of the test particle is so slow that we can replace (8.6) by $cdt/ds = 1$.

It is clear that these approximations are plausible on account of $v \sim v_{er}$ and of the order estimation in § 3 for $\eta \sim G\tilde{\mathcal{M}}/c^2 l$. Then, we have from (8.3) and (8.4) with (8.2)

$$l^2 \dot{\varphi} = H, \quad (8.7)$$

$$\ddot{l} - l \dot{\varphi}^2 + (G\tilde{\mathcal{M}}/l^3) - qh^2 l = 0, \quad (8.8)$$

where $\dot{l} = d(\Phi \cdot r)/dt$, for example.

We shall investigate the above two equations on the standpoint of the New-

tonian theory of gravitation. Since l is the proper radial distance for the smoothed-out universe, it is evident that (8.7) corresponds to the law of areal velocity. In the same manner (8.8) shows that not only Newtonian attraction due to variable mass but also another force proportional to the distance, which is attractive or repulsive according as $q < 0$ or > 0 , is operating simultaneously on the test particle. As was shown in § 3 (cf. (3.4)), these two kinds of force terms have the same order near the outer region of the local aggregation of matter specified by $l=L$, since $|q|$ is of the order $1 \sim 1/5$ (cf. (2.5)). In other words it is reconfirmed that the dimensional considerations made in § 3 are physically plausible. Moreover, if we tentatively consider the local gravitational field due to the Virgo cluster in place of the Supergalaxy, we have

$$G\tilde{M}/l^2 \sim 10^2 \times |q|h^2 l,$$

which shows that the effect of the cosmic expansion is almost negligible in the dynamics of a cluster of galaxies.

Discussion of a special case

Now let us investigate the solution of the quasi-Newtonian equations of motion (8.7) and (8.8) in such a case that

$$\tilde{M} = m\phi^{\alpha-1}(t), \quad (8.9)$$

where $m(>0)$ and α are constants. If we use the cartesian coordinates $x_i (i=1, 2, 3)$ in place of (r, θ, φ) , we can rewrite (8.7) and (8.8) as

$$\ddot{x}_i = -(G\tilde{M}/l^2)x_i - 2h\dot{x}_i, \quad (8.10)$$

where $l = \phi \cdot r = \phi(x_1^2 + x_2^2 + x_3^2)^{1/2}$. On the other hand, since $\phi(t)$ is given by (2.5), we have

$$h = h_0 \phi^{\gamma-2}, \quad (8.11)$$

where h_0 is the value of h in the present epoch $t=t_0$. Inserting (8.9) and (8.11) into (8.10), we obtain

$$\ddot{x}_i = -(Gm\phi^{\alpha-4}/r^3)x_i - 2h_0\phi^{\gamma-2}\dot{x}_i. \quad (8.12)$$

Now we shall take the following transformation of variables:

$$\gamma h_0 T = \phi^\gamma(t); \quad \xi_i = x_i \phi^\gamma(t). \quad (8.13)$$

Then (8.12) is reduced to

$$d^2\xi_i/dT^2 = -(Gm\xi_i/\xi^3)(\gamma h_0 T)^{(\alpha/\gamma-1)}. \quad (8.14)$$

In the above equation, if we formally regard T as the universal time and ξ_i 's as cartesian components of the position vector for a test particle, the equations are mathematically equivalent to the equations of motion of the one-body problem with the variable mass. It is generally impossible to solve (8.14) by quadrature, but

as is shown in Appendix II[†] it is not the case when $\alpha(\alpha-\gamma)=0$.

(i) *The case $\alpha=\gamma$* In this case, (8.14) is reduced to

$$d^2\tilde{\xi}_t/dT^2 = -Gm\tilde{\xi}_t/\tilde{\xi}^3, \quad (8.15)$$

which is the same as the equations for the ordinary one-body problem.

(ii) *The case $\alpha=0$* In this case, we have from (8.12)

$$d^2x_t/dT^{*2} = -Gmx_t/r^3, \quad (8.16)$$

with

$$\gamma h_0 T^* = \Phi^{-\gamma}(t). \quad (8.17)$$

Since (8.15) and (8.16) are of the same form, we have the following solution corresponding to an elliptic orbit :

$$(\alpha=\gamma) : \begin{cases} \tilde{\xi} = a(1-e \cos E), \\ nT = E - e \sin E, \end{cases} \quad (8.18)$$

$$(8.19)$$

$$(\alpha=0) : \begin{cases} r = a(1-e \cos E), \\ nT^* = E - e \sin E, \end{cases} \quad (8.20)$$

$$(8.21)$$

where

$$\begin{aligned} \tan\{(\varphi-\varphi_0)/2\} &= \sqrt{(1+e)/(1-e)} \cdot \tan(E/2), \\ n &\equiv (Gm/a^3)^{1/2}, \end{aligned} \quad (8.22)$$

with usual notations. Using (8.13) and (8.17), we can rewrite (8.18)–(8.21) in the original variables as follows :

$$\begin{cases} l = a\Phi^{1-\alpha}(t) \cdot [1-e \cos E], \\ n\Phi^{2\alpha-\gamma}(t)/\gamma h_0 = E - e \sin E, \end{cases} \quad (8.23)$$

$$(8.24)$$

where $\alpha=\gamma$ or 0.

(iii) *The case $\alpha=\gamma=1$* In this case, $\tilde{M}=m=\text{const.}$ and further $\Phi=t/t_0$ (the uniform expanding model-universe). Then (8.23) and (8.24) are reduced to

$$\begin{cases} l = a(1-e \cos E), \\ nt = E - e \sin E, \end{cases} \quad (8.25)$$

which is exactly the same as the solution of the one-body problem in Newtonian theory : this is seen directly from (8.7) and (8.8), since $q=0$ when $\gamma=1$ (cf. (2.5)).

(iv) *Another case, i.e. $\alpha=\gamma \neq 1$ or $\alpha=0$.* For simplicity's sake, we assume that

$$e=0, \text{ so that } E=(\varphi-\varphi_0). \quad (8.26)$$

Then we have from (8.23) and (8.24)

[†] The method in Appendix II is similar to that in the problem of two bodies with variable masses. η

$$l = a\Phi^{1-\alpha}(t), \quad (8.27)$$

$$\varphi - \varphi_0 = (n/\gamma h_0)\Phi^{2\alpha-\gamma}(t). \quad (8.28)$$

Eliminating $\Phi(t)$ from the above two equations, we obtain

$$\varphi - \varphi_0 = (n/\gamma h_0) (l/l_0)^{(2\alpha-\gamma)/(1-\alpha)}, \quad (8.29)$$

which denotes spiral orbits as shown schematically in Fig. 2.

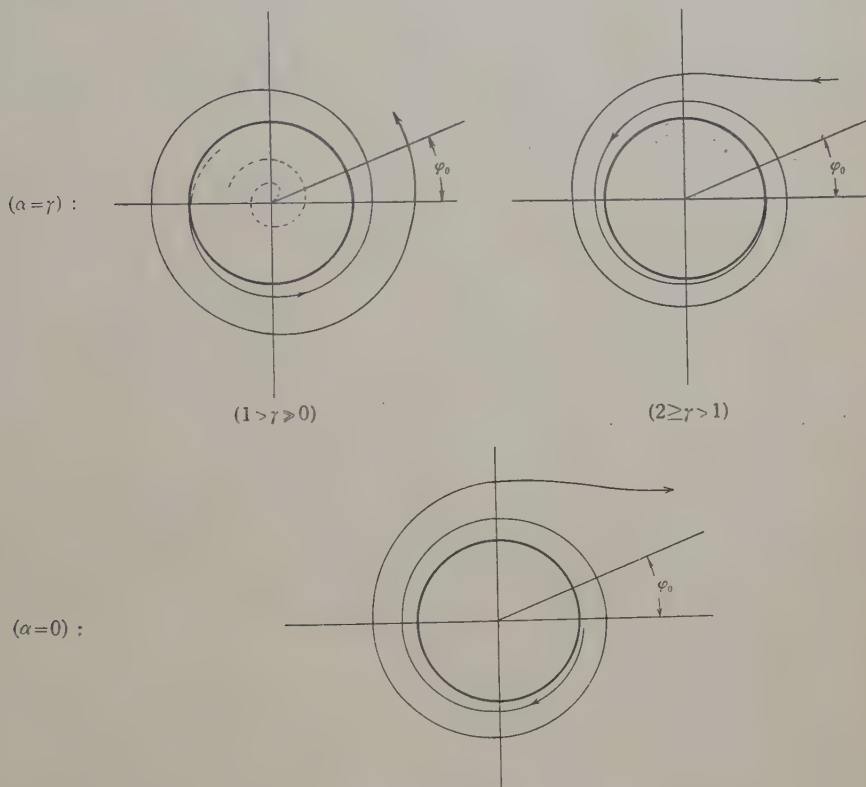


Fig. 2

Further, from (8.9) and (8.27), we have

$$\tilde{\mathcal{M}}l = ma = \text{const.}, \quad (8.30)$$

so that the radial velocity is outward or inward according as $\tilde{\mathcal{M}} < 0$ or > 0 , as illustrated by the arrow-symbols in Fig. 2. If we make use of (8.11) and (8.30), the differentiation of (8.28) with respect to the cosmic time t enables us to obtain

$$\omega = (G\tilde{\mathcal{M}}/l^3)^{1/2}, \quad (8.31)$$

where $\omega = |\dot{\varphi}|$, since $|(2\alpha-\gamma)/\gamma| = 1$. This is of the same form as Kepler's law in the ordinary one-body problem. From (8.27) and (8.31), we have

$$\left. \begin{aligned} v_T &\equiv l\omega = (G\mathfrak{M}/l)^{1/2}, \\ v_R &\equiv \dot{l} = (1-\alpha)hl, \end{aligned} \right\} \quad (8.32)$$

in which v_R is $(1-\alpha)$ times the recessional velocity " hl " for the smoothed-out universe. It can be easily proved that the above three expressions are numerically plausible (cf. § 3).

§ 9. Several remarks on our formalism hitherto developed

In this section, we shall examine several problems proper to the cosmological background space-time and a possible influence of the existence of the Supergalaxy upon Hubble's velocity-distance relation.

As was stated in I, if the cosmological background space-time is spatially euclidean, i.e. $A(r)=r$ (cf. § 2), the range of γ should be

$$1 > \gamma \geq 0, \quad (9.1)$$

if analyses of the velocity-distance relation due to Humason et al.⁸⁾ and Baum⁹⁾ are acceptable. In their analyses, however, the existence of the Supergalaxy, within which our Galaxy is situated, is not taken into consideration. Accordingly, it is necessary to reexamine whether the inequality relation (9.1) is appropriate, even if a possible influence of the rotational and expansive motion of the Supergalaxy is taken into account in the reduction of the velocity-distance relation.

Then, what is the matter with the other two* possible cases of $A(r)$? Though the integration of $d\tau = \phi(t)|dt|$ (cf. (2.4)) by quadrature is generally impossible in these two cases, we can deduce the following expression:

$$\phi\ddot{\phi} + (1-\gamma)\dot{\phi}^2 + \varepsilon(c/a)^2 = 0, \quad (9.2)**$$

where $\varepsilon=1$ or -1 according to the two alternative forms of $A(r)$; (2.5) corresponds to the case $\varepsilon=0$. Then, even in a case when $\varepsilon=\pm 1$, the formalism developed in this paper must hold almost in the same manner. Because, so far as the interior of the local aggregation of matter is concerned,** the metric can take the form (4.2), except the functional form of $\phi(t)$ stipulated by (9.2) with $\varepsilon=\pm 1$.

In short, it is almost compulsory to reexamine both the type of $A(r)$ and the numerical value of γ , as soon as the existence of the Supergalaxy is taken into consideration. We must remark, further, that the real Supergalaxy is not spherically symmetric but is a rotating system with the flattening of the order of $\sim 1/5$. Such a character also has to be taken into account in the reexamination as mentioned above. We shall deal with these problems in another paper.

* $A(r)=a \cdot \sin(r/a)$ or $a \cdot \sinh(r/a)$, where a is the radius of spatial curvature of the universe.

** This expression cannot be obtained in I.

*** Even in case $a < \infty$, i. e. $\varepsilon=\pm 1$, we can estimate that $L/a \leq 10$, where L is the linear dimension of the Supergalaxy.

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Appendix

(I) Now we shall assume that the spherical symmetry of the distribution of matter is denoted by the condition $\psi = \psi(r)$ and $\bar{\rho} = \bar{\rho}(r)$ instead of (7.1), where the argument t in ψ and $\bar{\rho}$ is omitted. Then (7.2)–(7.5) are valid as before. Therefore, inserting (7.4) into (4.12), we have

$$\partial^2 \psi_n / \partial x_i \partial x_m = -V(r) x_i x_m, \quad (\text{I.1})$$

with

$$V \equiv (1/\rho r^2) [(\partial/\partial r)(\dot{\psi} + h\psi)]^2. \quad (\text{I.2})$$

If we put

$$\psi_n = \psi_0(r) + \phi_n(x_i), \quad (\text{I.3})$$

where ψ_0 satisfies

$$(\partial/\partial r) \{ (1/r) (\partial\psi_0/\partial r) \} + rV = 0, \quad (\text{I.4})$$

then (I.1) is reduced to

$$\partial^2 \phi_n / \partial x_i \partial x_m = 0, \quad (\text{I.5})$$

whose general solution is

$$\phi_i = A_i(x_i, x_m) + B_i(x_n, x_l), \quad (\text{I.6})$$

where A 's and B 's are arbitrary functions of their arguments.

Next, inserting (7.4) and (I.3) with (I.6) into (4.10), we have

$$\partial^2 A_i(x_i, x_m) / \partial x_m^2 + \partial^2 B_m(x_l, x_n) / \partial x_l^2 = x_n^2 W(r) + Y(r), \quad (\text{I.7})$$

with

$$\left. \begin{aligned} W &\equiv \rho(U\dot{\psi}/r)^2 - V, \\ Y &\equiv [\tilde{p} + K(\psi)]\dot{\psi}^2 - (1/r)(\partial/\partial r)(r\partial\psi_0/\partial r), \end{aligned} \right\} \quad (\text{I.8})$$

where it should be remarked that we can prove without difficulty that $\tilde{p} = \tilde{p}(r)$. The condition that (I.7) is consistently valid becomes

$$\partial^2 A_i(x_i, x_m) / \partial x_m^2 + \partial^2 B_m(x_l, x_n) / \partial x_l^2 = \alpha - \beta(x_i^2 + x_m^2), \quad (\text{I.9})$$

with

$$W = \beta, \quad Y = \alpha - \beta r^2, \quad (\text{I.10})$$

where α and β are integration constants. Because (I.9) represents three equations for six unknown functions A 's and B 's, there exist many solutions with many degrees of freedom. As is easily seen from (I.6), however, these cannot contain x 's in the form of r . Therefore the second equation of (7.1) does not hold in general.

From this fact, it is easily seen that the variable r does not mean the radial coordinate.

(II) Writing (8.14) with dimensionless variables, we have

$$d^2 \xi_i / dT^2 = -(\xi_i / \xi^3) T^{(\alpha/\gamma-1)}, \quad (\text{II} \cdot 1)$$

where

$$T = \Phi^\gamma(t), \quad \xi_i = x_i \Phi^\gamma(t) / (Gm/\gamma^2 h_0^2)^{1/3}, \quad (\text{II} \cdot 2)$$

in place of (8.13). Now we shall assume that $\alpha \neq \gamma$. Then, putting

$$\xi_i^* = \nu^b \xi_i, \quad dT^* = -\nu^c dT, \quad (\text{II} \cdot 3)$$

with

$$\nu \equiv T^{(\alpha/\gamma-1)}, \quad (\text{II} \cdot 4)$$

we obtain from (II.1)

$$\begin{aligned} d^2 \xi_i^* / dT^{*2} = & -(\xi_i^* / \xi^{*3}) \nu^{1+3b-2c} + (c-2b)(\alpha/\gamma-1)(d\xi_i^* / dT^*) \nu^{-[c+\gamma/(\alpha-\gamma)]} \\ & - b \left(b + \frac{\gamma}{\alpha-\gamma} \right) (\alpha/\gamma-1)^2 \xi_i^* \nu^{-2[c+\gamma/(\alpha-\gamma)]}, \end{aligned} \quad (\text{II} \cdot 5)$$

where b and c are numerical constants ($\neq 0$). The condition that the second and third terms on the right-hand side of (II.5) vanish is

$$b = -\gamma/(\alpha-\gamma), \quad c = 2b = -2\gamma/(\alpha-\gamma). \quad (\text{II} \cdot 6)$$

Inserting the above formulae into (II.3), we have after necessary calculations

$$T^* = 1/T, \quad \xi_i^* = \xi_i/T, \quad (\text{II} \cdot 7)$$

without loss of generality. Then, (II.5) is reduced to

$$d^2 \xi_i^* / dT^{*2} = -(\xi_i^* / \xi^{*3}) T^{*(\alpha/\gamma-1)}, \quad (\text{II} \cdot 8)$$

from which we easily see that (II.8) is reduced to the differential equation of the same form as that of the one-body problem when $\alpha=0$. Thus in a case when $\alpha(\alpha-\gamma)=0$, it is proved that (II.1) can be solved with quadrature.

Next we shall assume that $\alpha(\alpha-\gamma) \neq 0$. Then we can perform a transformation of variables ($\xi_i^* \rightarrow \xi_i^{**}$, $T^* \rightarrow T^{**}$) similar to (II.3). However, if we are to search for such conditions that the equations corresponding to (II.5) be reduced to the same type as (II.8), we have

$$\left. \begin{aligned} \xi_i^{**} &= \xi_i^* / T^* = \xi_i, \\ T^{**} &= 1/T^* = T. \end{aligned} \right\} \quad (\text{II} \cdot 9)$$

Namely the transformation of variables from *-variables to **-variables is nothing but the inverse one of (II.7). In short, in a case when $\alpha(\alpha-\gamma) \neq 0$, the quadrature is generally impossible.

But the case of rectilinear motion is an exception, because the following solution is deduced without difficulty:

$$\xi = [(3\gamma)^2 / (\gamma + \alpha)(2\gamma - \alpha)]^{1/3} T^{(\gamma + \alpha)/3\gamma}, \quad (\text{II} \cdot 10)$$

provided that

$$-\gamma < \alpha < 2\gamma, \quad (\text{II} \cdot 11)$$

where $\xi_1 = \xi$, $\xi_2 = \xi_3 = 0$ (for example).

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On the Electromagnetic Structure of Nucleons and Their Mass Difference

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A semi-phenomenological methodology is proposed for the study of the structures of elementary particles, based on the problem of the electromagnetic structure of nucleon. It is convenient to separate the analyses of structure into two parts. One is the analysis of the outer structure which is characterized as the *quasi short distance* and defined by $r \sim 0.5y$.*** Though the present field theory could be applied in this region, it is shown that the shape independent feature also holds in so far as the present experimental information is concerned. The other is the presumption of the inner structure of the *extreme short distance*. We can extract a physical quantity which is effective in the analysis of this region by using a proposed trial model in harmony with the present experiments.

A presumption comes from the electromagnetic mass difference of nucleons. The analyses are devoted to three effects: i) effect of the change of the form factors at the extreme short distance, ii) effect of the neutron charge form factor, and iii) effect of the higher order corrections of the strong interactions. These effects have qualitatively good points for the explanation of the mass-difference. A small modification of the inner structure is especially quantitatively suitable to the experimental value.

Discussions are also devoted to problems of the theory of structure in the future.

§ 1. Introduction

Much information about the structure of elementary particles is being accumulated after the pioneer work on electron-proton scattering.¹⁾ Should the size of the structure appear in every phenomena it would be around 1 yukawa irrespective of the kinds of interactions. At the present stage, it is still difficult to decide whether these data show the true spreading of elementary particles themselves, or whether they can be explained by the present field theory, or whether they tell us the limit of validity of our theory.

At the very outer region of the nucleon structure, the decisive progress has been obtained by using the methodology of nuclear meson theory named the T-N-S method.²⁾ At the near outer region, however, even if we restrict the case of electromagnetic structure of nucleon which the experiments gave some clear infor-

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*** We use the units of $\hbar=c=1$ throughout this paper. The quantities with the dimension of length are expressed by unit of 1 yukawa= 10^{-13} cm.

mation, no decisive conclusion has yet been obtained in spite of many pertinacious attacks on the problem of meson theory^{3),4)} owing to many theoretical difficulties which are already pointed out in the T.N.S. article. On the other hand, some attempts for searching the limit of quantum electrodynamics are proposed,⁵⁾ but it is still very sceptical whether the proposed experiments will bring a clear cut solution.

Since the elementary particle would be a matter of complexity, its structure may reveal all properties which we cannot expect from the usual standpoint. One part of its structure could be accepted as the picture of clouds surrounding the elementary particle, but the other part might be the appearance of the true spreading of this particle⁶⁾ or the limit of validity of our theory. As the circumstances are not so simple as we imagined, we must take a more careful attitude than usually taken and must not have any fixed standpoint as already discussed by one of us (M.T.).²⁾

Besides all these cautious treatments, the first thing which we must take into consideration is to notice the manifoldness corresponding to the variety of phenomena which the structure itself reveals. We could not always expect the same structure information through every interaction, and also might not get the same effect in similar reactions which we suppose the same interaction plays.

The second thing which we must bear in mind is that we are striving to notice what knowledge we can get at certain distances and what conclusions at least we can draw from them. For instance, as the available energy which the experiments of electron-proton scattering have achieved is at most $q^2 \simeq 20 \text{ y}^{-2}$, where q is the momentum transfer, we can get only some data of structure down to the distance $r \simeq 2.1/q \simeq 0.5 \text{ y}$ by using our conversion relation as will be discussed in the next section. Knowledge about the structure down to this distance is not sufficient to warrant a deduction of the detail shape of the structure. Then if we have not rigid attitude for accepting the experimental data, we shall make the sufficient theoretical analyses of the structure in this region with much caution. The distance which characterised this region is just corresponding to $r \sim 1/3m_\pi \sim 1/m_K$, where m_π and m_K are pion and kaon masses respectively, then we may expect the current meson theory will give at least some qualitative successes. We shall call it the *quasi short distance* which is defined by $r \gtrsim 0.5 \text{ y}$ and more concretely we shall use the corresponding structure as the name of the *outer structure*.

On the contrary, about the problem of structure at a *very short distance*, defined as $r < 0.5 \text{ y}$, we have neither any experimental information nor any reliable theory. As we cannot expect a detailed analyses at the present stage, we must be satisfied with the semi-phenomenological analyses which are suitably taken.

One of the sources from which we may deduce the information about structure at this region would be the problem of the mass of elementary particle, specially the problem of mass-difference between nucleons.

Notwithstanding the fact that it has been considered necessary to explain the

mass-difference between nucleons by the electromagnetic field reaction, we could not get any success of this justification. And also the mass-differences between sigma triplets are too large to get any satisfactory answer.

The mass-difference of nucleons was at first analysed from the standpoint of the cohesive field.⁷⁾ This attempt seemed to give a good result at the beginning, but after obtaining the detailed analysis, we could not realize the experimental value regardless of the role assumed by this cohesive field.⁸⁾ An attempt of its explanation was made by introducing the anomalous magnetic moment interaction besides the usual interaction.⁹⁾ As the result is very sensitive about the cutoff procedure used, we obtained the result only by taking the weak cutoff. On the contrary, if we would use the exponential shape of the form factors with the experimentally confirmed mean square radius of $\langle r^2 \rangle = (0.8y)^2$, we could get only result with wrong sign and smaller magnitude comparing with the experimental one's.^{10), 11)} Also a similar result was obtained¹²⁾ for the same attempt of sigma's.¹³⁾ These results, as several authors have written, seem to give the impression that we shall have no success in this problem in harmony with the present knowledge from the electron-nucleon scattering. However, these conclusions have no ground, because all these attempts are standing on the careless treatments about the nucleon structure. We shall pick up the main defects of these attempts.

The first thing which we must bear in mind is the fact that the electron-nucleon experiments with the present available energy cannot always be restricted to any possible shape of form factors: the exponential shape is only one of the examples. We have started this analysis with this question, and also recently Hofstadter stated a similar question.¹⁴⁾ There still exists many possibilities of selecting form factors which can realize the experiments, and we can show that all requirements of the experiments are arranged to some concepts of shape-independence. As a small modification of the shape in the quasi small distance may give a distinguishable different contribution when we extrapolate it to the very short distance, we must make very cautious extrapolation of the form factors to the very short region $r < 0.5y$.

The second is that we must analyze at the very short distance all possible effects: these effects consist of i) the effect of the change of form factors at short distances which cannot affect the experimental results, ii) the effect of the charge form factor of neutron and iii) the effect of the higher order corrections of the strong interactions. Some of these parts were also considered by some authors.

The first part of this paper is devoted to reanalyse what knowledge is at least drawn from Stanford experiments at quasi short distance and we shall show that they can be replaced with some shape independent concepts (§ 2). Then we examine the physical quantities which are suitable for analyses of the structure at extremely short distance and introduce a trial model which is helpful to study them (§ 3). The mass-difference of nucleon is analysed by using this trial model and we can show that there exist some solutions to this problem (§ 4). The

mass problems of other elementary particles are also discussed qualitatively, with the theoretical problem of structure which we shall confront in the near future (§ 5, § 6).

§ 2. The outer structure and the concept of shape independence

The electromagnetic structure of nucleon at the quasi short distance defined by $r \gtrsim 0.5y$ has been analysed in detail through the experiments of electron-proton and electron-nucleus scatterings. According to these analyses, we have the following conclusion:¹⁵⁾

We assume the modification of the electromagnetic current of nucleon as:

$$e\bar{\psi}\Gamma_{\mu}(q)\psi = e\bar{\psi}\left[F_{1p}(q^2)\frac{1+\tau_3}{2} + F_{1n}(q^2)\frac{1-\tau_3}{2}\right]\gamma_{\mu}\psi \\ + \frac{e}{2M}\bar{\psi}\left[\mu_p F_{2p}(q^2)\frac{1+\tau_3}{2} + \mu_n F_{2n}(q^2)\frac{1-\tau_3}{2}\right]\cdot\sigma_{\mu\nu}q_{\nu}\psi, \quad (2.1)$$

where

$$\sigma_{\mu\nu} = \frac{1}{2i}(\gamma_{\mu}\gamma_{\nu} - \gamma_{\nu}\gamma_{\mu}) \quad (2.2)$$

and ψ and $\bar{\psi}$ are nucleon field quantities. $F_{1p}(q^2)$ and $F_{2p}(q^2)$ are the charge-form factors, while $F_{2p}(q^2)$ and $F_{2n}(q^2)$ the moment-form factors with the coefficients of anomalous magnetic moments $e\mu_p/2M$ and $e\mu_n/2M$, where suffixes p and n refer to proton and neutron respectively.

Then we have

i) The total charge of neutron is zero, while the total charge of proton is e and the total anomalous magnetic moments of proton and neutron are given as $e\mu_p/2M$ and $e\mu_n/2M$ respectively. This is equivalent to

$$F_{1n}(0) = 0 \quad \text{and} \quad F_{1p}(0) = F_{2p}(0) = F_{2n}(0) = 1. \quad (2.3)$$

ii) The charge-form factor of neutron is almost negligible up to $q^2 \simeq 9y^{-2}$ comparing with other form factors which are almost equal, up to $q^2 \lesssim 20y^{-2}$. This means:

$$F_{1n}(q^2) \simeq 0 \quad (2.4)$$

and

$$F_{1p}(q^2) \simeq F_{2p}(q^2) \simeq F_{2n}(q^2) \quad (\equiv F(q^2)). \quad (2.5)$$

iii) The mean square radius of the charge-form factor of neutron is considered as

$$|\langle r_n^2 \rangle| \lesssim (0.1y)^2 \quad (2.6)$$

and the others are given by

$$\langle r^2 \rangle = -6 \frac{dF(0)}{dq^2} \simeq [0.80 \pm 0.04]^2 y^2. \quad (2.7)$$

iv) The experimental results can be realized by using the exponential shape for the charge- and moment-form factors of proton and the moment-form factor of neutron.

However, the last statement seems very weak compared with others, because the form factor which realizes the experiments so far cannot be restricted only to the exponential shape as there are still many possibilities (almost infinite) for selecting the shape of form factors. This fact should be taken into consideration at the very beginning of the analyses. Only one question appeared from the standpoint of the current meson theory. As far as the present experimental information is concerned, we can decide the form factor neither as the exponential shape nor as the one which the meson theory may give. This question suggests that we should start this work. Moreover, Hofstadter recently raised some other examples of the form factor.

Even though the experiments are being done very carefully, the information available to us is not so sufficient enough to enter the details of form factor. In fact, if we arrange the conclusions sketched above, we can summarize them as the following three evidences:

- [A] The total amount of charge or moment distribution can be normalized either 1 or 0.
- [B] The mean square radius of the distribution is given a definite value, for instance, $\langle r^2 \rangle \simeq (0.8y)^2$ for the charge distribution of proton.
- [C] The partial amounts of distribution in the region $r \gtrsim 0.5y$ are almost constant and it becomes 60% at $r \simeq 0.5y$, with the exception of the neutron charge distribution.

In order to illustrate this, we can define the proton charge distribution as

$$4\pi r^2 \rho(r) = \frac{r^2}{2\pi^2} \int d\mathbf{q} F(q^2) \exp i\mathbf{q} \cdot \mathbf{r}. \quad (2.8)$$

The statement [A] is given by

$$\int_0^\infty 4\pi r^2 \rho(r) dr = F(0) = 1, \quad (2.9)$$

which corresponds to (2.3) and the statement [B] becomes

$$\int_0^\infty 4\pi r^4 \rho(r) dr = -6 \frac{dF(0)}{dq^2} = \langle r^2 \rangle \quad (2.10)$$

corresponding to (2.7). Both statements have been frequently discussed.

All efforts which Stanford's group has made by using the very high energy electron beams are concentrated to the statement [C]. In order to grasp the concrete idea, we raise the value of the *partial amount of distribution* defined by

$$Q(r) = \int_r^{\infty} 4\pi r^2 \rho(r) dr \quad (2.11)$$

at $r \simeq 0.5y$ for the various form factors which can realize the experiments. In Table I, the form factors which were excluded by experiments are also described by the mark \times for instructive purposes. The values of distribution at this point are also shown. From this table we can note that the partial amount of charge

Table I. Distribution and its partial amount of various form factors
(\times means the form factor which is excluded by experiments.)

	distribution $4\pi r^2 \rho(r)$	partial amount $Q(r)$	$Q(0.5y)$	$\frac{4\pi r^2 \rho(r)}{r=0.5y}$
Exponential	$\frac{r^2}{2b^3} e^{-(r/b)} \quad b = \left[\frac{\langle r^2 \rangle}{12} \right]^{1/2}$	$\left(1 + \frac{r}{b} + \frac{r^2}{2b^2} \right) e^{-(r/b)}$	0.62	$1.2y^{-1}$
Clementel - Villi	$\frac{r}{\eta b^2} e^{-(r/b)} \quad b = \left[\frac{\eta \langle r^2 \rangle}{6} \right]^{1/2}$ $\eta = \frac{5}{6}$	$\frac{1}{\eta} \left(1 + \frac{r}{b} \right) e^{-(r/b)}$	0.61	$1.3y^{-1}$
Yukawa \times	$\frac{r}{b^2} e^{-(r/b)} \quad b = \left[\frac{\langle r^2 \rangle}{6} \right]^{1/2}$	$\left(1 + \frac{r}{b} \right) e^{-(r/b)}$	0.50 \times	$1.0y^{-1}$
Yukawa + Zero core	$\frac{r}{b^2(1+d/b)} e^{-(r-d)/b} \quad r > d$ $d = 0.225y \quad b = 0.301y$	$\frac{1+(r/b)}{1+(d/b)} e^{-(r-d)/b} \quad r > d$	0.62	$1.1y^{-1}$
Singular Yukawa \times	$\frac{1}{b} e^{-(r/b)} \quad b = \left[\frac{\langle r^2 \rangle}{2} \right]^{1/2}$	$e^{-(r/b)}$	0.41 \times	$0.8y^{-1}$
Singular Yukawa + Zero core	$\frac{1}{b} e^{-(r-d)/b} \quad r > d$ $d = 0.44y \quad b = 0.40y$	$e^{-(r-d)/b} \quad r > d$	0.62	$2.1y^{-1}$
Two Yukawa (Trial model)*	$\frac{1-\lambda_1}{\lambda_0-\lambda_1} \frac{r}{b_0^2} e^{-(r/b_0)} - \frac{1-\lambda_1}{\lambda_0-\lambda_1} \frac{r}{b_1^2} e^{-(r/b_1)}$ $b_0 = \left[\frac{\lambda_0 \langle r^2 \rangle}{6} \right]^{1/2} \quad b_1 = \left[\frac{\lambda_1 \langle r^2 \rangle}{6} \right]^{1/2}$	$\frac{1-\lambda_1}{\lambda_0-\lambda_1} \left(1 + \frac{r}{b_0} \right) e^{-(r/b_0)} - \frac{1-\lambda_0}{\lambda_0-\lambda_1} \left(1 + \frac{r}{b_1} \right) e^{-(r/b_1)}$ $\lambda_0 = 0.80 \quad \lambda_1 = 0.07$	0.62	$1.3y^{-1}$

* This model will be discussed in next chapter as a trial model.

distribution is better than the magnitude of charge distribution for selecting the suitable form factor.

The statement [C] can be explained as follows: We can get

$$Q(r) = 1 - \frac{2}{\pi} \int_0^{\infty} dk F(k^2) k r^2 \left[\frac{\sin kr}{(kr)^2} - \frac{\cos kr}{kr} \right] \quad (2.12)$$

by using (2.12), (2.9), where the function with bracket in the last integrand has a sharp maximum at $kr=2.1$ if $r \neq 0$ and decreases rapidly at $kr \gtrsim 4.0$. Then if the function $F(k^2)k$ should be a decreasing function (not necessarily monotonic, especially at the region $kr \gtrsim 4.0$) its main contribution to the partial amount should come from the value at

$$kr=2.1 \quad (2.13)$$

and (2.12) can be approximated by

$$\begin{aligned} Q(r) &\simeq 1 - \frac{4.2}{\pi} F[(2.1)^2/r^2] \int_0^{4.0} dx \left[\frac{\sin x}{x^2} - \frac{\cos x}{x} \right] \\ &\simeq 1 - 1.6 F[(2.1)^2/r^2]. \end{aligned} \quad (2.14)$$

Therefore all form factors which can realize the experimental data up to $k^2=20y^{-2}$ have same definite values down to $r \simeq 0.47y$ and the partial amount of distribution at this point is given by

$$Q(0.47) \simeq 0.60 \quad (2.15)$$

in virtue of $F(20y^{-2}) \simeq 0.25$.

Also it is instructive to point out that *the relation (2.13) is more practical than the usual conversion $kr=1$ when we construct the picture in the usual space from the experimental information, that is, if we get the information up to $q^2 \simeq 20y^{-2}$, we can only have some knowledges down to $r \simeq 0.5y$ instead of $r \simeq 1/\sqrt{20}y \simeq 0.2y$. Then the usual arguments about the validity of quantum electrodynamics are incorrect because in order to obtain some information at $r \lesssim 0.3y$, we need the energy of 2 Bev, instead of 1 Bev.*

Now the statements made above can be expressed by (2.9), (2.10) and (2.14) and further, if we restrict only the monotonic decreasing form factor up to $q^2 \simeq 20y^{-2}$, we can summarize them as the shape independent requirements (2.9), (2.10) and (2.15). Then we can conclude that any monotonic decreasing form factor which satisfies these requirements is in agreement with the Stanford experiments.

It is notable that we do not use any detail shape of distribution in our requirements. Though the distribution seems to be on intuitive quantity, it is not the case for analyses of the structure. The value of distribution in the quasi short distance is too sensitively dependent on the particular type of the form factor to deduce any conclusion from it, as easily seen in Table I. While in the short distance its small variation gives a remarkable difference for the form factor. In its place by taking more effective physical quantities we would obtain the concepts of potential and field strength which are determined uniquely by knowing the partial amount of distribution.

Using the spectral representation of form factor with the weight function $g(m^2)$

$$F(q^2) = \frac{1}{\pi} \int dm^2 \frac{g(m^2)}{q^2 + m^2}, \quad (2.16)^*$$

the quantities of distribution, partial of distribution, potential and field strength are given by

$$4\pi r^2 \rho(r) = \frac{1}{\pi} \int dm^2 g(m^2) r e^{-mr}, \quad (2.17)$$

$$Q(r) = \frac{1}{\pi} \int dm^2 \frac{g(m^2)}{m^2} (1 + mr) e^{-mr}, \quad (2.18)$$

$$V(r) = \frac{1}{r} \frac{1}{\pi} \int dm^2 \frac{g(m^2)}{m^2} - \frac{1}{r} \frac{1}{\pi} \int dm^2 \frac{g(m^2)}{m^2} e^{-mr}, \quad (2.19)$$

$$E(r) = \frac{1}{r^2} \frac{1}{\pi} \int dm^2 \frac{g(m^2)}{m^2} - \frac{1}{r^2} \frac{1}{\pi} \int dm^2 \frac{g(m^2)}{m^2} (1 + mr) e^{-mr} \quad (2.20)$$

respectively.

We can get some information from the low energy experiments about the coefficients of expansion of q^2 of the form factor

$$F(q^2) \simeq \frac{1}{\pi} \int dm^2 \frac{g(m^2)}{m^2} - q^2 \frac{1}{\pi} \int dm^2 \frac{g(m^2)}{m^4}. \quad (2.21)$$

Information about the first term determines the total amount of distribution

$$Q(0) = \frac{1}{\pi} \int dm^2 \frac{g(m^2)}{m^2} \quad (2.22)$$

and correspondingly determines the leading standard values of potential and field strength

$$V_0(r) = Q(0)/r, \quad E_0(r) = Q(0)/r^2. \quad (2.23)$$

A determination of the second term restricts the totality of deviation of potential from the standard form irrespective of its detail deviation¹⁶⁾

$$\int 4\pi \delta V(r) r^2 dr = 4 \int dm^2 \frac{g(m^2)}{m^4}; \quad V(r) = V_0(r) - \delta V(r). \quad (2.24)$$

If the energy is increased, we can get information of field strength at a point of $r \simeq 2.1/q$, that is information about the deviation of field strength

$$\delta E(r) = \frac{Q(0)}{r^2} - E(r) \quad (2.25)$$

and correspondingly we can obtain the partial amount of distribution from

$$Q(r) = r^2 \delta E(r). \quad (2.26)$$

* We don't include the case in which the form factor tends to some finite (infinite) value at extreme high energy, but this case can be realized by the limiting case.

Then if we could get all the information about field strength down to r , we could determine the potential at r . Summarizing this information, we shall arrive at the point where the shape of distribution is presumed.

Though the procedure seems to be very trivial, this is necessary and essential for the analyses of structure, especially in the region where the experimental information is poor or beyond the present experimental techniques, because in the very limited region, the numbers of the effective physical quantities might be very much suppressed.

§ 3. The inner structure and the proposal of a trial model

Unfortunately, we have no suitable theory which can predict the inner structure of elementary particle at a very short distance beyond the region of the present experiments. Even if we should apply the present theory to this region, this is only an extrapolation which is less reliable. Then though we are obliged to postpone the detailed knowledge about the inner structure at our present stage, we can guess what properties are at least helpful in finding a clue to this problem.

Let us consider the *extreme short distance*, where the form factor (2.17) can be approximated by

$$F(q^2) \simeq \frac{1}{q^2} \frac{1}{\pi} \int dm^2 g(m^2) - \frac{1}{q^4} \frac{1}{\pi} \int dm^2 m^2 g(m^2). \quad (3.1)$$

Then our aim is to know the first (and if possible the second) coefficient of this expansion. Since we can show that the first coefficient represents the field strength due to the negative distribution in the neighbourhood of the center of the structure in virtue of the expansion of (2.20) by short distance $r_0 \simeq 0^*$ as

$$\frac{Q(0) - Q(r_0)}{r_0^2} \equiv - \frac{\Delta Q(r_0)}{r_0^2} \simeq \frac{1}{2\pi} \int dm^2 g(m^2), \quad (3.2)**$$

one of our aims is to get some information of this quantity. This seems to be possible, and we shall limit ourselves only to this first term at our present circumstances.

In order to know this quantity, we construct a trial model which is its representation, i.e., a trial model which represents it as a parameter. The role of a trial model thus introduced is not to deduce some detailed information about the inner structure, but to represent a characteristic which the true model would have.

* The expansion of (2.20) by short distance r is not allowed for the weak converging form factor such as the singular Yukawa shape. In this case we need some modification as will be done in Appendix II.

** Strictly speaking, this equation holds only approximately. For the case of small but finite r , we must take into account the further expansion terms as

$$\frac{1}{2\pi} \int dm^2 g(m^2) - \frac{r_0}{3\pi} \int dm^2 m g(m^2) + \frac{r_0^2}{8\pi} \int dm^2 m^2 g(m^2) - \dots$$

Then it must be understood that it is only a representative of various form factor models which are in conformity to this trend.

We can take one of them, as simple as possible; for instance, it must have at least four parameters such as

$$F(q^2) = \frac{C_0}{q^2 + m_0^2} + \frac{C_1}{q^2 + m_1^2}. \quad (3.3)$$

The requirements which a form factor must have are three statements [A], [B], [C] of low energy region, which correspond with (2.10), (2.11) and (2.16). Thus we can obtain the following form factor

$$F(q^2) = \frac{1 - \lambda_1}{\lambda_0 - \lambda_1} \frac{1}{1 + 2\lambda_0 a^2 q^2} - \frac{1 - \lambda_0}{\lambda_0 - \lambda_1} \frac{1}{1 + 2\lambda_1 a^2 q^2}, \quad (3.4)$$

where $a^2 = \langle r^2 \rangle / 12$ and λ_0, λ_1 are dimensionless parameters which satisfy

$$\lambda_0 \lambda_1 - \frac{3}{2}(\lambda_0 + \lambda_1) + \frac{5}{4} \approx 0, \quad \lambda_0 \geq \lambda_1 \geq 0, \quad (3.5)$$

due to statement [C]. λ_0, λ_1 must be positive in order to exclude the unsatisfactory singularity. By using this trial model, (3.2) is represented as

$$-\frac{4-Q(r_0)}{r_0^2} \approx -\frac{3}{\langle r^2 \rangle \lambda_0 \lambda_1} [1 - (\lambda_0 + \lambda_1)]. \quad (3.6)$$

One of the parameters λ_0 and λ_1 has some freedom, i.e., λ_0 runs a region of

$$5/6 \gtrsim \lambda_0 \gtrsim 1/2, \quad (3.7)$$

while λ_1 is determined by (3.5) within a region of $1/2 \geq \lambda_1 \geq 0$. The case of $\lambda_0 = \lambda_1 = 1/2$ corresponds to the exponential shape, while others of $\lambda_0 = 5/6, \lambda_1 = 0$ Clementel-Villi's shape.¹⁷⁾ Correspondingly, we can take any value for the quantity (3.6) within a region of

$$\infty \gtrsim 4-Q(r_0)/r_0^2 \gtrsim 0. \quad (3.7')$$

The distribution and partial amount of distribution are given by

$$4\pi r^2 \rho(r) = \frac{1 - \lambda_1}{\lambda_0 - \lambda_1} \frac{r}{b_0^2} e^{-(r/b_0)} - \frac{1 - \lambda_0}{\lambda_0 - \lambda_1} \frac{r}{b_1^2} e^{-(r/b_1)}, \quad (3.8)$$

$$Q(r) = \frac{1 - \lambda_1}{\lambda_0 - \lambda_1} \left(1 + \frac{r}{b_0}\right) e^{-(r/b_0)} - \frac{1 - \lambda_0}{\lambda_0 - \lambda_1} \left(1 + \frac{r}{b_1}\right) e^{-(r/b_1)}, \quad (3.9)$$

where

$$b_0 = [\lambda_0 \langle r^2 \rangle / 6]^{1/2} \quad \text{and} \quad b_1 = [\lambda_1 \langle r^2 \rangle / 6]^{1/2}.$$

If we start with the general form factor (2.16), we have

$$F(q^2) = \frac{1 - \lambda_1}{\lambda_0 - \lambda_1} \frac{1}{1 + 2\lambda_0 a^2 q^2} - \frac{1 - \lambda_0}{\lambda_0 - \lambda_1} \frac{1}{1 + 2\lambda_1 a^2 q^2} + F_1(q^2), \quad (3.10)$$

$$F_1(q^2) = -\frac{1}{\pi} \int dm^2 \frac{g(m^2)}{m^2} \frac{[1 - 2\lambda_0 m^2 a^2][1 - 2\lambda_1 m^2 a^2]}{[1 + q^2/m^2][1 + 2\lambda_0 a^2 q^2][1 + 2\lambda_1 a^2 q^2]}, \quad (3.11)$$

where

$$\frac{1}{\pi} \int dm^2 g(m^2) = -\frac{6}{\langle r^2 \rangle \lambda_0 \lambda_1} [1 - (\lambda_0 + \lambda_1)], \quad (3.12)$$

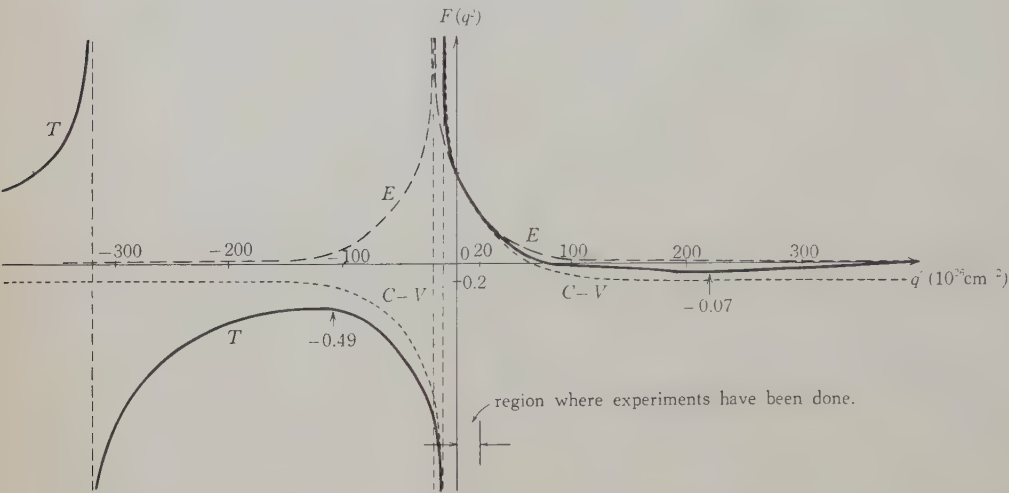


Fig. 1. The behaviors of the form factors in momentum space
Solid line means a trial model proposed here, broken line the
exponential shape and dotted line the Clementel-Villi shape.

Table II. Values of form factors

q^0	exponential	trial $\lambda_0=0.82$	Clementel-Villi
0	1.0000	1.0000	1.0000
2	0.8165	0.8186	0.8189
4	0.6792	0.6846	0.6852
6	0.5739	0.5817	0.5826
8	0.4913	0.5002	0.5013
10	0.4253	0.4341	0.4353
12	0.3718	0.3796	0.3806
14	0.3277	0.3338	0.3346
16	0.2911	0.2948	0.2954
18	0.2603	0.2614	0.2615
20	0.2341	0.2323	0.2320
22	0.2117	0.2068	0.2060
24	0.1923	0.1844	0.1830
26	0.1755	0.1644	0.1624
28	0.1608	0.1467	0.1439
30	0.1479	0.1307	0.1273

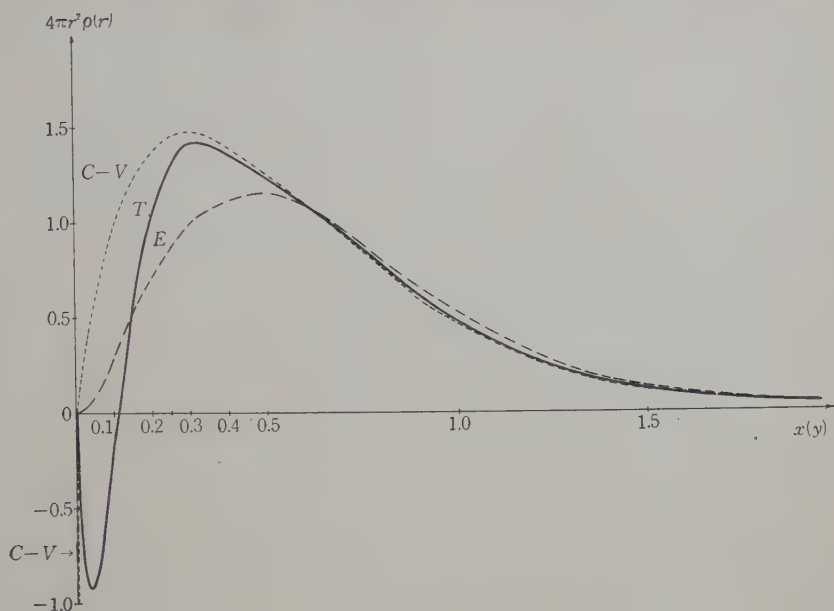


Fig. 2. Charge distributions of the form factors. Solid line means one of a trial model, while broken line and dotted line mean ones of exponential shape and of Clement-Villi shape respectively.

$$\frac{1}{\pi} \int dm^2 m^2 g(m^2) = - \left(\frac{6}{\langle r^2 \rangle \lambda_0 \lambda_1} \right)^2 [(\lambda_0 + \lambda_1) - (\lambda_0^2 + \lambda_0 \lambda_1 + \lambda_1^2)]. \quad (3.13)$$

As the remaining function $F_1(q^2)$ satisfies

$$F_1(0) = 0, \quad dF_1(0)/dq^2 = 0, \quad (3.14)$$

if we add a requirement of (3.5), all effects of $F_1(q^2)$ up to $q^2 \simeq 20y^{-2}$ may be negligible. Therefore even if we would assume

$$F_1(q^2) \simeq 0 \quad \text{for all } q^2 \quad (3.15)$$

as our trial model (3.4), the value of (3.12) (or (3.6)) might retain without change.

Though we also have quantities such as negative charge at origin

$$0.20 \geq \mathcal{L}_- Q(r_0) \geq 0$$

and at the distance where distribution becomes zero

$$r_0 \simeq \left[\frac{\langle r^2 \rangle}{6} \right]^{1/2} \frac{\sqrt{\lambda_0 \lambda_1}}{\sqrt{\lambda_0} - \sqrt{\lambda_1}} \log \left[\frac{1 - \lambda_0}{1 - \lambda_1} \cdot \frac{\lambda_0}{\lambda_1} \right], \quad 0.11 y^{-1} \gtrsim r_0 \gtrsim 0$$

by fully using a trial model, this information depends too sensitively on the details of a model used to be taken as reliable. The general features of this trial model

are shown in Fig. 1 and Table II for its momentum dependence and in Fig. 2 for its corresponding charge distribution. For the sake of comparison, we also raise both extreme cases, i.e., the exponential shape and Clementel-Villi's shape.

The differences between these shapes cannot be detectable at less than $q^2 = 30\text{y}^{-2}$. These differences may appear at about 1 Bev, through the higher order radiative corrections of Rosenbluth's formula, because the main difference will come from the intermediate states with high energy. The detailed estimations are now being done by other members of our group, $S-N$.¹⁸⁾

§ 4. Mass difference of nucleons

Since the structure of extremely short distance would be closely connected with the inherent natures of the elementary particle, the information could be taken by considering the critical problems where the elementary particle reveals its essential characters; such as the problems of mass, and anomalous magnetic moment, and the phenomena of its creation and annihilation.

In order to show this procedure, we take the problem of the electromagnetic mass difference between neutron and proton as an example, because we are in a better position to solve this problem than the others.

For the mass difference of nucleons, we must consider the following three effects which seem to be related with the structure at extreme or very short distances:

- i) the effect of changes of the charge- and moment-form factors of proton and of the moment-form factor of neutron at extremely short distances.
- ii) the effect of the charge-form factor of neutron.
- iii) the effect of the higher-order corrections of the strong interactions which is not included in the previous form factors.

i) *Effect of changes of form factors at extremely short distances*

Let us start with the electromagnetic interactions of nucleon (2.1), by assuming

$$F_{1p}(q^2) = F_{2p}(q^2) = F_{2n}(q^2) \equiv F(q^2), \quad F_{1n}(q^2) = 0. \quad (4.1)$$

The calculations are done by using the second order perturbation in e . There seems to be no modifications if we use a fashioned dispersion technique for this problem. Detailed discussions about the calculations which are based on different standpoints are raised in Appendix I with their results.

Using our trial model (3.4) for $F(q^2)$ with a fixed mean square radius $\langle r^2 \rangle = (0.80\text{y})^2$, we get the mass difference as a function of the field strength of negative distribution at near center $J_0 Q(r_0)/r_0^2$. The result is shown in Fig. 3, where the solid line is the result based on the calculation [a], which we consider to be the most reliable procedure and the dotted line is based on [b], which is not only numerically, but also theoretically less reliable.

It is very clear from this result that the mass difference can be realized in

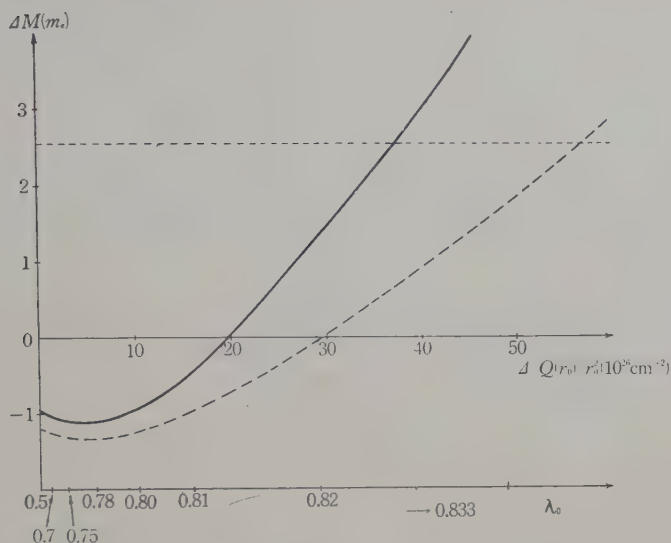


Fig. 3. Mass difference of nucleon due to the change of inner structure. Solid line means result of calculation [a], while dotted line result of calculation [b].

harmony with Stanford's experiments contrary to previous authors.

If the origin of the mass difference would come fully from this effect, we should have the field strength of negative charge at near center of nucleon as

$$\Delta Q(r_0)/r_0^2 \simeq 30 \sim 40 y^{-2}. \quad (4.2)$$

The corresponding values of negative charge amount and distance of its distribution are given by our model as

$$\Delta Q(r_0) \simeq 10 \sim 20\%, \quad r_0 \simeq 0.05 \sim 0.1 y. \quad (4.3)$$

The shape of its distribution has been raised in Fig. 2.

From this result, we can conclude that the exponential shape of form factors with $\langle r^2 \rangle = (0.80y)^2$ is unsuitable. If we want to realize (4.2) by using the exponential-like form factor, we get

$$F(q^2) = \frac{1}{[1 + 0.2 a^2 q^2]^2} - \frac{1.6 a^2 q^2}{[1 + 0.2 a^2 q^2]^2}, \quad (4.4)$$

which cannot fit the Stanford's data because it decreases too slowly at high energy. It is interesting to note that the contribution of the first term to root mean square radius is

$$\langle r_1 \rangle \simeq 0.45 \langle r \rangle \simeq 0.36 y, \quad (4.5)$$

which is just the same value deduced by Hiida-Sawamura.¹⁰⁾

ii) *Effect of the neutron charge-form factor*

The information about the charge-form factor of neutron is rare. However,

we can select the form factor from what is available within the experimental errors so as to give a good contribution to the mass difference. This statement was taken by Cini et al.¹¹⁾ There is no restriction for the choice of this form factor at our present stage.

A speculation which we present here depends on an assumption that the neutron may have a similar structure as the proton at near center, because the presumption that all elementary particles have similar nature at their center would be very reasonable.

We take the charge-form factor of neutron so as to give the value to the field strength at near center,

$$\frac{4-Q(r_0)}{r_0^2} \cong \frac{3}{\langle r^2 \rangle \lambda_0 \lambda_1} [1 - (\lambda_0 + \lambda_1)], \quad \langle r^2 \rangle = (0.80 y)^2, \quad (4.6)$$

then it becomes

$$F_{1n}(q^2) = \frac{1 - (\lambda_0 + \lambda_1)}{\lambda_0 - \lambda_1} \left[\frac{1}{1 + 2\lambda_0 a^2 q^2} - \frac{1}{1 + 2\lambda_1 a^2 q^2} \right], \quad (4.7)$$

where $a^2 = \langle r^2 \rangle / 12$. The mean square radius (if we define as $\langle r_n^2 \rangle = -6dF_{1n}(0)/dq^2$) is given by

$$\langle r_n^2 \rangle = [1 - (\lambda_0 + \lambda_1)] \langle r^2 \rangle. \quad (4.8)$$

Table III. Values of the assumed form factor of neutron charge

q^2	$F_{1n}(q^2) \quad \lambda_0=0.65$
0	0.0000
2	-0.0046
4	-0.0078
6	-0.0099
8	-0.0114
10	-0.0124
12	-0.0131
14	-0.0135
16	-0.0138
18	-0.0139
20	-0.0140
22	-0.0140
24	-0.0139
26	-0.0138
28	-0.0137
30	-0.0135

$$\langle r_n^2 \rangle = 0.02 \langle r^2 \rangle$$

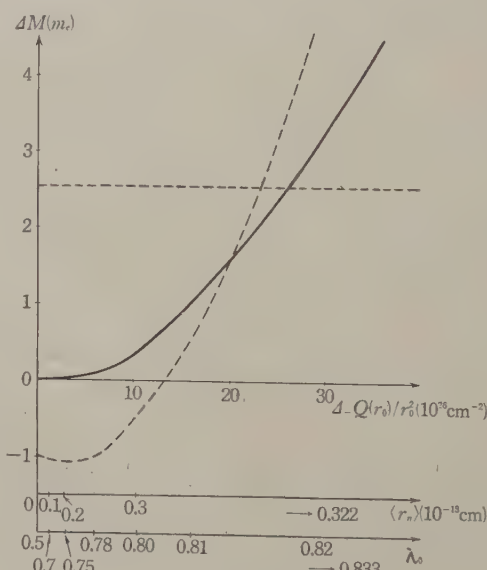


Fig. 4. Mass difference of nucleons due to effect of neutron charge-form factor. Solid line means it, while broken line the total sum of contribution of all form factors.

The effect of this form factor is shown in Fig. 4 with the solid line. The dotted line is the total contribution of i) and ii) by assuming the exactly same value of the field strength $\Delta Q(r_0)/r_0^2$ for both nucleons.

This assumption is based on a somewhat large mean square radius for neutron ($\sqrt{\langle r_n^2 \rangle} \sim 0.30y$), but it is not strictly excluded by the present uncertain result. However, as there is no reason for us to believe that the inner structure of neutron is just the same as that of proton, we can take a smaller value of m.s.r. such as $\langle r_n^2 \rangle \lesssim (0.1y)^2$. Then it is possible for us to conclude

$$\begin{aligned} \Delta Q(r_0)/r_0^2 &\lesssim 1y^{-2} && \text{for neutron charge-form factor,} \\ 20y^{-2} &\lesssim \Delta Q(r_0)/r_0^2 \lesssim 40y^{-2} && \text{for other form factors.} \end{aligned} \quad (4.9)$$

For this case, the effect of the charge-form factor of neutron is very small.

iii) Effect of the higher order corrections of strong interactions

Though the main part of the corrections of strong interactions is taken into account by using the form factors which enter the electromagnetic interactions of nucleons, this is not sufficient, because the pion-nucleon interaction gives at least some contribution, as sketched in Fig. 5, to the mass difference.

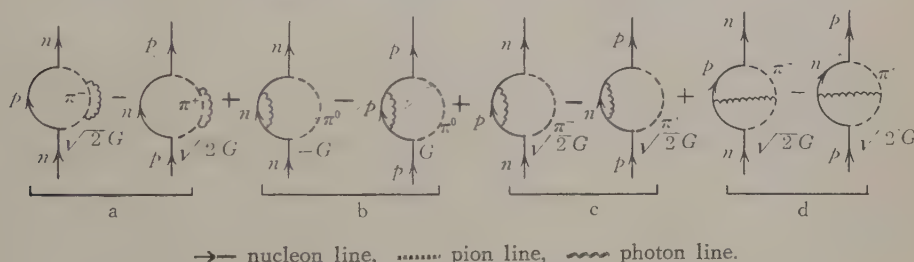


Fig. 5. Contributions due to pion-nucleon interaction which are not included in the form factors

Estimating the main contributions of these diagrams, we have

$$\Delta_a = 0, \quad (4.10)$$

$$\Delta_{b+c} \simeq -4M \frac{d}{dM} \left[\frac{G^2}{4\pi} \frac{M}{4\pi} \log \frac{4\Lambda^2}{M^2} \right], \quad (4.11)$$

$$\Delta_d \simeq \frac{e^2}{4\pi} \frac{G^2}{4\pi} \frac{3}{(8\pi)^2} (\mu_p - \mu_n) M \left[\frac{\Lambda^2}{M^2} \log \frac{4\Lambda^2}{M^2} \right], \quad (4.12)$$

where $4M$ means the mass difference of nucleons and $G^2/4\pi$ and Λ are constants of coupling and cut-off respectively.

As the results give divergent quantities which we cannot exactly evaluate, if we use $G^2/4\pi \simeq 15$ and $\Lambda \simeq M$, we get such large values as

$$\Delta_{b+c} \simeq 1.7 m_e, \quad \Delta_d \simeq 1.8 m_e. \quad (4.13)$$

However, since the perturbation calculations may give unreasonably large values

concerning the nucleon current interactions, as the case of nucleon anomalous moment and may sensitively depend on the cut off used,* it will be safe to estimate this result less than one-tenth of these values, that is,

$$A_{b+c+d} \lesssim 0.4 m_\pi. \quad (4.14)$$

The contributions from the kaon-nucleon interaction would be negligible owing to its relatively small coupling.¹²⁾

Even if we can presume these contributions to be small, there is no reason to conclude that the higher-order corrections are completely negligible. This uncertainty may affect our presumption about the inner structure of nucleon.

Summarizing the above effects, we can conclude as follows: If we take the following standpoint:

i) the effect of the corrections of strong interactions seems to be negligible or at most small.

ii) the effect of the neutron charge-form factor may be negligible or at most in the same order as the other form factors; the inner structure must give at least a non-zero value to the field strength of negative distribution at near center

$$\frac{A_- Q(r_0)}{r_0^2} > 0. \quad (4.15)$$

But more detailed estimations of the corrections of strong interactions are necessary. The effect of the charge form factor of neutron might be changed by a different standpoint, but it seems to be reasonable to take generally as

$$A_- Q(r_0)/r_0^2 \simeq 0. \quad (4.16)$$

(Schiff's model of neutron charge-form factor¹⁹⁾ gives zero.)

§ 5. Electromagnetic mass difference of other elementary particles

It is interesting to investigate the situations of the electromagnetic mass difference of other elementary particles in accordance with our method presented here. Since the detailed accounts will appear later, we only briefly sketch them.

i) Mass difference between hyperon-sigma triplets

Our method can also apply to this problem without any modification.²⁰⁾ In order to get the experimental mass difference $M_{\Sigma^-} - M_{\Sigma^0} \simeq 5 \text{ Mev}$, $M_{\Sigma^-} - M_{\Sigma^+} \simeq 7 \text{ Mev}$, we modify only the amounts of inner distribution assuming that the outer distribution is exactly similar to nucleons. As the information about these particles are very scarce, we must consider the following possibilities;

- i) Case of $\mu_- > 0$, $\mu_0 > 0$,
- ii) Case of $\mu_- > 0$, $\mu_0 < 0$,
- iii) Case of $\mu_- < 0$, $\mu_0 > 0$,

* A_{b+c} may become negative when the cut off increases.

iv) Case of $\mu_- < 0$, $\mu_0 < 0$,

where μ_- and μ_0 are anomalous magnetic moments of Σ^- and Σ^0 respectively. For cases of i) and ii) we can realize the experimental values only by assuming the same structure for these triplets with a restriction of $|\mu_0| < 1$. While for the case of iii) the same result is obtained by using the different structure for Σ^0 and for the case of iv) by using different structures for all triplets.

ii) *Mass differences between pions and between kaons*

For the boson case, as we have not yet such reliable information as we have about nucleon, we only take the simple form factor such as Yukawa's or exponential's.²¹⁾ In order to deduce the pion mass difference, it is sufficient to take the following hamiltonian,

$$H_\pi = ie(\phi^* \partial_\mu \phi - \partial_\mu \phi^* \cdot \phi) A_\mu G_1 + e^2 \phi^* \phi A_\mu^2 G_1^2, \quad (5.1)$$

where ϕ , ϕ^* and A_μ are field quantities of pion and photon respectively. For the sake of simplicity we omit the symbol of integration and the reader may understand it schematically. If we use the Yukawa shape for form factor G_1 , we can realize the experimental value by taking $\langle r^2 \rangle \simeq (0.5y)^2$.

On the contrary, the mass difference of kaon which has recently been reported is somewhat inadequate from this standpoint, because a similar interaction such as (5.1) cannot explain its mass difference $M_{K^0} - M_{K^+} \simeq 4.8$ Mev. Then we are obliged to introduce the following interactions;

$$H_{K^0} = ie(\phi^* \cdot \partial_\mu \phi - \partial_\mu \phi^* \cdot \phi) A_\mu G_1 + e^2 \phi^* \phi A_\mu^2 G_1^2 + \frac{ie\kappa}{m^2} (\partial_\mu \phi^* \cdot \partial_\nu \phi - \partial_\nu \phi^* \cdot \partial_\mu \phi) F_{\mu\nu} G_2. \quad (5.2)$$

The last new term which just corresponds to Pauli's terms of nucleons were once proposed by Nakano and Miyachi and it is justified in accordance with the nucleon Pauli terms. We can easily show that the introduction of this term is equivalent to the modification of the inner structure. But this is not sufficient for our problem, because we still have results in the wrong signs. The essential change we propose here is to introduce another interaction for neutral kaon such as

$$H_{K^0 \text{ neut}} = -\frac{ie\kappa_0}{m^2} (\partial_\mu \phi_0^* \cdot \partial_\nu \phi_0 - \partial_\nu \phi_0^* \cdot \partial_\mu \phi_0) F_{\mu\nu} G_2, \quad (5.3)$$

where ϕ_0 and ϕ_0^* are the field quantities of the neutral kaon. The reason for this possibility depends on essential different natures between kaon and pion, because we must distinguish the neutral kaon from the neutral anti-kaon, while we identify the neutral pion with its anti-pion. If we assume the form factor G_2 as exponential shape with $\langle r^2 \rangle \simeq (0.5y)^2$, we can explain the kaon mass difference by taking $\kappa \simeq \kappa_0 \sim 5$.

iii) *Mass differences between leptons*

Mass differences between neutrino, electron and muon are the most curious facts in this branch. The difference between neutrino and electron can be explained

by non-local modification of the electromagnetic interaction or by weak interactions as clarified collectively by Katayama, Taketani and Ferreira.²²⁾ The problem we must now consider is to find a way for explaining the mass difference between electron and muon.

If we start with massless leptons, the only way to explain their mass differences is to introduce Pauli's terms with different structures besides the usual minimal electromagnetic interaction.²³⁾ There is a reason for introducing such terms, because of the discrepancies between the experimental values of the anomalous magnetic moments and their theoretical values for both muon and electron. The discrepancies may be:

$$(\mu_{\text{exp}} - \mu_{\text{th}})e/2m_e \simeq 10^{-6} e/2m_e \quad \text{for electron,}$$

and

$$(\mu_{\text{exp}} - \mu_{\text{th}})e/2m_\mu \simeq 10^{-4} e/2m_\mu \simeq 10^{-6} e/2m_e \quad \text{for muon.}$$

This fact may compel us to introduce new Pauli terms with a universal constant $10^{-6}e/2m_e$. Then the main reason for the mass difference between muon and electron is attributed to the difference of their inner structure. Unfortunately, the explanation of this problem cannot be restricted to the form factor. It will be necessary to consider other phenomena which might be related to it.

§ 6. Discussions

The analysis presented here may become a help in understanding the structures of all elementary particles. As the clear determinations of the inner structures are achieved not only experimentally, but also by field theory, we must consider them by the semi-phenomenological way proposed here. Though our analysis is devoted only to the mass difference problem, a similar analysis must be accomplished for the other phenomena.

It seems to be very probable that we arrive at some proper concepts of the essential structure which we can understand as the core of elementary particle. Sooner or later we shall be compelled to construct the theory of structure. Then it is not vain to presume here the general features of this attempt.

Some existential standpoints of the structure of elementary particle have been taken from the non-local field theory²⁴⁾ and the non-linear field theory.²⁵⁾ Unfortunately, we have not succeeded in constructing the dynamical theory of the structure itself, though the non-local field theory was based on the existence of the elementary particle structure and though the non-linear field theory could verify it classically.^{26),27)} If we should try to formulate them along the line of the quantum theory, we might be confronted by many difficulties; for instance, the necessity of indefinite metric,²⁸⁾ the violation of causality,²⁹⁾ the non-existence of displacement operator,³⁰⁾ and so on.

However, it is very doubtful whether we can consider the elementary particle

structure quantum-theoretically, because we cannot accept its partial creation or annihilation.

The only applicable object of quantum theory is the structure as a whole and at the inner region of the structure we may have completely different mechanics. Therefore it is probable that at this region we may be obliged to throw away some parts of observable quantities (which usually are acceptable) in the outer region as meaningless. But we hope there still exist some observable quantities applicable in the inner region as similar as in the outer region. For instance, this paper has shown that the concept of field strength may a more important role in this region than the concepts of distribution and potential. In the inner region of structure, the weight of each point or the space-time order of each part would be less meaningful and in their stead only effects of structure as a whole might be important.

The mechanics governed in this region would be those of the unquantized field or of the anti-quantized field which may play a role of cut off on the usual quantization at short distance. There is no necessity, neither for the definiteness of metric nor for causality. The mechanics would be very similar to the ones of collective movement or of fluid instead of particles. Attempts to this approach must be concentrated to the question of how we can localize many difficulties appearing in the present stage to this region in order to construct the new mechanics. This question seems to be very closely connected with the modification of space-time structure in our elementary particle world.

Acknowledgement

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Appendix I. Calculations of mass difference of nucleons

For the mass difference, our calculations are based on the second order perturbation of the electromagnetic interactions of nucleons which contain the form factors. Concerning the electromagnetic interactions this procedure would give good results, and even if we would use a dispersion technique, the result might not be improved so far as the lowest configuration approximation is concerned. Unfortunately, however, we have no satisfactory theory including the so-called non-local interaction. Judging from this situation we are obliged to take the three following standpoints tentatively, one of which will be verified in the near future.

- a) *Case of the form factors which depend only on the square of spatial momentum at the rest system of nucleon*

In this case, we define our form factors as

$$F(q^2) \equiv F(\mathbf{q}^2) = \frac{1}{\pi} \int dm^2 \frac{g(m^2)}{q^2 + m^2} \quad (\text{A} \cdot 1)$$

with real spectral function $g(m^2)$. Though the result becomes convergent by this form factor, there is still a question about the gauge invariance at short distance where the structure plays an important role. The relativistic property may be satisfied if we take its argument as

$$\pi^2 = q^2 - \frac{(pq)^2}{p^2}, \quad (\text{A} \cdot 2)$$

where p is the four-momentum of nucleon, because it becomes \vec{q}^2 at the rest system of nucleon.

The result was already calculated by Hiida and Sawamura, which are given as

$$\begin{aligned} \delta M_{11} &= -\frac{e^2}{4\pi} \frac{M}{\pi} \left[\frac{1}{2} L^{(0)} - I^{(0)} - I^{(1)} \right], \\ \delta M_{12} &= -\frac{e^2}{4\pi} \frac{3\mu M}{\pi} I^{(1)}, \\ \delta M_{22} &= -\frac{e^2}{4\pi} \frac{5\mu^2 M}{4\pi} I^{(1)}, \end{aligned} \quad (\text{A} \cdot 3)$$

where δM_{11} , δM_{12} and δM_{22} mean the self-energies due to the charge-charge, charge-moment, and moment-moment interactions, and coefficients will be tabulated later.

b) *Case of the form factors which are defined regular on the complex momentum plane*

We introduce the form factors

$$F(q^2) = \frac{1}{\pi} \int dm^2 \frac{g(m^2)}{q^2 + m^2 - i\epsilon}; \quad g(m^2) \text{ real}, \quad (\text{A} \cdot 4)$$

which correspond exactly to Feynman's relativistic cut-off procedure. However, if we intend to make a consistent theory of this procedure, we are compelled to introduce some kind of indefinite metric in our formulation, because our hamiltonian becomes non-hermitic due to the complex character of the form factor. This would be a fatal nature to the non-local interaction theory in order to get the finite result without destroying the gauge invariant requirement.

The calculation was done by Cini et al., and they got

$$\begin{aligned} \delta M_{11} &= -\frac{e^2}{4\pi} \frac{M}{8\pi} [L^{(1)} + 16W^{(0)} + 4W^{(1)} - 2W^{(2)}], \\ \delta M_{12} &= -\frac{e^2}{4\pi} \frac{3\mu M}{8\pi} [L^{(1)} + 8W^{(1)} - 2W^{(2)}], \\ \delta M_{22} &= -\frac{e^2}{4\pi} \frac{\mu^2 M}{64\pi} [-2D^{(0)} + 6L^{(1)} + L^{(2)} + 64W^{(1)} - 8W^{(2)} - 2W^{(3)}], \end{aligned} \quad (\text{A} \cdot 5)$$

with coefficients which will be tabulated later.

c) *Case of the form factors which are defined only on the real plane*

In order to maintain the relativistic and gauge invariant requirements, we also take the form factors as

$$F(q^2) = \frac{1}{\pi} P \int dm^2 \frac{g(m^2)}{q^2 + m^2}, \quad (\text{A} \cdot 6)$$

where we define it by principal value at $q^2 = -m^2$. Though this procedure seems to be more satisfactory than from a formal view-point, unfortunately we cannot get any finite result irrespective of the shape of form factors, because of the incompleteness of cancellations of divergent terms. We cannot get any counter term which cancels out the fluctuation energy of photon. Then we only raise the result without discussing their numerical values,

$$\begin{aligned} \delta M_{11} &= -\frac{e^2}{4\pi} \frac{\Lambda I}{8\pi} [4K_\infty^2 + 4 \log 2K_\infty \cdot D^{(0)} + 2L^{(0)} + 3L^{(1)} + 32\tilde{W}^{(0)} + 8\tilde{W}^{(1)} - 4\tilde{W}^{(2)}], \\ \delta M_{12} &= -\frac{e^2}{4\pi} \frac{\mu M}{4\pi} [\log 2K_\infty \cdot D^{(0)} + 8\tilde{W}^{(1)} - 2\tilde{W}^{(3)}], \\ \delta M_{22} &= -\frac{e^2}{4\pi} \frac{\mu^2 M}{16\pi} [\log 2K_\infty \cdot (3D^{(0)} + D^{(1)}) + 32\tilde{W}^{(1)} - 4\tilde{W}^{(2)} - \tilde{W}^{(3)}], \end{aligned} \quad (\text{A} \cdot 7)$$

where K_∞ means dimensionless divergent quantity.

All coefficients which appear in (A·3), (A·5) are expressed by the following 15 quantities :

$$\begin{aligned} D^{(\alpha)} &= \frac{1}{\pi^2} \int d\lambda d\lambda' g(\lambda) g(\lambda') \frac{1}{2} (\lambda^\alpha + \lambda'^\alpha), \quad \alpha = 0, 1, \\ L^{(\alpha)} &= \frac{1}{\pi^2} \int d\lambda d\lambda' g(\lambda) g(\lambda') \frac{\lambda^\alpha \log \lambda - \lambda'^\alpha \log \lambda'}{\lambda - \lambda'}, \quad \alpha = 0, 1, 2, \\ I^{(\alpha)} &= \frac{1}{\pi^2} \int d\lambda d\lambda' g(\lambda) g(\lambda') \frac{\lambda^\alpha i(\lambda) - \lambda'^\alpha i(\lambda')}{\lambda - \lambda'}, \quad \alpha = 0, 1, \\ W^{(\alpha)} &= \frac{1}{\pi^2} \int d\lambda d\lambda' g(\lambda) g(\lambda') \frac{\lambda^\alpha \omega(\lambda) - \lambda'^\alpha \omega(\lambda')}{\lambda - \lambda'}, \quad \alpha = 0, 1, 2, 3, \\ \tilde{W}^{(\alpha)} &= \frac{1}{\pi^2} \int d\lambda d\lambda' g(\lambda) g(\lambda') \frac{\lambda^\alpha \tilde{\omega}(\lambda) - \lambda'^\alpha \tilde{\omega}(\lambda')}{\lambda - \lambda'}, \quad \alpha = 0, 1, 2, 3, \end{aligned} \quad (\text{A} \cdot 8)$$

where

$$\begin{aligned} i(\lambda) &= 4\omega(4\lambda), \\ \omega(\lambda) &= \begin{cases} \frac{1}{\sqrt{\lambda(4-\lambda)}} \tan^{-1} \sqrt{\frac{4-\lambda}{\lambda}} & \text{for } 4 > \lambda, \\ \frac{1}{\sqrt{\lambda(\lambda-4)}} \log \frac{\sqrt{\lambda} + \sqrt{\lambda-4}}{2} & \text{for } 4 < \lambda \end{cases} \end{aligned} \quad (\text{A} \cdot 9)$$

and

$$\tilde{\omega}(\lambda) = \begin{cases} \frac{1}{\sqrt{\lambda(4-\lambda)}} \left[\tan^{-1} \sqrt{\frac{4-\lambda}{\lambda}} - \frac{\pi}{4} \right] & \text{for } 4 > \lambda, \\ \frac{1}{\sqrt{\lambda(\lambda-4)}} \log \frac{\sqrt{\lambda} + \sqrt{\lambda-4}}{2} & \text{for } 4 < \lambda. \end{cases}$$

In these expressions, we take $\lambda = m^2/M^2$.

The weight function for our trial model is given by

$$\frac{1}{\pi} g(\lambda) = \frac{1-\lambda_1}{\lambda_0-\lambda_1} \frac{1}{2\lambda_0 M^2 a^2} \delta\left(\lambda - \frac{1}{2\lambda_0 M^2 a^2}\right) - \frac{1-\lambda_0}{\lambda_0-\lambda_1} \frac{1}{2\lambda_1 M^2 a^2} \delta\left(\lambda - \frac{1}{2\lambda_1 M^2 a^2}\right). \quad (\text{A} \cdot 10)$$

Appendix II. A trial model of superposition of singular Yukawa shapes

The general arguments described in this text cannot exactly apply to the case of singular Yukawa shape of form factor, because it decreases weakly at $q^2 \rightarrow \infty$. Then the discussions of (2.17)–(2.21) do not hold. In order to apply them, it is necessary to enforce the damping character of form factor by using some device. Therefore for the superposition of singular Yukawa shapes, we need more parameters than those of the Yukawa shape.

Let us take the superposition of three singular Yukawa shapes

$$F(q^2) = \sum_{i=1}^3 \frac{C_i}{qb_i} \tan^{-1} qb_i, \quad (\text{A} \cdot 11)$$

where $b_i^2 = \lambda_i \langle r^2 \rangle / 2$. Then the required condition which we propose here is

$$\sum_{i=1}^3 \frac{C_i}{b_i} = 0, \quad (\text{A} \cdot 12)$$

which can remove the singularity of the field strength of the negative distribution at near center.

The partial amount of distribution and the field strength are given by

$$Q(r) = \sum_{i=1}^3 C_i e^{-(r/b_i)},$$

$$-\delta E(r_0) = \frac{-\Delta Q(r_0)}{r_0^2} = \frac{1}{\langle r^2 \rangle} \sum_{i=1}^3 \frac{C_i}{\lambda_i}. \quad (\text{A} \cdot 13)$$

The requirements of [A], [B] and [C] are expressed as

$$\sum_{i=1}^3 C_i = 1,$$

$$\sum_{i=1}^3 \lambda_i C_i = 1, \quad (\text{A} \cdot 14)$$

$$\sum_{i=1}^3 \frac{\tan^{-1}(6\lambda_i)^{1/2}}{(6\lambda_i)^{1/2}} C_i = \frac{1}{4}.$$

Even if we use (A.12), (A.13) and (A.14) for adjusting parameters, there still remains one parameter to be free, which seems to be useful in getting other information.

A trial model proposed here is very similar to those stated in this text. In fact, as this model becomes

$$F(q^2) = C_1 \frac{1}{qb_1} \tan^{-1} \frac{(b_1 - b_3)q}{1 + b_1 b_3 q^2} + C_2 \frac{1}{qb_2} \tan^{-1} \frac{(b_2 - b_3)q}{1 + b_2 b_3 q^2} \quad (\text{A.15})$$

in virtue of (A.12), we can approximate it as

$$F(q^2) \simeq C_1 \frac{b_1 - b_3}{b_1} \frac{1}{1 + b_1 b_3 q^2} + C_2 \frac{b_2 - b_3}{b_2} \frac{1}{1 + b_2 b_3 q^2}, \quad (\text{A.16})$$

which is nothing else but a trial model as stated previously.

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Mass Difference between the Sigma Hyperons

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The problem of the mass differences of the triplet sigma is discussed within the general standpoint of the electromagnetic structures of elementary particles.

As the present data about these particles are very poor, we consider many possibilities for the values of the anomalous magnetic moments, especially those which according to meson theory have a reasonable magnitude and sign. Assuming the external distributions of charge and anomalous magnetic moments, as well as the magnitudes of these moments, the inner structures are guessed in order to explain the mass differences of the triplet.

Effects of higher order corrections due to strong interactions are also discussed.

§ 1. Introduction

In the Gell-Mann-Nishijima scheme, the hyperon Σ is considered as a triplet state which consists of Σ^+ , Σ^- , Σ^0 ; these component particles have somewhat large mass differences as follows:*

$$\begin{aligned}\delta M_{-+} &= M_{\Sigma^-} - M_{\Sigma^+} = 6.84 \pm 0.40 \text{ Mev}, \\ \delta M_{-0} &= M_{\Sigma^-} - M_{\Sigma^0} = 4.45 \pm 0.63 \text{ Mev}.\end{aligned}\tag{1}$$

In a similar way as for the neutron-proton mass difference, which was considered by Feynman and Speisman,¹⁾ many authors tried to explain the above mass differences by assuming that they are due to electromagnetic interactions. In this way Marshak et al.²⁾, Katsumori³⁾, by the use of cut-off *à la* Feynman could explain the mass differences δM_{-+} and δM_{-0} . For doing this, they have used extremely high values for the anomalous magnetic moments (a.m.m.) of the sigmas, what does not seem reasonable. Indeed, Marshak et al.²⁾ has shown that if we do not want an a.m.m. of Σ^+ and Σ^- greater in absolute value than 4, in hyperon magnetons, then μ_0 , the a.m.m. of Σ^0 , must be greater than 1.5. On the other hand, as for the moment μ_0 the pions do not contribute, while the kaons do, we need a coupling constant $g_K^2/4\pi \sim 3$, which is higher than the value generally accepted. And more, μ_- the a.m.m. of Σ^- , is always assumed positive, what is not clear from the standpoint of meson theory.

As is well known, perturbation theory gives wrong values for the a.m.m. of

* Latest data of sigma's mass-differences (July 1959) are

$\delta M_{-+} = 6.76 \pm 0.33 \text{ Mev}$, $\delta M_{-0} = 4.45 \pm 0.4 \text{ Mev}$.

the nucleons, but if we neglect the nucleon current and take into account only the pion current, then we can obtain good results. By analogy we expect that for the determination of the anomalous moments μ_+ and μ_- of Σ^+ and Σ^- , a good result can be obtained if we use a perturbation treatment where the hyperon current is neglected. Therefore we should get for μ_+ a positive value of the order of the proton's a.m.m. and for μ_- a negative one of the order of neutron's a.m.m. This situation has not been examined by the previous authors.

Similar considerations were made by Kato and Takeda.⁴⁾ They calculated δM_{-} by supposing that Σ^- and Σ^+ have charge and a.m.m. distributions given by an exponential form factor, with a root mean square (r.m.s.) radius equal to 0.8 yukawa (1 yukawa = 10^{-13} cm), which is taken from Stanford experiments on $e-N$ scattering, and also assuming some reasonable a.m.m. values such as $\mu_0 = 0.08$ and $\mu_- = -1.74$. Their result was $\delta M_{-} \sim 0.16 m_e$ which is a rather small value compared with the experimental one.

Bransden and Moorhouse⁵⁾ suggested that the relatively high value of δM_{+} could be due to the appreciable mass difference between K^0 and K^+ . On these lines they have calculated δM_{+} by perturbation theory, assuming $\langle j_{\Sigma N K}^2 \rangle / 4\pi = 4$. This assumption however is not consistent with the value $\langle j_{K}^2 \rangle / 4\pi \sim 0.3$ deduced from the cross sections of $K^- - p$ scattering at energies of 100 Mev.⁶⁾ For this reason we do not believe that the mass-difference of the intermediate kaons could give such an appreciable contribution to the δM_{+} and δM_0 mass differences.

Recently Hiida-Sawamura⁷⁾ and Cini et al.⁸⁾ tried to explain the mass difference between neutron and proton by the use of exponential form factors, for the nucleons, obtained from Stanford experiments, and they got a negative result. Nevertheless, by making a critical analysis of the results of Stanford experiments, Katayama, Taketani and co-workers⁹⁾ (this paper is hereafter denoted by [A]) have shown that it is possible to obtain a correct neutron-proton mass difference by a convenient modification of the form factors in the region of high energies, where experimental data are not yet available. They have taken, as a possibility, form factors which are linear combinations of two Yukawa functions, which, within experimental errors, can explain the Stanford results in the range of energies used up to now. The form factors, however, differ appreciably from the generally accepted exponential form factor for higher energies.

The purpose of the present article is to explain the mass differences between the hyperons sigma, by the introduction of suitable electromagnetic structures for these particles and consideration of different sets of values for the anomalous magnetic moments, especially those which, in analogy to the nucleon case, without having a very high absolute value, have a convenient sign. This possibility was not developed successfully by the previous authors.^{2), 3), 4)} Assuming that Σ^+ and Σ^- have similar structures as the nucleons, in the region of low momentum transfer studied in the Stanford experiments, and examining some possibilities for Σ^0 , we can determine the inner structures in such a way as to explain the observed

δM_{-+} and δM_{-0} mass differences.

According to some authors,¹⁰⁾ the principle of charge independence in strong interactions which apparently holds in a certain level corresponding to great distances, could break down at short distances. Taketani suggested that this breakdown of the principle of charge independence at high energies could be responsible for the appreciable mass difference between the various components of each multiplet in the Gell-Mann-Nishijima scheme. However, even if there are some small fluctuations in the charge independence of strong interactions, these could be properly taken into account, in a phenomenological way, in the electromagnetic form factors which describe the inner structure of elementary particles.

§ 2. Effects of change of the form factors at high energy region

In what follows, we shall use the following hamiltonian for the interaction of the hyperons, $\bar{\psi}(x)$, $\psi(x)$ with the electromagnetic field $A_\mu(x)$, $F_{\mu\nu}(x)$:

$$H = H_1 + H_2 + H_3 + H_4 + H_5 \quad (2)$$

where:

$$\begin{aligned} H_1 &= -ie \int dx dy \bar{\psi}(x) \gamma_\mu \psi(x) F_1(x-y) A_\mu(y), \\ H_2 &= -\mu \frac{e}{4M} \int dx dy \bar{\psi}(x) \sigma_{\mu\nu} \psi(x) F_2(x-y) F_{\mu\nu}(y), \\ H_3 &= \frac{e^2}{M^2} \int dx dy dz d\omega \bar{\psi}(x) \psi(y) F_3(xyz\omega) F_{\mu\nu}(z) F_{\mu\nu}(\omega), \\ H_4 &= \frac{ie^2}{M^2} \int dx dy dz d\omega \bar{\psi}(x) \gamma_5 [F_{12}(z) F_{34}(\omega) + \dots] \psi(y) F_4(xyz\omega), \\ H_5 &= \frac{ie^2}{M^2} \int dx dy dz d\omega \bar{\psi}(x) \gamma_i \gamma_m \gamma_n \psi(y) F_{is} \frac{\partial F_{mn}}{\partial \omega_s} F_5(xyz\omega). \end{aligned} \quad (3)$$

μ is the a.m.m. and M is the mass of the hyperon. Although the theory is based in a non-local interaction which does not satisfies the requirements of microcausality, it is possible that (2) could be derived from a more fundamental theory where the causality principle holds.

Neglecting H_3 , H_4 and H_5 , whose contributions we shall discuss in § 3, we obtain for the self-mass of the hyperon up to e^2 :

$$\delta M = \delta M_{11} + \delta M_{12} + \delta M_{22} \quad (4)$$

where:

$$\delta M_{11} = \frac{2ie^2}{(2\pi)^4} \int F_1^2(q^2) \frac{iq + M}{-2pq + q^2} \frac{1}{q^2} d^4q, \quad (5)$$

$$\delta M_{12} = \frac{3ie^2}{(2\pi)^4} \frac{\mu}{M} \int \frac{F_1(q^2) F_2(q^2)}{-2pq + q^2} d^4q, \quad (6)$$

$$\delta M_{22} = \frac{ie^2}{(2\pi)^4} \frac{\mu^2}{4M^2} \int \frac{F_2^2(q^2) [4ipqq + 4Mq^2 - 3iq^2q]}{[-2pq + q^2]q^2} d^4q \quad (7)$$

with $q^2 = \vec{q}^2 - q_0^2$ and $q = \gamma \cdot q$.

Calculations of Hiida-Sawamura,⁷⁾ Cini et al.⁸⁾ have shown that with the exponential form factor we cannot explain the positive value of the mass difference between neutron and proton. However, according to [A] this would mean that, at high momentum transfers the electromagnetic form factors should deviate appreciably from the exponential one, while for low momenta they would practically coincide. It is well known that, for example, the Villi-Clementel¹¹⁾ model nicely explains the results of Stanford experiments up to now. In this model, the proton consists of a negative point charge at its center surrounded by a positively charged cloud of Yukawa type. Deviations from the exponential form factor would occur, to give an example, for electrons colliding with protons with an energy 1.1 Bev, for which energy we would have a zero diffraction at 140°.

Katayama, Taketani and co-workers⁹⁾ have considered the problem of the inner structure of the nucleons from a more general standpoint. They have taken a certain class of form factors which explain all available experimental data for $e-N$ scattering (which correspond to distances greater than 0.5y) and tried to determine the inner structure by attacking the problem of neutron-proton mass difference.

The form factor of the nucleon must satisfy the following conditions:

- 1) The total charge must be equal to one, i.e. $F(0) = 1$.
- 2) The r.m.s. radius must be given by $\langle r^2 \rangle = (0.80y)^2$.
- 3) For $r \geq 0.5y$ it should give more or less 60% of the total charge. This means that the Stanford experiments give only information about the amount of charge for $r \geq 0.5y$, but nothing cannot say about the details of inner parts.

A sufficiently general choice, satisfying 1) and 2), is the superposition of two Yukawa form factors:

$$F(q^2) = \frac{1 - \lambda_1}{\lambda_0 - \lambda_1} \frac{1}{1 + (\lambda_0/A)(q^2/M^2)} - \frac{1 - \lambda_0}{\lambda_0 - \lambda_1} \frac{1}{1 + (\lambda_1/A)(q^2/M^2)} \quad (8)$$

where $A = 6/\langle r^2 \rangle M^2$, λ_0 and λ_1 being dimensionless parameters. Condition 3) reads:

$$\lambda_0 \lambda_1 - \frac{3}{2}(\lambda_0 + \lambda_1) + \frac{5}{4} = 0, \quad (9)$$

λ_0 lying between 1/2 and 5/6; for $\lambda_0 = \lambda_1 = 1/2$ (8) reduces to the exponential form factor and $\lambda_0 = 5/6$, $\lambda_1 = 0$ to the Villi-Clementel's.

As, according to the pion theory, Σ^+ and Σ^- have external charge and a.m.m. distributions very similar to the nucleons, it is natural to extend to the sigmas, in the low energy region, the form factors which were obtained in Stanford experiments for the nucleons, with a r.m.s. radius of the same order, i.e. 0.8y.

The internal region could then be guessed by the consideration of the mass differences of the sigmas.

Therefore, describing the charge and a.m.m. distributions of Σ^- and Σ^+ by the form factor (8) with the requirement (9), Σ^- will correspond to a positively charged core surrounded by a negatively charged cloud, while the opposite would occur for Σ^+ . The parameter λ_0 (or λ_1) gives the amount of charge in the core, as well as its radius, and therefore informs us about the inner region. For the Σ^0 we can suppose it to have only an a.m.m. distribution. No charge distribution is considered for this particle.

We shall restrict ourselves to the consideration of 3-dimensional form factors, i.e. $F(q^2) = F(\vec{q}^2)$. Under such circumstances, using Eqs. (4) ~ (7), we get the following expressions for the mass differences:

$$\delta M_{-+} = \frac{e^2}{4\pi} \frac{M}{\pi} \left[\frac{1}{2} L_{-}^{(0)} - I_{-}^{(0)} - I_{-}^{(1)} - \frac{1}{2} L_{+}^{(0)} + I_{+}^{(0)} + I_{+}^{(1)} + 3(\mu_{-} I_{-}^{(1)} + \mu_{+} I_{+}^{(1)}) - \frac{5}{4}(\mu_{-}^2 I_{-}^{(1)} - \mu_{+}^2 I_{+}^{(1)}) \right], \quad (10)$$

$$\delta M_{-0} = \frac{e^2}{4\pi} \frac{M}{\pi} \left[\frac{1}{2} L_{-}^{(0)} - I_{-}^{(0)} + (3\mu_{-} - 1.25\mu_{-}^2 - 1) I_{-}^{(1)} + 1.25\mu_{-}^2 I_{-}^{(1)} \right], \quad (11)$$

where $L^{(0)}$, $I^{(0)}$, $I^{(1)}$ for the form factor (8) with condition (9) are well defined functions of λ_0 ; They are given in the Appendix. We also give the expression for $I^{(1)}$ which corresponds to the exponential form factor. The lower indices $-$, $+$ and 0 refer to the Σ^- , Σ^+ and Σ^0 particles respectively.

As the present experimental information about the sigmas is very poor, many possibilities exist as far as the structure and values of anomalous magnetic moments are concerned. We shall consider here only a few instances of structures which can explain the observed mass differences δM_{-+} and δM_{-0} , and limiting ourselves to the case $|\mu_0| < 1$. This last choice is connected to the restrictions imposed by meson theory. We shall then stress the possibility of a negative μ_{-} , with a value of the order of -2 , which is the interesting case from the standpoint of pion theory, as was discussed in the introduction. On the other hand, we shall assume that Σ^+ and Σ^- have the same structure as the nucleons given by Eqs. (8) and (9), with the same r.m.s. radius.*

Case I. Same structure for Σ^+ , Σ^- and Σ^0

The structure will be described by the form factors (8) and (9) with the same λ_0 for all the three Σ . Therefore the charge distributions of Σ^- and Σ^+ are perfectly symmetrical. In this case, δM_{-+} is given by

* Really in this article we have used a value for the r.m.s. radius of Σ somewhat smaller than 0.8y. But the general conclusions will not be affected. We would like to express our thanks to Professors Y. Katayama, M. Taketani and Messrs. D. R. de Oliveira, S. Ragusa for putting at our disposal their numerical data on the neutron-proton mass difference.

$$\delta M_{-+} = (e^2/4\pi) 2\mu_0 I^{(1)} [3 - 2,5(\mu_- - \mu_0)], \quad (12)$$

using (10), where we have taken $2\mu_0 = \mu_+ + \mu_-$, the functions $L^{(0)}$, $I^{(0)}$ and $I^{(1)}$ having the same value for all the three particles.

Now, as $I_-^{(1)}$ is a positive definite quantity, in order to obtain $\delta M_{-+} > 0$ we must have $\mu_- > 1.2 + \mu_0$ for $\mu_0 < 0$ and $\mu_- < 1.2 + \mu_0$ for $\mu_0 > 0$. We have studied

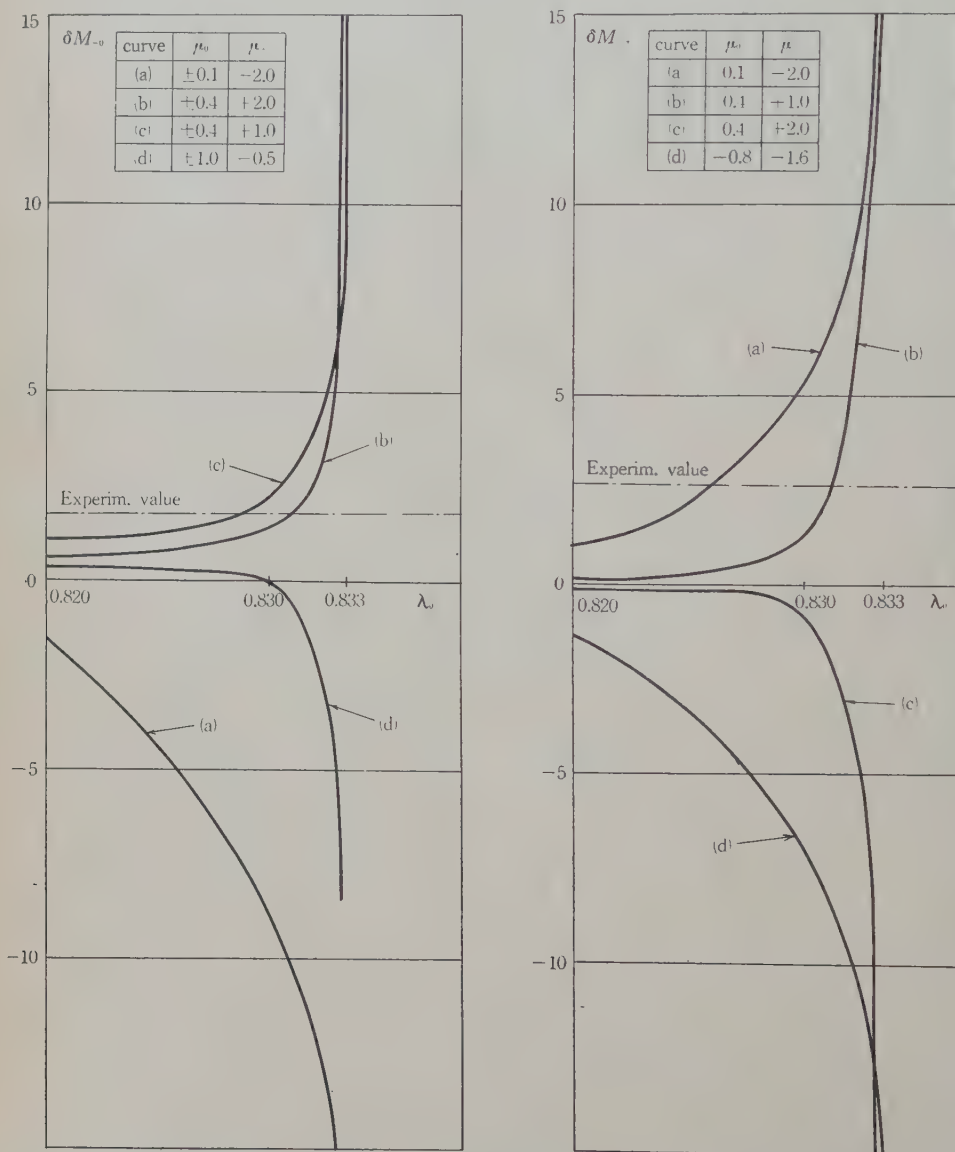


Fig. 1. Plotting of δM_{-+} and δM_{-0} as functions of the internal parameter λ_0 , in units of $e^2/4\pi \times M/\pi = 1/137 M/\pi$. For the δM_{-+} case, the choice $\mu_0 = -0.8$, $\mu_- = 2$ gives a curve very similar to (a).

graphically the behaviour of δM_{-+} and δM_{-0} as a function of the internal parameter λ_0 , for different sets of values of μ_0 and μ_- , especially those which correspond to $|\mu_0| \lesssim 1$. In Fig. 1 we give some of them. It is not possible to realize $\delta M_{-0} > 0$ with $|\mu_0| \leq 1$ and μ_- negative with an absolute value greater than 0.5. For instance, the possibility $|\mu_0| \lesssim 1$ and $\mu_- \sim -2$ which seems interesting from the point of view of meson theory, cannot explain the positive value of δM_{-0} , while for realizing the experimental δM_{-+} we must have $\mu_0 > 0$.

Case II. Different structure for Σ^0

As in the case I, we could not explain δM_{-0} by assuming the possibility $|\mu_0| \leq 1$ and μ_- negative with an absolute value ≥ 0.5 , we shall now consider the situation in which Σ^0 has a different a.m.m. structure from Σ^+ and Σ^- .

The simplest modification would be to assume the same radius and external structure, given by (8) and (9), for the three particles, but making the internal structure of the Σ^0 different from the other two.

As a possibility we can take $\mu_0 = 0.2$ and $\mu_- = -1.8$. To explain the experimental value of δM_{-+} , we then need a common structure for Σ^- and Σ^+ , characterized by $\lambda_0 = 0.83$, and in order to explain the value of δM_{-0} we shall need to ascribe to Σ^0 a value of λ_0 slightly greater than 0.833.

Another possibility would be to suppose that Σ^0 has not the same r.m.s. radius than Σ^+ and Σ^- . We can describe the a.m.m. distribution of Σ^0 by an exponential form factor for example; confining ourselves only to values of $\mu_- \sim -2$ and $|\mu_0| \leq 1$, we have obtained the following values (see Table I):

Table I.

μ_0	μ_-	λ_0	δM_{-+}	δM_{-0}	$r^{(0)}/r^{(n)}$
0.2	-1.8	0.83	15	experimen- tal value	1/35
0.75	-2.2	0.82	14.8	"	1/6
0.75	-2.5	0.82	16	"	1/6.5
0.8	-4.3	0.81	14	"	1/7
0.8	-2.0	0.82	15	"	1/8

δM_{-+} given in electron mass unit, $r^{(n)} = 0.8y = \text{r.m.s. of nucleon}$, $r^{(0)} = \text{r.m.s. of } \Sigma^0$ (exponential form factor).

The last four rows are especially interesting, because there $r^{(0)}$ is not very small. Its order may be the right one, as for the a.m.m. form factor of Σ^0 the pions do not contribute. We should therefore expect for Σ^0 an r.m.s. radius smaller than the corresponding one for the nucleon, as a consequence of the higher mass of the kaon relatively to the pion.

Case III. Different structures for Σ^+ , Σ^- and Σ^0

As for $\mu_0 > 0$ and $\mu_- < 0$, with $|\mu_-| > \mu_0$ we cannot explain δM_{-+} , under the hypothesis that Σ^- and Σ^+ have the same structure, we shall now suppose that the triplet Σ has different structures.

This can be realized, for instance, by supposing that the hyperons Σ have

the same radius and external structure, differing only in their internal structures. The corresponding form factor could be taken as (8) and (9), and then we should have a different λ_0 for each component of the multiplet. In this model the charge distributions of Σ^+ and Σ^- are not perfectly symmetric.

As an example with $\mu_0=0.1$ and $\mu_-=-2$, the experimental value of δM_{-+} can be explained by taking $\lambda_{0-}=0.81$ for Σ^- and λ_{0+} between 0.82 and 0.83 for Σ^+ . To explain the observed δM_{-0} mass difference we should need a value λ_{0+} of the hyperon Σ^0 , slightly greater than 0.833.

Another possibility would be to suppose that Σ^0 has not the same radius and external distribution than Σ^+ and Σ^- . Taking an exponential form factor, as an example, we obtain the following Table:

Table II.

μ_0	μ_-	μ_+	λ_{0+}	λ_{0-}	δM_{-+}	$r^{(o)}/r^{(n)}$	δM_{-0}
0.1	-2	2.6	between 0.82~0.83	0.81	experimental value	1/10	experimental value
-0.5	-2.2	1.2	0.83	0.80	13.5	1/8	12~15
-0.8	-2.6	1	between 0.83~0.833	0.78	experimental value	1/4	14

δM_{-+} and δM_{-0} given in electron mass unit.

By the arguments given above, it may be that the r.m.s. radius of Σ^0 is reasonable.

§ 3. Effects of the higher order effects corrections of the strong interactions

Our problem now is to discuss the influence of the terms like H_3 , H_4 and H_5 which appear in the interaction (2) upon the hyperon mass differences δM_{-+} and δM_{-0} . These terms mean the consideration of electromagnetic self-energy correc-

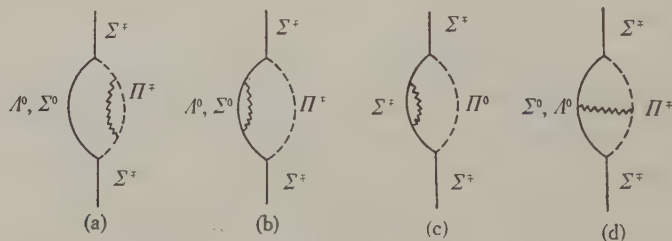


Fig. 2.

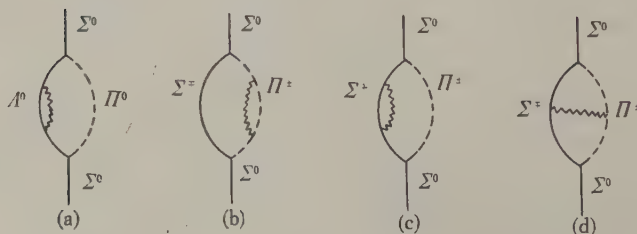


Fig. 3.

tions in the intermediate states in which the hyperon Σ dissociates [Figs. 2, 3 and 4].

Up to the order e^2 only H_3 contributes to the self-mass.

We shall study the contribution of these graphs to δM_{-+} and δM_{-0} , by using relativistic perturbation theory and assuming global symmetry.

a) *Influence of Pion interactions*

The corresponding graphs are raised in Figs. 2 and 3:

Figs. 2(a) and 2(b) give $\delta M_{-+}=0$, while 2(c) gives

$$(\delta M_{-+})_{(2c)} \sim \frac{g_{\Sigma\Sigma\pi}^2}{(4\pi)^2} (M_{\Sigma^-} - M_{\Sigma^+}) = \frac{15}{4\pi} (M_{\Sigma^-} - M_{\Sigma^+}), \quad (13)$$

where $\Sigma_{\Sigma^-} - \Sigma_{\Sigma^+}$ is the mass difference of the intermediary Σ^\pm .

Now Fig. 2(a) which contains only the intermediary Λ^0 and 3(a) gives

$$(\delta M_{-0})_{(2a)-(3a)} \sim \frac{g_{\Sigma\Lambda\pi}^2}{48\pi^2} \frac{M_\pi}{M} \delta M_\pi \sim \frac{1}{20} (M_{\Sigma^-} - M_{\Sigma^0}), \quad (14)$$

where δM_π is the mass difference between π^- and π^0 . The quantity (14) is negligible.

Fig. 2(b) which contains only the intermediary Σ^0 and 3(c) gives

$$(\delta M_{-0})_{(2b)-(3c)} \sim -\frac{g_{\Sigma\Sigma\pi}^2}{(4\pi)^2} (M_{\Sigma^-} - M_{\Sigma^0}) = -\frac{15}{4\pi} (M_{\Sigma^-} - M_{\Sigma^0}). \quad (15)$$

Considering Fig. 2(d) and 3(d), in which we keep only the terms of higher order divergence, and if we make a cut-off of the order of baryon mass, we obtain

$$(\delta M_{\Sigma^0})_{(2d)-(3d)} \sim 0.8 (\mu(\Lambda^0) - \mu(\Sigma^0)) \quad (16)$$

in electron mass units. Taking $|\mu(\Lambda^0)| \sim |\mu(\Sigma^0)| \lesssim 1$, we see that the contribution of Fig. 2(d) and Fig. 3(d) gives at most 10% for δM_{-0} .

A similar contribution to δM_{-+} appears through Fig. 2(d) as

$$(\delta M_{-+})_{(2d)} \sim 1.6 (\mu(\Lambda^0) + \mu(\Sigma^0)).$$

From (13) and (15) we see that the contribution to δM_{-+} and δM_{-0} could be of the same order as the electromagnetic mass differences. But, as is well known, the relativistic perturbation theory, when applied to the various processes as nuclear forces, scattering of pions and nucleons at low energies, anomalous magnetic moments of the nucleons, etc. gives wrong results. We can get reasonable results for the above phenomena if we neglect the nucleon current (static theory) and make a cut-off in the momenta of the intermediary virtual states. Extending the static model to the interactions baryon-pion, we would obtain values for δM_{-+} and δM_{-0} which are hundred times smaller than those given by (13) and (15). Then, it may be that the true value would be in between, that is, ten or twenty times smaller. Therefore, according to these arguments, the effects of higher order pion interactions would give a very small contribution to δM_{-+} and

δM_{-0} , of 20% at most. But to have a better answer we should have a good method of calculation for strong interactions.

b) *Influence of kaon interactions*

The corresponding graphs are raised in Fig. 4

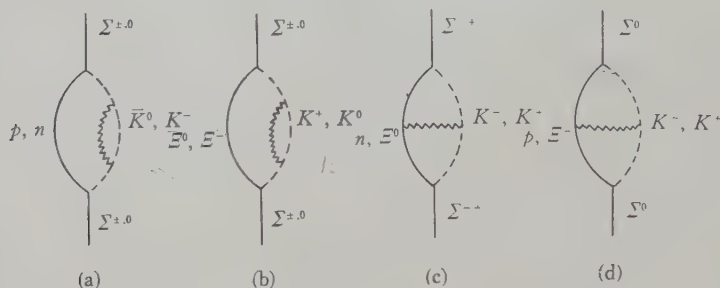


Fig. 4.

Fig. (4a)–(4b) were considered by Brahsden and Moorhouse,⁵⁾ who claimed that it would be possible to explain the experimental δM_{-+} and δM_{-0} through the mass difference of \bar{K}^0 and K^- . In the more favorable case, which corresponds to assume the coupling of the kaons with nucleons as pseudoscalar and to cascade particles as scalar, we have

$$\delta M_{-+} = [M(\bar{K}^0) - M(K^-)] \left[0.037 \frac{g_{\Sigma NK}^2}{4\pi} + 0.445 \frac{g_{\Sigma \Xi K}^2}{4\pi} \right] \quad (17)$$

by neglecting the intermediate $N-P$ and $\Xi^- - \Xi^0$ mass differences. To explain the experimental value of δM_{-+} they have taken $g_{\Sigma NK}^2/4\pi = 4$ and $g_{\Sigma \Xi K}^2/4\pi = 3$; these values seem to be extremely high. In fact, in order to explain the observed cross sections for scattering at low energies of the order of 100 Mev, we need coupling constants ten times smaller than the value assumed by Brahsden and Moorhouse.

Similar considerations are valid for δM_{-0} . Fig. 4(c) gives, with $(g_{\Sigma NK}^2/4\pi) = g_{\Sigma \Xi K}^2/4\pi \sim 0.3$,

$$(\delta M_{-+})_{(4c)} \sim 2 \cdot 10^{-2} [\mu(n) - \mu(\Xi^0)]. \quad (18)$$

while Figs. 4(c) and 4(d) give

$$(\delta M_{-0})_{(4c)-(4d)} \sim 2 \cdot 10^{-2} \left[\mu(n) - \frac{\mu(p) + \mu(\Xi^-)}{2} \right] \quad (19)$$

in electron mass units. We see, from (19) and (20), that even if the a.m.m. of the cascade particle is very high, let us say 10, even then δM_{-+} and δM_{-0} obtained from Figs. 4(c) and 4(d) are less than one electron mass.

§ 4. Conclusions

The point of view of this paper as well as of Marshak et al.,²⁾ Katsumori³⁾

and Kato and Takeda,⁴⁾ is that the mass differences δM_{-+} and δM_{-0} are due to electromagnetic interactions. The authors of references (2) and (3) have used a particular set of form factors to describe the electromagnetic structure of Σ , namely the Feynman form factors and the straight cut-off respectively. However, with these types of structures we cannot explain δM_{-+} and δM_{-0} if we do not assume also very high anomalous magnetic moments with convenient signs. In fact, an interesting case which has been studied in this paper is that one in which the anomalous magnetic moment is smaller than 1 for Σ^0 and of the order of -2 for Σ^- . This happens when we use a perturbation treatment for the a.m.m. of the sigma, where in analogy to the nucleon case we neglect the hyperon current. This situation was also considered by Kato and Takeda⁴⁾ for the $\Sigma^- - \Sigma^+$ mass difference, by extending to these particles the Stanford form factors (exponential) obtained for the nucleons. They have got a very small value for δM_{-+} and so in order to obtain higher values we should have a very small r.m.s. radius.

In this paper we have shown that in order to get a right result for δM_{-+} and δM_{-0} with reasonable values for the a.m.m. of the sigma, we need not take very small r.m.s. radius, it is sufficient to modify the inner region of the electromagnetic structures, some external behaviour being assumed. That is, if we know the outer region of the electromagnetic form factor, we can then, by consideration of problems like the mass differences, get some knowledge about the inner structure. This is the procedure which the Japanese school of Nuclear Forces has followed in order to gain some information about the high energy region of nuclear forces,¹²⁾ after the low energy phenomena were reasonably understood. The same standpoint was followed in the problem of the nucleon mass difference in reference 9).

To reach more definite conclusions about the sigma mass differences it would be necessary to have experimental data for the anomalous magnetic moments and for the electromagnetic form factors. Although the experimental situation is very poor, we can raise some plausible hypothesis from meson theory, viz., to assume $|\mu_0| \lesssim 1$ as well as that Σ^- and Σ^+ have, in the low energy region, the same electromagnetic structures as the nucleons. We then consider the following possibilities for explaining the mass differences δM_{-+} and δM_{-0} :

- i) $\mu_- > 0, \mu_0 > 0,$ ii) $\mu_- > 0, \mu_0 < 0;$
- iii) $\mu_- < 0, \mu_0 > 0,$ iv) $\mu_- < 0, \mu_0 < 0.$

The cases i) and ii) can be realised by assuming that the sigmas have the same structure. The case iii) for $|\mu_-| \gtrsim 0.5$ only works if we assume a different structure for Σ^0 , while for case iv) we need to assume different structures for the three particles.

From the arguments given in this paper as well as in [A], the mass difference between the components of a baryon multiplet is essentially determined by the inner structure, i.e. the region of high energy phenomena. In this region the strong, intermediary and weak interactions have an important role, and they are supposed to be taken into account by convenient electromagnetic form factors. It

would be interesting to see more clearly how these various interactions in the high energy, where the law of charge independence could not be valid, influence on the mass differences of the components of baryon and meson multiplets.

Evidently many other possibilities exist to explain the sigma mass differences. For instance, we could ascribe to Σ^0 a charged form factor, which could give an appreciable contribution to ∂M_{-+} and ∂M_{-0} , but this will not be considered here.

Acknowledgements

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Appendix

For the form factor (8) :

$$\begin{aligned} L^{(0)} &= \frac{A}{(\lambda_0 - \lambda_1)^2} \left[\frac{(1 - \lambda_1)^2}{\lambda_0^2} + 2 \frac{(1 - \lambda_1)(1 - \lambda_0)}{(\lambda_0 - \lambda_1)} \log \frac{A}{\lambda_0} + (\lambda_0 \leftrightarrow \lambda_1) \right], \\ I^{(0)} &= \frac{A}{(\lambda_0 - \lambda_1)^2} \left[-\frac{1}{2} \frac{(1 - \lambda_1)}{(\lambda_0 - A)} + (1 - \lambda_1) \left(2 \frac{1 - \lambda_0}{\lambda_0 - \lambda_1} - \frac{1 - \lambda_1}{2\lambda_0} \frac{\lambda_0 - 2A}{\lambda_0 - A} \right) \Omega\left(\frac{A}{\lambda_0}\right) + (\lambda_0 \leftrightarrow \lambda_1) \right], \\ I^{(1)} &= \frac{A^2}{(\lambda_0 - \lambda_1)^2} \left[-\frac{1}{2} \frac{(1 - \lambda_1)^2}{\lambda_0(\lambda_0 - A)} + \frac{(1 - \lambda_1)}{\lambda_0} \left(2 \frac{1 - \lambda_0}{\lambda_0 - \lambda_1} - \frac{1 - \lambda_1}{2\lambda_0} \frac{\lambda_0 - 2A}{\lambda_0 - A} + \frac{1 - \lambda_1}{\lambda_0} \right) \Omega\left(\frac{A}{\lambda_0}\right) + (\lambda_0 \leftrightarrow \lambda_1) \right], \end{aligned}$$

where

$$\Omega\left(\frac{A}{\lambda_0}\right) = 4\omega\left(\frac{A}{\lambda_0}\right),$$

with

$$\omega(\lambda) = \begin{cases} \frac{1}{\sqrt{\lambda(4-\lambda)}} \arctg \sqrt{\frac{4-\lambda}{\lambda}} & \text{for } \lambda < 4, \\ \frac{1}{\sqrt{\lambda(\lambda-4)}} \log \frac{1}{2} (\sqrt{\lambda} + \sqrt{\lambda-4}) & \text{for } \lambda > 4. \end{cases}$$

If we take an exponential form factor

$$F(q^2) = \frac{1}{(1 + 1/2A \cdot q^2/M^2)^2},$$

we have

$$I^{(1)} = \frac{A^2}{12(1-2A)^3} \left[(3-20A-32A^2) + 12(1-8A+32A^2)\omega(8A) \right], \text{ etc.}$$

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Two-Nucleon Potential with the "One-Pion-Exchange Tail". II

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The p - p scattering below 100 Mev is analysed on the basis of a static potential with the "one-pion-exchange tail" proposed recently from our analysis around 100 Mev. This potential can reproduce all available data in this energy range quite satisfactorily. Some discussions on the depolarization parameter are given.

As a part of the systematic approach to the pion theory of nuclear forces according to the Taketani theory,¹⁾ a nucleon-nucleon potential with the "one-pion-exchange tail" in the triplet odd state has recently been proposed, based on the analysis of the p - p scattering data at 90 Mev.²⁾ It is gratifying that the intermediate part (around the pion Compton wave length) of this potential, determined semi-phenomenologically, also possesses the main features expected from the static pion theory of nuclear forces.³⁾ Taking the proposed potential as the basis, we report in this note on the results of analysis of all the available p - p scattering data below 100 Mev.

We have chosen the following triplet odd potential :*

$$V(x) = V_c(x) + V_T(x)S_{12},$$

with

$$\begin{aligned} V_c &= V_c^{(1)}(x), & V_T &= V_T^{(1)}(x); & x &\geq 1.0, \\ V_c &= -20 \text{ Mev}, & V_T &= V_T^{(1)}(x=1.0); & 1.0 > x &\geq 0.7, \\ V_c &= -100 \text{ Mev}, & V_T &= 0; & 0.7 &\geq x > 0.32, \\ V &= \infty; & & & 0.32 &\geq x. \end{aligned} \tag{1}$$

* Similar analysis has also been carried out assuming a potential which has the same form as (1) except that the hard core radius = 0.34 and $V_T^{(1)}(x=1.0)$ extends right down to the hard core. The differential cross sections calculated from this potential together with the same singlet phase shifts as given in Table II are practically indistinguishable from those reported here. The polarization at 95 Mev turned out to be slightly smaller (by 10% around 45°) than that shown in Fig. 2.

Table I. Triplet odd nuclear Blatt-Biedenharn phase shifts as calculated from the potential (1)

$E(\text{Mev})$	3P_0	3P_1	3F_3	3H_5	3P_2	3F_2	ϵ_2	3F_4	3H_4	ϵ_4
18.2	0.200	-0.069	-0.002	-0.000	0.020	-0.004	-0.474	0.000	-0.000	-0.723
19.8	0.218	-0.074	-0.003	-0.000	0.023	-0.005	-0.475	0.000	-0.000	-0.723
30.14	0.318	-0.101	-0.006	-0.000	0.040	-0.007	-0.473	0.002	-0.001	-0.722
39.4	0.388	-0.120	-0.009	-0.001	0.056	-0.009	-0.468	0.003	-0.002	-0.720
70	0.521	-0.157	-0.020	-0.003	0.103	-0.012	-0.449	0.008	-0.004	-0.713
95	0.563	-0.170	-0.029	-0.005	0.133	-0.012	-0.438	0.013	-0.006	-0.706

Here the inter-nucleon distance x is measured in the pion Compton wave length 1.415×10^{-13} cm. $V_C^{(1)}$ and $V_T^{(1)}$ are respectively the central and tensor part of the well-known one-pion-exchange potential (OPEP) with $g^2/4\pi = 0.08$ in the state concerned.

To be consistent with this potential we expect that the singlet even potential is also given by OPEP for $x \gtrsim 1.7$. For $1.7 \gtrsim x \gtrsim 0.7$ the deviation from the OPEP may be estimated from the two-pion-exchange potential.³⁾ These considerations are sufficient to determine the singlet G -wave phase shift ${}^1\delta_4$ with enough accuracy below 100 Mev. The singlet D -wave phase shift ${}^1\delta_2$ slightly depends on the details of the potential at $0.7 \lesssim x \lesssim 1.7$, hence it is considered to be an adjustable parameter within certain limit. The singlet S -wave phase-shift ${}^1\delta_0$ critically depends on the largely unknown interactions at $x \lesssim 0.7$ and is treated as a free parameter.

The triplet odd phase shifts calculated from the potential (1) are given in Table I. These phase shifts and ${}^1\delta_4$ determined by the one-pion-exchange tail of the singlet even potential are now fixed and the phase shift analysis of the p - p data for ${}^1\delta_0$ and ${}^1\delta_2$ are made at 18.2, 19.8, 30.14, 39.4, 70, and 95 Mev. The best fits shown in Fig. 1, Fig. 2 and Fig. 3 have been attained

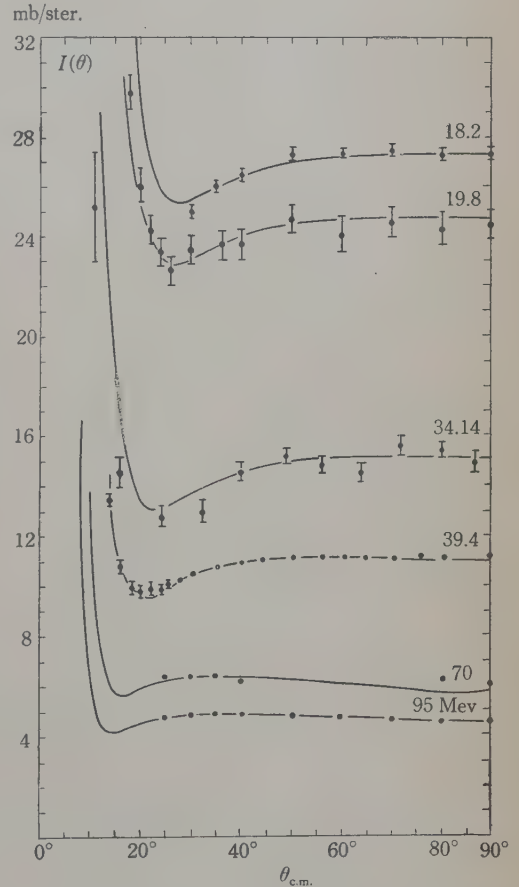


Fig. 1. Angular distribution
Experimental data are taken from refs. 5) and 6).

Table II. Singlet even nuclear Blatt-Biedenharn phase shifts (in radian) which give the best fits combined with those calculated from (1)

E (Mev)	1S_0	1D_2	1G_4
18.2	0.88	0.009	0.000
19.8	0.86	0.010	0.000
30.14	0.74	0.014	0.000
39.4	0.65	0.025	0.001
70	0.40	0.070	0.004
95	0.26	0.094	0.006

for the singlet even phase shifts given in Table II.

The agreement in Fig. 1, Fig. 2 and Fig. 3 is very satisfactory. Good agreement was already anticipated in the Taketani theory; namely, it was expected that the one-pion-exchange potential and the static two-pion-exchange potential, together with some suitable phenomenological inner interactions, may be able to reproduce experimental data up to 100 Mev. Some calculations were made and qualitative agreement was obtained.⁴⁾ But the results of the present paper are most systematic and comprehensive ones, having been calculated by SILLIAC. We have already discussed the physical reasoning of the potential (1) and showed in detail how it can reproduce experimental data at 90 Mev.²⁾ The same discussions can also be applied to the lower energy data concerned in the present paper.*

It is significant that $^1\partial_0$ and $^1\partial_2$ form a smooth function of energy and are consistent with the following characteristics of the interaction: hard core at

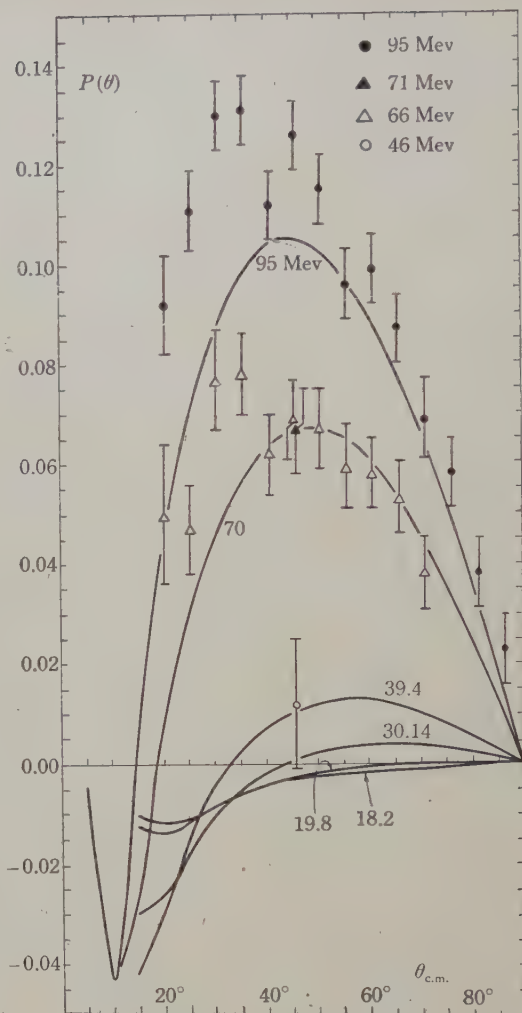


Fig. 2. Polarization
Experimental data are taken from ref. 5).

* Below 10 Mev, agreement between theory and experiment is obtained in somewhat different way, since the most important term to be discussed is the Coulomb-nuclear interference one. Besides, effects of the vacuum polarization and others have to be taken into account correctly. Such discussions will be done elsewhere. It should be pointed out that at about 4 Mev the triplet P -wave phase shifts calculated with the potential (1) are consistent with those by MacGregor,⁷⁾ type I, with small magnitude of $^1\partial_2$.

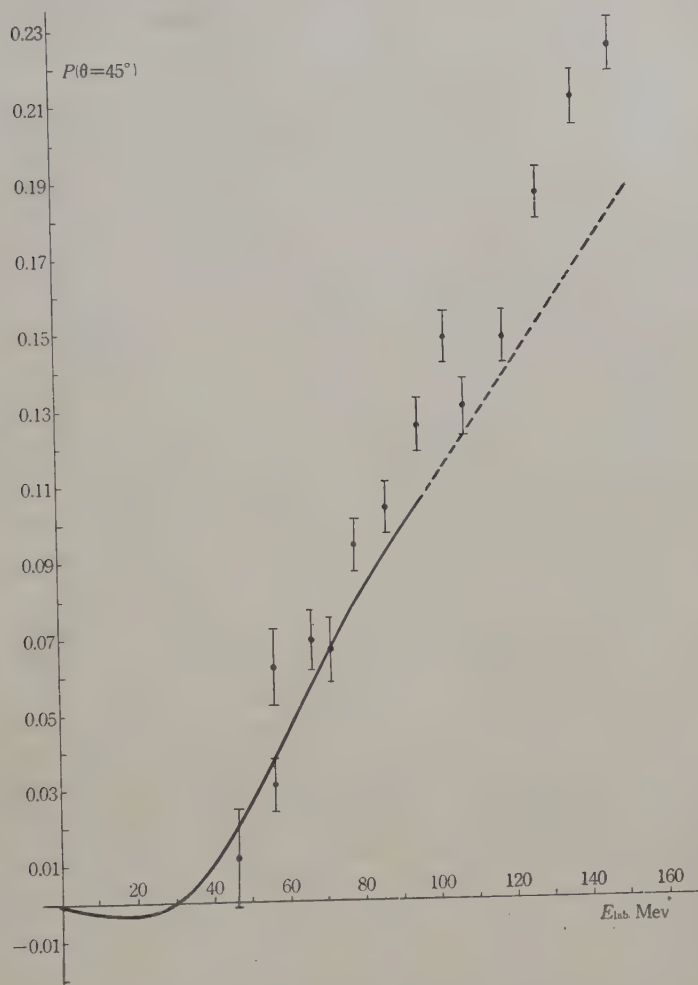


Fig. 3. Energy dependence of polarization at $\theta=45^\circ$
Experimental data are taken from ref. 5).

$x \simeq 0.4$ followed by a very strong attraction gradually reducing to OPEP at $x \simeq 1.6$.^{*} As such we conclude that the present work provides a consistent picture of the p - p interaction below 100 Mev.

We have also calculated the triple scattering and the spin correlation parameters using the phase shifts listed in Tables I and II. Of these we show the depolarization $D(\theta)$ and its energy dependence in Fig. 4 and Fig. 5. The potential (1) predicts a negative $D(80^\circ)$ below 100 Mev. Inspection of the expression for $D(\theta)$

^{*} For example, the following potential gives phase shifts very close to those given in Table II: Hard core at 0.42 followed by a square well of depth 430 Mev to $x=0.7$, then another well of depth 20 Mev reducing to the OPEP at $x=1.6$.

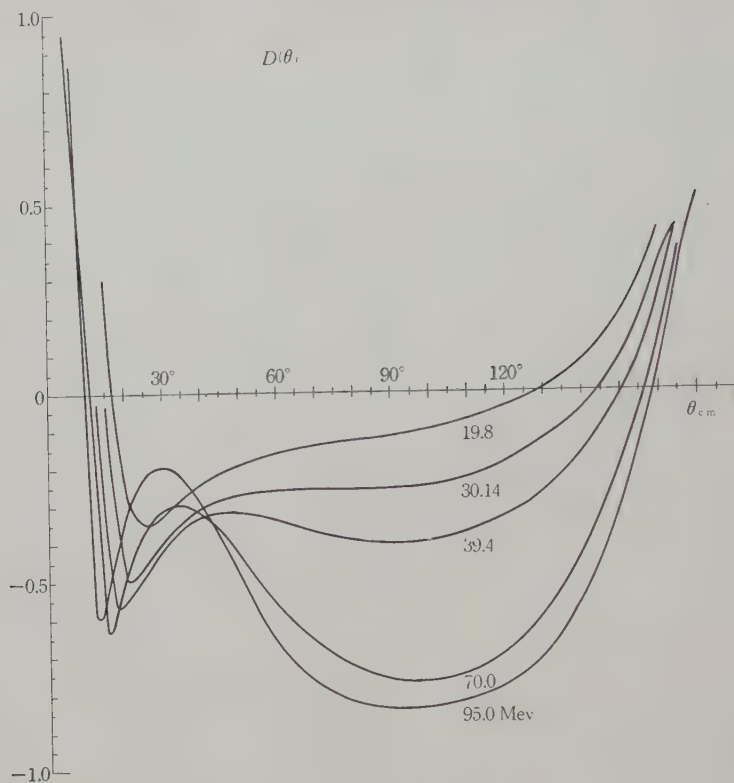
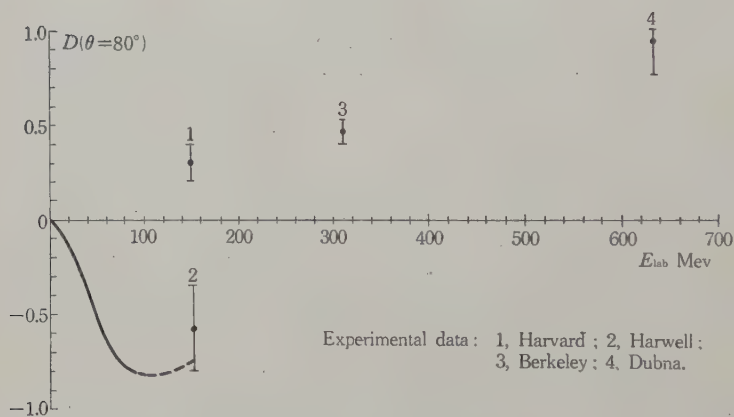


Fig. 4. Depolarization


Fig. 5. Energy dependence of depolarization at $\theta=80^\circ$
Experimental data are taken from ref. 9).

in terms of the phase shifts⁸⁾ shows that this negative $D(80^\circ)$ is due to the positive tensor potential in the triplet odd state which gives ${}^3\delta_0 \gg 0$ and $\epsilon_2 < 0$.²⁾ However, $D(80^\circ)$ begins to increase around 100 Mev mainly because the cross section remains constant above this energy.

At present we have two contradicting experimental data on $D(\theta)$ at 150 Mev.⁹⁾ A positive $D(80^\circ)$ indicated by Harvard group seems to require the modification of our potential even around the pion range. On the other hand, the correction of much shorter range (velocity dependent or otherwise), if any, will suffice to explain the negative $D(80^\circ)$ obtained by Harwell group. Thus, the deviation of the experimental data from our predictions in Fig. 5 indicates the extent of invalidity of the static approximation or a new pion-nucleon interaction other than $\sigma \cdot \nabla$ coupling, or both.* Precise measurements of $D(\theta)$ and other triple scattering and spin correlation parameters are highly desirable for the energy range from several tens to 150 Mev.

We are now planning extensive analyses at higher energies including $D(\theta)$. The implication of our potential on the n - p scattering also remains to be seen.

We are grateful to Professor S. T. Butler for some comments. All numerical works reported here have been carried out on SILLIAC within the Adolph Basser Computing Laboratory, University of Sydney.

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* It may well be that our potential (1) already contains some part of such effects in an "averaged" form as far as the cross section and the polarization are concerned. We expect, however, that such effects will show themselves up in different ways in different experimental quantities thus providing a possibility of being "seen".

Letters to the Editor

The opinions expressed in these columns do not necessarily reflect those of the Board of Editors. Communications should be submitted in duplicate and should be held to within 100 lines (pica type) on standard size letter paper (approx. 21×30 cm), so that each letter will be arranged into two pages when printed. Do not forget to count in enough space for formulas, figures or tables.

Effects of the Weak Interaction on the Hydrogen Energy Levels

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The phenomena treated by the strong interactions which have mainly to do with the structure and the character of the hyperons and the mesons can be characterized by the length r_0 about $10^{-13}\text{cm} \sim 10^{-14}\text{cm}$, i.e. $r_0 \simeq 1/\mu \sim 1/M$ (μ is the pion mass and M the nucleon mass). On the other hand, as is well known, the ratio of the coupling constants of the weak and the strong interactions is about $10^{-13} \sim 10^{-15}$ with appropriate normalization of the length dimension of the coupling constants,¹⁾ and so it is generally assumed that the reactions of the weak to the strong interactions would be of negligible order in the present low energy phenomena provided that contributions of the high energy parts are small. But as was stressed by Umezawa²⁾ it seems to be true that the weak interactions are all unrenormalizable, i.e. the second kind interactions, contrary to the strong ones

which are renormalizable, the first kind. Then the weak interactions would become stronger in the inner region in spite of the smallness of the coupling constant if they were the elementary interactions. This fact might have received considerable changes by the reactions, but at least the weak interactions would become to compete with the strong interactions in some regions. We have no satisfactory method to treat the interactions at present, especially the unrenormalizable ones. But from considerations stated above it is very interesting to examine the strength of the weak interactions in the short range, and especially to discuss the possibility of observation of effects from the weak ones which are generally considered as hidden by the effect of the strong interactions. These problems have already been discussed by Taketani³⁾, Umezawa²⁾ and Blokhintsev⁴⁾ from general and somewhat different points of view, and they have all obtained the conclusion that in the inner regions, $r_0 \simeq 10^{-14} \sim 10^{-23}\text{cm}$, of the elementary particles one cannot neglect the contributions of the weak interactions. In this note we consider the energy levels of the hydrogen atom as the phenomenon to examine the above mentioned effect. The reason why we adopt this phenomenon is that it is the bound state

problem, in which case effects of the potential become remarkable contrary to the scattering problems, when the potential is attractive and its depth is large enough even if the range is very small. Thus we have calculated the potential between the proton and the electron through the β -decay interactions and the energy level shift by this potential. We can consider some other effects to shift the energy level, for example, the charge distribution of the nucleons,⁵⁾ the electric dipole moment of the electron⁶⁾ and so on. Therefore it might be rather doubtful to be able to observe the effect alone treated in this note, but it would be worth while to point out existence of the level shift of this type.

The calculation has been performed by the covariant lowest order perturbation with the V - A type⁷⁾ of the β -decay interaction, introducing the cut-off momentum K , neglecting the total energy momentum of the system $P_\mu (-P_\mu^2 \simeq M^2)$ compared with K , and retaining only the term of the highest power as to K . Then the potential energy $V(\mathbf{r})$ in the frame in which the proton is at rest is given by

$$V(\mathbf{r}) = - (1 + (\boldsymbol{\sigma}^{(p)} \cdot \boldsymbol{\sigma}^{(e)})) \times (f^2 K^2 / 4\pi) \delta(\mathbf{r}) \quad (1)$$

where $\boldsymbol{\sigma}^{(p)}$ and $\boldsymbol{\sigma}^{(e)}$ are spin operators related to the proton and the electron respectively, and f is the Fermi coupling constant, i.e. putting $f^2 = f_0^2 / \mu^4$, $f_0^2 \simeq 10^{-14}$. $V(\mathbf{r})$ in (1) is the δ -function type interaction, and so this potential contributes merely to the S -states as in the case of the Lamb shift. The level shift ΔE_n can be estimated by this $V(\mathbf{r})$

treated as perturbation, using the wave function $\psi_n(\mathbf{r})$ of the hydrogen atom of the Schrödinger equation for the triplet state in which case (1) is attractive, resulting that

$$\Delta E_n = - (2f^2 K^2 / \pi) (me^2 / n)^3 \quad (2)$$

which must be compared with the energy level under the Coulomb potential, $E_n = -me^4 / 2n^2$.

Now in the case of the Lamb shift the energy levels have been measured to the order of the megacycle per second. If we think that this has been explained well only by the present quantum electrodynamics, ΔE_1 must be smaller than the order of 0.1 Mc/sec.⁸⁾ Then $r_0 = 1/K \gtrsim 10^{-17}$ cm, which is consistent with that obtained by Umezawa, and the weak interactions may give observable contributions to the strong interactions at $r_0 \lesssim 10^{-17}$ cm. But it is plausible at present to cut off the contributions of the weak interactions at $1/K \simeq 10^{-17}$ cm in order to avoid discrepancy between theory and experiment, although there exist some doubts as to the convergence of the perturbation series with this K .

In conclusion the authors would like to express their thanks to Professors K. Aizu and H. Umezawa for their valuable discussions, and to Professor S. Ogawa for his pointing out an important error in the manuscript of the previous note.

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Spontaneous Magnetization, A Generalized Square Lattice

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The spontaneous magnetization of a generalized square ising lattice is proposed. It reduces to that of the aniso-

tropic triangular, honeycomb and square lattices in the special cases.

Although Yang¹⁾ has obtained the exact expressions for the spontaneous magnetization of the square ising lattice immediately by means of the concrete algebraic calculation, the same calculation has not yet been made on any other two-dimensional lattice.

But, fortunately, the expressions for the spontaneous magnetization of several types of lattices (namely for the triangular, honeycomb and Kagomé) have been found by Potts²⁾ and one of us (Naya)³⁾ without any concrete calculation.

By generalizing these expressions, we can obtain the spontaneous magnetization of the square lattice with four parameters L_1 , L_2 , L_3 and L_4 , repeated with period two in two directions (we call this the generalized square lattice Fig. 1)

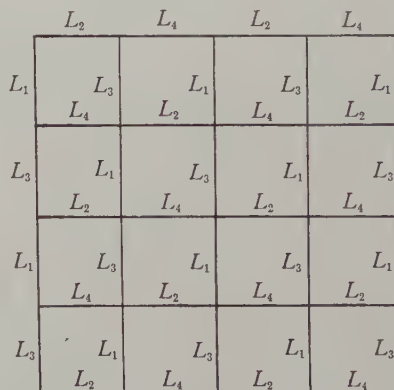


Fig. 1.

$$I = [1 - (s_1^2 + s_2^2 + s_3^2 + s_4^2 + 2s_1s_2s_3s_4 + 2c_1c_2c_3c_4 + 2) / \{(s_1s_2s_3)^2 + (s_2s_3s_4)^2 + (s_3s_4s_1)^2 + (s_4s_1s_2)^2 + 2s_1s_2s_3s_4 + 2(s_1s_2s_3s_4)^2 + 2c_1c_2c_3c_4s_1s_2s_3s_4\}]^{1/8}, \quad (1)$$

where $s_i \equiv \sinh 2L_i$, $c_i \equiv \cosh 2L_i$ ($i=1, 2, 3, 4$).

We can also rewrite it, with use of the parameters of the dual lattice (L_1^* , L_2^* , L_3^* , L_4^*),

$$I = \left(1 - \frac{(s_1^* s_2^* s_3^* s_4^*) (s_1^2 + s_2^2 + s_3^2 + s_4^2 + 2s_1 s_2 s_3 s_4 + 2c_1 c_2 c_3 c_4 + 2)}{(s_1 s_2 s_3 s_4) (s_1^{2*} + s_2^{2*} + s_3^{2*} + s_4^{2*} + 2s_1^* s_2^* s_3^* s_4^* + 2c_1^* c_2^* c_3^* c_4^* + 2)} \right)^{1/8}. \quad (2)$$

The low temperature expansion of (1) is given in terms of $x_i = \exp(-2L_i)$ by

$$I = 1 - 2(x_1 x_2 x_3 x_4) - 2 \sum (x_i x_j x_k)^2 + 2(x_1 x_2 x_3 x_4)^2 - 6x_1 x_2 x_3 x_4 \sum x_i^2 x_j^2 + 4x_1 x_2 x_3 x_4 \sum x_i^2 x_j^2 x_k^2 - 4 \sum x_i^2 x_j^2 x_k^2 - 30(x_1 x_2 x_3 x_4)^2 \sum x_i^2 + O(x^{12}), \quad (3)$$

in which the summations on the right hand side run for i, j and k from 1 to 4, excluding the congruence of them.

We can assure the correctness of the expression (1) in several viewpoints. In the first place, the low temperature expansion is quite in coincidence with the results obtained by counting the low temperature configurations. In the

second place, it is clear that (1) becomes unity at absolute zero of temperature and vanishes at the Curie point determined by the equation

$$c_1 c_2 c_3 c_4 + s_1 s_2 s_3 s_4 = \sum s_i s_j.$$

On the other hand, the formula (1) reduces to the spontaneous magnetization of the anisotropic triangular lattice in the limit $L_4 \rightarrow \infty$,

$$I = \left(1 - \frac{1}{(s_1 s_2)^2 + (s_2 s_3)^2 + (s_3 s_1)^2 + 2(s_1 s_2 s_3)^2 + 2c_1 c_2 c_3 s_1 s_2 s_3} \right)^{1/8},$$

which is equivalent to that given by Potts²⁾ but written in a form slightly different from his. (1) also tends to the spontaneous magnetization of the anisotropic honeycomb lattice when $L_4 \rightarrow 0$,

$$I = \left(1 - \frac{s_1^2 + s_2^2 + s_3^2 + 2c_1 c_2 c_3 + 2}{(s_1 s_2 s_3)^2} \right)^{1/8}.$$

Furthermore, the formula (1) gives the spontaneous magnetization of the rectangular lattice when $L_1 = L_3$, $L_2 = L_4$ or when $L_3 \rightarrow 0$, $L_4 \rightarrow \infty$,

$$I = (1 - 1/(s_1^2 s_2^2))^{1/8},$$

which has been given by Onsager, Kaufman and Potts²⁾ and proved by Chang.⁴⁾ We can now believe that ex-

pression (1) for the spontaneous magnetization of the generalized square lattice is correct.

Appendix I

The spontaneous magnetization for the anisotropic Kagomé lattice is

$$I = 1/3 \sum (A/A_i)^{1/2} [1 - 16/A^6 \times \{A^2 \sum s_i^2 s_j^2 A_i^2 A_j^2 + 8s_1^2 s_2^2 s_3^2 A_1 A_2 A_3 + s_1 s_2 s_3 (A_1 A_2 A_3)^{1/2} (s_1^2 + A_1) (s_2^2 + A_2) \times (s_3^2 + A_3)\}]^{1/8},$$

where

$$A = \sum s_i^2 s_j^2 + 2c_1 c_2 c_3 s_1 s_2 s_3 + 2(s_1 s_2 s_3)^2$$

$$A_i = A + s_i^2, \quad c_i = \cosh 2L_i, \quad s_i = \sinh 2L_i.$$

Appendix II

The Curie temperature of the generalized square lattice has been derived by T. Utiyama in terms of the Gudermannian functions, $\text{gd}(2L) = \arctan(\sinh 2L)$, as follows,

$$\text{gd}(2L_1) + \text{gd}(2L_2) + \text{gd}(2L_3) + \text{gd}(2L_4) = \pi.$$

This is a generalization of the Onsager-relation

$$\text{gd}(2L) = \pi/z,$$

where z means the coordination number for the two-dimensional lattices.

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A Note on the Phenomenological Theory of Unstable Particles

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Some months ago there appeared a paper by M. Ida in which among others "positive" and "negative" resonances as defined by the author were discussed¹⁾. It seems to us that it is not quite trivial to apply a similar idea, as presented below, to the theory of unstable particles. As H. Araki et al. and M. Lévy pointed out we have as yet two

principal methods of defining these particles.²⁾

1. Through the poles of propagators analytically continued onto the complex energy plane.
2. Through the properties of the scattering process of the stable particles, e.g., the well-known possibility of interpreting the resonances, as being a consequence of formation of an unstable intermediate state.

Let us consider the second, phenomenological approach. It has, among others, a known defect consisting of the great difficulty connected with the definition in this manner of a three-body decaying particle. But it seems that even in the case of a two-body decaying particle serious doubts appear. Consider the scattering of two stable particles, a and b , and assume for simplicity that the elastic scattering is the only possible one. We work in the centre of mass system. Let $|t, l\rangle$ correspond to the spherical wave packet describing the interacting particle with the definite relative angular momentum l and $|t, l\rangle_0$ to the free wave packet corresponding to the same particle and let $|t_0, l\rangle = |t_0, l\rangle_0$. The quantity

$$\tau_R = \lim_{\substack{t \rightarrow +\infty \\ t_0 \rightarrow -\infty}} \frac{{}_0\langle t, l | r | t, l \rangle_0 - \langle t, l | r | t, l \rangle}{{}_0\langle t, l | v_r | t, l \rangle_0} \quad (1)$$

is the mean retardation acceleration of interacting wave packet relative to the non interacting one. When τ_R is very large the interpretation imposes itself that in the scattering process the essential role is played by an intermediate compound state. An easy calculation leads to the expression

$${}_0\langle t, l | v_r | t, l \rangle_0 \cdot \tau_R = \langle t, l | i\hat{S} \frac{\partial \hat{S}^*}{\partial \hat{p}} | t, l \rangle \quad (2)$$

where \hat{S} is the scattering S -operator and \hat{p} is the operator of radial momentum; here $\partial \hat{S}^* / \partial \hat{p}$ means a differentiation of the operator \hat{S}^* with respect to the operator \hat{p} . The dependence of the right-hand side of (2) on the time is only seeming. Performing the limiting transition from the wave packet to the stationary state, we obtain

$$\tau_R = \langle E, l | i\hat{S} \frac{\partial \hat{S}^*}{\partial E} | E, l \rangle = 1/2 \frac{\partial \delta_l(E)}{\partial E},$$

$$\hat{S} = e^{2i\delta} \quad (3)$$

which is true in the c.m.s. In an arbitrary reference system we write

$$\tau_R = \langle E, l | i\hat{S} \frac{\partial \hat{S}^*}{\partial \hat{p}} | E, l \rangle, \quad \hat{p} = (p_\mu p^\mu)^{1/2}.$$

The δ -function due to the energy conservation by the operator \hat{S} does not appear in (3) because of our normalization connected with the limiting transition performed. As it is known, in the case of a positive resonance, near the point $E=E_0$ where E_0 is the mass of the intermediate unstable particle τ_R reaches a maximum, and is equal to the life-time of the intermediate state. It can be easily verified by using the formula

$$\delta_l = \delta_l^{(0)} + \arctg \frac{\Gamma}{E - E_0}.$$

Here it is necessary to note that the resonance is shifted from the point $E=E_0$ and the mass of the unstable particle is set better by the maximum of τ_R than by the maximum of the scattering cross section, because

we can consider $\delta_l^{(0)}$ a slowly varying function of energy. Similarly the points corresponding to the negative resonances of M. Ida are maxima of the cross section, but as we have for them $\tau_R < 0$ beside $\partial \tau_R / \partial E = 0$ they cannot be interpreted as connected with the creation of unstable particle. We see that τ_R is an essential quantity in the phenomenological theory. We can, however, imagine a local maximum of τ_R which does not correspond to the resonance. For τ_R to be large at least a part of particles entering the range of action of the potential must form a compound state. Since in the discussed case there is no resonance, certainly not all the particles from this intermediary state, and thus $\tau_R < \tau$ where τ denotes the life time of the compound state. If all the interacting particles passed through the intermediary unstable state we should have $\tau = \tau_R \cdot 4\pi(2l+1)/(p^2\sigma_l)$ and the sufficiently high extrema of the quantity $\tau_R \cdot 4\pi \cdot (2l+1)/(p^2\sigma_l)$ would give us the spectrum of unstable particles. It is difficult, of course, to formulate a sharp criterion that would decide whether the given state is a state of unstable particle or whether it is not. One can give various criteria of the type $\tau \gg \tau_0$. In order to determine τ_0 it is generally necessary to postulate something about interaction, e.g., the range of action of the potential. We cannot, however, exclude the possibility of the interacting particles being able to interact in two ways—first by going through a compound state and second in which the interacting particle suffers the almost immediate scattering. Since the investigation of the scatter-

ing process does not permit to discriminate between the mentioned two kinds of interaction, we believe that in this case it must be agreed that the phenomenological approach meets here with a very serious obstacle. It is neither very clear to us, how to interpret physically a step in the energy dependence of the cross section, which naturally produces a maximum of τ_R ; this phenomenon is not connected with the unstable particle if we define the latter by means of an integral criterion.

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The Role of Electron-Phonon Interaction in the Impurity Conduction of Semiconductors

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The impurity conduction in semiconductors has been investigated from various points of view, especially stimulated by the proposition of the impurity band model by Hung.¹⁾ First of all, it is to be noted that the concentration of impurity atoms can be varied in as wide a range as 10^5 , and one may take it as a typical phenomena through which one can conveniently investigate the effect of electron-electron correla-

tion or electron-lattice interaction on the validity of the conventional band model, as a function of interatomic distance (therefore, as a function of band effective mass), because it is technically so difficult to vary the lattice constants of usual crystals through appreciable ranges. The second problem, which may be as interesting as the first, is the effect of randomness on the band model (that is, how one is justified in using the band model in spite of random distribution of impurities, and if so, in what sense), and on the transport properties. In this and the following letters, we discuss some of these points, confining ourselves to the so-called low-concentration range.

According to Fritzsche and Lark-Horovitz,²⁾ the electric resistivity ρ of *p*-type germanium crystal with majority impurity concentration n and minority impurity concentration nK (K being called the degree of compensation) varies with absolute temperature T as $\exp(\varepsilon/\kappa T)$ at low temperatures, and ε is approximately independent of n and K so far as $7 \cdot 10^{14} < n < 2 \cdot 10^{16}/\text{cc}$ and $0.3 < K < 0.8$. Secondly, ρ , as a function of K , has a minimum at some value K_0 (seemingly smaller than 0.5), and ρ increases as large as fourteen times, as K increases from 0.33 to 0.6. The first of the above two facts can be explained by taking into account the electron-phonon interaction as follows.

Let us consider a hypothetical periodic array of impurities, and electrons which are bound in their hydrogenlike $1s$ -states. Neglecting the electron correlation, these electrons will move freely in the $1s$ impurity band, the band width being of

the order of the resonance energy

$$W(R) = -2\varepsilon_b(1 + \alpha R) \exp(-\alpha R) \quad (1)$$

between neighbouring impurities separated by R , where ε_b and α^{-1} are the binding energy and the radius of the impurity 1s-state. On the other hand, the electron-phonon interaction causes the energy shift U of the 1s impurity state, which can be calculated as

$$U = -E_1^2 \alpha^3 / 16\pi c_{ii}, \quad (2)$$

if the deformation potential approximation is made use of. (c_{ii} is an appropriate elastic constant). Making use of the variation principle, we can conclude that the electron in the impurity band moves freely (the band model is valid), or is trapped at some impurity (the localized model) according as

$$z|W(R)| \gtrless |U| \quad (3)$$

where z is the number of the nearest neighbours in the assumed impurity lattice (such a possibility was already suggested by Kurosawa³⁾). It can also be shown that this transition from the one type to the other as R increases is rather abrupt, the situation which is quite different from the well-known case of the interaction between an electron and polar mode vibration in ionic crystal (in the latter case the effective mass is varied instead of R).

When R is larger than this critical value R_c , or the impurity concentration n is sufficiently smaller than $n_c \equiv R_c^{-3}$, it does not matter whether the impurities are arranged in the periodic lattice or at random, because the electron will migrate from one site to another, losing

its memory (i.e., from which site it came previously) after each hopping. Making use of the theory of multiphonon process,⁴⁾ but with the resonance energy $W(R)$ instead of the non-adiabatic term H' whose effect is negligible in the present case, it is easily shown that the time rate Γ of hopping between a pair of sites separated by R is given by

$$\Gamma = \frac{\sqrt{\pi}}{2\hbar} W(R)^2 (\varepsilon \cdot \kappa T)^{-1/2} \times \exp(-\varepsilon/\kappa T). \quad (4)$$

One can also show that, if $\alpha R \gg 1$, the activation energy ε is equal to $U/2$, being independent of R . By making this ε to correspond to that ε mentioned above, the observed T -dependence of \mathcal{Q} as well as the independence of ε upon n and K can be explained.

The formula (2) gives ε several times smaller than the observed value ($\varepsilon \sim 0.55 \cdot 10^{-8}$ ev), but the discrepancy will be reduced by taking into account the transverse acoustic vibration. Making use of the observed values of ε and $\varepsilon_b (\sim 10^{-2}$ ev), $\alpha^{-1} \sim 45 \text{ \AA}$ and $z = 6$, the inequality (3) gives the critical concentration $n_c \sim 4 \cdot 10^{16}/\text{cc}$, which is consistent with the upper bound of the above mentioned n -range in which ε was observed to be independent of n and K . Above n_c , the phonon does not play a primary rôle in the impurity conduction, and the random distribution of impurities will be essential in causing finite resistivity. However, it is not yet clear whether the observed transition from low to high concentration ranges, accompanied by a rather abrupt change of \mathcal{Q} ,⁵⁾ really corresponds to this n_c , or

is due to the overlap of higher impurity bands ($2s$ or $2p$) for which $z|W'(R)| > |U'|$ must already be valid owing to the effectively smaller $\alpha' (\sim \alpha/2)$, although the values of n_c estimated above coincides with the observed transition concentration.

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The Spatial and Time-Fluctuation of Coulomb Energy in the Low Concentration Impurity Conduction

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In the preceding letter, we discussed how and when the concept of hopping becomes meaningful as a separable elementary process in the impurity conduction of semiconductors, and as the result, we could explain the temperature dependence of \mathcal{Q} , and also the independence of ε upon n and K , together with the n -range ($n < n_c$) for the validity of the latter. Remaining problems to be solved are (1) the remarkable K -dependence of \mathcal{Q} and (2) the K -range for the validity of (n, K)-independence of ε . We are now confronted with a com-

plicated transport phenomena, where the Coulomb interaction between localized electrons plays an essential rôle.

Taking n -type semiconductor for simplicity, we have nK negatively charged acceptors (A^-) and nK positively charged vacant-sites of donor (D^+), as the result of compensation. D^+ can migrate from donor to donor through the hopping of electrons mentioned above, whereas A^- are always fixed. For small values of $K (\ll 1)$, each A^- governs a number of donors, and a D^+ is usually trapped around A^- and occasionally migrate to another A^- as the result of several successive hoppings from donor to donor, as has been discussed by a number of authors,¹⁾ who ascribed the observed activation energy in \mathcal{Q} - T curve to the difference of energies at the donor sites near and far from A^- .

For larger values of K , this model (a number of donors being governed by each A^-) is no longer valid. Moreover, in order to explain the large \mathcal{Q} - K dependence, it seems necessary to take account of time fluctuation of site energy in addition to the spatial (from site to site) fluctuation mentioned above. We shall now outline the procedure very briefly.

At any given instant, the Coulomb interaction E_i between an electron localized at the i -th donor site on the one hand, and all A^- and D^+ on the other hand, can be determined if the spatial distributions of the latter are given. The probability that an electron hops from the i -th site to the j -th site is given by (4) of the preceding letter, except for that ε in the exponential is to be replaced by

$$\varepsilon_{ij}^* = \varepsilon \{1 + (E_j - E_i)/4\varepsilon\}^2. \quad (1)$$

This means that the probability of the hopping ($i \rightarrow j$) is small if $E_j - E_i > \kappa T$. On the other hand, E_j must be $> E_F - O(\kappa T)$ (j to be vacant) and at the same time E_i must be $< E_F + O(\kappa T)$ (i to be occupied) in order that the hopping actually takes place. Combining these, we can conclude that the hopping takes place with appreciable probability when both E_i and E_j are within $O(\kappa T)$ of the Fermi energy E_F at that instant.

Now the site energy E_i fluctuates from time to time, because D^+ surrounding it are always migrating. Let us denote its time average and mean deviation by \bar{E}_i and Δ_i , respectively. \bar{E}_i is different from site to site, depending on the spatial distribution of surrounding acceptors and donors. Denote its (normalized) distribution function by $P(\bar{E})$. The Fermi energy mentioned above can be determined from

$$\int_{-\infty}^{E_F} P(E) dE = 1 - K, \quad (2)$$

if an explicit form of $P(E)$ is known.

Now, the i -th donor site can contribute to the hopping only if $\bar{E}_i - \Delta_i < E_F < \bar{E}_i + \Delta_i$ (it is assumed that $\Delta_i \gg \kappa T$), and in that case, only during the time fraction of $\kappa T/\Delta_i$.

Assuming that Δ_i is a function of \bar{E}_i only, donor sites can be divided into three groups, according to the values of \bar{E}_i , as follows.

(I) $\bar{E}_i < E'$ fraction being f_1 ,

(II) $E' < \bar{E}_i < E''$ fraction being m ,

(III) $E'' < \bar{E}_i$ fraction being f_2 .

Here E' and E'' are given by $E' + \Delta(E')$

$= E_F = E'' - \Delta(E'')$. The site of group (I) is always occupied by an electron, while that of group (III) is always vacant. A site of group (II) is occupied during some time-intervals, and vacant during other time-intervals. The ratio of these time fractions is given, on the average, by

$$\int_{E'}^{E_F} P(E) dE : \int_{E_F}^{E''} P(E) dE.$$

This, in turn, enables one to determine the mean time-fluctuation $\Delta(\bar{E}_i)$ of site energy E_i as a function of \bar{E}_i , because, on the one hand, Δ_i originates from those surrounding donors which belong to the group (II), and on the other hand, Δ_i and \bar{E}_i have some correlation, both depending upon the spatial distribution of surrounding donors and acceptors.

Through this self-consistent procedure, which takes into account the time as well as spatial fluctuation of site energy, we can determine the fraction m of the group (II) as a function of K . The result of preliminary calculation shows that m tends to zero as $K \rightarrow 0$ or $K \rightarrow 1$, and has a maximum value near unity at $K \sim 0.3$.

Since it is only the group (II) which actually contributes to transport phenomena, the effective mean distance between neighboring (active) donors is $m^{-1/3}R$ instead of R ($\equiv n^{-1/3}$) itself. According to (4) and (1) of the preceding letter, Ω is expected to be proportional to $\exp(2\alpha R m^{-1/3})$. Because $\alpha R \gg 1$ for the low concentration with which we concern ourselves now, a rather small decrease in m as K in-

creases from 0.33 to 0.6 may well cause the increase in ϱ by an order of magnitude.

For the values of K near zero or unity, the above procedure, in which we have cut off the hopping probability to zero on the outside of $O(\kappa T)$ around E_F , is expected to be inappropriate. Making use of the principle of the least Joule's heat in the theory of electrical circuit, we can devise a variation method to calculate actual resistivity. In the case of $K \ll 1$, this procedure leads to the activation energy of ϱ - T curve, which consists of two parts: the first is ε discussed in the preceding letter, and the second is essentially that activation energy discussed by Mott, Price and Kasuya.¹⁾

Although the main features of the impurity conduction at low concentration seem to be explainable qualitatively by the above considerations, there remain a number of problems to be solved, especially interesting from the mathematical point of view. These will be pointed out in detail in the full report of the above work.

The author wishes to express his sincere thanks to Prof. T. Matsubara for valuable discussions.

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On the Polarization of High Energy Nucleon Elastically Scattered from Light Nuclei

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Recently, many authors,^{1,2)} calculated the differential cross sections and polarizations of high energy nucleons elastically scattered from light nuclei, using the impulse approximation with the nucleon-nucleon scattering phase shifts. It is because the impulse approximation is equivalent to the use of the optical model potential in the first Born approximation which may be justified for the small angle scattering from light nuclei.³⁾ The ordinary procedure is that, given several alternative sets of the nucleon-nucleon scattering phase shifts, the optical model potentials corresponding to these sets of the phase shifts are derived, and then the polarization of the nucleon scattered in small angles are calculated. The comparison of the results with the observed polarization may distinguish various nuclear phase shifts sets rejecting some of them.

The best way of studying the nucleon-nucleon interactions by the use of these optical model potentials is to investigate the polarizations of the nucleons elastically scattered from light nuclei, because when one calculate the polarizations of these nucleons through the optical model potential, the final results are independent of the detailed structure of the target nucleus and to a certain

extent free from the limitation of the Born approximation. That is, the expression for the polarization does not contain the nuclear form factors which depend on the detailed structure of the target nucleus; the expression depends only on the nucleon-nucleon scattering phase shifts.

The main purposes of the present note are the following:

- 1) to construct the optical model potential by the use of the nucleon-nucleon scattering phase shifts derived from the meson theoretical potential,⁴⁾
- 2) to calculate the polarization of the nucleons elastically scattered from light nuclei through this potential,

- 3) to compare the results with experimental data⁵⁾ and with the results basing on the Gammel-Thaler⁶⁾ and Signell-Marshak phase shifts.⁷⁾

In the present calculation, we use the method adopted by Ohnuma²⁾ in his calculations of the polarization of the protons elastically scattered from C^{12} . In this calculation, the Coulomb phase shifts are omitted from the nuclear amplitude, and this omission is compensated by neglecting the phase factor $\exp[-2i\eta \ln \sin^2 \theta/2]$ which normally multiplies the Coulomb amplitude in the case of pure Coulomb scattering.

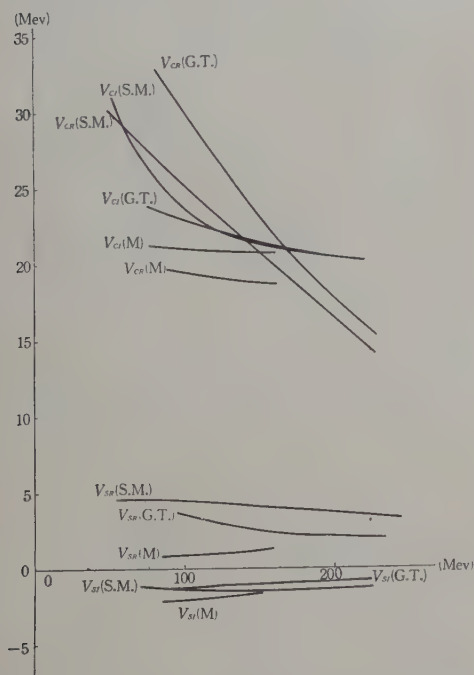


Fig. 1. Parameters of the optical model potential V_{CR} , V_{CI} , V_{SR} and V_{SI} calculated by the use of the nucleon-nucleon scattering phase shifts of mesontheoretical potential (M), Gammel-Thaler (G. T.) and Signell-Marshak (S. M.)

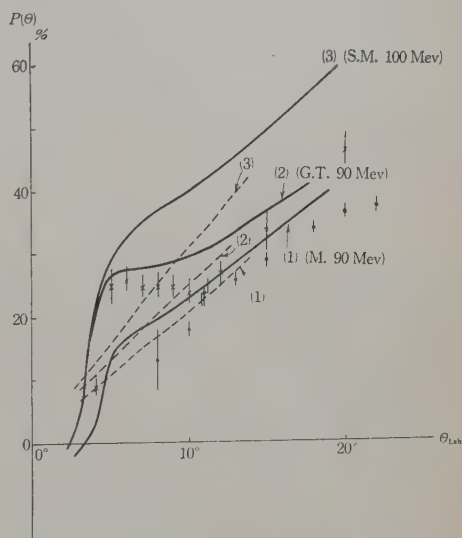


Fig. 2. Polarizations of the nucleons elastically scattered from He^4 and C^{12} . (1) (M) at 90 Mev, (2), (G. T.) at 90 Mev and (3) (S. M.) at 100 Mev.

Solid curves refer to the polarizations of proton-light nuclei scattering and dashed curves to the polarizations of neutron-light nuclei scattering, the latter are obtained by setting $e \rightarrow 0$ in the expression of polarization.

Experimental points \bullet indicate the values measured on $p-He^4$ (96 Mev) by Gotow and \times the values on $p-C^{12}$ (95 Mev) by Dickson-Salter.

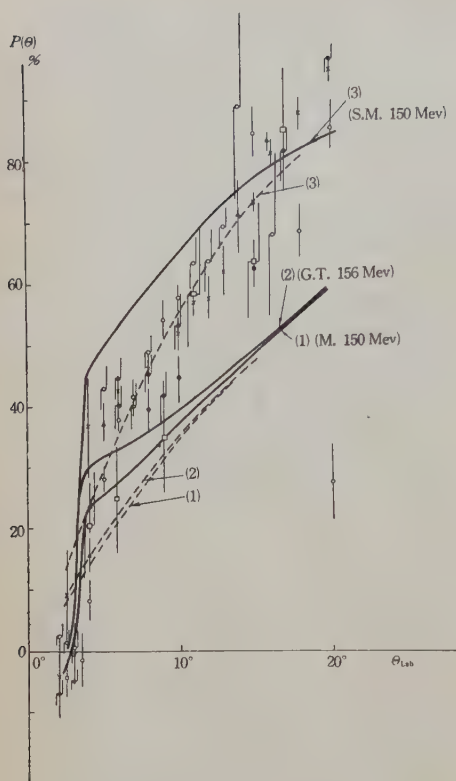


Fig. 3. Polarizations of the nucleons elastically scattered from C^{12} , O^{16} , Ca^{40} and He^4 .

(1) (M.) at 150 Mev, (2) G. T.) at 156 Mev and (3) (S. M.) at 150 Mev.

Solid and dashed curves correspond to the polarizations of proton and neutron-light nuclei scattering respectively.

Experimental points \bullet indicate the values measured on p - He^4 (147 Mev) by Wilson, et al., \square the values on n - C^{12} (155 Mev) by Harding and \times , \triangle , \circ , the values of p - C^{12} , p - O^{16} and p - Ca^{40} (155 Mev) measured by Alphonse, et al, respectively.

The optical model potential is related to the isotopic spin average of n - n and n - p scattering spin matrix coefficients for forward scattering times the nuclear form factor. And for quantities specifying the optical model potential V_{OR} , V_{OI} , V_{SR} and V_{SI} are given

as the functions of the nucleon-nucleon scattering phase shifts at the definite incident energy. It is important to note that these four quantities depend on $r_0 = R/A^{1/3}$ where R is radius and A is mass number of the target nucleus, even if the nucleon density in the target nucleus is assumed A -independent. But the expression of the polarization is independent of the assumed nucleon density or the charge distribution and radius R of the target nucleus, moreover, independent of r_0 .

And, the expression predicts that the nucleons elastically scattered from all nuclei with $J=0$ and $T=0$ or α -particle type nuclei, should polarize equally. This prediction is checked by available experimental data in Figs. 2 and 3.

The difference between the results based on the meson theoretical phase shifts and those based on the Gammel-Thaler and Signell-Marshak phase shifts are observed at very small scattering angles. This difference is mainly owing to the difference between the parameters V_{SR} corresponding to those sets of phase shifts, since the polarization is roughly proportional, at small scattering angles, to

$$\frac{(V_{CI} V_{SR} + V_{CR} |V_{SI}|) / \sin \theta}{V_{CI}^2 + V_{CR}^2}.$$

In conclusion, we would like to emphasize that the set of phase shifts derived from the meson-theoretical potential reproduces the experimental polarization of nucleon-light nuclei scattering.

The author would like to express his deep gratitude to Dr. J. Iwadare who suggested this problem, for his sympathetic

tic understanding, guidance and continuous support. Also the author would like to thank Prof. M. Kobayashi for his kind encouragement.

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Integral Equation for Pair Distribution Function

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One of the authors has proposed the hyper-netted chain (HNC) approxi-

mation to consider by means of the Fourier transformation as many graphs as possible in the virial expansion formulas for the free energy and the pair distribution function (p.d.f.).¹⁾ He has shown also that the p.d.f. in the HNC approximation satisfies an integral equation^{2,3)} which is similar to Born-Green's integral equation.⁴⁾

In this note we shall show that the technique used in the recent derivation of the formulas in the HNC approximation^{2,3)} can be applied to get an exact integral equation for the p.d.f. The HNC approximation will appear as the zeroth approximation to solve this equation.

We define the function $w(r)$ which is related with the p. d. f. $g(r)$ by

$$\ln g(r) = -\frac{\phi(r)}{kT} + w(r). \quad (1)$$

$w(r)$ can be expressed in a virial expansion form.⁵⁾

$$w(r_{12}) = \sum_{m=1}^{\infty} \frac{\rho^m}{m!} \int \cdots \int d\mathbf{r}_3 \cdots d\mathbf{r}_{m+2} \times \sum_{\substack{m+2 \geq i > j \geq 3 \\ m+2 \geq k \geq 3 \\ 2 \geq \kappa \geq 1}}^{(W)} \prod b_{ij} b_{k\kappa} \quad (2)$$

where

$$b_{ij} \equiv b(r_{ij}) = e^{-\phi(r_{ij})/kT} - 1.$$

$\sum^{(W)}$ denotes that the sum is taken over all products (or graphs) in which each particle of the set $\{\mathbf{r}_3, \cdots, \mathbf{r}_{m+2}\}$ is connected to \mathbf{r}_1 and \mathbf{r}_2 by an independent path and also in which the particles $\{\mathbf{r}_3, \cdots, \mathbf{r}_{m+2}\}$ are connected among themselves independently of \mathbf{r}_1 and \mathbf{r}_2 .

We express $w(r_{12})$ symbolically by $w \Big|_0$

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or by $\bigcirc \xrightarrow{W} \bigcirc$.^{*} The other symbols appearing in the following have the analogous meanings.

To classify the graphs appearing in

$$\begin{aligned} \bigcirc \xrightarrow{W} \bigcirc &= \bigcirc \xrightarrow{X} \bigcirc + \bigcirc \xrightarrow{Z} \bullet \xrightarrow{Z} \bigcirc + \bigcirc \xrightarrow{Z} \bullet \xrightarrow{Z} \bullet \xrightarrow{Z} \bigcirc + \dots \\ &\equiv \bigcirc \xrightarrow{X} \bigcirc + \bigcirc \xrightarrow{Z_s} \bigcirc, \end{aligned} \quad (3f)$$

where a black circle means an “ s -point,” over the coordinate corresponding to which the integration is to be performed. $\bigcirc \xrightarrow{X} \bigcirc$ and $\bigcirc \xrightarrow{Z} \bigcirc$ are symbols expressing $x(r_{12})$ and $z(r_{12})$ which are defined by (2) if we replace $\sum^{(W)}$ there by $\sum^{(X)}$ and $\sum^{(Z)}$, respectively, where $\sum^{(X)}$ means that the sum is taken over the graphs having no s -point among those in $\sum^{(W)}$ and $\sum^{(Z)}$ means that the sum is taken over the graphs which have no s -point and for which each particle of the set $\{\mathbf{r}_3, \dots, \mathbf{r}_{m+2}\}$ is connected to \mathbf{r}_1 and \mathbf{r}_2 by an independent path. It is to be noted that the graphs in $\sum^{(Z)}$ need not be connected among themselves if we remove \mathbf{r}_1 and \mathbf{r}_2 .

The graphs in $\sum^{(Z)}$ can be grouped together by the numbers of parts which is divided when we remove \mathbf{r}_1 and \mathbf{r}_2 , that is

$$\begin{aligned} \bigcirc \bigcirc &= \bigcirc \bigcirc + \bigcirc \bigcirc \bigcirc W + \bigcirc \bigcirc \bigcirc W W + \dots \\ &+ \bigcirc \bigcirc \bigcirc + W \bigcirc \bigcirc W + W \bigcirc \bigcirc W W + \dots, \end{aligned} \quad (4f)$$

$\sum^{(W)}$, we introduce the notion “ s -point,” which is the point by which the graph can be separated into two independent parts. We group the graphs in $\sum^{(W)}$ together by the numbers of s -points, namely

where $\bigcirc \bigcirc$ denotes $b(r_{12})$. In (4f) we have taken account, in particular, of the fact that $\bigcirc \bigcirc$ does not include $\bigcirc \bigcirc$ and that $\bigcirc \bigcirc$ has no s -points.

Following the analysis in the foregoing works,^{1,2,3)} we can express the contents of (3f) as

$$W(k) = X(k) + \frac{\rho Z(k)^2}{1 - \rho Z(k)}. \quad (3)$$

The contents of (4f) are written as

$$z(r) = b(r)e^{w(r)} + x(r) + e^{w(r)} - 1 - w(r),$$

which is rewritten by using (1) as

$$z(r) = g(r) - 1 - w(r) + x(r). \quad (4)$$

In (3) and (4), $W(k)$, $X(k)$ and $Z(k)$ are Fourier transforms of $w(r)$, $x(r)$ and $z(r)$ respectively.

^{*} To be more exact, it might be better to use the symbol $\bigcirc \xrightarrow[r_1 \quad r_2]{W} \bigcirc$ rather than $\bigcirc \xrightarrow{W} \bigcirc$.

As defined above,

$$x(r_{12}) = \sum_{m=2}^{\infty} \frac{\rho^m}{m!} \int \cdots \int d\mathbf{r}_3 \cdots d\mathbf{r}_{m+2} \times \sum_{\substack{(X) \\ m+2 \geq l \geq j \geq 3 \\ m+2 \geq k \geq 3 \\ 2 \geq i \geq 1}} \prod b_{ij} b_{k\kappa}. \quad (5)$$

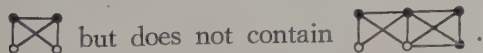
Taking notice of the relation

$$z(r) + z_s(r) = z(r) + w(r) - x(r) = g(r) - 1$$

(cf. (4) and (3f)), we can rewrite (5) as follows:

$$x(r_{12}) = \sum_{m=2}^{\infty} \frac{\rho^m}{m!} \int \cdots \int d\mathbf{r}_3 \cdots d\mathbf{r}_{m+2} \times \sum_{\substack{(X') \\ m+2 \geq l \geq j \geq 3 \\ m+2 \geq k \geq 3 \\ 2 \geq i \geq 1}} \prod \{g(r_{ij}) - 1\} \{g(r_{k\kappa}) - 1\}. \quad (6)$$

Here $\sum^{(X')}$ means that the sum is to be taken over all those products in $\sum^{(X)}$ which consist of junctions alone (A junction means a point at which three or more lines meet.), except for \mathbf{r}_1 and \mathbf{r}_2 , and for which there are no parts which are connected to any part of the graph including \mathbf{r}_1 and \mathbf{r}_2 by only two points. For example, $\sum^{(X')}$ contains



but does not contain



In principle a set of (1), (3), (4) and (6) can be used for determining the p.d.f. $g(r)$ exactly. In practice we must restrict the types of graphs appearing on the right-hand side of (6) within some special types. If we take $x(r) = 0$ as the zeroth approximation, the set of (1), (3) and (4) is reduced to

$$W(k) = \rho Z(k)^2 / \{1 - \rho Z(k)\}$$

and

$$Z(r) = e^{-\phi(r)/kT + w(r)} - 1 - w(r),$$

which are just the equations in the HNC approximation. The next approximation will be to approximate $x(r)$ by the contribution corresponding to the

graph . Such a type of integral

has been treated by Nijboer and van Hove.⁶⁾ In this way we get a method to improve the HNC approximation systematically and we can in principle reach the exact p.d.f.

More detailed calculations and discussions on this subject will be published in this journal.

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On the Collective Excitation of Spherical Nuclei

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In the previous paper¹⁾ (hereafter referred to as I), we have attempted to clarify the mechanism of the nuclear collective excitation from the standpoint of the particle excitation, and the fundamental idea was illustrated by taking

the simplified two-dimensional harmonic oscillator shell model with the quadrupole-quadrupole interaction.

Recently Belyaev²⁾ has investigated the effects of the pairing interaction on nuclear properties. One of the important results is the existence of such a kind of vibrational motions in spherical nuclei that is caused by the particles in the outermost partly-filled shell.

Such a type of collective excitations could not result from our simplified model in I. In this note, however, we shall show that our method developed in I can also produce the collective mode if we take the pairing interaction into account.

Let us consider the following simplified Hamiltonian :

$$H = H_0 + H_{Q-Q}, \quad H_0 = \sum_{m=-\Omega}^{+\Omega} (\epsilon - \lambda) a_m^+ a_m - G \sum_{m, m' > 0}^{\Omega} a_m^+ a_{-m}^+ a_{-m'} a_{m'}, \quad (1)$$

$$H_{Q-Q} = -\frac{1}{2} k \sum_{\substack{m_1, m_2 \\ m_1', m_2' = -\Omega}}^{+\Omega} \langle m_1 | Q_{2M} | m_1' \rangle \times \langle m_2 | Q_{2M}^* | m_2' \rangle a_{m_1}^+ a_{m_1'} a_{m_2}^+ a_{m_2'},$$

where

$$\begin{aligned} \langle m | Q_{2M} | m' \rangle &= \langle nljm | r^2 Y_M^{(2)}(\theta, \varphi) | nljm' \rangle \\ &= \langle nl | r^2 | nl \rangle \langle l j || Y^{(2)} || l j \rangle / \sqrt{5} \\ &\times (-1)^{j-m'} (jjm - m' | 2M). \end{aligned}$$

Here we adopt the $j-j$ coupling shell model, and consider only the particles in the outermost partly-filled shell with a set of quantum numbers (n, l, j) for simplicity. The second term in H_0 represents the simplified pairing interaction³⁾ and H_{Q-Q} the quadrupole-quadrupole interaction. λ is the chemical

potential and $\Omega \equiv (2j+1)/2$.

The ground state of H_0 is represented by the vacuum state $|c_0\rangle$ with respect to the Bogolubov Fermion α_m , i.e., $\alpha_m |c_0\rangle = 0$. α_m are derived from a_m by the Bogolubov transformation :

$$\begin{aligned} a_m &= u_m \alpha_m + v_m \alpha_{-m}^+, \\ u_m &= u_{-m} = u \\ v_m &= -v_{-m} = \begin{cases} +v & m > 0 \\ -v & m < 0 \end{cases} \end{aligned} \quad (2)$$

where

$$\begin{aligned} u^2 &= \frac{1}{2} \{ 1 - (\epsilon - \lambda) / \sqrt{(\epsilon - \lambda)^2 + \Delta^2} \} \\ &= (1 - N/2\Omega), \\ v^2 &= \frac{1}{2} \{ 1 - (\epsilon - \lambda) / \sqrt{(\epsilon - \lambda)^2 + \Delta^2} \} \\ &= N/2\Omega, \end{aligned}$$

and

$$\Delta^2 = (G\Omega/2)^2 \{ 1 - (1 - N/\Omega)^2 \}$$

(N : the particle number in the shell under consideration).

By the transformation (2), H_0 becomes approximately

$$H_0 = U + \sum_{m=-\Omega}^{+\Omega} E \alpha_m^+ \alpha_m, \quad (3)$$

where U is the vacuum energy and $E = \sqrt{(\epsilon - \lambda)^2 + \Delta^2} = G\Omega/2$.

Now, let us consider the transformed H_{Q-Q} and leave only the following pair (quasi-particle) excitation terms in the same way as in I :

$$\begin{aligned} H'_{Q-Q} &= -\frac{1}{2} K \sum_{M=-2}^{+2} \{ C_{2M}^+ - (-1)^M C_{2-M} \} \\ &\times \{ C_{2M} - (-1)^M C_{2-M}^+ \}, \end{aligned} \quad (4)$$

where

$$K = k | \langle nl | r^2 | nl \rangle \langle l j || Y^{(2)} || l j \rangle |^2 / 5$$

and

$$\begin{aligned}
 C_{2M} &= \sum_{mm'} (-1)^{-m'} (jjm - m' | 2M) \\
 &\quad \times u_m v_m \alpha_{-m'} \alpha_m, \\
 C_{2M}^+ &= \sum_{mm'} (-1)^{+m'} (jjm - m' | 2M) \\
 &\quad \times u_m v_m \alpha_m^+ \alpha_{-m'}^+.
 \end{aligned} \quad (5)$$

If we make an approximation in the same way as in I:

$$\begin{aligned}
 [C_{2M}, C_{2M}^+] &\simeq [C_{2M}, C_{2M}^+] |c_0\rangle \\
 &= \delta_{MM'} \cdot 2u^2 v^2,
 \end{aligned} \quad (6)$$

we obtain the eigenvalue equation for the collective excitation from the following equations:

$$\begin{aligned}
 \hbar\omega \langle \Psi | C_{2M}^+ | \Psi_0 \rangle \\
 + \langle \Psi | [C_{2M}, H_0 + H'_{Q-Q}] | \Psi_0 \rangle &= 0, \\
 \hbar\omega \langle \Psi | (-1)^M C_{2-M} | \Psi_0 \rangle \\
 + \langle \Psi | [(-1)^M C_{2-M}, H_0 + H'_{Q-Q}] | \Psi_0 \rangle &= 0,
 \end{aligned}$$

where $\hbar\omega = E_c - E_0$, and $|\Psi\rangle$ is an eigenstate of $H_0 + H'_{Q-Q}$ with eigenvalue E_c and $|\Psi_0\rangle$ the ground state of $H_0 + H'_{Q-Q}$ with energy E_0 . Then the eigenvalue equation for the collective excitation becomes

$$1 = \left\{ \frac{1}{\hbar\omega + 2E} - \frac{1}{\hbar\omega - 2E} \right\} (2Ku^2v^2). \quad (7)$$

Solving Eq. (7), we have

$$\hbar\omega = GQ \sqrt{1 - \theta_N / \theta_{N_0}} \quad (8)$$

where $\theta_N = \{1 - (1 - N/Q)^2\}$

and $\theta_{N_0} = GQ/4K$.

The quadrupole vibration energy (8) is just equivalent to Eq. (163) in Belyaev's paper,²⁾ which has been obtained by use of the "Cranking Model." θ_{N_0} represents the ratio of the strength of the pairing interaction to that of the quadrupole-quadrupole interaction. If $\theta_{N_0} > 1$, the nucleus remains spherical for any occupation. If $\theta_{N_0} < \theta_N$, i.e., $Q(1 - \sqrt{1 - \theta_{N_0}}) < N < Q(1 + \sqrt{1 - \theta_{N_0}})$ the spherical nucleus is unstable and becomes the deformed nucleus.

Here it is interesting to see that we have obtained Eq. (8) by the method which does not necessarily depend on the "adiabatic approximation," underlying Belyaev's derivation. Actually, the adiabatic description of the vibration is valid only if $\hbar\omega \ll GQ$, i.e., near the point of instability.

The systematic investigation of the nuclear collective motion in the more realistic case from the standpoint of the particle excitations is going to be prepared.

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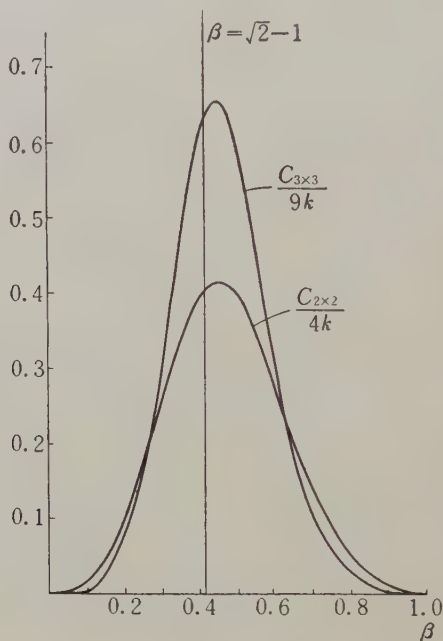
Errata

On the Theory of Cooperative Phenomena

Shigetoshi KATSURA

Prog. Theor. Phys. **11** (1954) 476 referred to as I. Proc. Int. Conf. Theor. Phys. (Kyoto, 1953), p. 534 referred to as II.

" $\beta=0.42$ " in I (p. 482, line 8; p. 491, line 26) and in II (p. 534, line 34 and Table 1) should be corrected as " $\beta=0.450$ (2×2) and 0.442 (3×3)". " $J/kT_c=0.87$ " in II (Table 1) should be corrected as " $J/kT_c=0.799$ (2×2) and 0.816 (3×3)". Fig. 6 in I and Fig. 4 in II should be replaced by the following figure.

Fig. 6. C/Nk

In this point the author is indebted to Dr. S. G. Brush's remark (private communication).

"(2×2)" in I (p. 489, line 27 and caption of Fig. 16) and in II (caption of Fig. 10 and 11) should be read ($2\times\infty$)".

" $\Xi^{1/9}$ " and " α " in the ordinate and abscissa in Fig. 1 in I (p. 478) should be read " $\beta\Xi^{1/9}$ " and " $\alpha/2$ " respectively.

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Thermodynamic Functions of the Relativistic Thomas-Fermi Atom at Low Temperatures

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The thermodynamic functions of the relativistic Thomas-Fermi atom, at low temperatures and high pressures, have been obtained in terms of the boundary and initial parameters of the Thomas-Fermi and 'perturbation' equations. Finally these functions are expressed in terms of a parameter y_0 which depends on the density of the material. Our final results are, however, valid only when the densities are very high such as those occurring in the interiors of white dwarf stars.

§ 1. Introduction

Expressions for the thermodynamic functions of the non-relativistic Thomas-Fermi atom at low temperatures have been obtained by Gilvarry³⁾ in terms of the atomic radius and the boundary and initial parameters associated with the solutions of the 'unperturbed' Thomas-Fermi equation and the 'first order perturbation equation'. For the sake of generality we have obtained, in this paper, the expressions for these functions by using the relativistic Thomas-Fermi model of the atom (see Gilvarry²⁾). Our final results are true only in the case when pressure is very high or the atomic radius very small. In the extreme relativistic approximation ($\rho_m \gg 10^6$ g/cc) our expressions are identical with those obtained by Kothari and Singh⁶⁾ for the relativistic degenerate Fermi-Dirac gas, apart from a term due to electrostatic interactions. In the non-relativistic approximation ($\rho_m \ll 10^6$ g/cc) our expressions go over to those obtained by Gilvarry³⁾ and March.⁷⁾

§ 2. Helmholtz free energy and virial theorem

We shall first obtain an expression for Helmholtz free energy F for the neutral atom and then derive the 'virial theorem' in the case of the relativistic Thomas-Fermi atom. Let us label by i the various stationary states of the electrons in the atom. The probability ρ_i that any one of the states (say the i th state) characterised by the total energy E_i , be occupied by an electron is given by:

$$\rho_i = \left[1 + \exp \frac{E_i - E_m}{kT} \right]^{-1}, \quad (1)$$

where $E_m - mc^2 \equiv kT\eta$ is the 'chemical potential' per particle and $\sum_i \rho_i = Z$ (total number of electrons in the atom) determines E_m . The entropy S is given by

$$S = -k \sum_i \{ (1 - \rho_i) \ln(1 - \rho_i) + \rho_i \ln \rho_i \}, \quad (2)$$

where the summation is to extend over all the states 'i'. From (1) and (2) we have

$$TS = -kT \sum_i \ln(1 - \rho_i) - ZkT\eta + \sum_i \rho_i (E_i - mc^2). \quad (3)$$

Since the total inter-electron interaction energy E_{ee} appears twice in the summation as a result of double counting of interactions,

$$\sum_i \rho_i (E_i - mc^2) = E_T + E_{ee}, \quad (4)$$

where E_T is the total energy of the electrons of the atom at temperature T .

Using (3) and (4) we have

$$\begin{aligned} F &\equiv E_T - TS = ZkT\eta - E_{ee} + kT \sum_i \ln(1 - \rho_i) \\ &= ZkT\eta - E_{pot} + E_{en} + kT \sum_i \ln(1 - \rho_i). \end{aligned} \quad (5)$$

Replacing the sum over various states, labelled by i , in the last term of (5) by integration, we have

$$F = ZkT\eta - E_{pot} + E_{en} + kT \int_{E_l}^{\infty} d\tau \int \ln(1 - \rho) n(r, E) dE, \quad (6)$$

where $n(r, E)dE$ is the number of states per unit volume within the energy range dE and E_l is the lower limit for E . For the 'density of states' we shall use the expression derived by Rudkjøbing¹⁰⁾ on the basis of Dirac equation for the electron in a central potential field $V(r)$, which is best suited for our purpose. We have

$$n(r, E) dE = \frac{8\pi}{h^3 c^3} \left[(E + eV)^2 - m^2 c^4 - \left(re \frac{dV}{dr} \right)^2 \right]^{1/2} (E + eV) dE, \quad (7)$$

where r is the distance of the point in question from the nucleus of the atom. Thus we have

$$F = ZkT\eta - E_{pot} + E_{en} - \frac{8\pi}{3h^3 c^3} \int_{E_l}^{\infty} d\tau \int \frac{\{ (E + eV)^2 - m^2 c^4 - (re \cdot dV/dr)^2 \}^{3/2}}{1 + \exp[(E - E_m)/kT]} dE. \quad (8)$$

We now proceed to derive the virial theorem. For the Helmholtz free energy density we have, according to (8),

$$f = \rho kT\eta - \epsilon_{ee} - \frac{8\pi}{3h^3 c^3} \int_{E_l}^{\infty} \frac{\{ (E + eV)^2 - m^2 c^4 - (re \cdot dV/dr)^2 \}^{3/2}}{1 + \exp[(E - E_m)/kT]} dE, \quad (9)$$

where ϵ_{ee} is the inter-electron interaction energy density.

Hence,

$$\nabla f = kT \eta \nabla \rho + \frac{1}{2} e V_e \nabla \rho + \frac{1}{2} e \rho \nabla V_e - \nabla I, \quad (10)^*$$

where I is given by

$$I \equiv \frac{8\pi}{3h^3 c^3} \int_{E_1}^{\infty} \frac{\{(E + eV)^2 - m^2 c^4 - (re \cdot dV/dr)^2\}^{3/2}}{1 + \exp[(E - E_m)/kT]} dE.$$

Also we have

$$\nabla f = kT \eta \nabla \rho + \frac{1}{2} e V_e \nabla \rho - \frac{1}{2} e \rho \nabla V_e - e \rho \nabla V_N, \quad (11)$$

where the second term on the right-hand side takes account of the double counting (implicit in the first term) of the effect on f of a change in ρ and the last two terms take account of the effect on f of a change in the potential $V(r)$. (See Gilvarry⁴). From (10) and (11) we have

$$\nabla I = e \rho \nabla (V_e + V_N) = e \rho \nabla V. \quad (12)$$

Since I_b = the value of I at the atomic boundary = P^0 , we have

$$3Pv = I_b \int \mathbf{r} \cdot d\mathbf{S} = \int I \mathbf{r} \cdot d\mathbf{S}, \quad (13)$$

$d\mathbf{S}$ being a directed element of the surface area and v the atomic volume. From Gauss's theorem (13) becomes

$$3Pv = 3 \int I d\tau + \int \mathbf{r} \cdot \nabla I d\tau;$$

which, using (12), gives

$$3Pv = 3 \int I d\tau + \int \mathbf{r} \cdot e \rho(r) \nabla V(r) d\tau. \quad (14)$$

The second term on the right-hand side is, according to a lemma by Duffin¹ (see also Gilvarry⁴), equal to the total potential energy of the electrons in the atom. Hence we have

$$3Pv = \frac{8\pi}{h^3 c^3} \int_{E_1}^{\infty} d\tau \int \frac{\{(E + eV)^2 - m^2 c^4 - (re \cdot dV/dr)^2\}^{3/2}}{1 + \exp[(E - E_m)/kT]} dE + E_{pot}. \quad (15)$$

Eq. (15) is the required form of the virial theorem. From (8) and (15) we have

* We have $E_{ee} = \int \epsilon_{ee} d\tau = -\frac{e}{2} \int \rho V_e d\tau$, and $E_{en} = \int \epsilon_{en} d\tau = -e \int \rho V_N d\tau$,

V_e and V_N being the electrostatic potentials at r due to the electron distribution and nuclear charge respectively, so that $V = V_e + V_N$.

$$F = ZkT\eta + E_{en} - Pv - \frac{2}{3}E_{pot}. \quad (16)$$

§ 3. Thermodynamic functions for low temperatures

It was pointed out by Marshak and Bethe⁸⁾ that when kT is small compared to the maximum kinetic energy of the electrons near the boundary of the atom, i.e. when

$$(E_m + eV_b - mc^2)/kT \gg 1,$$

the effect of temperature can be treated by the perturbation method. In an earlier paper,⁹⁾ where we regarded temperature as a perturbation, we derived the relativistic Thomas-Fermi equation in the form:

$$\frac{1}{x} \frac{d^2 \Phi}{dx^2} = \lambda \left\{ \left(\frac{\Phi}{x} \right)^2 + \beta \frac{\Phi}{x} - \left(\frac{d\Phi}{dx} - \frac{\Phi}{x} \right)^2 \right\}^{3/2} \\ \times \left[1 + \zeta k^2 T^2 \frac{(\Phi^2/x)^2 \cdot (1 + \beta \Phi/x) + (\beta^2/8) - (1/2)(d\Phi/dx - \Phi/x)^2}{\{(\Phi/x)^2 + \beta \cdot \Phi/x - (d\Phi/dx - \Phi/x)^2\}^2} \right], \quad (17)$$

where Φ is a dimensionless variable (called the 'perturbed' TF function) given by

$$\frac{Ze^2}{r} \Phi(x) \equiv E_m + eV(r) - mc^2,$$

and $a_0 x \equiv r$; and where a_0 = Bohr radius,

$$\lambda \equiv \frac{32\pi^2 Z^3 e^6}{3h^3 c^3} = 4\alpha^3 Z^2 / 3\pi = 1.65 \times 10^{-7} Z^2,$$

$$\beta \equiv 2mc^2 a_0 / Ze^2 = 3.76 \times 10^4 \frac{1}{Z},$$

and

$$\zeta \equiv \pi^2 \beta^2 / (2mc^2)^2 = 5.19 \times 10^{21} \frac{1}{Z^2}. \quad (18)$$

We shall be interested here, in the terminology of Gilvarry, in the first order temperature perturbation. Hence we may express Φ as

$$\Phi = \phi + \zeta k^2 T^2 \chi, \quad (19)$$

where ϕ is the 'unperturbed' TF function and χ may be referred to as the 'first order perturbation function'. ϕ satisfies the following differential equation:

$$\frac{1}{x} \frac{d^2 \phi}{dx^2} = \lambda \left\{ \left(\frac{\phi}{x} \right)^2 + \beta \frac{\phi}{x} - \left(\frac{d\phi}{dx} - \frac{\phi}{x} \right)^2 \right\}^{3/2}, \quad (20)$$

which has been dealt with in the earlier paper.⁹⁾

χ satisfies the following differential perturbation equation:

$$\begin{aligned} \frac{d^2\chi}{dx^2} = & \lambda \frac{\phi^3}{x^2} \left[\frac{3\chi}{\phi} z^{3/2} - \frac{3}{2} z^{1/2} \left\{ \frac{\beta x}{\phi^2} \chi + 2 \left(\frac{x}{\phi} \frac{d\phi}{dx} - 1 \right) \right. \right. \\ & \times \left. \left(\frac{x}{\phi} \frac{d\chi}{dx} - \frac{\chi}{\phi^2} x \frac{d\phi}{dx} \right) \right\} + \frac{(\phi/x)^2 z + (\beta^2/8)}{(\phi/x)^4 z^{1/2}} \left. \right], \end{aligned} \quad (21)$$

where

$$z \equiv 1 + \frac{\beta x}{\phi} - \left(\frac{x}{\phi} \frac{d\phi}{dx} - 1 \right)^2.$$

The boundary conditions for (17) are:

$$(d\phi/dx)_{x_0} = \phi_b/x_0$$

and

$$\phi_i = 1, \quad (22)$$

which give the following boundary conditions for Eqs. (20) and (21) respectively:

$$(d\phi/dx)_{x_0} = \phi_b/x_0; \quad \phi_i = 1, \quad (23)$$

and

$$(d\chi/dx)_{x_0} = \chi_b/x_0; \quad \chi_i = 0. \quad (24)$$

We shall follow the method of Marshak and Bethe⁸⁾ and assume that the temperature perturbation does not alter the atomic volume.

The potential energy of the electrons in the atom can be calculated by using Duffin's lemma.¹⁾ Thus

$$E_{pot} = e \int \rho \mathbf{r} \cdot \nabla V(r) d\tau,$$

which, expressing $V(r)$ in terms of $\phi(x)$, gives for a spherically symmetric atom,

$$E_{pot} = \frac{Z^2 e^2}{a_0} \left[\phi_i' - \frac{1}{2} \frac{\phi_b^2}{x_0} + \frac{1}{2} \int_0^{x_0} \left(\frac{d\phi}{dx} \right)^2 dx \right]. \quad (25)$$

The pressure can be expressed in terms of the boundary parameters as

$$P = p + \zeta k^2 T^2 p_1, \quad (26)$$

where $p = \pi m^4 c^5 / 3h^3 \cdot f(y)$, being the pressure corresponding to the case of complete degeneracy⁹⁾ and p_1 is given by

$$p_1 \equiv \frac{\pi m^4 c^5}{3h^3} \left[y f'(y) \frac{\chi_b}{\phi_b} \left(1 - \frac{1}{2} \frac{\beta x_0 / \phi_b}{1 + (\beta x_0 / \phi_b)} \right) + y (y^2 + 1)^{1/2} \frac{16}{\beta^2} \right]. \quad (27)$$

Here

$$f(y) \equiv (2y^3 - 3y)(y^2 + 1)^{1/2} + 3 \operatorname{Sinh}^{-1} y,$$

and

$$y \equiv \frac{2}{\beta x_0} \phi_b \left(1 + \frac{\beta x_0}{\phi_b} \right)^{1/2}. \quad (28)$$

Further, E_{en} can be expressed as $ZV_e(0)$ where $V_e(0)$ is the potential at the nucleus on account of the electron distribution alone. So,

$$V_e(0) = \left(V - \frac{Ze}{r} \right)_{r=0} = \frac{d}{dr} (rV(r))_{r=0}.$$

Hence

$$E_{en} = \frac{Z^2 e^2}{a_0} \left(\phi_i' - \frac{\phi_b}{x_0} \right). \quad (29)$$

Also

$$kT\eta \equiv E_m - mc^2 = \frac{Ze^2}{a_0} \frac{\phi_b}{x_0}, \quad (30)$$

since $V(r_0) = 0$ for the neutral atom.

Using (25), (26), (27), (28), (29) and (30), Eq. (16) becomes

$$\begin{aligned} F = & \left[\frac{1}{3} \frac{Z^2 e^2}{a_0} \left(\phi_i' + \frac{\phi_b^2}{x_0} - \int_0^{x_0} (\phi')^2 dx \right) - \frac{4\pi^2 m^4 c^5}{9h^3} a_0^3 x_0^3 f(y) \right] \\ & - \frac{1}{2} k^2 T^2 \left[\frac{4\pi^2 m^4 c^5}{9h^3} a_0^3 x_0^3 \left\{ y f'(y) \frac{\chi_b}{\phi_b} \left(1 - \frac{1}{1 + \beta x_0' \phi_b} \right) + \frac{16}{\beta^2} y (1 + y^2)^{1/2} \right\} \right. \\ & \left. - \frac{Z^2 e^2}{3a_0} \left\{ \chi_i' + \frac{2\chi_b \phi_b}{x_0} - 2 \int_0^{x_0} \chi' \phi' dx \right\} \right]. \quad (31) \end{aligned}$$

With the free energy expressed in terms of the boundary and initial parameters, all other thermodynamic functions of the neutral atom can be calculated.

In order to express F in terms of x_0 we are required to know the boundary and initial parameters viz. ϕ_b , χ_b , ϕ_i and χ_i as functions of x_0 . This would require us to solve (20) and (21) subject to (23) and (24) respectively. We have here considered, for simplicity, only the case when atomic volume is small. In such a case it is possible to expand $\phi(x)$ and $\chi(x)$ in Taylor's series about the boundary point, as

$$\phi(x) = \phi(x_0) + \sum_{n=1}^{\infty} t_n (x_0 - x)^n, \quad (32)$$

and

$$\chi(x) = \chi(x_0) + \sum_{n=1}^{\infty} s_n (x_0 - x)^n. \quad (32a)$$

The coefficients up to t_5 in (32) have been given in an earlier paper.⁹⁾ The condition $\phi(0) = 1$ enables one to express $\phi(x_0)$ as a function of x_0 . As a result we

have, for (28),

$$y = y_0 \left\{ 1 - \frac{3}{10} \left(\frac{\lambda}{3} \right)^{1/3} \left(1 + \frac{1}{2y_0^2} - \frac{6}{5} \left(\frac{\lambda}{3} \right)^{1/3} + O\left(\frac{1}{y_0^3} \right) + O(\lambda^{2/3}) \right) \right\} \quad (33)$$

for

$$y_0 \gg 1, \quad (\text{extreme relativistic approx.})$$

and

$$y = y_0 \left\{ 1 - \frac{3^{2/3}}{10} \frac{\lambda^{1/3}}{y_0} (1 + \frac{1}{2} y_0^2 + O(y_0^4)) + O\left(\frac{\lambda^{2/3}}{y_0^2} \right) + O(\lambda^{2/3}) \right\}, \quad (33a)$$

for

$$\left(\frac{324}{\pi} Z^2 \right)^{1/3} \frac{\alpha}{10} \ll y_0 \ll 1, \quad (\text{non-relativistic approx.})$$

where

$$y_0 = \frac{2 \cdot 3^{1/3}}{\lambda^{1/3} \beta x_0} = (\rho_m / B)^{1/3},$$

and

$$B = \frac{8\pi m^3 c^3 (A/Z) m_H}{3h^3} = (0.982 A/Z) \times 10^6 \text{ g/cc}, \quad (34)$$

ρ_m being the density of the material.

For the coefficients up to s_3 in (32a) we have

$$\begin{aligned} s_1 &= -\chi_0/x_0, \\ s_2 &= \frac{\lambda}{2} \frac{\phi_b^3}{x_0^2} \left[\frac{3\chi_b}{\phi_b} \left(1 + \frac{\beta x_0}{\phi_b} \right)^{3/2} \left(1 - \frac{\sigma}{2} \right) + \frac{(\phi_b/x_0)^2 \cdot (1 + \beta x_0/\phi_b) + \beta^2/8}{(\phi_b/x_0)^4 \cdot (1 + \beta x_0/\phi_b)^{1/2}} \right], \\ s_3 &= -\frac{\lambda}{6} \frac{\phi_b^3}{x_0^3} \left[\frac{3\chi_b}{\phi_b} \left(1 + \frac{\beta x_0}{\phi_b} \right)^{3/2} \left(1 - \frac{\sigma}{2} \right) + \frac{(\phi_b/x_0)^2 \cdot (1 + \beta x_0/\phi_b) + \beta^2/8}{(\phi_b/x_0)^4 \cdot (1 + \beta x_0/\phi_b)^{1/2}} \right], \end{aligned} \quad (35)$$

where

$$\sigma = \frac{\beta x_0/\phi_b}{1 + \beta x_0/\phi_b}.$$

The condition $\chi(0)=0$ may be used to express $\chi(x_0)$ or χ_b in terms of x_0 . We have only taken account of the first three terms in the expansion. This gives

$$\frac{\chi_b}{\phi_b} = -\frac{4}{3} \frac{y^2 + \frac{1}{2}}{\beta^2 y^4 (1 - \sigma/2)}. \quad (36)$$

Thus, using Taylor's series expansion for the functions, we have, for F and P in the extreme relativistic approximation ($y_0 \gg 1$ or $\rho_m \gg 0.982 A/Z \times 10^6 \text{ g/cc}$), the following expressions:

$$F = \frac{8\pi m^4 c^5}{h^3} v y_0^3 \left[\left\{ \frac{y_0}{4} - \frac{1}{3} + \frac{1}{4y_0} + O\left(\frac{1}{y_0^3}\right) - \frac{3}{10} \alpha y_0 \left(\frac{4Z^2}{9\pi}\right)^{1/3} \right. \right. \\ \left. \times \left(1 - \frac{1}{y_0} + O\left(\frac{1}{y_0^2}\right)\right) + y_0 O(\alpha^2 Z^{4/3}) \right\} \\ \left. - \frac{\pi^2}{6} \left(\frac{kT}{mc^2}\right)^2 \left\{ \frac{1}{y_0} + \frac{1}{2y_0^3} - \frac{1}{8y_0^5} + O\left(\frac{1}{y_0^7}\right) \right\} \right], \quad (37)$$

and

$$P = \frac{hc}{8} \left(\frac{Z\rho_m}{Am_H} \right)^{4/3} \left(\frac{3}{\pi} \right)^{1/3} \left[\left\{ 1 - \frac{1}{y_0^2} + O\left(\frac{1}{y_0^4}\right) \right. \right. \\ \left. - \frac{6}{5} \left(\frac{4Z^2}{9\pi}\right)^{1/3} \alpha \left(1 + \frac{1}{y_0^2} + O\left(\frac{1}{y_0^4}\right)\right) + O(\alpha^2 Z^{4/3}) \right\} \\ \left. + \frac{2}{3} \pi^2 \left(\frac{kT}{mc^2}\right)^2 \frac{1}{y_0^2} \left(1 + \frac{3}{2y_0^2} - \frac{5}{8y_0^4} + O\left(\frac{1}{y_0^6}\right)\right) \right]. \quad (38)$$

It may be seen that apart from the terms containing α (where α is the fine structure constant), these expressions are almost identical with those obtained by Kothari and Singh⁴⁾ for the relativistic degenerate Fermi-Dirac gas in the extreme relativistic approximation. (The parameter y_0 in our case may be identified with the parameter x used by Kothari and Singh.) One can also note that, howsoever high the pressure be, the contribution of electrostatic terms in these expressions always remains appreciable.

The corresponding functions in the non-relativistic approximation ($0.00342 Z^{2/3} \ll y_0 \ll 1$ or $0.0394 AZ \ll \rho_m \ll 0.982 \cdot A/Z \times 10^6$ g/cc) may be expressed as:

$$F = \frac{3h^2}{10m} \left(\frac{3}{8\pi} \right)^{2/3} \left(\frac{Z\rho_m}{Am_H} \right)^{5/3} v \left[1 - 3 \left(\frac{4Z^2}{9\pi} \right)^{1/3} \frac{\alpha}{y_0} - \frac{5}{28} y_0^2 + \frac{5}{72} y_0^4 \right. \\ \left. + O\left(\frac{\alpha^2}{y_0^2} Z^{4/3}\right) + O(y_0^6) \right] \\ - k^2 T^2 v \frac{\pi^2 m}{2h^2} \left(\frac{8\pi}{3} \right)^{2/3} \left(\frac{Z\rho_m}{Am_H} \right)^{1/3} \left(1 + \frac{y_0^2}{2} - \frac{y_0^4}{8} + O(y_0^6) \right), \quad (39)$$

and

$$P = \frac{h^2}{5m} \left(\frac{3}{8\pi} \right)^{2/3} \left(\frac{Z\rho_m}{Am_H} \right)^{5/3} \left[\left\{ 1 - \frac{3}{2} \left(\frac{4Z^2}{9\pi} \right)^{1/3} \frac{\alpha}{y_0} + \frac{9}{20} \left(\frac{4Z^2}{9\pi} \right)^{2/3} \frac{\alpha^2}{y_0^2} \right. \right. \\ \left. - \frac{5}{14} y_0^2 + \frac{5}{24} y_0^4 + O\left(\frac{\alpha^3}{y_0^3} Z^2\right) + O(y_0^6) \right\} \\ \left. + 5 \frac{\pi^2 m^2}{h^4} k^2 T^2 \left(\frac{8\pi}{3} \right)^{4/3} \left(\frac{Am_H}{Z\rho_m} \right)^{4/3} \left\{ 1 + \frac{y_0^4}{8} + O(y_0^6) \right\} \right]. \quad (40)$$

It may be observed that in the non-relativistic limit ($y_0 \ll 1$) these thermodynamic

functions go over to those given by Gilvarry³⁾ and March.⁷⁾ The terms involving y_0^2 and y_0^4 are thus the relativistic corrections.

Concluding, I wish to express my sincere thanks to Professor D. S. Kothari and Professor F. C. Auluck for their constant interest and encouragement during the course of this work.

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Diamagnetism of Electrons in a Weak Periodic Potential

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A method recently developed by Blatt, Matsubara and May¹⁾ for evaluating the magnetic response of a system in a *non gauge-invariant* approximation is illustrated by a simple example: that of electrons moving in a weak periodic lattice. The result is in (quantitative) agreement with more detailed gauge-invariant calculations by other authors. The calculation is also of interest in that it provides an example where use of the "London gauge", $\text{div} \mathbf{A} = 0$, leads to unphysical results.

§ 1. Introduction

Recently Blatt, Matsubara and May¹⁾ (referred to as BMM) have developed a new technique for evaluating the magnetic response of a system described by an approximate hamiltonian which is *not* gauge-covariant.*** Briefly this technique is as follows: In general the linear magnetic response of a system has the form

$$\mathbf{M}(\mathbf{q}) = K(\mathbf{q}) \mathbf{B}(\mathbf{q}) \quad (1.1)$$

where \mathbf{M} is the magnetization, \mathbf{B} the magnetic induction (in Fourier space), and the kernel, K , involves only the field-free eigenvalues and eigenvectors of the hamiltonian describing the system. There are in fact an infinite number of forms for $K(\mathbf{q})$

$$K(\mathbf{q}) = K_B(\mathbf{q}) + \alpha(\mathbf{q}) \frac{G(\mathbf{q})}{q^3} \quad (1.2)$$

with $K_B(\mathbf{q})$ and $G(\mathbf{q})$ given by explicit formulae ((2.6) and (2.7)), and $\alpha(\mathbf{q})$ an arbitrary function of \mathbf{q} . Now if $K(\mathbf{q})$ has been calculated from an exact gauge-covariant hamiltonian, then $G(\mathbf{q})$ vanishes identically, and all the forms (1.2) are identical; on the other hand, if a *non* gauge-covariant approximate hamiltonian $H^{(0)}$ has been employed, then $G^{(0)}(\mathbf{q}) \neq 0$, and an indeterminate result is obtained for the kernel, depending on the parameter $\alpha(\mathbf{q})$.

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*** A hamiltonian $H(\mathbf{A})$ is called gauge-covariant if for any scalar function Λ there exists a unitary transformation U_Λ such that

$$U_\Lambda^\dagger H(\mathbf{A}) U_\Lambda = H(\mathbf{A} + \text{grad} \Lambda).$$

However, there is still one well-defined value of α , α_1 say, for which $K^{(0)}(\alpha_1)$ is *closest* to the exact value K ; if we characterize the difference between true and approximate hamiltonians by an expansion parameter ε ,

$$H = H^{(0)} + \varepsilon H^{(1)} + \dots \quad (1.3)$$

and expand the exact kernel around $K^{(0)}(\alpha)$, in powers of ε ,

$$K = K^{(0)}(\alpha) + \varepsilon K^{(1)}(\alpha) + \varepsilon^2 K^{(2)}(\alpha) + \dots \quad (1.4)$$

then we choose $\alpha = \alpha_1$ by the demand that

$$K^{(1)}(\alpha_1) = 0. \quad (1.5)$$

The ensuing kernel $K^{(0)}(\alpha_1)$ is thus a stationary approximation to the exact kernel, and the ambiguity due to the violation of gauge-covariance has disappeared.

Our aim in this paper is to illustrate the above technique by a simple example—that of electrons moving in a weak periodic lattice.

The exact hamiltonian for such a system is

$$\left(-\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}) \right) \Psi_k(\mathbf{r}) = E(\mathbf{k}) \Psi_k(\mathbf{r}) \quad (1.6)$$

where the potential $V(\mathbf{r})$ has the translational symmetry of the lattice: that is,

$$V(\mathbf{r} + \boldsymbol{\rho}_n) = V(\mathbf{r}), \quad (1.7)$$

$\boldsymbol{\rho}_n$ being any one of the lattice vectors. The electron wave-functions then have the form²⁾

$$\Psi_k(\mathbf{r}) = \exp(i\mathbf{k} \cdot \mathbf{r}) u_k(\mathbf{r}) \quad (1.8)$$

with the function $u_k(\mathbf{r})$ having the same periodicity as the lattice.

In the theory of metals, approximate wave-functions

$$\Psi_k^{(0)}(\mathbf{r}) = \exp(i\mathbf{k} \cdot \mathbf{r}) u_0(\mathbf{r}) \quad (1.9)$$

are frequently employed when dealing with a weak potential. The energy values in the approximation are

$$E^{(0)}(\mathbf{k}) = \frac{\hbar^2 k^2}{2m} + E_0 \quad (1.10)$$

and E_0 , $u_0(\mathbf{r})$ are determined from the equation

$$\left[-\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}) \right] u_0(\mathbf{r}) = E_0 u_0(\mathbf{r}). \quad (1.11)$$

The expansion parameter ε in this scheme is defined by

$$H = H^{(0)} + \varepsilon H^{(1)} \quad (1.12)$$

where H is the gauge-covariant hamiltonian (1.6) and $H^{(0)}$ the approximate hamiltonian corresponding to the above eigenvectors and eigenvalues, (1.9) and (1.10).

This approximation scheme is an improvement on the free electron one, in

that the factor $u_0(\mathbf{r})$ in Eq. (1.9) contains the essential features due to the lattice. Quantities such as the cohesive forces in metals due to conduction electrons may be calculated satisfactorily by use of the method. However, since the approximation is better for small wave-numbers \mathbf{k} than for large ones, it may be expected to lead to non gauge-invariant results for magnetic quantities; this is found to be the case (Section 2). This difficulty is then overcome by applying the BMM technique (Section 3).

Our attention is confined to evaluating the kernel in the limit of small q ($q \ll k_F$) since this region determines the essential nature of the magnetic response (larger q are pertinent to penetration phenomena, which shall not concern us).

The results are in quantitative agreement with a more detailed and fully gauge-covariant calculation by Tani.³⁾ The example is also of interest because it provides an example where the prescription⁴⁾ of calculating everything in the "London gauge" ($\text{div } \mathbf{A} = 0$ or $\alpha(\mathbf{q}) \equiv 1$) leads to nonsense.

§ 2. Calculation of $K^{(0)}(\mathbf{q})$

The quantities $K_B(\mathbf{q})$ and $G(\mathbf{q})$ mentioned in Eq. (1.2) are defined in the following manner (see BMM): For a system of N particles, with charges e_i and masses m_i , the field-free eigenfunctions of the hamiltonian H ,

$$H|k\rangle = E_k|k\rangle \quad (2.1)$$

are used to construct quantities $S_{\mu\nu}(\mathbf{q})$ and L :

$$L = \sum_i e_i^2 / m_i c^2, \quad (2.2)$$

$$S_{\mu\nu}(\mathbf{q}) = -\frac{1}{c^2} \sum_{\mathbf{k}, \mathbf{k}'} \langle \mathbf{k} | j_\mu(\mathbf{q}) | \mathbf{k}' \rangle \langle \mathbf{k}' | j_\nu(-\mathbf{q}) | \mathbf{k} \rangle F_1(E_k, E_{k'}), \quad (2.3)$$

where $\mathbf{j}(\mathbf{q})$ is defined as

$$\mathbf{j}(\mathbf{q}) \equiv \sum_i \frac{e_i}{2m_i} (\mathbf{p}_i \exp(-i\mathbf{q} \cdot \mathbf{x}_i) + \exp(-i\mathbf{q} \cdot \mathbf{x}_i) \mathbf{p}_i) \quad (2.4)$$

and $F_1(E_k, E_{k'})$ is the difference quotient

$$F_1(E_k, E_{k'}) \equiv \frac{F_0(E_k) - F_0(E_{k'})}{E_k - E_{k'}} \quad (2.5)$$

with $F_0(E)$ being the field-free distribution function of the system. $K_B(\mathbf{q})$ and $G(\mathbf{q})$ are then defined by

$$K_B(\mathbf{q}) = \frac{1}{q^2} \sum_{\mu\nu} \left\{ \frac{1}{2} \delta_{\mu\nu} - \frac{3}{2} \frac{q_\mu q_\nu}{q^2} \right\} S_{\mu\nu}(\mathbf{q}), \quad (2.6)$$

$$G(\mathbf{q}) = \sum_{\mu\nu} \frac{q_\mu q_\nu}{q^2} S_{\mu\nu}(\mathbf{q}) - L. \quad (2.7)$$

Thus for our example of electrons in a weak periodic potential, we begin the calculation of the zeroth order kernel, $K^{(0)}(\mathbf{q})$, by writing down the appropriate expression for the matrix elements of $\mathbf{j}(\mathbf{q})$:

$$\langle \mathbf{k} | \mathbf{j}_\mu(\mathbf{q}) | \mathbf{k}' \rangle = \frac{e}{2m} (k + k')_\mu \int d\mathbf{r} |u_0(\mathbf{r})|^2 \exp(i(\mathbf{k}' - \mathbf{k} - \mathbf{q}) \cdot \mathbf{r}) \quad (2.8)$$

$$= \frac{e}{2m} (k + k')_\mu X(\mathbf{k}' - \mathbf{k} - \mathbf{q}). \quad (2.9)$$

(Use has been made of the fact that $u_0(\mathbf{r})$ must be either even or odd in \mathbf{r} .)

Then, after the substitution $\mathbf{k} \rightarrow \mathbf{k} - (\mathbf{k}' + \mathbf{q})/2$, $\mathbf{k}' \rightarrow \mathbf{k} + (\mathbf{k}' + \mathbf{q})/2$, Eq. (2.3) gives $S_{\mu\nu}^{(0)}(\mathbf{q})$ as

$$S_{\mu\nu}^{(0)}(\mathbf{q}) = -\frac{e^2}{m^2 c^2} \sum_{\mathbf{k}, \mathbf{k}'} k_\mu k_\nu |X(\mathbf{k}')|^2 F_1 \left\{ \frac{(\mathbf{k} + (\mathbf{k}' + \mathbf{q})/2)^2}{2m}, \frac{(\mathbf{k} - (\mathbf{k}' + \mathbf{q})/2)^2}{2m} \right\} \quad (2.10)$$

with $F_0(E)$ the Fermi distribution function.

At this stage, some simplifying assumptions are made as to the nature of $|u_0(\mathbf{r})|^2$; since this calculation is done essentially to illustrate the BMM technique, such mathematical simplifications are unimportant. We put

$$|u_0(\mathbf{r})|^2 = C \left\{ 1 + \beta \sum_{\mathbf{r}_n} \exp(-(\mathbf{r} - \mathbf{r}_n)^2/b^2) \right\}. \quad (2.11)$$

This function provides a two-parameter fit (in terms of a strength parameter β , and a range b) to the solution of Eq. (1.11) for any specific $V(\mathbf{r})$; the free electron case corresponds to the limit $\beta \rightarrow 0$. We also assume $b \ll a$, a being the lattice spacing. The normalization constant C is given by

$$C \int |u_0(\mathbf{r})|^2 d\mathbf{r} = CV \left(1 + \beta \left[\frac{\sqrt{\pi} b}{a} \right]^3 \right) = 1. \quad (2.12)$$

Substitution of Eq. (2.11) into (2.8) leads, in the limit of infinite volume, to the result

$$X(\mathbf{k}) = CV \left\{ \delta(\mathbf{k}) + \beta \left(\frac{\sqrt{\pi} b}{a} \right)^3 \sum_{\mathbf{\kappa}_n} \delta(\mathbf{k} - \mathbf{\kappa}_n) \exp(-k^2 b^2/4) \right\} \quad (2.13)$$

and

$$|X(\mathbf{k})|^2 = \delta(\mathbf{k}) + \left[CV \beta \left(\frac{\sqrt{\pi} b}{a} \right)^3 \right]^2 \sum_{\mathbf{\kappa}_n \neq 0} \delta(\mathbf{k} - \mathbf{\kappa}_n) \exp(-k^2 b^2/2). \quad (2.14)$$

The sum over $\mathbf{\kappa}_n$ extends over all vectors of the reciprocal lattice; $\delta(\mathbf{k})$ is defined to be 1 if $\mathbf{k} = 0$, zero otherwise.

Comparison of Eqs. (2.10) and (2.14) now shows that $S_{\mu\nu}^{(0)}$ can be separated into two parts, the first ($S_{\mu\nu}^{(0)}(1)$) corresponding to free electrons and the second ($S_{\mu\nu}^{(0)}(2)$) containing perturbation effects due to the lattice

$$S_{\mu\nu}^{(0)}(1) = -\frac{e^2}{m^2 c^2} \sum_{\mathbf{k}} k_\mu k_\nu F_1 \left\{ \frac{(\mathbf{k} + \mathbf{q}/2)^2}{2m}, \frac{(\mathbf{k} - \mathbf{q}/2)^2}{2m} \right\}, \quad (2.15)$$

$$S_{\mu\nu}^{(0)}(2) = -\frac{e^2}{m^2 c^2} \left\{ \frac{\beta(\sqrt{\pi} b/a)^3}{1 + \beta(\sqrt{\pi} b/a)^3} \right\}^2 \sum_k \sum_{\kappa_n \neq 0} k_\mu k_\nu \exp(-\kappa_n^2 b^2/2) \\ \times F_1 \left\{ \frac{(\mathbf{k} + (\boldsymbol{\kappa}_n + \mathbf{q})/2)^2}{2m}, \frac{(\mathbf{k} - (\boldsymbol{\kappa}_n + \mathbf{q})/2)^2}{2m} \right\}. \quad (2.16)$$

The expression for $L^{(0)}$, Eq. (2.2), is independent of the approximation used

$$L^{(0)} = L = (e^2/mc^2) N. \quad (2.17)$$

Eqs. (2.15) and (2.17) now describe the Landau⁵⁾ diamagnetism of a free electron gas. It is easy to verify that

$$\sum_{\mu\nu} \frac{q_\mu q_\nu}{q^2} S_{\mu\nu}^{(0)}(1) - L^{(0)} = 0, \quad (2.18)$$

$$\lim_{q \rightarrow 0} \frac{1}{2q^2} \sum_{\mu\nu} \left(\delta_{\mu\nu} - \frac{3q_\mu q_\nu}{q^2} \right) S_{\mu\nu}^{(0)}(1) = -\frac{e^2 N}{4mc^2} \cdot \frac{1}{k_F^2}. \quad (2.19)$$

(We have introduced the Fermi wave number, k_F , which is related to the number of electrons by $N = V k_F^3/3\pi^2$.)

Corresponding equations with $S_{\mu\nu}^{(0)}(2)$ may be obtained by: (a) assuming $2k_F \lesssim \kappa_1$, i.e. $k_F a \lesssim \pi$ (if there are ν conduction electrons per lattice ion, this corresponds to the assumption $\nu \lesssim \pi/3$), and (b) replacing the sum over reciprocal lattice vectors κ_n by $(a/2\pi)^3 \int d\kappa$ (this is permissible for $b/a \ll 1$, which has already been assumed). The consequent results are

$$\lim_{q \rightarrow 0} \sum_{\mu\nu} \frac{q_\mu q_\nu}{q^2} S_{\mu\nu}^{(0)}(2) = \frac{\pi\sqrt{2\pi}}{12} \cdot \frac{e^2 N}{mc^2} \cdot \frac{\beta^2(b/a)^3}{[1 + \beta(\sqrt{\pi} b/a)^3]^2} \left\{ 1 + \mathcal{O}\left(\frac{b}{a}\right)^2 \right\}, \quad (2.20)$$

$$\lim_{q \rightarrow 0} \frac{1}{q^2} \sum_{\mu\nu} \left(\frac{1}{2} \delta_{\mu\nu} - \frac{3}{2} \frac{q_\mu q_\nu}{q^2} \right) S_{\mu\nu}^{(0)}(2) = \frac{\pi\sqrt{2\pi}}{20} \cdot \frac{e^2 N}{mc^2} \cdot b^2 \\ \times \frac{\beta^2(b/a)^3}{[1 + \beta(\sqrt{\pi} b/a)^3]^2} \cdot \left\{ 1 + \mathcal{O}\left(\frac{b}{a}\right) \right\}. \quad (2.21)$$

Collecting all these results, and neglecting terms of relative order $(b/a)^2$ and $\beta^2(b/a)^3$,

$$\lim_{q \rightarrow 0} \frac{G^{(0)}}{q^3} = \sigma_1 \beta^2 \left(\frac{b}{a} \right)^3 \left(\frac{L}{q^2} \right) \quad (2.22)$$

$$\lim_{q \rightarrow 0} K_B^{(0)} = -\frac{L}{4k_F^2} \left\{ 1 - \sigma_2 \beta^2 \left(\frac{b}{a} \right)^3 (bk_F)^2 \right\} \quad (2.23)$$

with numerical factors, $\sigma_1 = \pi\sqrt{2\pi}/12$, $\sigma_2 = \pi\sqrt{2\pi}/5$.

Thus the approximate eigenfunctions have led to a result which violates the gauge identity $G \equiv 0$. Referring back to Eq. (1.2) we see that $K^{(0)}$ will now depend on a parameter α :

$$K^{(0)}(\alpha) = K_B^{(0)} + \alpha \frac{G^{(0)}}{q^2}. \quad (2.24)$$

The technique described in the introduction is now applied to choose a best approximate kernel from the set (2.24): $\alpha = \alpha_1$ is selected according to the criterion that $K^{(1)}(\alpha_1)$ shall vanish.

§ 3. Calculation of α_1

The function $S_{\mu\nu}^{(1)}$, and thence $K^{(1)}(\alpha)$, is found by extending Eq. (2.3) to get

$$S_{\mu\nu}^{(1)}(\mathbf{q}) = -\frac{1}{c^2} \sum_{\mathbf{k}, \mathbf{k}', \mathbf{k}''} \langle \mathbf{k} | j_\mu(\mathbf{q}) | \mathbf{k}' \rangle \left\{ \langle \mathbf{k}' | H^{(1)} | \mathbf{k}'' \rangle \langle \mathbf{k}'' | j_\nu(-\mathbf{q}) | \mathbf{k} \rangle \right. \\ \left. + \langle \mathbf{k}' | j_\nu(-\mathbf{q}) | \mathbf{k}'' \rangle \langle \mathbf{k}'' | H^{(1)} | \mathbf{k} \rangle \right\} \\ \times F_2(E(\mathbf{k}), E(\mathbf{k}'), E(\mathbf{k}'')). \quad (3.1)$$

The matrix elements of $H^{(1)}$ are readily found from Eq. (1.12):

$$\langle \mathbf{k} | H^{(1)} | \mathbf{k}' \rangle = \frac{i\hbar^2}{4m} \int d\mathbf{r} \exp(-i(\mathbf{k}-\mathbf{k}') \cdot \mathbf{r}) (\mathbf{k}-\mathbf{k}') \cdot \nabla |u_0(\mathbf{r})|^2 \quad (3.2)$$

or, using the particular form (2.11),

$$\langle \mathbf{k} | H^{(1)} | \mathbf{k}' \rangle = -\frac{\pi^{3/2} \hbar^2}{4m} \beta \frac{b}{a^3} V C \sum_{\kappa_n} (\kappa_n b)^2 \exp(-\kappa_n^2 b^2/4) \delta(\mathbf{k}-\mathbf{k}'-\kappa_n). \quad (3.3)$$

Thus, substituting from Eqs. (2.9), (2.13) and (3.3) into (3.1), we get

$$S_{\mu\nu}^{(1)}(\mathbf{q}) = 2 \left(\frac{e}{2mc} \right)^2 \sum_{\mathbf{k} \mathbf{k}' \mathbf{k}''} \sum_{\kappa_n \kappa_{n'}} \frac{\pi^3}{4m} \beta^2 \left(\frac{b}{a} \right)^4 \frac{(CV)^3}{a^2} (\kappa_n b)^2 \exp\{-(\kappa_n^2 + \kappa_{n'}^2) b^2/4\} \\ \times (k+k')_\mu (k+k')_\nu \left\{ \begin{aligned} &\delta(\mathbf{k}-\mathbf{k}'+\mathbf{q}) \delta(\mathbf{k}'-\mathbf{k}''+\kappa_n) \delta(\kappa_n-\kappa_{n'}) \\ &+ \delta(\mathbf{k}-\mathbf{k}''+\mathbf{q}) \delta(\mathbf{k}'-\mathbf{k}''-\kappa_n) \delta(\kappa_n-\kappa_{n'}) \end{aligned} \right\} \\ \times F_2 \left\{ \frac{k^2}{2m}, \frac{k'^2}{2m}, \frac{k''^2}{2m} \right\}. \quad (3.4)$$

Making use of the fact that $S_{\mu\nu}$ is a symmetric tensor, this may be reduced, after some manipulation, to

$$S_{\mu\nu}^{(1)} = A \sum_{\kappa_n} (\kappa_n b)^2 \exp(-\kappa_n^2 b^2/2) \sum_{\mathbf{k}} \left(\mathbf{k} + \frac{\mathbf{q}}{2} \right)_\mu \left(\mathbf{k} + \frac{\kappa_n + \mathbf{q}}{2} \right)_\nu \\ \times F_2 \left\{ \frac{k^2}{2m}, \frac{(\mathbf{k}+\mathbf{q})^2}{2m}, \frac{(\mathbf{k}+\kappa_n)^2}{2m} \right\}. \quad (3.5)$$

A is a constant,

$$A \equiv \frac{e^2}{mc^2} \cdot \frac{\pi^3}{m^2} \cdot \beta^2 \left(\frac{b}{a} \right)^4 \frac{(CV)^3}{a^2}. \quad (3.6)$$

We now proceed to evaluate $G^{(1)}/q^2$ and $K_B^{(1)}$ in the limiting case of $q \ll k_F$. First $G^{(1)}$ is found from

$$\lim_{q \rightarrow 0} \sum_{\mu\nu} S_{\mu\nu}^{(1)} \frac{q_\mu q_\nu}{q^2} = mA \sum_{\kappa_n} (\kappa_n b)^2 \exp(-\kappa_n^2 b^2/2) \sum_k \times \frac{(\mathbf{k} \cdot \mathbf{q}) ([\mathbf{k} - (\kappa_n/2)] \cdot \mathbf{q})}{q^2 \kappa_n \cdot (\mathbf{k} - (\kappa_n/2))} \left[F_0' \left(\frac{k^2}{2m} \right) - F_1 \left(\frac{k^2}{2m}, \frac{(\mathbf{k} - \kappa_n)^2}{2m} \right) \right] \quad (3.7)$$

$$\approx m^2 A \sum_{\kappa_n} (\kappa_n b)^2 \exp \{ -(\kappa_n b)^2/2 \} \cdot \frac{2N}{\kappa_n^2}. \quad (3.8)$$

A partial integration has been performed to reduce $F_0'(k^2/2m)$ to $F_0(k^2/2m)$. Replacing the sum over κ_n by an integral,

$$\lim_{q \rightarrow 0} \sum_{\mu\nu} S_{\mu\nu}^{(1)} \frac{q_\mu q_\nu}{q^2} \approx \frac{2A}{(2\pi)^{3/2}} N m^2 b^3 \left(\frac{a}{b} \right)^3. \quad (3.9)$$

That is (since $L^{(1)}=0$, $L^{(0)}$ being exact)

$$\lim_{q \rightarrow 0} G^{(1)}/q^2 = \frac{\pi^2}{\sqrt{2\pi}} \beta^2 \left(\frac{b}{a} \right)^3 \frac{L}{q^2}. \quad (3.10)$$

A similar, but more complicated calculation gives

$$\lim_{q \rightarrow 0} \sum_{\mu\nu} \left(\frac{1}{2} \delta_{\mu\nu} - \frac{3}{2} \frac{q_\mu q_\nu}{q^2} \right) S_{\mu\nu}^{(1)} \approx m^3 A \sum_{\kappa_n} (\kappa_n b)^2 \exp(-(\kappa_n b)^2/2) \times \left\{ \frac{N}{\kappa_n^2} \cdot \frac{q^2}{k_F^2} \right\} \left\{ -1 + \frac{2(\kappa_n \cdot \mathbf{q})^2}{\kappa_n^2 q^2} \right\}. \quad (3.11)$$

(The complications arise from the fact that we now have to consider terms up to order q^2/k_F^2 .) Simplifying this expression, we get

$$\lim_{q \rightarrow 0} K_B^{(1)} = -\frac{\pi^3}{6\sqrt{2\pi}} \beta^2 \left(\frac{b}{a} \right)^3 \frac{L}{k_F^2}. \quad (3.12)$$

α_1 is now found from the condition

$$K^{(1)}(\alpha_1) = K_B^{(1)} + \alpha_1 \frac{G^{(1)}}{q^2} = 0. \quad (3.13)$$

Thence, for small q ,

$$\alpha_1 = \frac{1}{6} \frac{q^2}{k_F^2}. \quad (3.14)$$

§ 4. Conclusion

The above approximation scheme for electrons moving in a (weak) periodic potential thus finally gives a slightly modified Landau diamagnetism, with a kernel

$$K(\mathbf{q}) = K^{(0)}(\mathbf{q}, \alpha_1(\mathbf{q})) \quad (4.1)$$

$$= -\frac{e^2 N}{4mc^2} \cdot \frac{1}{k_F^2} \left\{ 1 - \tau \beta^2 \left(\frac{b}{a} \right)^3 \left(1 + \mathcal{O} \left(\frac{b}{a} \right)^2 \right) \right\}. \quad (4.2)$$

(τ is a numerical constant, $\pi\sqrt{2\pi}/18$.)

This is in qualitative agreement with the result first derived by Peierls.⁶⁾ If the assumed $u_0(\mathbf{r})$ given by Eq. (2.11) is fitted to the actual ground state Bloch function for sodium, as given by Wigner and Seitz,⁷⁾ the ensuing values of the parameters β and b imply a diminution of the Landau diamagnetism for sodium by roughly 0.07%. This compares favourably with a more detailed and fully gauge-covariant calculation recently made by Tani,³⁾ (using the sound wave formalism to describe the lattice), who gets a correction of 0.06%.

In problems such as the one considered above, where non gauge-invariant approximations are employed, many authors have followed the course of using the "London gauge", $\text{div } \mathbf{A} = 0$. It is worth noting that in this example, such a choice (which corresponds to choosing $\alpha_1(\mathbf{q}) \equiv 1$ as opposed to the BMM variational choice of $\alpha_1(\mathbf{q})$) leads to a singular kernel such as that for superconductors, only with the wrong sign (cf. Eqs. (2.22), (2.23) and (2.24))—this is completely unphysical.

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Green Function Method for Electron Gas. I**—General Formulation—**

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The Green function method is applied to the problems of electron gas. It is emphasized that there is a perfect parallelism between the Green function methods in zero-temperature problems and in finite-temperature problems of systems in thermal equilibrium. Further it is pointed out that the Green function method is also useful for the calculation of transport quantities.

§ 1. Introduction

Much advances have been made recently in the study of many-body problems in quantum mechanics and quantum statistics. Particularly noteworthy is the computation of the exact correlation energy of an electron gas at high density by Gell-Mann and Brueckner¹⁾ (GB). The high density result of GB has subsequently been derived by other methods by Sawada et al.,²⁾ Hubbard,³⁾ and others.⁴⁾ The field theoretical techniques are quite useful in summing certain subseries of terms in the so-called "linked-cluster expansion". Recently Galickij and Migdal,⁵⁾ and Klein and Prange⁶⁾ have applied the Green function method to many-body problems in quantum mechanics. This method is quite suitable for the study of many-body problems. From the one-particle Green function one may compute the energy and damping of quasiparticles and from the two-particle Green function one may obtain the potential energy due to two-body forces and the energy spectrum of excitations which are not describable as a sum of energies of quasi-particles.

Montroll and Ward⁷⁾ have developed a generalization of the cluster integral theory of Mayer to deal with the quantum statistics of many-particle systems. They have shown that in the case of an electron gas the classical limit of the contribution of ring integrals to the grand partition function yields the Debye-Hückel theory, while the low temperature limit leads to the GB result.

Matsubara⁸⁾ has applied a formalism of quantum field theory to the calculation of the grand partition function in quantum statistics. Matsubara's method has been developed recently by Bloch and Dominici,⁹⁾ Fradkin,¹⁰⁾ and others.¹¹⁾ The applicability of the field theoretical techniques to quantum statistics is based on the existence of the theorem corresponding to that of Wick¹²⁾ in quantum field theory. There is a close correspondence, which was first pointed out by Matsubara, between the Green function methods in quantum mechanics and in quantum statistics.

The present paper is mainly concerned with an exposition of the methods. We obtain the two-particle Green function of a dense electron gas and derive the result of GB. From the Bethe-Salpeter equation for the two-particle Green function for an electron-hole pair the dispersion relation of plasmons is derived. This method is applied to the problem of the exciton and the plasmon in insulators. The improvement of the dispersion relation of plasmons by including higher order effects will be carried out in II. The calculation of the diamagnetic susceptibility of a dense electron gas, which will be published in III, is based on the two-particle Green function obtained in § 2. In § 3 we show that the grand partition function of an electron gas at high density can be calculated by the use of the two-particle Green function which is obtained in quite the same way as in § 2 and that the result of Montroll and Ward is derived quite simply. In § 4 we introduce Green functions which are dependent on time and temperature and we point out that these Green functions are suitable for the calculation of transport coefficients. We emphasize that there is a close resemblance between the Green function methods in zero temperature problems and in finite-temperature problems of systems in thermal equilibrium and in the problems of transport processes.

§ 2. Time-dependent Green functions

The Hamiltonian for the system of electrons is

$$H = H_0 + H_1, \quad (2.1)$$

where

$$H_0 = \frac{\hbar^2}{2m} \int \nabla \phi^*(\mathbf{r}) \cdot \nabla \phi(\mathbf{r}) d\mathbf{r}, \quad (2.2a)$$

$$H_1 = \frac{1}{2} \iint \phi^*(\mathbf{r}) \phi^*(\mathbf{r}') v(\mathbf{r} - \mathbf{r}') \phi(\mathbf{r}') \phi(\mathbf{r}) d\mathbf{r} d\mathbf{r}', \quad (2.2b)$$

$$v(\mathbf{r} - \mathbf{r}') = \frac{1}{V} \sum_{\mathbf{k}} v(\mathbf{k}) \exp(i\mathbf{k}(\mathbf{r} - \mathbf{r}')), \quad v(\mathbf{k}) = 4\pi e^2/k^2. \quad (2.3)$$

$\phi(\mathbf{r})$, $\phi^*(\mathbf{r})$ are the field operators which satisfy the well-known anticommutation relations

$$\begin{aligned} [\phi(\mathbf{r}), \phi^*(\mathbf{r}')]_{+} &= \delta(\mathbf{r} - \mathbf{r}'), \\ [\phi(\mathbf{r}), \phi(\mathbf{r}')]_{+} &= [\phi^*(\mathbf{r}), \phi^*(\mathbf{r}')]_{+} = 0. \end{aligned} \quad (2.4)$$

The one- and two-particle Green functions are defined as

$$G(x_1, x_2) = i \langle T \{ \phi(x_1) \phi^*(x_2) \} \rangle, \quad (2.5)$$

$$G(x_1, x_2; x_3, x_4) = i \langle T \{ \phi(x_1) \phi(x_2) \phi^*(x_3) \phi^*(x_4) \} \rangle, \quad (2.6)$$

where x represents a point in space and time, and T is the chronological ordering symbol of Wick. $\phi(x)$, $\phi^*(x)$ are the Heisenberg operators; $\phi(x) = \exp[i/\hbar \cdot Ht]$

$\cdot \psi(\mathbf{r}) \exp[-i/\hbar \cdot Ht]$. The average is taken over the ground state function of the Hamiltonian H . From (2.1) and (2.6) we obtain

$$E_0 = \langle H \rangle$$

$$= \varepsilon_0 + \frac{i}{2} \int_0^g \frac{dg}{g} \iint \lim_{t' \rightarrow t+} G(\mathbf{r}t, \mathbf{r}'t; \mathbf{r}t', \mathbf{r}'t') v(\mathbf{r} - \mathbf{r}') d\mathbf{r} d\mathbf{r}', \quad (g \equiv e^2) \quad (2.7)$$

where ε_0 is the ground state energy without interaction. As is well known, (2.5) and (2.6) may be written in the form

$$G(x_1, x_2) = i \langle T \{ \psi(x_1) \psi^*(x_2) S \} \rangle_0 / \langle S \rangle_0, \quad (2.8)$$

$$G(x_1, x_2; x_3, x_4) = i \langle T \{ \psi(x_1) \psi(x_2) \psi^*(x_3) \psi^*(x_4) S \} \rangle_0 / \langle S \rangle_0, \quad (2.9)$$

where $\psi(x) = \exp[i/\hbar \cdot H_0 t] \psi(\mathbf{r}) \exp[-i/\hbar \cdot H_0 t]$ and S is the S matrix; the average is taken over the ground state function of the unperturbed Hamiltonian H_0 .

As was shown by Schwinger,¹³⁾ and Gell-Mann and Low,¹⁴⁾ the equation for the two-particle Green function has the form

$$G(x_1, x_2; x_3, x_4) = iG(x_1, x_3)G(x_2, x_4) - iG(x_1, x_4)G(x_2, x_3) \\ + i/\hbar \cdot \int \cdots \int G(x_1, x_5)G(x_2, x_6)\Gamma(x_5, x_6; x_7, x_8)G(x_7, x_8; x_3, x_4)dx_5dx_6dx_7dx_8, \quad (2.10)$$

where Γ is the irreducible interaction part. At high density limit we may obtain the two-particle Green function without solving the integral equation (2.10). Let us define the effective interaction $\tilde{v}(x_1 - x_2)$ by³⁾

$$\tilde{v}(x_1 - x_2) = v(x_1 - x_2) + 1/i\hbar \cdot \iint v(x_1 - x_3)G_0(x_3, x_4)G_0(x_4, x_2)\tilde{v}(x_4 - x_2)dx_3dx_4 \quad (2.11)$$

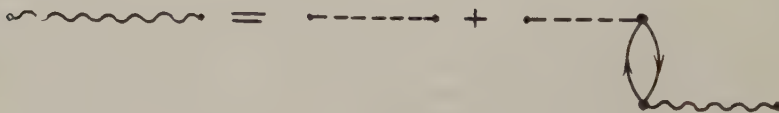


Fig. 1. Wavy and dotted lines represent the effective and Coulomb interactions respectively

where $v(x_1 - x_2) = v(\mathbf{r}_1 - \mathbf{r}_2) \delta(t_1 - t_2)$. The Feynman diagram corresponding to (2.11) is shown in Fig. 1. $G_0(x, x')$ is the one-particle Green function without interaction and is given by

$$G_0(x, x') = \frac{1}{2\pi V} \sum_{\mathbf{k}} \int G_0(\mathbf{k}, \varepsilon) \exp(i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}') - i\varepsilon(t - t')) d\varepsilon, \quad (2.12)$$

where

$$G_0(\mathbf{k}, \varepsilon) = \frac{-1}{\varepsilon - \varepsilon_0(\mathbf{k})/\hbar + i\delta\theta(k)} \quad (\delta \rightarrow +0) \quad (2.13)$$

and

$$\begin{aligned} \varepsilon_0(\mathbf{k}) &= \hbar^2 k^2 / 2m, \\ \theta(\mathbf{k}) &= 1 - 2n_0(\mathbf{k}). \end{aligned} \quad (2.14)$$

Here $n_0(k)$ is the occupation number of unperturbed state. We can solve (2.11) at once by introducing the Fourier transform of $\mathcal{V}(x_1 - x_2)$:

$$\mathcal{V}(x_1 - x_2) = \frac{1}{2\pi V} \sum_{\mathbf{k}} \int \mathcal{V}(\mathbf{q}, \omega) \exp\{i\mathbf{q} \cdot (\mathbf{r}_1 - \mathbf{r}_2) - i\omega(t_1 - t_2)\} d\omega. \quad (2.15)$$

We get

$$\mathcal{V}(\mathbf{q}, \omega) = v(\mathbf{q}) / \{1 + v(\mathbf{q})Q(\mathbf{q}, \omega)\}, \quad (2.16)$$

where

$$\begin{aligned} Q(\mathbf{q}, \omega) &= \frac{i}{2\pi\hbar V} \sum_{\mathbf{k}} \int G_0(\mathbf{k}, \varepsilon) G_0(\mathbf{q} + \mathbf{k}, \omega + \varepsilon) d\varepsilon \\ &= \frac{1}{\hbar V} \sum_{\mathbf{k}} \frac{n_0(\mathbf{q} + \mathbf{k}) - n_0(\mathbf{k})}{\omega + \{\varepsilon_0(\mathbf{k}) - \varepsilon_0(\mathbf{q} + \mathbf{k})\}/\hbar - i\delta\{n_0(\mathbf{k}) - n_0(\mathbf{q} + \mathbf{k})\}}. \end{aligned} \quad (2.17)$$

Then the two-particle Green function is given by

$$\begin{aligned} G(x_1, x_2; x_3, x_4) &= iG_0(x_1, x_3)G_0(x_2, x_4) - iG_0(x_1, x_4)G_0(x_2, x_3) \\ &+ i/\hbar \cdot \iint G_0(x_1, x_5)G_0(x_2, x_6) \mathcal{V}(x_5 - x_6) \{iG_0(x_5, x_3)G_0(x_6, x_4)\} dx_5 dx_6 \\ &+ i/\hbar \cdot \iint G_0(x_1, x_5)G_0(x_2, x_6) v(x_5 - x_6) \{-iG_0(x_5, x_4)G_0(x_6, x_3)\} dx_5 dx_6. \end{aligned} \quad (2.18)$$

Inserting (2.18) into (2.7), we get the result of GB for the energy of the ground state. The first term of the right-hand side of (2.18) does not contribute to the energy. The second term gives the exchange energy (Fig. 2a). The contribution from the third term is represented by the diagram shown in Fig. 2b. The last term gives the second order exchange energy which is denoted as $\varepsilon_b^{(2)}$ in GB's paper (Fig. 2c).

The Green function which yields the result of GB (except the last term in (2.18)) may also be obtained by solving the following Bethe-Salpeter equation,



Fig. 2a

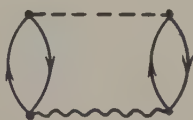


Fig. 2b



Fig. 2c

$$G(x_1, x_2; x_3, x_4) = -iG_0(x_1, x_4)G_0(x_2, x_3)$$

$$-i/\hbar \cdot \iint G_0(x_1, x_5)G_0(x_5, x_3)v(x_5-x_6)G(x_6, x_2; x_6, x_4)dx_5dx_6. \quad (2.19)$$

The interaction part of the second term in (2.19) is described by the diagram shown in Fig. 3a. The dispersion relation of plasmons can be derived from (2.19). Let $t_1, t_3 > t_2, t_4$. Then, using the completeness of the set of eigenfunctions Ψ_n , we rewrite (2.6) as

$$G(x_1, x_2; x_3, x_4) = -i \sum_n \chi_n(x_1, x_3) \bar{\chi}_n(x_2, x_4), \quad (2.20)$$

where

$$\chi_n(x_1, x_3) = (\Psi_0, T\{\phi(x_1)\phi^*(x_3)\}\Psi_n), \quad (2.21a)$$

$$\bar{\chi}_n(x_2, x_4) = (\Psi_0, \bar{T}\{\phi(x_2)\phi^*(x_4)\}\Psi_n)^*. \quad (2.21b)$$

Here \bar{T} orders the operators in the reversed chronological order to T . Following Gell-Mann and Low's procedure,¹⁴⁾ we obtain the following equation for the bound state (plasmon state) amplitude $\chi(x_1, x_3)$ for a pair of an electron and a hole,

$$\chi(x_1, x_3) = -i/\hbar \cdot \iint G_0(x_1, x_5)G_0(x_5, x_3)v(x_5-x_6)\chi(x_6, x_6)dx_5dx_6. \quad (2.22)$$

For simultaneous time $t_1=t_3$ the function $\chi(x_1, x_3)$ has the physical meaning of wave function describing the behavior of the electron-hole pair and may be written as

$$\chi(\mathbf{r}_1 t, \mathbf{r}_3 t) = \exp\{i\mathbf{k} \cdot (\mathbf{r}_1 + \mathbf{r}_3) - i\omega t\} \frac{1}{V} \sum_{\mathbf{p}} \exp\{i\mathbf{p} \cdot (\mathbf{r}_1 - \mathbf{r}_3)\} f_k^{pl}(\mathbf{p}), \quad (2.23)$$

where $\hbar\omega$ is the excitation energy, and $\hbar\mathbf{k}$ and $\hbar\mathbf{p}$ are the total and relative momentum. Inserting (2.23) into (2.22), we get the dispersion relation of Bohm and Pines

$$\begin{aligned} 1 &= \frac{1}{2\pi\hbar i V} v(\mathbf{k}) \sum_{\mathbf{p}} \int G_0\left(\mathbf{p} + \frac{\mathbf{k}}{2}, \varepsilon\right) G_0\left(\mathbf{p} - \frac{\mathbf{k}}{2}, \varepsilon - \omega\right) d\varepsilon \\ &= \frac{4\pi e^2}{m V} \sum_{\mathbf{p}} \frac{n_0(\mathbf{p})}{(\omega - \hbar\mathbf{k} \cdot \mathbf{p}/m)^2 - \hbar^2 k^4/4m^2}. \end{aligned} \quad (2.24)$$

The improvement of the dispersion relation by including the higher order effects will be discussed in II. It should be noted that the normalization of the bound state amplitude is derived from the sum rule (see Appendix). We can also obtain the dispersion relation of plasmons in solids, which was derived by Kanazawa,¹⁵⁾ and Nozières and Pines,¹⁶⁾ by expanding $\phi(\mathbf{r})$, $\phi^*(\mathbf{r})$ in terms of Bloch functions.



Fig. 3a



Fig. 3b

Next, we apply the above-mentioned method to the problem of exciton and plasmon in insulators. Adding the interaction part which is described by the diagram shown in Fig. 3b to (2.19), we get the equations for the singlet and triplet states of an electron-hole pair

$$\begin{aligned} \chi(x_1, x_2) = & -i/\hbar \cdot \iint G_0(x_1, x_3) \dot{G}_0(x_4, x_2) v(x_3 - x_4) \chi(x_4, x_4) dx_3 dx_4 \\ & + i/\hbar \cdot \iint G_0(x_1, x_3) G_0(x_4, x_2) v(x_3 - x_4) \chi(x_3, x_4) dx_3 dx_4, \quad (2.25a) \end{aligned}$$

(for singlet state)

$$\chi(x_1, x_2) = i/\hbar \cdot \iint G_0(x_1, x_3) G_0(x_4, x_2) v(x_3 - x_4) \chi(x_3, x_4) dx_3 dx_4. \quad (2.25b)$$

(for triplet state)

As the unperturbed Hamiltonian we adopt that of the Hartree-Fock approximation and denote the eigenfunctions (Bloch functions) and eigenvalues by $\varphi_{np\sigma}(\mathbf{r})$ and $\varepsilon_n(\mathbf{p})$ respectively, where n , p and σ denote the band index, the reduced wave vector and the spin. The Hamiltonian is

$$H = H_0 + H_1, \quad (2.26)$$

$$H_0 = \sum_{np\sigma} \varepsilon_n(\mathbf{p}) a_{np\sigma}^* a_{np\sigma}, \quad (2.27a)$$

$$\begin{aligned} H_1 = & \frac{1}{2} \sum_{\substack{n_1 n_2 n_1' n_2' \\ p_1 p_2 k \\ \sigma_1 \sigma_2 \sigma_1' \sigma_2'}} \left\langle \begin{matrix} n_1 & n_2 \\ p_1 + k & p_2 - k \end{matrix} \middle| V \middle| \begin{matrix} n_2' & n_1' \\ p_2 & p_1 \end{matrix} \right\rangle a_{n_1 p_1 + k \sigma_1}^* a_{n_2 p_2 - k \sigma_2}^* a_{n_2' p_2 \sigma_2'} a_{n_1' p_1 \sigma_1'} \\ & - \sum_{\substack{n_1 n_2 \\ p_1 p_2 \\ \sigma \sigma'}} \left\{ \left\langle \begin{matrix} n_1 & n_2 \\ p_1 & p_2 \end{matrix} \middle| V \middle| \begin{matrix} n_2 & n_1' \\ p_2 & p_1 \end{matrix} \right\rangle - \left\langle \begin{matrix} n_1 & n_2 \\ p_1 & p_1 + p_2 \end{matrix} \middle| V \middle| \begin{matrix} n_1' & n_2 \\ p_1 + p_2 & p_2 \end{matrix} \right\rangle \delta_{\sigma \sigma'} \right\} N_{n_2 p_2 \sigma} a_{n_1 p_1 \sigma'}^* a_{n_1' p_1 \sigma'}, \quad (2.27b) \end{aligned}$$

where

$$\left\langle \begin{matrix} n_1 & n_2 \\ p_1 & p_2 \end{matrix} \middle| V \middle| \begin{matrix} n_3 & n_4 \\ p_3 & p_4 \end{matrix} \right\rangle = \int \varphi_{n_1 p_1}^*(\mathbf{r}_1) \varphi_{n_2 p_2}^*(\mathbf{r}_2) \frac{e^2}{|\mathbf{r}_1 - \mathbf{r}_2|} \varphi_{n_3 p_3}(\mathbf{r}_2) \varphi_{n_4 p_4}(\mathbf{r}_1) d\mathbf{r}_1 d\mathbf{r}_2, \quad (2.28)$$

and N_{np} is the occupation number of unperturbed state, and $N_{np}=1$ and $N_{np}=0$ correspond to the valence band and conduction band respectively.

We expand $\chi(x, x')$ for simultaneous times in terms of φ_{np} :

$$\chi_{k\omega}(\mathbf{r}t, \mathbf{r}'t) = \sum_{n_1 n_2 p} \varphi_{n_1 p + k/2}(\mathbf{r}) \varphi_{n_2 p - k/2}^*(\mathbf{r}') \exp(-i\omega t) f_{k\omega}^{n_1 n_2}(\mathbf{p}). \quad (2.29)$$

Considering the spin states, we write f as $^M f$ where $^1 f$ and $^3 f$ correspond to the singlet and triplet respectively. (2.25a) and (2.25b) lead to

$$\begin{aligned} ^M f_{k\omega}^{n_1 n_2}(\mathbf{p}) = & \left[\sum \left\langle \begin{matrix} n_1 & n_2' \\ p + k/2 & p' - k/2 \end{matrix} \middle| V \middle| \begin{matrix} n_2 & n_1' \\ p - k/2 & p' + k/2 \end{matrix} \right\rangle \right. \\ & \left. - 2\delta_M \sum \left\langle \begin{matrix} n_1 & n_2' \\ p + k/2 & p' - k/2 \end{matrix} \middle| V \middle| \begin{matrix} n_1' & n_2 \\ p' + k/2 & p - k/2 \end{matrix} \right\rangle \right] \end{aligned}$$

$$\times \frac{N_{n_1 p+k/2} - N_{n_2 p-k/2}}{\hbar\omega - \varepsilon_{n_1}(\mathbf{p} + \mathbf{k}/2) + \varepsilon_{n_2}(\mathbf{p} - \mathbf{k}/2)} \cdot Mf_{\hbar\omega}^{n_1' n_2'}(\mathbf{p}'), \quad (2.30)$$

where

$$\begin{aligned} \delta_M &= 1 \quad \text{for } M=1, \\ &= 0 \quad \text{for } M=3. \end{aligned} \quad (2.31)$$

Rewriting (2.30) by using the abbreviations c and v for the indices corresponding to the conduction and valence band, respectively, we have

$$\begin{aligned} Mf_{\hbar\omega}^{cv}(\mathbf{p}) &= -\frac{1}{\hbar\omega - \varepsilon_c(\mathbf{p} + \mathbf{k}/2) + \varepsilon_v(\mathbf{p} - \mathbf{k}/2)} \\ &\times \left[\sum \left\{ \left\langle \begin{smallmatrix} c & v' \\ p+k/2 & p'-k/2 \end{smallmatrix} \middle| V \middle| \begin{smallmatrix} v & c' \\ p-k/2 & p'+k/2 \end{smallmatrix} \right\rangle \right. \right. \\ &- 2\delta_M \left\langle \begin{smallmatrix} c & v' \\ p+k/2 & p'-k/2 \end{smallmatrix} \middle| V \middle| \begin{smallmatrix} c' & v \\ p'+k/2 & p-k/2 \end{smallmatrix} \right\rangle \left. \right\} \cdot Mf_{\hbar\omega}^{c'v'}(\mathbf{p}') \\ &- \sum \left\{ \left\langle \begin{smallmatrix} c & c' \\ p+k/2 & p'-k/2 \end{smallmatrix} \middle| V \middle| \begin{smallmatrix} v & v' \\ p-k/2 & p'+k/2 \end{smallmatrix} \right\rangle \right. \\ &- 2\delta_M \left\langle \begin{smallmatrix} c & c' \\ p+k/2 & p'-k/2 \end{smallmatrix} \middle| V \middle| \begin{smallmatrix} v' & v \\ p'+k/2 & p-k/2 \end{smallmatrix} \right\rangle \left. \right\} \cdot Mf_{\hbar\omega}^{v'v}(\mathbf{p}') \Big], \quad (2.32a) \end{aligned}$$

$$\begin{aligned} Mf_{\hbar\omega}^{vc}(\mathbf{p}) &= \frac{1}{\hbar\omega + \varepsilon_v(\mathbf{p} - \mathbf{k}/2) - \varepsilon_c(\mathbf{p} + \mathbf{k}/2)} \\ &\times \left[\sum \left\{ \left\langle \begin{smallmatrix} v & v' \\ p+k/2 & p'-k/2 \end{smallmatrix} \middle| V \middle| \begin{smallmatrix} c & c' \\ p-k/2 & p'+k/2 \end{smallmatrix} \right\rangle \right. \\ &- 2\delta_M \left\langle \begin{smallmatrix} v & v' \\ p+k/2 & p'-k/2 \end{smallmatrix} \middle| V \middle| \begin{smallmatrix} c' & c \\ p'+k/2 & p-k/2 \end{smallmatrix} \right\rangle \left. \right\} \cdot Mf_{\hbar\omega}^{c'v'}(\mathbf{p}') \\ &- \sum \left\{ \left\langle \begin{smallmatrix} v & c' \\ p+k/2 & p'-k/2 \end{smallmatrix} \middle| V \middle| \begin{smallmatrix} c & v' \\ p-k/2 & p'+k/2 \end{smallmatrix} \right\rangle \right. \\ &- 2\delta_M \left\langle \begin{smallmatrix} v & c' \\ p+k/2 & p'-k/2 \end{smallmatrix} \middle| V \middle| \begin{smallmatrix} v' & c \\ p'+k/2 & p-k/2 \end{smallmatrix} \right\rangle \left. \right\} \cdot Mf_{\hbar\omega}^{v'c}(\mathbf{p}') \Big]. \quad (2.32b) \end{aligned}$$

These eigenvalue equations are the same as those derived by Horie.¹⁷⁾

§ 3. Temperature-dependent Green functions

The one- and two-particle Green functions are defined as

$$\bar{G}(x_1, x_2) = \langle T \{ \psi(x_1) \bar{\psi}(x_2) \} \rangle, \quad (3.1)$$

$$\bar{G}(x_1, x_2; x_3, x_4) = \langle T \{ \psi(x_1) \psi(x_2) \bar{\psi}(x_3) \bar{\psi}(x_4) \} \rangle, \quad (3.2)$$

where

$$\psi(x) \equiv \psi(\mathbf{r}, t) = \exp(\mathcal{H}t) \psi(\mathbf{r}) \exp(-\mathcal{H}t), \quad (3.3a)$$

$$\bar{\psi}(x) = \exp(\mathcal{H}t) \psi^*(\mathbf{r}) \exp(-\mathcal{H}t). \quad (3.3b)$$

and

$$\mathcal{H} = H - \mu N; \quad (3.4)$$

μ and N are the chemical potential and the total number operator respectively. In this section t is the variable corresponding to inverse temperature (multiplied by k) and T is the corresponding ordering symbol. The average is the grand canonical one:

$$\langle \dots \rangle = \text{Tr} \{ \exp(-\beta \mathcal{H}) \dots \} / \Xi, \quad (3.5)$$

where $\beta = (kT)^{-1}$, k and T being the Boltzmann constant and temperature respectively. Ξ is the grand partition function:

$$\Xi = \text{Tr} \exp(-\beta \mathcal{Q}) \equiv \exp(-\beta \Omega), \quad (3.6)$$

where \mathcal{Q} is the grand potential. From (3.2) and (3.6) we get

$$\Omega = \Omega_0 - \frac{1}{2} \int_0^g \frac{dg}{g} \iint \lim_{\beta_2 \rightarrow \beta_1+} \bar{G}(\mathbf{r}\beta_1, \mathbf{r}'\beta_1; \mathbf{r}\beta_2, \mathbf{r}'\beta_2) v(\mathbf{r}-\mathbf{r}') d\mathbf{r} d\mathbf{r}', \quad (3.7)$$

where Ω_0 is the grand potential without interaction, i.e. the grand potential of a perfect gas: $\Xi_0 = \text{Tr} \exp(-\beta \mathcal{H}_0) = \exp(-\beta \Omega_0)$. (3.1) and (3.2) can be written as

$$\bar{G}(x_1, x_2) = \langle T \{ \psi(x_1) \bar{\psi}(x_2) U(\beta) \} \rangle_0 / \langle U(\beta) \rangle_0, \quad (3.8)$$

$$\bar{G}(x_1, x_2; x_3, x_4) = \langle T \{ \psi(x_1) \psi(x_2) \bar{\psi}(x_3) \bar{\psi}(x_4) U(\beta) \} \rangle_0 / \langle U(\beta) \rangle_0, \quad (3.9)$$

where

$$\psi(x) = \exp(\mathcal{H}_0 t) \psi(\mathbf{r}) \exp(-\mathcal{H}_0 t), \quad \bar{\psi}(x) = \exp(\mathcal{H}_0 t) \psi^*(\mathbf{r}) \exp(-\mathcal{H}_0 t), \quad (3.10)$$

$$U(\beta) = T \left\{ \exp \left(- \int_0^\beta H_1(t) dt \right) \right\}, \quad (3.11)$$

$$H_1(t) = \exp(\mathcal{H}_0 t) H_1 \exp(-\mathcal{H}_0 t), \quad (3.12)$$

and

$$\langle \dots \rangle_0 = \text{Tr} \{ \exp(-\beta \mathcal{H}_0) \dots \} / \Xi_0. \quad (3.13)$$

Comparing these formulae with those in § 1, we see that there is a close correspondence between the time-dependent and the temperature-dependent Green functions. Therefore we may use Feynman diagrams in the same way as in § 1. The integral equation for the two-particle Green function has the form

$$\begin{aligned} \bar{G}(x_1, x_2; x_3, x_4) = & -\bar{G}(x_1, x_3) \bar{G}(x_2, x_4) + \bar{G}(x_1, x_4) \bar{G}(x_2, x_3) \\ & - \int \dots \int_0^\beta \dots \int \bar{G}(x_1, x_5) \bar{G}(x_2, x_6) \bar{F}(x_5, x_6; x_7, x_8) \bar{G}(x_7, x_8; x_3, x_4) dx_5 dx_6 dx_7 dx_8. \end{aligned} \quad (3.14)$$

Now we calculate the grand potential in the approximation corresponding to GB's approximation or Mayer's ring approximation in classical statistics. Our approximation corresponds to the following simplification in (3.14)

$$\begin{aligned}\bar{G}(x_1, x_2) &\rightarrow \bar{G}_0(x_1, x_2) = \langle T\{\psi(x_1)\bar{\psi}(x_2)\} \rangle_0 \\ &= \begin{cases} 1/V \cdot \sum_k (1-f_k) \exp\{i\mathbf{k} \cdot (\mathbf{r}_1 - \mathbf{r}_2) - (\varepsilon_0(k) - \mu)(t_1 - t_2)\} & (t_1 > t_2), \\ -1/V \cdot \sum_k f_k \exp\{i\mathbf{k} \cdot (\mathbf{r}_1 - \mathbf{r}_2) - (\varepsilon_0(k) - \mu)(t_1 - t_2)\} & (t_1 < t_2), \end{cases} \\ (f_k &= 1/[\exp(\beta(\varepsilon_0(k) - \mu)) + 1]) \end{aligned} \quad (3.15)$$

$$\bar{G}(x_7, x_8; x_3, x_4) \rightarrow -\bar{G}_0(x_7, x_3)\bar{G}_0(x_8, x_4), \quad (3.16)$$

$$\bar{\Gamma}(x_5, x_6; x_7, x_8) \rightarrow \bar{\mathcal{V}}(x_5 - x_6) \delta(x_5 - x_7) \delta(x_6 - x_8), \quad (3.17)$$

where $\delta(x_1 - x_2) = \delta(\mathbf{r}_1 - \mathbf{r}_2) \delta(t_1 - t_2)$. $\bar{\mathcal{V}}(x_5 - x_6)$ is the effective interaction determined by the following equation,

$$\bar{\mathcal{V}}(x_5 - x_6) = v(x_5 - x_6) - \iiint_0^\beta v(x_5 - x) \bar{G}_0(x, x') \bar{G}_0(x', x) \bar{\mathcal{V}}(x' - x_6) dx dx', \quad (3.18)$$

where

$$v(x_5 - x_6) = v(\mathbf{r}_5 - \mathbf{r}_6) \delta(t_5 - t_6). \quad (3.19)$$

Making the approximations given above and adding the term corresponding to the second order exchange effect, we get the two-particle Green function as follows:

$$\begin{aligned}\bar{G}(x_1, x_2; x_3, x_4) &= -\bar{G}_0(x_1, x_3)\bar{G}_0(x_2, x_4) + \bar{G}_0(x_1, x_4)\bar{G}_0(x_2, x_3) \\ &+ \iiint_0^\beta \bar{G}_0(x_1, x_5)\bar{G}_0(x_2, x_6) \bar{\mathcal{V}}(x_5 - x_6) \bar{G}_0(x_5, x_3)\bar{G}_0(x_6, x_4) dx_5 dx_6 \\ &- \iiint_0^\beta \bar{G}_0(x_1, x_5)\bar{G}_0(x_2, x_6) v(x_5 - x_6) \bar{G}_0(x_5, x_4)\bar{G}_0(x_6, x_3) dx_5 dx_6. \end{aligned} \quad (3.20)$$

In the limit of classical statistics the second and fourth terms expressing the exchange effects are unnecessary.

Now we calculate the effective interaction $\bar{\mathcal{V}}$. Taking account of the periodicity of the one-particle Green function that $\bar{G}(\mathbf{r}, \tau) = -\bar{G}(\mathbf{r}, \tau + \beta)$, where $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ and $\tau = t_1 - t_2$, we may change the definition of $v(x_1 - x_2)$ as follows,

$$v(x_1 - x_2) = v(\mathbf{r}_1 - \mathbf{r}_2) \sum_n \delta(t_1 - t_2 + n\beta). \quad (3.21)$$

Then, by making the Fourier transform the integral equation for $\bar{\mathcal{V}}$ becomes

$$\bar{\mathcal{V}}(\mathbf{q}, \omega_n) = v(\mathbf{q}, \omega_n) - v(\mathbf{q}, \omega_n) \bar{Q}(\mathbf{q}, \omega_n) \bar{\mathcal{V}}(\mathbf{q}, \omega_n), \quad (3.22)$$

where

$$\bar{v}(\mathbf{q}, \omega_n) = \frac{1}{2} \int_{-\beta}^{\beta} d\tau \int d\mathbf{r} \bar{v}(\mathbf{r}, \tau) \exp(-i\mathbf{q} \cdot \mathbf{r} + i\omega_n \tau), \quad (3.23)$$

$$(\omega_n = n\pi/\beta; n = \text{integer})$$

$$v(\mathbf{q}, \omega_n) = v(\mathbf{q}) \{1 + (-1)^n\}/2, \quad (3.24)$$

$$\bar{Q}(\mathbf{q}, \omega_n) = \frac{1}{V\beta} \sum_{\mathbf{k}} \sum_{n'} \bar{G}_0(\mathbf{k}, \omega_{n'}) \bar{G}_0(\mathbf{k} + \mathbf{q}, \omega_n + \omega_{n'}), \quad (3.25)$$

$$\begin{aligned} \bar{G}_0(\mathbf{k}, \omega_n) &= \frac{1}{2} \int_{-\beta}^{\beta} d\tau \int d\mathbf{r} \bar{G}_0(\mathbf{r}, \tau) \exp(-i\mathbf{q} \cdot \mathbf{r} + i\omega_n \tau) \\ &= \begin{cases} \frac{1}{\epsilon_0(\mathbf{k}) - \mu - i\omega_n} & \text{for } n: \text{ odd,} \\ 0 & \text{for } n: \text{ even.} \end{cases} \end{aligned} \quad (3.26)$$

From (3.22) we get

$$\bar{v}(\mathbf{q}, \omega_n) = v(\mathbf{q}, \omega_n) / \{1 + v(\mathbf{q}, \omega_n) \bar{Q}(\mathbf{q}, \omega_n)\}. \quad (3.27)$$

Using the above expression for $G_0(k, \omega_n)$, we have

$$\bar{Q}(\mathbf{q}, \omega_n) = \frac{1}{V} \sum_{\mathbf{k}} \frac{f_{\mathbf{k}+\mathbf{q}} - f_{\mathbf{k}}}{\epsilon_0(\mathbf{k} + \mathbf{q}) - \epsilon_0(\mathbf{k}) - i\omega_n}. \quad (3.28)$$

The following two limiting cases are interesting.

i) Low temperature limit (GB limit): in this case $f_{\mathbf{k}} \rightarrow n_0(\mathbf{k})$ and the corresponding $\bar{Q}(\mathbf{q}, \omega_n)$ was calculated by GB in the limit of $q \ll k_0$:

$$\bar{Q}^{\text{GB}}(\mathbf{q}, \omega_n) = \frac{mk_0}{\pi^2 \hbar^2} R(u), \quad (3.29)$$

$$R(u) = 1 - u \tan^{-1} u^{-1}, \quad (3.30)$$

$$u = m\omega_n / \hbar^2 k_0 q. \quad (3.31)$$

ii) Classical limit (Debye-Hückel limit): in this case using $f_{\mathbf{k}} \rightarrow \exp[\beta(\mu - \epsilon_0(k))]$, we easily get

$$\bar{Q}^{\text{DH}}(\mathbf{q}, \omega_n) = -\beta n \delta_{n,0} + O(q^2), \quad (3.32)$$

(n : electron density).

From (3.27) we see that, as in the case of the ground state, the interaction is effectively screened by the "polarization" of the medium. This screening is statistical, while in the case of the ground state dynamical. This screening was first pointed out by Matsubara.⁸⁾ In particular $[-4\pi e^2 \bar{Q}^{\text{DH}}(0, 0)]^{-1/2} = [4\pi e^2 n \beta]^{-1/2} = \lambda_D$ is called Debye's shielding length.

Using (3.7) and (3.20), we finally get the following expression for the grand potential :

$$\mathcal{Q} = \mathcal{Q}_0 + \mathcal{Q}_{ex}^{(1)} + \mathcal{Q}_{ex}^{(2)} + \mathcal{Q}_R, \quad (3.33)$$

where

$$\mathcal{Q}_0 = -\frac{1}{\beta} \sum_k \ln[1 + \exp\{-\beta(\varepsilon_0(k) - \mu)\}], \quad (3.34a)$$

$$\mathcal{Q}_{ex}^{(1)} = -\frac{1}{2V} \sum_{k, k'} v(k - k') f_k f_{k'}, \quad (3.34b)$$

$$\mathcal{Q}_{ex}^{(2)} = \frac{1}{2V^2} \sum_{k_1, k_2, q} v(q) v(k_1 - k_2 - q) \frac{(1 - f_{k_1})(1 - f_{k_2}) f_{k_1 - q} f_{k_2 + q}}{\varepsilon_0(k_1) + \varepsilon_0(k_2) - \varepsilon_0(k_1 - q) - \varepsilon_0(k_2 + q)}, \quad (3.34c)$$

$$\mathcal{Q}_R = \frac{1}{2\beta} \sum_q \sum_n^{\text{even}} [\ln\{1 + v(q) \bar{Q}(q, \omega_n)\} - v(q) \bar{Q}(q, \omega_n)]. \quad (3.34d)$$

Taking the suitable limiting procedure and using (3.29) and (3.32) for $\bar{Q}(q, \omega_n)$, we get from (3.33) GB's result for the correlation energy in the low temperature limit and DH's equation of state in the limit of classical statistics.⁷⁾

The higher order effects in the above two limits have different characters from each other. In the low temperature limit the exchange correction to the polarization effect is the next higher order contribution, while in the classical limit it is needed to include the higher order effects of two-body scattering, that is, the "ladder type diagrams" make the contribution to the next higher order. We will discuss this problem later in this series.

§ 4. Time- and temperature-dependent Green functions

We define one- and two-particle Green functions which depend on time and temperature as

$$\hat{G}(x_1, x_2) = i \text{Tr} \exp(-\beta \mathcal{H}) \hat{T} \{ \psi(x_1) \bar{\psi}(x_2) \} / \text{Tr} \exp(-\beta \mathcal{H}), \quad (4.1)$$

$$\hat{G}(x_1, x_2; x_3, x_4) = i \text{Tr} \exp(-\beta \mathcal{H}) \hat{T} \{ \psi(x_1) \psi(x_2) \bar{\psi}(x_3) \bar{\psi}(x_4) \} / \text{Tr} \exp(-\beta \mathcal{H}). \quad (4.2)$$

Here

$$\bar{\psi}(x_1) \equiv \bar{\psi}(\mathbf{r}_1, \lambda_1) = \exp(i/\hbar \cdot \mathcal{H} \lambda_1) \psi^*(\mathbf{r}_1) \exp(-i/\hbar \cdot \mathcal{H} \lambda_1), \quad (4.3)$$

where $\lambda_1 \equiv t_1 - i\hbar\beta_1$. \hat{T} is defined in the analogous way as Wick's chronological operator :

$$\begin{aligned} \hat{T} \{ \psi(x_1) \bar{\psi}(x_2) \} &= +\psi(x_1) \bar{\psi}(x_2) \quad \text{if } \lambda_1 > \lambda_2 \\ &= -\bar{\psi}(x_2) \psi(x_1) \quad \text{if } \lambda_1 \leq \lambda_2 \end{aligned} \quad (4.4)$$

where $\lambda_1 > \lambda_2$ means that $t_1 > t_2$ and $\beta_1 \geq \beta_2$, or $t_1 \geq t_2$ and $\beta_1 > \beta_2$.

We consider operators $U(\lambda, \lambda_0)$ and $U^+(\lambda, \lambda_0)$ which satisfy the following equations,*

$$i\hbar(\partial U(\lambda, \lambda_0)/\partial \lambda) = \mathcal{H}_1(\lambda) U(\lambda, \lambda_0), \quad (4.5a)$$

$$-i\hbar(\partial U^+(\lambda, \lambda_0)/\partial \lambda) = U^+(\lambda, \lambda_0) \mathcal{H}_1(\lambda), \quad (4.5b)$$

with

$$U(\lambda_0, \lambda_0) = U^+(\lambda_0, \lambda_0) = 1. \quad (4.6)$$

Here

$$\mathcal{H}_1(\lambda) = \exp(i/\hbar \cdot \mathcal{H}_0 \lambda) \cdot \mathcal{H}_1 \cdot \exp(-i/\hbar \cdot \mathcal{H}_0 \lambda). \quad (4.7)$$

From (4.5a), (4.5b) and (4.6) we get

$$U^+(\lambda, \lambda_0) U(\lambda, \lambda_0) = 1. \quad (4.8)$$

The solution of (4.5a) is given by

$$U(\lambda, \lambda_0) = \sum_{n=0}^{\infty} \left(\frac{-i}{\hbar} \right)^n \int_{\lambda_0}^{\lambda} d\lambda_1 \int_{\lambda_0}^{\lambda_1} d\lambda_2 \cdots \int_{\lambda_0}^{\lambda_{n-1}} d\lambda_n \mathcal{H}_1(\lambda_1) \mathcal{H}_1(\lambda_2) \cdots \mathcal{H}_1(\lambda_n), \quad (4.9)$$

which may be rewritten as follows,

$$U(\lambda, \lambda_0) = \sum_{n=0}^{\infty} \left(\frac{-i}{\hbar} \right)^n \frac{1}{n!} \int_{\lambda_0}^{\lambda} \cdots \int_{\lambda_0}^{\lambda} d\lambda_1 \cdots d\lambda_n \hat{T}[\mathcal{H}_1(\lambda_1) \cdots \mathcal{H}_1(\lambda_n)], \quad (4.10)$$

where the integration over λ is to be performed along a path shown in Fig. 4. Using the relation

$$\exp(i/\hbar \cdot \mathcal{H}_0 \lambda) \mathcal{H}_1(\lambda') \exp(-i/\hbar \cdot \mathcal{H}_0 \lambda) = \mathcal{H}_1(\lambda + \lambda'),$$

we find

$$\begin{aligned} & \exp(i/\hbar \cdot \mathcal{H}_0 \lambda) U(\lambda', \lambda_0) \\ & \times \exp(-i/\hbar \cdot \mathcal{H}_0 \lambda) = U(\lambda + \lambda', \lambda + \lambda_0). \end{aligned} \quad (4.11)$$

It then follows from the equations

$$\exp(-i/\hbar \cdot \mathcal{H} \lambda) = \exp(-i/\hbar \cdot \mathcal{H}_0 \lambda) U(\lambda, 0), \quad (4.12a)$$

$$\exp(i/\hbar \cdot \mathcal{H} \lambda) = U^+(\lambda, 0) \exp(i/\hbar \cdot \mathcal{H}_0 \lambda), \quad (4.12b)$$

and from using (4.11), that

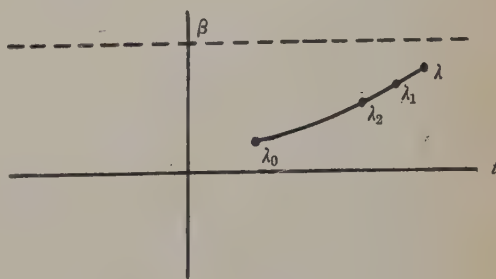


Fig. 4

* U^\dagger is not the Hermitian conjugate of U .

$$U(\lambda, \lambda_0) = \exp(i/\hbar \cdot \mathcal{H}_0 \lambda) \exp(-i/\hbar \cdot \mathcal{H}(\lambda - \lambda_0)) \exp(-i/\hbar \cdot \mathcal{H}_0 \lambda_0). \quad (4.13)$$

Thus we have

$$U(\lambda, \lambda') U(\lambda', \lambda_0) = U(\lambda, \lambda_0). \quad (4.14)$$

If we set $\lambda = \lambda_0$ in (4.14), we obtain

$$U(\lambda_0, \lambda) U(\lambda, \lambda_0) = 1. \quad (4.15)$$

From (4.8) and (4.15) we get

$$U^{-1}(\lambda, \lambda_0) = U^+(\lambda, \lambda_0) = U(\lambda_0, \lambda). \quad (4.16)$$

Gell-Mann and Low¹⁴⁾ have shown that

$$\Psi_n = C_n U(0, \pm \infty) \varphi_n / (C_n U(0, \pm \infty) \varphi_n), \quad (4.17)$$

where Ψ_n and φ_n are the normalized eigenfunctions of the total Hamiltonian and the unperturbed Hamiltonian respectively. C_n is the normalization constant. For a typical order $\lambda_1 > \lambda_2$ we have

$$\begin{aligned} \hat{G}(x_1, x_2) &= i \sum_N \sum_n (\Psi_n^N, \exp(-\beta \mathcal{H}) \phi(x_1) \bar{\phi}(x_2) \Psi_n^N) / \sum_N \sum_n (\Psi_n^N, \exp(-\beta \mathcal{H}) \Psi_n^N) \\ &= i \sum_N \sum_n (\varphi_n^N, U(\infty, 0) \exp(-\beta \mathcal{H}_0) U(-i\hbar\beta, 0) U^{-1}(\lambda_1, 0) \phi(x_1) U(\lambda_1, 0) \\ &\quad \times U^{-1}(\lambda_2, 0) \bar{\phi}(x_2) U(\lambda_2, 0) U(0, -\infty) \varphi_n^N) / \sum_N \sum_n (\varphi_n^N, U(\infty, 0) \\ &\quad \times \exp(-\beta \mathcal{H}_0) U(-i\hbar\beta, 0) U(0, -\infty) \varphi_n^N). \end{aligned} \quad (4.18)$$

Here

$$\phi(x) = \exp(i/\hbar \cdot \mathcal{H}_0 \lambda) \phi(\mathbf{r}) \exp(-i/\hbar \cdot \mathcal{H}_0 \lambda). \quad (4.19)$$

Using the relation

$$\exp(\beta \mathcal{H}_0) U(\infty, 0) \exp(-\beta \mathcal{H}_0) = U(\infty - i\hbar\beta, -i\hbar\beta), \quad (4.20)$$

we rewrite (4.18) as

$$\begin{aligned} \hat{G}(x_1, x_2) &= i \text{Tr} \exp(-\beta \mathcal{H}_0) U(\infty - i\hbar\beta, -i\hbar\beta) U(-i\hbar\beta, 0) U(0, \lambda_1) \\ &\quad \cdot \phi(x_1) U(\lambda_1, 0) U(0, \lambda_2) \bar{\phi}(x_2) U(\lambda_2, 0) U(0, -\infty) / \text{Tr} \\ &\quad \cdot \exp(-\beta \mathcal{H}_0) U(\infty - i\hbar\beta, -i\hbar\beta) U(-i\hbar\beta, 0) U(0, -\infty) \\ &= i \text{Tr} \exp(-\beta \mathcal{H}_0) \hat{T} \{ \phi(x_1) \bar{\phi}(x_2) U(\infty - i\hbar\beta, -\infty) \} / \text{Tr} U(\infty - i\hbar\beta, -\infty) \\ &= i \langle \hat{T} \{ \phi(x_1) \bar{\phi}(x_2) U(\infty - i\hbar\beta, -\infty) \} \rangle_0 / \langle U(\infty - i\hbar\beta, -\infty) \rangle_0. \end{aligned} \quad (4.21)$$

The above equation holds also for $\lambda_1 \leq \lambda_2$. We have a similar expression for the two-particle Green function.

The proof of the fact that the average of a normal product vanishes is quite the same as that given by Matsubara⁸⁾ and Thouless¹⁸⁾ for the temperature-dependent Green functions. We shall briefly recapitulate the proof. We decompose $\bar{\phi}(x)$ and $\phi(x)$ into two parts:

$$\begin{aligned}\bar{\phi}(x) &= \bar{\phi}_+(x) + \bar{\phi}_-(x), \\ \phi(x) &= \phi_+(x) + \phi_-(x),\end{aligned}\tag{4.22}$$

where

$$\begin{aligned}\bar{\phi}_+(x) &= V^{-1/2} \sum_k (1-g_k) a_k^* \exp\{-i\mathbf{k}\cdot\mathbf{r} + i(\epsilon_0(k) - \mu)\lambda\}, \\ \bar{\phi}_-(x) &= V^{-1/2} \sum_k g_k a_k^* \exp\{-i\mathbf{k}\cdot\mathbf{r} + i(\epsilon_0(k) - \mu)\lambda\}, \\ \phi_+(x) &= V^{-1/2} \sum_k g_k a_k \exp\{i\mathbf{k}\cdot\mathbf{r} - i(\epsilon_0(k) - \mu)\lambda\}, \\ \phi_-(x) &= V^{-1/2} \sum_k (1-g_k) a_k \exp\{i\mathbf{k}\cdot\mathbf{r} - i(\epsilon_0(k) - \mu)\lambda\}.\end{aligned}\tag{4.23}$$

The normal product of $F(\bar{\phi}\phi)$, where F is a functional of $\bar{\phi}$ and ϕ , is defined by making the substitution (4.22) in F , and reordering each term so that $\bar{\phi}_-$ and ϕ_- come first, followed by $\bar{\phi}_+$ and ϕ_+ , with a sign of \pm according to whether the permutation is even or odd. The contraction $\phi^* \bar{\phi}^*$ is given by

$$\begin{aligned}\phi^*(x) \bar{\phi}^*(x') &= \hat{T}[\phi(x) \bar{\phi}(x')] - N[\phi(x) \bar{\phi}(x')] \\ &= \begin{cases} [\phi_-(x), \bar{\phi}_+(x')]_+ & \lambda > \lambda' \\ -[\phi_+(x), \bar{\phi}_-(x')]_+ + [\phi_-(x), \bar{\phi}_+(x')]_+ & \lambda \leq \lambda'. \end{cases}\end{aligned}$$

If $g_k^2 = 1 - f_k$, we find that

$$\begin{aligned}\phi^*(x) \bar{\phi}^*(x') &= \sum_k (1-f_k) \exp\{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}') - i(\epsilon_0(k) - \mu)(\lambda - \lambda')\} \quad \text{if } \lambda > \lambda' \\ &\quad - \sum_k f_k \exp\{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}') - i(\epsilon_0(k) - \mu)(\lambda - \lambda')\} \quad \text{if } \lambda \leq \lambda' \\ &= -i\hat{G}_0(x, x')\end{aligned}\tag{4.24}$$

$$= -i\hat{G}_0(x, x')\tag{4.25}$$

and

$$\begin{aligned}\langle N[\phi(x) \bar{\phi}(x')] \rangle_0 &= \sum_k \{2g_k(1-g_k)(1-f_k) + (1-g_k)^2(1-f_k) - g_k^2 f_k\} \\ &\quad \times \exp\{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}') - i(\epsilon_0(k) - \mu)(\lambda - \lambda')\} = 0.\end{aligned}\tag{4.26}$$

Further, it can be shown that normal products of higher order vanish.¹⁸⁾ We can also prove the "linked cluster expansion" in the same way as in the case of the time-dependent Green functions or temperature-dependent Green functions. Therefore we may apply the same techniques as in § 2 and § 3 to the calculation of the Green functions (4.1) and (4.2), using Feynman diagrams.

The Green functions (4.1) and (4.2) would be useful for the calculation of transport quantities. The expressions for the transport quantities are given by a formalism developed by Kubo.¹⁹⁾ In the case that the external electromagnetic field is represented by a vector potential $\mathbf{A}(\mathbf{r}, t)$, the response current is given by

$$i^{\alpha}(\mathbf{r}, t) = \text{Tr } \rho j_1^{\alpha}(\mathbf{r}, t) - \frac{1}{i\hbar c} \sum_{\beta=1}^3 \text{Tr} \int_{-\infty}^0 dt' \int d\mathbf{r}' \rho [j_0^{\alpha}(\mathbf{r}), j_0^{\beta}(\mathbf{r}', t')] A^{\beta}(\mathbf{r}', t+t'),\tag{4.27}$$

where

$$\rho = \exp(-\beta \mathcal{H}) / \text{Tr} \exp(-\beta \mathcal{H}), \quad (4.28)$$

$$\mathbf{j}_0(\mathbf{r}) = e/2m \cdot [-\hbar/i \nabla \phi^*(\mathbf{r}) \cdot \phi(\mathbf{r}) + \phi^*(\mathbf{r}) \hbar/i \nabla \phi(\mathbf{r})], \quad (4.29a)$$

$$\mathbf{j}_1(\mathbf{r}, t) = (-e^2/mc) \mathbf{A}(\mathbf{r}, t) \phi^*(\mathbf{r}) \phi(\mathbf{r}) \quad (4.29b)$$

and

$$\mathbf{j}_0(\mathbf{r}, t) = \exp(i/\hbar \cdot \mathcal{H}_0 t) \mathbf{j}_0(\mathbf{r}) \exp(-i/\hbar \cdot \mathcal{H}_0 t). \quad (4.30)$$

In the case that the external electric field is represented by a scalar potential $\varphi(\mathbf{r}, t)$, the response current is given by

$$\begin{aligned} i^\alpha(\mathbf{r}, t) &= \text{Tr} \int_0^\infty d\tau \int_0^\beta d\beta' \int d\mathbf{r}' \rho \text{div} \mathbf{j}_0(\mathbf{r}', -i\hbar\beta') j_0^\alpha(\mathbf{r}, \tau) \varphi(\mathbf{r}', t+\tau) \\ &= \text{Tr} \int_0^\infty d\tau \int_0^\beta d\beta' \int d\mathbf{r}' \rho j_0^\alpha(\mathbf{r}, \tau - i\hbar\beta) \text{div} \mathbf{j}_0(\mathbf{r}', -i\hbar\beta') \varphi(\mathbf{r}', t+\tau). \end{aligned} \quad (4.31)$$

The right-hand sides of (4.27) and (4.31) can be expressed in terms of Green functions as follows,

$$\begin{aligned} i^\alpha(\mathbf{r}, t) &= \frac{-ie^2}{mc} A^\alpha(\mathbf{r}, t) \hat{G}(\mathbf{r}0, \mathbf{r}0) - \frac{1}{i\hbar c} \int_{-\infty}^0 dt' \int d\mathbf{r}' \sum_{\beta=1}^3 \left[\frac{e^2 \hbar^2}{4m^2} \right. \\ &\quad \cdot \lim_{\substack{\tau'' \rightarrow \tau \\ \tau''' \rightarrow \tau'}} (\nabla_\alpha - \nabla_{\alpha''}) (\nabla_{\beta'} - \nabla_{\beta'''}) \hat{G}(\mathbf{r}0, \mathbf{r}'t'; \mathbf{r}''0_+, \mathbf{r}'''t'_+) - c.c. \left. \right] A^\beta(\mathbf{r}', t+t'), \end{aligned} \quad (4.27)'$$

$$\begin{aligned} i^\alpha(\mathbf{r}, t) &= \frac{e^2 \hbar^2}{4m^2} \sum_{\beta=1}^3 \int_0^\infty d\tau \int_0^\beta d\beta' \int d\mathbf{r}' \nabla_{\beta'} \lim_{\substack{\tau'' \rightarrow \tau \\ \tau''' \rightarrow \tau'}} (\nabla_\alpha - \nabla_{\alpha''}) (\nabla_{\beta'} - \nabla_{\beta'''}) \\ &\quad \cdot \hat{G}(\mathbf{r}, \tau - i\hbar\beta, \mathbf{r}', -i\hbar\beta'; \mathbf{r}'', (\tau - i\hbar\beta)_+, \mathbf{r}''', (-i\hbar\beta')_+) \varphi(\mathbf{r}', t+\tau), \end{aligned} \quad (4.31)'$$

where $\lambda_+ = \lambda + \delta (\delta \rightarrow +0)$.

For a classical gas we obtain the Drude conductivity, if we set $\hat{G}(x_1, x_2; x_3, x_4) = i\hat{G}_0(x_1, x_3) \hat{G}_0(x_2, x_4) - i\hat{G}_0(x_1, x_4) \hat{G}_0(x_2, x_3)$ in (4.31), where \hat{G}_0 is given by (4.24).

The Green function $\hat{G}(x_1, x_2)$ is defined only in regions I and III shown in Fig. 5. To make a Fourier

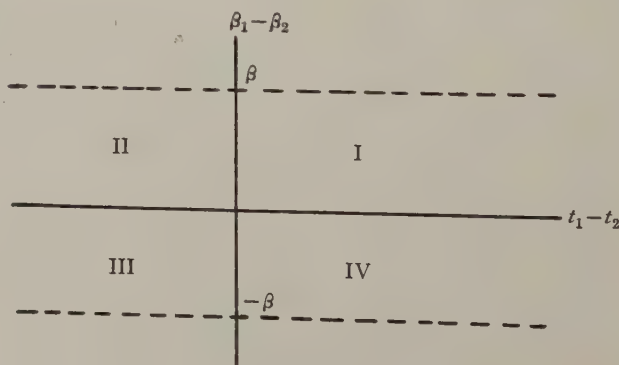


Fig. 5

transform of $\hat{G}_0(x_1, x_2)$ it is expedient that we extend the definition of $\hat{G}_0(x_1, x_2)$ to regions II and IV as follows: In regions I and II

$$\hat{G}_0(k, \lambda_1 - \lambda_2) = i(1 - f_k) \exp\{-i(\varepsilon_0(k) - \mu)(\lambda_1 - \lambda_2)\} \quad (4.32a)$$

and in regions III and IV

$$\hat{G}_0(k, \lambda_1 - \lambda_2) = -if_k \exp\{-i(\varepsilon_0(k) - \mu)(\lambda_1 - \lambda_2)\}. \quad (4.32b)$$

Then the Fourier transform of $\hat{G}_0(x_1, x_2)$ is given by

$$\hat{G}_0(k, \varepsilon, \omega_n) = \frac{1}{2\pi i \beta} \frac{1}{i\omega_n - (\varepsilon_0(k) - \mu)} \delta(\varepsilon - \varepsilon_0(k)). \quad (4.33)$$

It is desirable that we define an ordering symbol, which is different from \hat{T} , so that the definition of G covers the whole region. Details will be published later.

Appendix

We derive the normalization condition for $f_k^{pl}(\mathbf{p})$. We make use of the \hat{f} -sum rule for the oscillator strength:

$$\sum_n \hat{f}_{on}(k) = N, \quad (A.1)$$

where

$$\hat{f}_{on}(k) = \frac{2m}{\hbar k^2} \omega_{no} |\langle \Psi_0, \rho_k \Psi_n \rangle|^2, \quad \left(\rho_k = \frac{1}{V} \sum_p a_{p+k}^* a_p \right), \quad (A.2)$$

which may be rewritten as

$$\frac{\hbar k^2}{2m} N = \omega_{pl}(k) \left| \sum_p f_{kw}^{pl}(\mathbf{p}) \right|^2 + \sum_q \omega_q \left| \sum_p f_{kw}^{sq}(\mathbf{p}) \right|^2. \quad (A.3)$$

Here $\omega_q = \{\varepsilon_0(\mathbf{q} + \mathbf{k}/2) - \varepsilon_0(\mathbf{q} - \mathbf{k}/2)\}/\hbar$. The scattering state amplitude $f_{kw}^{sq}(\mathbf{p})$ is determined by the equation

$$f_k^{sq}(\mathbf{p}) = \delta_{pq} [n_0(\mathbf{q} - \mathbf{k}/2) (1 - n_0(\mathbf{q} + \mathbf{k}/2))]^{1/2} - \frac{v(k)}{\hbar} \frac{n_0(\mathbf{p} + \mathbf{k}/2) - n_0(\mathbf{p} - \mathbf{k}/2)}{\omega_q - \omega_p - i\delta \{n_0(\mathbf{p} + \mathbf{k}/2) - n_0(\mathbf{p} - \mathbf{k}/2)\}} \sum_{p'} f_k^{sq}(\mathbf{p}'). \quad (A.4)$$

Inserting the explicit expression for $f_{kw}^{sq}(\mathbf{p})$ into (A.3) and rearranging terms, we get

$$\begin{aligned} \frac{\hbar k^2}{2m} N = & \omega_{pl}(k) \left| \sum_p f_{kw}^{pl}(\mathbf{p}) \right|^2 + \sum_q \omega_q n_0(\mathbf{q} - \mathbf{k}/2) (1 - n_0(\mathbf{q} + \mathbf{k}/2)) \\ & + \sum_q \omega_q n_0(\mathbf{q} - \mathbf{k}/2) (1 - n_0(\mathbf{q} + \mathbf{k}/2)) \left\{ \frac{1}{|1 + f(\omega_q + i\delta)|^2} - 1 \right\}, \end{aligned} \quad (A.5)$$

where

$$f(\omega_q + i\delta) = v(k) \sum_p n_0(\mathbf{p}) (1 - n_0(\mathbf{p} + \mathbf{k})) \left[\frac{1}{\omega_p + \omega_q + i\delta} + \frac{1}{\omega_p - \omega_q - i\delta} \right]. \quad (A.6)$$

The direct calculation shows that the second term of the right-hand side in (A.5) $= \hbar k^2 N/2m$. Making use of the techniques of Sawada et al., we get

$$\text{the third term in (A.5)} = \frac{\hbar}{v(\mathbf{k})} \cdot \frac{1}{2\pi i} \int_{C_1} dz \cdot z \left\{ \frac{1}{1+f(z)} - f(z) \right\}, \quad (\text{A.7})$$

where the contour C_1 is shown in Fig. 5. Changing the contour C_1 to C_2 and C_3 (Fig. 6) and calculating the residue, we find

$$|\sum_p f_k^{pl}(\mathbf{p})|^2 = \frac{\hbar}{v(\mathbf{k})} e^2 \frac{\partial \omega_{pl}(\mathbf{k})}{\partial e^2}. \quad (\text{A.8})$$

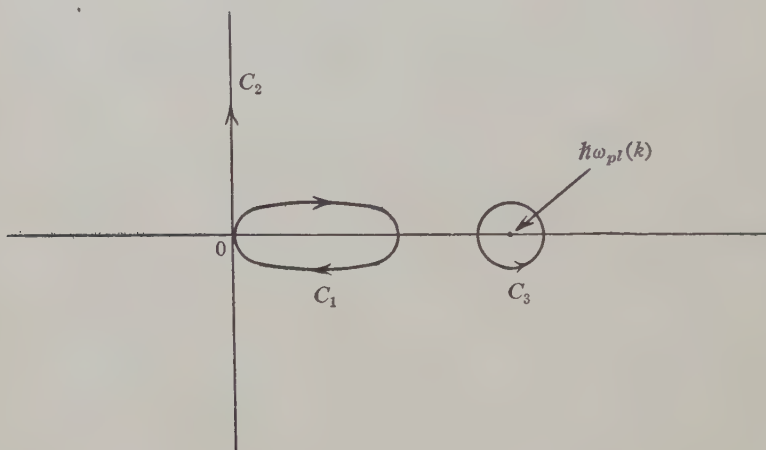


Fig. 6

It is easy to rewrite the above expression as

$$\sum_p |f_k^{pl}(\mathbf{p})|^2 \{n_0(\mathbf{p}-\mathbf{k}/2) - n_0(\mathbf{p}+\mathbf{k}/2)\} = 1. \quad (\text{A.9})$$

The correlation energy can be written in terms of $f_k^{sc}(\mathbf{p})$ and $f_k^{pl}(\mathbf{p})$:

$$E_{\text{corr.}} = \frac{1}{2V} \int_0^g \frac{dg}{g} \left[\sum_k v(\mathbf{k}) |\sum_p f_k^{pl}(\mathbf{p})|^2 + \sum_k v(\mathbf{k}) |\sum_p f_k^{sc}(\mathbf{p})|^2 \right] - \frac{N}{2} v(0). \quad (\text{A.10})$$

It is easy to show that, using the above normalization condition for $f_k^{pl}(\mathbf{p})$, we are led to GB's result.

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Green Function Method for Electron Gas. II*— Dispersion Relation of Plasmons —*

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The shift of plasmon energy due to the electron exchange is calculated. Our result is the same as that obtained by Nozières and Pines. The analysis of the experimental values of Watanabe referring to Bohm-Pines' values is made and the agreement between our theory and the experiment seems to be good at high densities.

§ 1. Introduction

Several authors¹⁾ attempted to improve Bohm-Pines' dispersion relation of plasmons by including the effect of electron exchange and reached the conclusions that the exchange effect gives a certain correction to the term of order k^2 (k is the wave number of a plasmon). The corrections, however, vary considerably with the authors and the characters of the approximations in their theories are not quite obvious. Recently DuBois²⁾ has calculated the shift of plasmon energy due to the electron exchange, using the Feynman-Dyson techniques of field theory. In the present paper we obtain the dispersion relation by the method described in I.³⁾ The correction to the term of order k^2 is the same as that obtained by Nozières and Pines,⁴⁾ who derived the shift of the plasmon frequency by Bohm-Pines' theory.

§ 2. Dispersion relation

The two-particle Green function^{3)*}

$$G(x_1, x_2; x_3, x_4) = i \langle \Psi_0, T \{ \psi(x_1) \psi(x_2) \psi^*(x_3) \psi^*(x_4) \} \Psi_0 \rangle \quad (2.1)$$

satisfies the Bethe-Salpeter equation;

$$\begin{aligned} G(x_1, x_2; x_3, x_4) = & iG(x_1, x_3)G(x_2, x_4) - iG(x_1, x_4)G(x_2, x_3) \\ & + i \int \cdots \int G(x_1, x_5)G(x_6, x_3)\Gamma(x_5, x_6; x_7, x_8)G(x_7, x_2; x_8, x_4) dx_5 dx_6 dx_7 dx_8. \end{aligned} \quad (2.2)$$

Let $t_1, t_3 > t_2, t_4$. Then, we have

$$G(x_1, x_2; x_3, x_4) = -i \sum_S \chi_S(x_1, x_3) \bar{\chi}_S(x_2, x_4), \quad (2.3)$$

* Throughout this paper the atomic unit, $m = \hbar = 1$, is used and the notations in I are followed.

where

$$\chi_s(x_1, x_3) = (\Psi_0, T\{\psi(x_1)\psi^*(x_3)\}\Psi_s). \quad (2.4)$$

Applying Gell-Mann and Low's procedure to (2.2), we have the following equation for the bound state amplitude $\chi_s(x_1, x_3)$ of an electron and a hole,

$$\chi_s(x_1, x_3) = i \int G(x_1, x_5) G(x_6, x_3) \Gamma(x_5, x_6; x_7, x_8) \chi_s(x_7, x_8) dx_5 dx_6 dx_7 dx_8. \quad (2.5)$$

The interaction part Γ is given by the diagrams shown in Fig. 1 in lowest order. Then we get

$$\begin{aligned} \chi_s(x_1, x_3) = & -2i \int G(x_1, x_5) G(x_6, x_3) v(x_5 - x_6) \chi(x_5, x_6) dx_5 dx_6 \\ & + i \int G(x_1, x_5) G(x_6, x_3) v(x_5 - x_6) \chi_s(x_5, x_6) dx_5 dx_6, \end{aligned} \quad (2.6)$$

where $v(x - x') = v(\mathbf{r} - \mathbf{r}') \delta(t - t')$.

The first and the second terms on the right-hand side in (2.6) correspond to the diagrams shown in Fig. 1 (a) and Fig. 1 (b) respectively. The

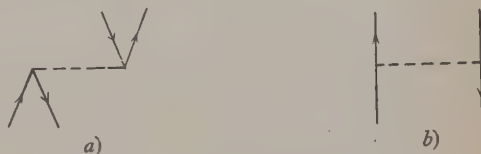


Fig. 1. First order interaction part

factor 2 of the first term is due to the fact that in exciting a new pair an electron of either spin is available. The Fourier transform of (2.6) is given by*

$$\begin{aligned} \chi_{kw}(p) = & iG\left(p + \frac{k}{2}\right) G\left(p - \frac{k}{2}\right) \left\{ -2v(k) \int \chi_{kw}(p) \frac{dp}{(2\pi)^4} \right. \\ & \left. + \int v(q) \chi_{kw}(p+q) \frac{dq}{(2\pi)^4} \right\}, \end{aligned} \quad (2.7)$$

which was first derived by Galickij and Migdal.⁵⁾ If we neglect the second term in parentheses in (2.7), substitute the non-interacting one-particle Green function $G_0(p)$ for $G(p)$ and integrate the both sides over p , we get the Bohm-Pines dispersion relation⁶⁾

$$\begin{aligned} 1 = & -2iv(k) \int G_0\left(p + \frac{k}{2}\right) G_0\left(p - \frac{k}{2}\right) \frac{dp}{(2\pi)^4} \\ = & -2v(k) \int \frac{n_0(\mathbf{p} + \mathbf{k}/2) - n_0(\mathbf{p} - \mathbf{k}/2)}{\omega - \varepsilon_0(\mathbf{p} + \mathbf{k}/2) + \varepsilon_0(\mathbf{p} - \mathbf{k}/2)} \frac{d\mathbf{p}}{(2\pi)^3} \\ = & 4\pi e^2 \int \frac{2n_0(\mathbf{p})}{(\omega - \mathbf{k} \cdot \mathbf{p})^2 - (k^4/4)} \frac{d\mathbf{p}}{(2\pi)^3}. \end{aligned} \quad (2.8)$$

* Hereafter p, q and k denote the four-dimensional energy-momentum vectors, while the bold-faced \mathbf{p}, \mathbf{q} and \mathbf{k} represent the ordinary momenta, for instance, $p = (p\varepsilon)$, $k = (k\omega)$, $dp = d\mathbf{p} d\varepsilon$, etc. Confusion of the four-dimensional vector with the magnitude of the momentum will not occur, because the four-dimensional vector always appears in the argument of G or χ .

In order to improve the dispersion relation we must take into account (i) the exchange energy and (ii) the ladder type diagram. The Green function $G(\mathbf{p}, \varepsilon)$ may be written as

$$G(\mathbf{p}, \varepsilon) = G_0(\mathbf{p}, \varepsilon) + G_0(\mathbf{p}, \varepsilon) \Sigma(\mathbf{p}, \varepsilon) G(\mathbf{p}, \varepsilon), \quad (2.9)$$

or

$$G(\mathbf{p}, \varepsilon) = 1 / \{ G_0^{-1}(\mathbf{p}, \varepsilon) - \Sigma(\mathbf{p}, \varepsilon) \} = 1 / \{ \varepsilon_0(\mathbf{p}) - \varepsilon - \Sigma(\mathbf{p}, \varepsilon) \}, \quad (2.10)$$

where $\Sigma(\mathbf{p}, \varepsilon)$ is the irreducible self-energy part. The first order self-energy which is represented by the diagram shown in Fig. 2 is given by

$$\Sigma^{(1)}(\mathbf{p}) = \int v(\mathbf{p} - \mathbf{p}') n_0(\mathbf{p}') \frac{d\mathbf{p}'}{(2\pi)^3}. \quad (2.11)$$

If we substitute $\varepsilon_0(\mathbf{p}) - \Sigma^{(1)}(\mathbf{p})$ for $\varepsilon_0(\mathbf{p})$ in (2.8), we find that a term of $\ln k$ appears in the dispersion relation. This logarithmic divergence is cancelled by the second term in parentheses in (2.7), which represents the effect of the ladder type diagram. The ladder type diagrams give, in general, contributions of higher order in r_s than the pair diagrams and may be regarded as a small perturbation. Then we may solve (2.7) by successive approximation. Taking the first approximation, we get



Fig. 2. $\Sigma^{(1)}$

$$\begin{aligned} \chi_{k\omega}(\mathbf{p}) = & -2iv(\mathbf{k}) G\left(\mathbf{p} + \frac{\mathbf{k}}{2}\right) G\left(\mathbf{p} - \frac{\mathbf{k}}{2}\right) \left\{ 1 + i \int v(\mathbf{q}) G\left(\mathbf{p} + \mathbf{q} + \frac{\mathbf{k}}{2}\right) \right. \\ & \times G\left(\mathbf{p} + \mathbf{q} - \frac{\mathbf{k}}{2}\right) \frac{d\mathbf{q}}{(2\pi)^4} \left. \right\} \int \chi_{k\omega}(\mathbf{p}') \frac{d\mathbf{p}'}{(2\pi)^4}. \end{aligned} \quad (2.12)$$

Within our approximation (to the order of $r_s \ln r_s$ or r_s) it is permitted to substitute $G_0(\mathbf{p}, \varepsilon)$ for $G(\mathbf{p}, \varepsilon)$ in the second term in parentheses of (2.12). Then we have the dispersion relation

$$\begin{aligned} 1 = & -2v(\mathbf{k}) \left\{ i \int G\left(\mathbf{p} + \frac{\mathbf{k}}{2}\right) G\left(\mathbf{p} - \frac{\mathbf{k}}{2}\right) \frac{d\mathbf{p}}{(2\pi)^4} + i^2 \iint G_0\left(\mathbf{p} + \frac{\mathbf{k}}{2}\right) G_0\left(\mathbf{p} - \frac{\mathbf{k}}{2}\right) \right. \\ & \times v(\mathbf{q}) G_0\left(\mathbf{p} + \mathbf{q} + \frac{\mathbf{k}}{2}\right) G_0\left(\mathbf{p} + \mathbf{q} - \frac{\mathbf{k}}{2}\right) \frac{d\mathbf{p}}{(2\pi)^4} \frac{d\mathbf{q}}{(2\pi)^4} \left. \right\} \\ = & -2v(\mathbf{k}) \left\{ \int \frac{n_0(\mathbf{p} + \mathbf{k}/2) - n_0(\mathbf{p} - \mathbf{k}/2)}{\omega - \mathbf{k} \cdot \mathbf{p} + \Sigma^{(1)}(\mathbf{p} + \mathbf{k}/2) - \Sigma^{(1)}(\mathbf{p} - \mathbf{k}/2)} \frac{d\mathbf{p}}{(2\pi)^3} \right. \\ & + \iint \frac{n_0(\mathbf{p} + \mathbf{k}/2) - n_0(\mathbf{p} - \mathbf{k}/2)}{\omega - \mathbf{k} \cdot \mathbf{p}} \\ & \times v(\mathbf{q}) \frac{n_0(\mathbf{p} + \mathbf{q} + \mathbf{k}/2) - n_0(\mathbf{p} + \mathbf{q} - \mathbf{k}/2)}{\omega - \mathbf{k} \cdot (\mathbf{p} + \mathbf{q})} \frac{d\mathbf{p}}{(2\pi)^3} \frac{d\mathbf{q}}{(2\pi)^3} \left. \right\} \end{aligned} \quad (2.13)$$

instead of (2.8).

Expanding the terms in parentheses in (2.13) in powers of e^2 and k^2 and retaining terms up to orders e^2 and k^4 , we get

$$1 = -2v(\mathbf{k}) \left[\int \frac{n_0(\mathbf{p} + \mathbf{k}/2) - n_0(\mathbf{p} - \mathbf{k}/2)}{\omega - \mathbf{k} \cdot \mathbf{p}} \frac{d\mathbf{p}}{(2\pi)^3} \right. \\ \left. + \frac{2}{\omega^4} \iint (n_0(\mathbf{p}) - n_0(\mathbf{p} - \mathbf{k})) n_0(\mathbf{p} + \mathbf{q}) \right. \\ \left. \times v(\mathbf{q}) \left\{ 3 \left(\mathbf{k} \cdot \mathbf{p} - \frac{\mathbf{k}^2}{2} \right) (\mathbf{k} \cdot \mathbf{q}) + (\mathbf{k} \cdot \mathbf{q})^2 \right\} \frac{d\mathbf{p}}{(2\pi)^3} \frac{d\mathbf{q}}{(2\pi)^3} \right]. \quad (2.14)$$

If we compare the above equation with (2.8), we see that the second term in the bracket is just the correction term to the dispersion relation of BP. Using the following relation,

$$n_0(\mathbf{p}) - n_0(\mathbf{p} - \mathbf{k}) = -\frac{(\mathbf{k} \cdot \mathbf{p})}{p} \delta(p - p_0) + \frac{k^2}{2p} \delta(p - p_0) \\ - \frac{(\mathbf{k} \cdot \mathbf{p})^2}{2p^3} (\delta(p - p_0) - p\delta'(p - p_0)) + \dots,$$

and performing the integrations over \mathbf{p} and \mathbf{q} , we find that the correction term in (2.14) yields $-3/20 \cdot \omega_p^4/\omega^4 \cdot k^2/p_0^2$ up to order k^2 . The dispersion relation now reads

$$\omega^2 = \omega_p^2 + \frac{3}{5} p_0^2 k^2 \left(1 - \frac{1}{4} \frac{\omega_p^2}{p_0^4} \right) + O(k^4), \quad (2.15)$$

which is to be compared with that of BP;

$$\omega^2 = \omega_p^2 + \frac{3}{5} p_0^2 k^2 + O(k^4). \quad (2.16)$$

Thus the correction factor due to the electron exchange is*

$$1 - \frac{1}{4} \frac{\omega_p^2}{p_0^4} = 1 - \frac{\alpha}{3\pi} r_s = 1 - 0.0553 r_s, \quad (2.17)$$

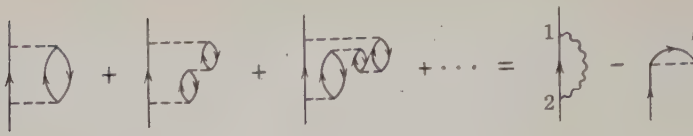
which agrees with that of NP.⁽⁴⁾

§ 3. Further remarks

As mentioned in § 2, when the ladder type diagram is ignored the exchange energy gives a $\ln k$ term to the plasmon frequency. To avoid this difficulty we may as well include the electron correlation, which means here the action of the polarization cloud back on the electron. We consider the contributions to the self energy from the diagrams shown in Fig. 3. Then we have

$$\Sigma^{(2)}(\mathbf{p}, \varepsilon) = -i \int \frac{d\mathbf{k} d\omega}{(2\pi)^4} G_0(\mathbf{p} + \mathbf{k}, \varepsilon + \omega) v(\mathbf{k}) \left(1 - \frac{1}{\varepsilon(\mathbf{k}, \omega)} \right), \quad (3.1)$$

* The correction factor obtained by DuBois is $1 - \frac{\alpha}{12\pi} r_s$. He has taken into account the same diagrams as ours. The discrepancy might be due to his trivial mistake.

Fig. 3. $\Sigma^{(2)}$

where

$$\begin{aligned} \epsilon(\mathbf{k}, \omega) &= 1 + 2iv(\mathbf{k}) \int \frac{d\mathbf{p}}{(2\pi)^4} G_0\left(\mathbf{p} + \frac{\mathbf{k}}{2}\right) G_0\left(\mathbf{p} - \frac{\mathbf{k}}{2}\right) \\ &= 1 + 2v(\mathbf{k}) \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{n_0(\mathbf{p} + \mathbf{k}/2) - n_0(\mathbf{p} - \mathbf{k}/2)}{\omega - \mathbf{p} \cdot \mathbf{k} + i\delta(n_0(\mathbf{p} + \mathbf{k}/2) - n_0(\mathbf{p} - \mathbf{k}/2))} \end{aligned} \quad (3.2)$$

is the complex dielectric constant. Since we are concerned with the self energy up to order r_s , we may replace $\epsilon(\mathbf{k}, \omega)$ in (3.1) by static dielectric constant $\epsilon(\mathbf{k}, 0)$;⁷⁾

$$\epsilon(\mathbf{k}, 0) = 1 + \frac{2\alpha r_s}{\pi} \frac{p_0^2}{k^2} \left\{ 1 + \frac{4p_0^2 - k^2}{4p_0 k} \ln \frac{2p_0 + k}{2p_0 - k} \right\}. \quad (3.3)$$

Then the sum of $\Sigma^{(1)}$ and $\Sigma^{(2)}$ yields

$$\Sigma(\mathbf{p}) = \Sigma^{(1)} + \Sigma^{(2)} = \frac{1}{2} \int \frac{d\mathbf{k}}{(2\pi)^3} v(\mathbf{k}) \left(1 - \frac{1}{\epsilon(\mathbf{k}, 0)} \right) + \int \frac{d\mathbf{k}}{(2\pi)^3} n_0(\mathbf{p} + \mathbf{k}) \frac{v(\mathbf{k})}{\epsilon(\mathbf{k}, 0)}, \quad (3.4)$$

where use has been made of the relation

$$i \int_{-\infty}^{\infty} \frac{d\varepsilon}{2\pi} G_0(\mathbf{p}, \varepsilon) = n_0(\mathbf{p}) - \frac{1}{2}.$$

The first term in (3.4) is simply an additive constant which has no effect on the plasmon frequency and the second one may be interpreted as the exchange energy of the screened Coulomb potential $v(\mathbf{k})/\epsilon(\mathbf{k}, 0)$. This interpretation is most easily understood by the language of diagrams; the neglect of retardation corresponds to that the times t_1 and t_2 of the vertices 1 and 2 in Fig. 3 are set equal as shown in Fig. 4. This diagram just corresponds to the first one in Fig. 3 in NP, where they evaluated the shift of the plasmon frequency caused by H_{sr} in the BP theory.

Fig. 4. $\Sigma^{(1)} + \Sigma^{(2)}$

Our screened potential is essentially (at least, up to order r_s) equivalent to H_{sr} . If we substitute $\Sigma(\mathbf{p})$ for $\Sigma^{(1)}(\mathbf{p})$ in (2.13), we find that the logarithmic divergence disappears, as anticipated at the start.

At the same time, we must consider the effect of the ladder type diagrams, the second term in parentheses in (2.13). Here using "bare" Coulomb potential in the ladder is, also, the origin of the logarithmic divergence, which is, however, eliminated by introduction of the screening of the potential. This means, within our approximation, $v(\mathbf{q})$ must be replaced by $v(\mathbf{q})/\epsilon(\mathbf{q}, 0)$. The ladder type diagram corresponds to the second one in Fig. 3 in NP. After these modifications are both taken into account, we again get the correction factor (2.17).

It can be shown that the contribution from the plasmon state to the correlation energy is not of the simple form $\frac{1}{2}[\hbar\omega - (\hbar\omega)_{\epsilon=0}]dk/(2\pi)^3$ in our case. This might be due to the fact that the separation of the bound state from the scattering state is not clear near the cut-off momentum owing to the damping of plasmons. In the theory of BP the damping of plasmons arises mainly through the short range collisions of electrons (NP). Therefore the damping is associated with the non-RPA corrections to the plasmon frequency. DuBois has calculated the damping and discussed the determination of k_c .

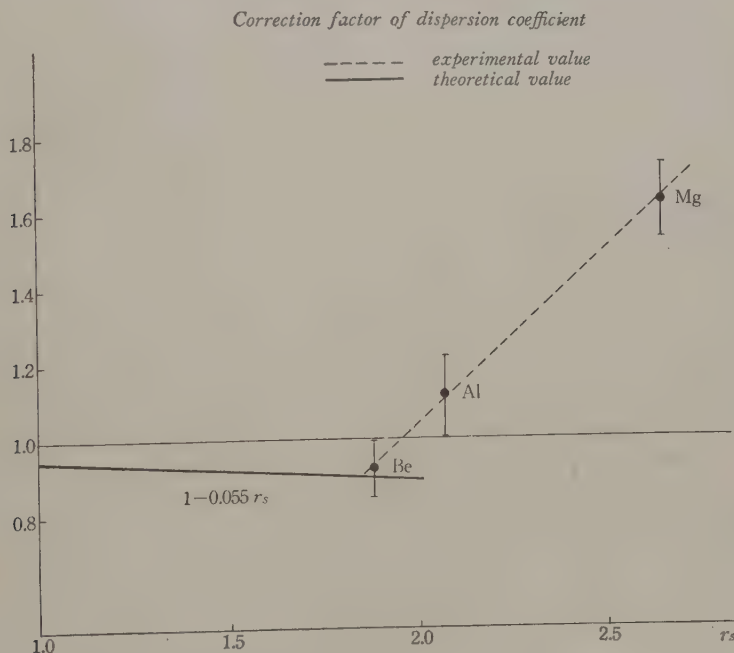


Fig. 5

Finally we add some comments on the comparison with experiment (Fig. 5). First we plot the ratios of Watanabe's experimental values⁸⁾ of the dispersion co-

efficients to BP's theoretical values as a function of r_s . Because of very singular structure of the Brillouin zone, Ge is put out of consideration. On the other hand, for Be, Al and Mg the effective mass is nearly equal to the true mass and we may regard the effect of the positive ion lattice as comparably small. In fact, the above-mentioned ratios stand in a straight line within experimental error and we entertain a hope that this curve can be approximately described by the function of r_s alone. Then we plot the theoretical correction curve (2.17). At high densities such as in Be the agreement between theory and experiment seems to be rather good, whereas at lower densities the discrepancy becomes large and we will have to proceed to the evaluation of corrections up to higher orders.

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Green Function Method for Electron Gas. III

—Diamagnetism—

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The Green function method is applied to the calculation of the diamagnetic susceptibility of a dense electron gas. The exact high density value for the correction to the Landau diamagnetism is calculated.

§ 1. Introduction

The influence of Coulomb interaction between electrons on the diamagnetism was treated by March and Donovan,¹⁾ and Kanazawa.²⁾ March and Donovan, and Fletcher and Larson³⁾ inserted one-electron energy spectrum, which was obtained by Bohm-Pines theory, into the formula for the diamagnetic susceptibility for the quasi-bound electrons:⁴⁾

$$\chi = -\frac{e^2 k_0}{12(\pi\hbar c)^2} \left[\frac{2}{3} \frac{d^2 E}{dk^2} + \frac{1}{3k} \frac{dE}{dk} \right]_{k=k_0}. \quad (1.1)$$

The effect of the long-range part of the Coulomb interactions including the effect of subsidiary conditions was investigated by Kanazawa in the scheme of Bohm-Pines theory. All these works show that there is a small correction to the diamagnetic susceptibility due to the Coulomb interactions.

Wentzel⁵⁾ used an equivalent Hamiltonian which gives the correct high density value for the correlation energy and calculated the diamagnetic susceptibility. His conclusion is that there is no correction to the Landau value of non-interacting electron gas. His argument, however, is valid only in so far as the exchange effects are omitted. If we take the exchange effects into account, there remains a finite correction,⁶⁾ and in this case the equivalent Hamiltonian formalism of Wentzel cannot be applied. The diagrams which were taken into account by Wentzel are shown in Fig. 1a and Fig. 1b. The contribution from the process shown in Fig. 1a gives the Landau diamagnetism and the contributions from the processes shown in Fig. 1b vanish. Wentzel did not consider the contributions from other processes which are shown, for example, by Fig. 1c or Fig. 3c. In this paper we investigate the contributions from these processes and derive the exact high density formula for the correction to the Landau diamagnetism, using the Green function method.

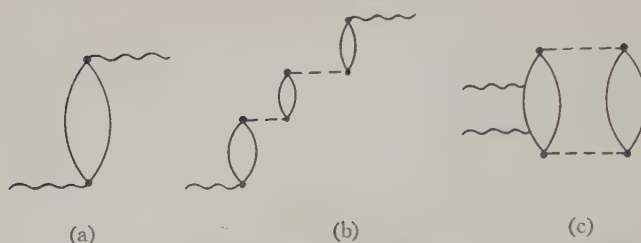


Fig. 1. Diagrams contributing to susceptibility
Wavy line and dotted line represent magnetic perturbations and Coulomb interactions respectively.

In § 2 we express the diamagnetic susceptibility in terms of the two-particle Green function, which is calculated in a consistent approximation in § 3 and § 4.

Since temperature dependence of the diamagnetism is expected to be small, we calculate the susceptibility at zero temperature. The interaction Hamiltonian with the magnetic field is treated as a small perturbation, and consequently our formulation is not applicable to the case of strong magnetic field (de-Haas van Alphen effect).

§ 2. Magnetic susceptibility in terms of Green function

We consider an N -electron system in a box of unit volume. As usual it is embedded in the uniform positive charge. We apply a static magnetic field, which is expressed in terms of the vector potential $\mathbf{A}(\mathbf{r}) = \sum_{\mathbf{q}} \mathbf{A}(\mathbf{q}) \exp(i\mathbf{q} \cdot \mathbf{r})$. Then the Hamiltonian and the current operators are: ($\hbar=1$)

$$H = H_0 + H_c + H' + H'' \quad (2.1)$$

$$H_0 = \sum_p \epsilon_p^0 a_p^\dagger a_p, \quad \epsilon_p^0 = p^2/2m \quad (2.2a)$$

$$H_c = \frac{1}{2} \sum_{\substack{p, p', k \\ k \neq 0}} V(k) a_{p+k}^\dagger a_{p'-k}^\dagger a_{p'} a_p \quad (2.2b)$$

$$H' = -\frac{1}{c} \int d\mathbf{r} \mathbf{j}_0(\mathbf{r}) \cdot \mathbf{A}(\mathbf{r}) \quad (2.2c)$$

$$H'' = -\frac{1}{2c} \int d\mathbf{r} \mathbf{j}_1(\mathbf{r}) \cdot \mathbf{A}(\mathbf{r}) \quad (2.2d)$$

$$\mathbf{j}_0(\mathbf{r}) = -\frac{ie}{2m} \{ \nabla \psi^\dagger(\mathbf{r}) \psi(\mathbf{r}) - \psi^\dagger(\mathbf{r}) \nabla \psi(\mathbf{r}) \} \quad (2.3)$$

$$\mathbf{j}_1(\mathbf{r}) = -\frac{e^2}{mc} \mathbf{A}(\mathbf{r}) \psi^\dagger(\mathbf{r}) \psi(\mathbf{r}) \quad (2.4)$$

where $\psi(\mathbf{r}) = \sum_p a_p \exp(i\mathbf{p} \cdot \mathbf{r})$, $V(k) = 4\pi e^2/k^2$, a_p and a_p^\dagger are the annihilation and creation operators of electrons. We have omitted spin indices for simplicity. The expectation value of the current $\mathbf{j}_0(\mathbf{q}) + \mathbf{j}_1(\mathbf{q})$, which are the Fourier components

of the current operators (2.3) and (2.4), is written in the form:

$$i_{\alpha}(\mathbf{q}) = \langle \Psi(t) | j_{\alpha}(\mathbf{q}, t) | \Psi(t) \rangle = \sum_{\beta} K^{\alpha\beta}(\mathbf{q}) A_{\beta}(\mathbf{q}) + O(A^2) \quad (2.5)$$

where we have referred to the interaction representation with $H' + H''$ as the interaction Hamiltonian which is switched on adiabatically in the infinite past. $\Psi(t)$ is the Schroedinger function in this representation which is obtained adiabatically from Ψ_0 , the ground state of $H_0 + H_c$, and

$$\mathbf{j}(\mathbf{q}, t) = \exp[i(H_0 + H_c)t] \mathbf{j}(\mathbf{q}) \exp[-i(H_0 + H_c)t].$$

Gauge invariance and the condition of continuity require that⁷⁾

$$K^{\alpha\beta}(\mathbf{q}) = (q^2 \delta_{\alpha\beta} - q_{\alpha} q_{\beta}) K(\mathbf{q}). \quad (2.6)$$

Then the susceptibility is given by

$$\chi = \frac{1}{c} \lim_{q \rightarrow 0} K(\mathbf{q}). \quad (2.7)$$

Thus our problem is to calculate the expectation value of the current of our system.

As is seen from (2.5) and (2.7), we need to calculate $\mathbf{i}(\mathbf{q})$ only to the first order of A . Therefore in the calculation of $\mathbf{i}_1(\mathbf{q}) = \langle \Psi(t) | \mathbf{j}_1(\mathbf{q}, t) | \Psi(t) \rangle$ we may replace $\Psi(t)$ by Ψ_0 and we have

$$\mathbf{i}_1(\mathbf{r}, t) = \langle \Psi_0 | \mathbf{j}_1(\mathbf{r}, t) | \Psi_0 \rangle = -\frac{e^2}{mc} \mathbf{A}(\mathbf{r}) \langle \Psi_0 | \psi^{\dagger}(\mathbf{r}) \psi(\mathbf{r}) | \Psi_0 \rangle = -\frac{ne^2}{mc} \mathbf{A}(\mathbf{r}).$$

Therefore

$$\mathbf{i}_1(\mathbf{q}, t) = -\frac{ne^2}{mc} \mathbf{A}(\mathbf{q}), \quad K_1^{\alpha\beta} = -\frac{ne^2}{mc} \delta_{\alpha\beta}. \quad (2.8)$$

This is the so-called London diamagnetic term, which is almost cancelled by the paramagnetic part.

Next we consider the paramagnetic current \mathbf{i}_0 :

$$\begin{aligned} i_0^{\alpha}(\mathbf{r}, t) &= \langle \Psi(t) | j_0^{\alpha}(\mathbf{r}, t) | \Psi(t) \rangle \\ &= -i \int_{-\infty}^t dt' \langle \Psi_0 | [j_0^{\alpha}(\mathbf{r}, t), H'(t')] | \Psi_0 \rangle \\ &= -\frac{i}{c} \int d\mathbf{r}' \int_{-\infty}^t dt' \langle \Psi_0 | [j_0^{\alpha}(\mathbf{r}, t), j_0^{\beta}(\mathbf{r}', t')] | \Psi_0 \rangle A_{\beta}(\mathbf{r}'), \end{aligned} \quad (2.9)$$

since

$$\Psi(t) = T \exp \left[-i \int_{-\infty}^t dt' (H'(t') + H''(t')) \right] \Psi_0$$

and higher order terms in A are neglected. From (2.3),

$$\begin{aligned} \mathbf{j}_0(x) \mathbf{j}_0(x') &= \frac{e^2}{4m^2} \{ -\nabla \psi^{\dagger}(x) \psi(x) \nabla' \psi^{\dagger}(x') \psi(x') \\ &\quad - \psi^{\dagger}(x) \nabla \psi(x) \psi^{\dagger}(x') \nabla' \psi(x') \} \end{aligned}$$

$$\begin{aligned}
& +\mathcal{F}\phi^+(x)\phi(x)\phi^+(x')\mathcal{F}'\phi(x') \\
& +\phi^+(x)\mathcal{F}\phi(x)\mathcal{F}'\phi^+(x')\phi(x')\}, \quad (2.10) \\
& (x=(\mathbf{r}, t)).
\end{aligned}$$

Introducing the two-particle Green function

$$G_2(1, 2; 3, 4) = i\langle \Psi_0 | T\phi(1)\phi(2)\phi^+(3)\phi^+(4) | \Psi_0 \rangle$$

where $\phi(1) = \phi(\mathbf{r}_1, t_1)$, we get ($t_1 = t > t' = t_2$)

$$\begin{aligned}
& \langle \Psi_0 | j_0^\alpha(x_1) j_0^\beta(x_2) | \Psi_0 \rangle \\
& = \frac{-ie^2}{4m^2} \lim_{\substack{r'_{1,2} \rightarrow r_{1,2} \\ t'_{1,2} \rightarrow t_{1,2} + 0}} (\mathcal{F}_1^\alpha - \mathcal{F}'_1^\alpha) (\mathcal{F}_2^\beta - \mathcal{F}'_2^\beta) G_2(x_1, x_2; x'_1, x'_2) \quad (2.11)
\end{aligned}$$

and

$$\langle \Psi_0 | j_0^\beta(x_2) j_0^\alpha(x_1) | \Psi_0 \rangle = \langle \Psi_0 | j_0^\alpha(x_1) j_0^\beta(x_2) | \Psi_0 \rangle^*. \quad (2.12)$$

From (2.9) to (2.12) we get the final expression of i_0 in terms of G_2 :

$$\begin{aligned}
i_0^\alpha(\mathbf{r}_1, t_1) & = \frac{e^2}{4m^2 c} \int_{-\infty}^t dx_2 \lim_{\substack{r'_{1,2} \rightarrow r_{1,2} \\ t'_{1,2} \rightarrow t_{1,2} + 0}} (\mathcal{F}_1^\alpha - \mathcal{F}'_1^\alpha) (\mathcal{F}_2^\beta - \mathcal{F}'_2^\beta) G_2(x_1, x_2; x'_1, x'_2) \\
& \times A_\beta(\mathbf{r}_2) + \text{c.c.} \quad (2.13)
\end{aligned}$$

Thus the whole information of diamagnetism is contained in the two-particle Green function for the system.

§ 3. Pair theory approximation

For a high-density electron gas the two-particle Green function, which gives the correct high density value for the correlation energy, has been obtained in I:

$$\begin{aligned}
G_2(1, 2; 3, 4) & = -iG_0(1, 4)G_0(2, 3) \\
& - \int dx_5 dx_6 G_0(1, 5)G_0(2, 6) \mathcal{V}^0(5, 6)G_0(5, 3)G_0(6, 4) \\
& + \int dx_5 dx_6 G_0(1, 5)G_0(2, 6) V(5, 6)G_0(5, 4)G_0(6, 3), \quad (3.1)
\end{aligned}$$

where $\mathcal{V}^0(x, x')$ is the effective interaction, the Fourier transform of which is $\mathcal{V}^0(k, \omega) = V(k)/\varepsilon(k, \omega)$, $\varepsilon(k, \omega)$ being the complex dielectric constant and $V(x, x') = V(\mathbf{r} - \mathbf{r}')\delta(t - t')$. G_0 is the unperturbed one-particle Green function and is given by

$$G_0(x, x') = \sum_p \int \frac{d\varepsilon}{2\pi} G_0(\mathbf{p}, \varepsilon) \exp i(\mathbf{p} \cdot \mathbf{r} - \varepsilon t), \quad (3.2)$$

$$G_0(\mathbf{p}, \varepsilon) = -[\varepsilon - \varepsilon_p^0 + i\delta(1 - 2n_p^0)]^{-1} \quad (3.3)$$

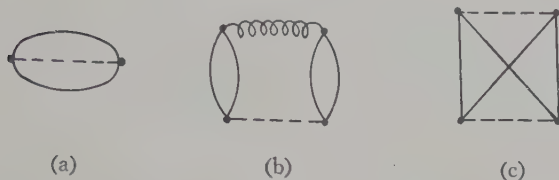


Fig. 2. Diagrams for correlation energy
Helical line represents the effective interaction.

where n_p^0 is the occupation number of the unperturbed state. The contribution of each term of (3.1) to the correlation energy is shown graphically in Fig. 2a, 2b and 2c. In this section we will use (3.1) for G_2 . Inserting (3.2) and (3.3) into (3.1) we get

$$\begin{aligned} & \lim_{r_{1,2} \rightarrow r_{1,2}} (\nabla_1^\alpha - \nabla_1'^\alpha) (\nabla_2^\beta - \nabla_2'^\beta) G_2(\mathbf{r}_1 t_1, \mathbf{r}_2 t_2; \mathbf{r}'_1 t_1^+, \mathbf{r}'_2 t_2^+) \\ &= +i \sum_{p_1 p_2} \int \frac{d\varepsilon_1 d\varepsilon_2}{(2\pi)^2} G_0(\mathbf{p}_1 \varepsilon_1) G_0(\mathbf{p}_2 \varepsilon_2) \\ & \times (p_1 + p_2)_\alpha (p_1 + p_2)_\beta \exp[i(\mathbf{p}_1 - \mathbf{p}_2) \cdot (\mathbf{r}_1 - \mathbf{r}_2) - i(\varepsilon_1 - \varepsilon_2)(t_1 - t_2)] \\ & + \sum_{p_1 p_2 q} \int \frac{d\varepsilon d\varepsilon' d\omega}{(2\pi)^3} G_0(\mathbf{p}_1 \varepsilon) G_0(\mathbf{p}_2 \varepsilon') \tilde{V}(\mathbf{q}, \omega) G_0(\mathbf{p}_1 - \mathbf{q}, \varepsilon - \omega) \\ & \times G_0(\mathbf{p}_2 + \mathbf{q}, \varepsilon' + \omega) (2p_1 - q)_\alpha (2p_2 + q)_\beta \exp[i\mathbf{q} \cdot (\mathbf{r}_1 - \mathbf{r}_2) - i\omega(t_1 - t_2)] \\ & - \sum_{p_1 p_2 q} \int \frac{d\varepsilon d\varepsilon' d\omega}{(2\pi)^3} G_0(\mathbf{p}_1 \varepsilon) G_0(\mathbf{p}_2 \varepsilon') V(\mathbf{q}) G_0(\mathbf{p}_1 - \mathbf{q}, \varepsilon' + \omega) \\ & \times G_0(\mathbf{p}_2 + \mathbf{q}, \varepsilon - \omega) (p_1 + p_2 + q)_\alpha (p_1 + p_2 - q)_\beta \\ & \times \exp[i(\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{q}) \cdot (\mathbf{r}_1 - \mathbf{r}_2) - i\omega(t_1 - t_2)]. \end{aligned} \quad (3.4)$$

From (3.4) and (2.13) we obtain

$$i_0^\alpha(\mathbf{q}) = \frac{1}{c} \int \frac{d\omega}{2\pi} \frac{1}{\omega - i\delta} F^{\alpha\beta}(\mathbf{q}, \omega) A_\beta(\mathbf{q}) + \text{c.c.} \quad (3.5)$$

Therefore

$$K_0^{\alpha\beta}(\mathbf{q}) = \frac{2}{c} \int \frac{d\omega}{2\pi} \left[P \frac{1}{\omega} \text{Re} F^{\alpha\beta}(\mathbf{q}, \omega) - \pi \delta(\omega) \text{Im} F^{\alpha\beta}(\mathbf{q}, \omega) \right]. \quad (3.6)$$

Here

$$F^{\alpha\beta}(\mathbf{q}, \omega) = F_0^{\alpha\beta}(\mathbf{q}, \omega) + F_1^{\alpha\beta}(\mathbf{q}, \omega) + F_2^{\alpha\beta}(\mathbf{q}, \omega) \quad (3.7)$$

with

$$F_0^{\alpha\beta}(\mathbf{q}, \omega) = \frac{e^2}{4m^2} \sum_{\mathbf{p}} \int \frac{d\varepsilon}{2\pi} (2\mathbf{p}-\mathbf{q})_{\alpha} (2\mathbf{p}-\mathbf{q})_{\beta} G_0(\mathbf{p}, \varepsilon) G_0(\mathbf{p}-\mathbf{q}, \varepsilon-\omega) \quad (3.8)$$

$$F_1^{\alpha\beta}(\mathbf{q}, \omega) = -i \frac{e^2}{4m^2} \sum_{\mathbf{p}_1 \mathbf{p}_2} \int \frac{d\varepsilon d\varepsilon'}{(2\pi)^2} \mathcal{V}(\mathbf{q}, \omega) (2\mathbf{p}_1-\mathbf{q})_{\alpha} (2\mathbf{p}_2+\mathbf{q})_{\beta} \\ \times G_0(\mathbf{p}_1, \varepsilon) G_0(\mathbf{p}_1-\mathbf{q}, \varepsilon-\omega) G_0(\mathbf{p}_2, \varepsilon') G_0(\mathbf{p}_2+\mathbf{q}, \varepsilon'+\omega) \quad (3.9)$$

$$F_2^{\alpha\beta}(\mathbf{q}, \omega) = i \frac{e^2}{4m^2} \sum_{\mathbf{p}_1 \mathbf{p}_2} \int \frac{d\varepsilon d\varepsilon'}{(2\pi)^2} V(\mathbf{p}_1-\mathbf{p}_2-\mathbf{q}) (2\mathbf{p}_1-\mathbf{q})_{\alpha} (2\mathbf{p}_2+\mathbf{q})_{\beta} \\ \times G_0(\mathbf{p}_1, \varepsilon) G_0(\mathbf{p}_1-\mathbf{q}, \varepsilon-\omega) G_0(\mathbf{p}_2, \varepsilon') G_0(\mathbf{p}_2+\mathbf{q}, \varepsilon'+\omega). \quad (3.10)$$

Here we notice that the terms $F_0^{\alpha\beta}$, $F_1^{\alpha\beta}$ and $F_2^{\alpha\beta}$ come from the first, the second and the third term of (3.1) respectively, and so it is evident that their contributions to $K_0^{\alpha\beta}$ are represented diagrammatically as in Fig. 3a, 3b and 3c.* We write $K_0^{\alpha\beta}$ as

$$K_0^{\alpha\beta}(\mathbf{q}) = K_{(0)}^{\alpha\beta} + K_{(1)}^{\alpha\beta} + K_{(2)}^{\alpha\beta}. \quad (3.11)$$

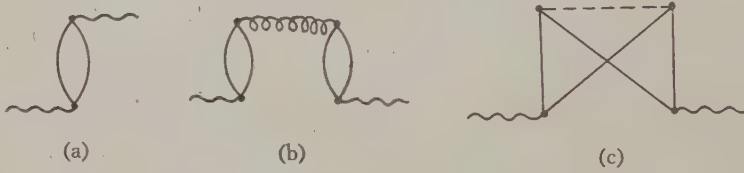


Fig. 3. Contribution from (3.1) or (3.11)

If we put $V(k) \rightarrow 0$ for the moment, then the second and the third terms of (3.11) vanish. Therefore $K_{(0)}^{\alpha\beta}$ must give, when combined with $K_1^{\alpha\beta}$ of (2.8), the usual Landau diamagnetism. This is actually the case, as is shown in Appendix I.

Next we consider $K_{(1)}^{\alpha\beta}$. Comparing Fig. 3b with Fig. 1b, we see at once that this is the term investigated by Wentzel. In other words, to take into account only the first and the second terms in (3.1) corresponds to Wentzel's approximation, in which H' is reduced to the parts involving only pair creation and annihilation terms. In our formulation the proof that $K_{(1)}^{\alpha\beta}$ vanishes is quite easy, if we rewrite (3.9) as follows:

$$F_1^{\alpha\beta}(\mathbf{q}, \omega) = -i \mathcal{V}(\mathbf{q}, \omega) L_{\alpha}(\mathbf{q}, \omega) L_{\beta}(\mathbf{q}, \omega) (e/2m)^2, \\ L_{\alpha}(\mathbf{q}, \omega) = \sum_{\mathbf{p}} \int \frac{d\varepsilon}{2\pi} (2\mathbf{p}+\mathbf{q})_{\alpha} G_0(\mathbf{p}, \varepsilon) G_0(\mathbf{p}+\mathbf{q}, \varepsilon+\omega). \quad (3.12)$$

Using

$$\int \frac{d\varepsilon}{2\pi} G_0(\mathbf{p}, \varepsilon) G_0(\mathbf{p}+\mathbf{q}, \varepsilon+\omega) = i \left[\frac{n_{\mathbf{p}+\mathbf{q}}^0 (1-n_{\mathbf{p}}^0)}{\omega_{\mathbf{p}, \mathbf{q}} - \omega + i\delta} - \frac{n_{\mathbf{p}}^0 (1-n_{\mathbf{p}+\mathbf{q}}^0)}{\omega_{\mathbf{p}, \mathbf{q}} - \omega - i\delta} \right] \quad (3.13)$$

* $\mathbf{i}(\mathbf{q}) = -(1/c) \partial E(\mathbf{A}) / \partial \mathbf{A}(-\mathbf{q})$, where $E(\mathbf{A}) = E_0 + \frac{1}{2} \chi \mathbf{E}(\mathbf{q} \times \mathbf{A}(\mathbf{q})) \cdot (\mathbf{q} \times \mathbf{A}(-\mathbf{q}))$. Fig. 3 represents the diagrams which contribute to the second term.

we get

$$L_{\alpha}(\mathbf{q}, \omega) = i \sum_p n_p^0 (1 - n_{p+q}^0) (2p + q)_{\alpha} \left[\frac{-1}{\omega_{p,q} - \omega - i\delta} + \frac{-1}{\omega_{p,q} + \omega - i\delta} \right]$$

where $\omega_{p,q} = \epsilon_{p+q}^0 - \epsilon_p^0$. Thus $L(\mathbf{q}, \omega) = L(\mathbf{q}, -\omega)$. Returning to (3.12) and changing the integration variables as $\mathbf{p} \rightarrow -\mathbf{p} - \mathbf{q}$, $\epsilon \rightarrow \epsilon - \omega$, we find that $L(\mathbf{q}, \omega) = -L(\mathbf{q}, -\omega) = -L(\mathbf{q}, \omega) = 0$.

Therefore in our approximation (3.1), the term which gives a correction to the Landau diamagnetism is only $K_{(2)}^{\alpha\beta}$, which corresponds to the diagram shown in Fig. 3c. The calculation is performed in Appendix II and the result is:

$$K_{(2)}^{\alpha\beta} = \frac{1}{c} \left(\frac{e}{m} \right)^2 \sum_{pp'} V(\mathbf{p} - \mathbf{p}') (p + \frac{1}{2}q)_{\alpha} (p' + \frac{1}{2}q)_{\beta} \frac{n_{p+q}^0 - n_p^0}{\omega_{p,q}} \cdot \frac{n_{p'+q}^0 - n_{p'}^0}{\omega_{p',q}}. \quad (3.14)$$

Here we encounter with two difficulties. Firstly, (3.14) is not gauge-invariant. From the form of (2.6), gauge-invariance is guaranteed if $\sum_{\alpha,\beta} q_{\alpha} q_{\beta} K^{\alpha\beta}(\mathbf{q}) = 0$, which does not hold. Secondly the integral of (3.14) diverges.

These difficulties indicate clearly that our approximation of G_2 is not sufficient for the problem of diamagnetism. We must improve the two-particle Green function (3.1) by taking into account the higher order effects. This, however, is not surprising. The difficulty of gauge-invariance is rather evident at the beginning, because in the case of diamagnetism there is another term besides (3.14) which is linear in $V(k)$. From perturbation theoretic point of view, they together form the correct first order correction to the ideal gas value. Therefore it is not surprising that a part thereof alone, (3.14), is not gauge-invariant. In the case of correlation energy, in contrast to our case, all terms which are in the same order in the r_s expansion are all contained in (3.1). But now this is not the case. Thus we must include, besides (3.14), all terms that are of the same order in r_s . To this purpose we first replace G_0 in the first term of (3.1) by G_1 , which we define as the one-particle Green function including exchange self-energy:

$$G_1(\mathbf{p}, \epsilon)^{-1} = G_0(\mathbf{p}, \epsilon)^{-1} - \Sigma_{ex}(\mathbf{p}) \quad (3.15)$$

$$\Sigma_{ex}(\mathbf{p}) = \sum_k V(k) n_{p+k}^0.$$

In other words, we replace, in the expression for the Landau diamagnetism (A2), ϵ_p^0 by $E_p = \epsilon_p^0 - \Sigma_{ex}(\mathbf{p})$. Next we expand it in $V(k)$ and retain only the first order. Then we get

$$K_{(2)}^{\alpha\beta} = - \frac{1}{c} \left(\frac{e}{m} \right)^2 \sum_{pp'} V(\mathbf{p} - \mathbf{p}') (p + \frac{1}{2}q)_{\alpha} (p' + \frac{1}{2}q)_{\beta} \frac{(n_{p+q}^0 - n_p^0)(n_{p'+q}^0 - n_{p'}^0)}{\omega_{p,q}^2}. \quad (3.16)$$

Combining (3.14) and (3.16) we obtain

$$K_2^{\alpha\beta} = K_{(2)}^{\alpha\beta} + K_{(2)}^{\prime\alpha\beta} = \frac{1}{c} \left(\frac{e}{m} \right)^2 \sum_{pp'} V(\mathbf{p} - \mathbf{p}') \frac{(n_{p'-q}^0 - n_p^0)(n_{p'-q}^0 - n_{p'}^0)}{\omega_{p,q}} \\ \times (p + \frac{1}{2}q)_\alpha \left[\frac{(p' + \frac{1}{2}q)_\beta}{\omega_{p',q}} - \frac{(p + \frac{1}{2}q)_\beta}{\omega_{p,q}} \right]. \quad (3.17)$$

This expression is evidently gauge-invariant as expected and thus the first difficulty has been removed. But the second difficulty still remains. We will investigate it in the next section.

§ 4. Elimination of divergence

In the preceding section we have looked for the correction which is linear in $V(k)$ and we have found it to be divergent. This is rather an expected result, since if we calculate the diamagnetic susceptibility in a simple manner using (1.1), it diverges in the Hartree-Fock approximation. As is well known in the calculation of the correlation energy, these divergences are removed by replacing $V(k)$ by the effective interaction $\mathcal{V}^i(\mathbf{k}, \omega)$. Physically it means to take into account the correlation effect, or the screening of Coulomb potential, which removes the unnatural distribution of the energy level on the Fermi surface. In our calculation this effect is taken into account by replacing $G_0(x-x')$ in the first term of (3.1) by $G(x-x')$, which includes the polarization part besides the exchange energy, and replacing $V(x-x')$ in the third term of (3.1) by $\mathcal{V}^i(x-x')$: i.e.,

$$G_2(1, 2; 3, 4) = -iG(1, 4)G(2, 3) \quad (4.1a)$$

$$+ \int dx_5 dx_6 G_0(1, 5) G_0(2, 6) \mathcal{V}^i(5, 6) G_0(5, 4) G_0(6, 3), \quad (4.1b)$$

where we have omitted the second term of (3.1) which gives no contribution. The Fourier transform of $G(x-x')$ in our case is, as is well known,

$$G(\mathbf{p}, \varepsilon)^{-1} = G^0(\mathbf{p}, \varepsilon)^{-1} - \Sigma_{ex}(\mathbf{p}) - \Sigma_c(\mathbf{p}, \varepsilon). \quad (4.2)$$

$\Sigma_{ex}(\mathbf{p})$ is given in (3.15), and

$$\Sigma_c(\mathbf{p}, \varepsilon) = \sum_{\mathbf{k}} \int \frac{d\omega}{2\pi i} V(\mathbf{k}) G_0(\mathbf{p} + \mathbf{k}, \varepsilon + \omega) \frac{Q(\mathbf{k}, \omega)}{1 + Q(\mathbf{k}, \omega)} \quad (4.3)$$

where $Q(\mathbf{k}, \omega) = \varepsilon(\mathbf{k}, \omega) - 1$.

Here we make a simplifying assumption:

$$\mathcal{V}^i(\mathbf{k}, \omega) \longrightarrow \mathcal{V}^i(\mathbf{k}, 0), \quad (4.4)$$

then (4.3) is integrated straightforwardly and we get

$$\Sigma_{ex}(\mathbf{p}) + \Sigma_c(\mathbf{p}, \varepsilon) = \sum_{\mathbf{k}} V(\mathbf{k}) n_{p+\mathbf{k}}^0 \left\{ 1 - \frac{Q(\mathbf{k}, 0)}{1 + Q(\mathbf{k}, 0)} \right\} = \sum_{\mathbf{k}} \mathcal{V}^i(\mathbf{k}, 0) n_{p+\mathbf{k}}^0 \quad (4.5)$$

which is independent of ε . Comparing (4.2) with (3.15) we see at once that

the contribution of the first term of (4.1) is $K_{(0)}^{\alpha\beta} + (3.16)$, except that $V(p-p')$ is replaced by $\mathcal{V}(p-p', 0)$. Contribution of (4.1b) is calculated in the same fashion as Appendix II, and the result is just (3.14) where also $V(p-p')$ is replaced by $\mathcal{V}(p-p', 0)$. Thus we come to the final result:

$$K^{\alpha\beta}(\mathbf{q}) = K_L^{\alpha\beta}(\mathbf{q}) + K_C^{\alpha\beta}(\mathbf{q}), \quad (4.6)$$

$$K_C^{\alpha\beta}(\mathbf{q}) = \frac{1}{c} \left(\frac{e}{m} \right)^2 \sum_{\mathbf{p}\mathbf{p}'} \mathcal{V}(\mathbf{p}-\mathbf{p}', 0) (n_{\mathbf{p}+\mathbf{q}}^0 - n_{\mathbf{p}}^0) (n_{\mathbf{p}'+\mathbf{q}}^0 - n_{\mathbf{p}'}^0) \\ \times \frac{(\mathbf{p} + \frac{1}{2}\mathbf{q})_\alpha}{\omega_{\mathbf{p},\mathbf{q}}} \left[\frac{(\mathbf{p}' + \frac{1}{2}\mathbf{q})_\beta}{\omega_{\mathbf{p}',\mathbf{q}}} - \frac{(\mathbf{p} + \frac{1}{2}\mathbf{q})_\beta}{\omega_{\mathbf{p},\mathbf{q}}} \right]. \quad (4.7)$$

Expression (4.7) is gauge-invariant and finite. Its calculation is elementary but somewhat lengthy, which we give in Appendix III.

In Appendix III we use a further approximation of replacing $\mathcal{V}(k, 0)$ by $\mathcal{V}'(k)$, where $(\mathbf{q} = \mathbf{k}/k_0)$

$$\mathcal{V}(k, 0) = \frac{4\pi e^2}{k_0^2} \left\{ q^2 + \frac{2\alpha r_s}{\pi} \left[1 - \frac{q^2}{12} + O(q^4) \right] \right\}^{-1}, \quad (4.8a)$$

$$\mathcal{V}'(k) = \frac{4\pi e^2}{k_0^2} (q^2 + 2\varepsilon)^{-1}, \quad \varepsilon = \frac{\alpha r_s}{\pi}. \quad (4.8b)$$

Then the result is:

$$K_C^{\alpha\beta}(\mathbf{q}) = (q^2 \delta_{\alpha\beta} - q_\alpha q_\beta) K_C(q), \quad (4.9a)$$

$$K_C(q) = \frac{4\pi e^4}{(2\pi)^6 c} \left[q^{-2} I - \frac{1}{8} J \right] \quad (4.9b)$$

where I vanishes independent of ε , and

$$J = A \log \varepsilon + B + C\varepsilon \log \varepsilon + \dots, \quad (4.9c)$$

$$A = \frac{8\pi^2}{9}, \quad B = \frac{8\pi^2}{9} (4 - \log 2). \quad (4.9d)$$

§ 5. Discussion

In our calculation we have made four approximations:

- (i) approximation (4.1)
- (ii) the approximation made in deriving the second term of (4.7) which originates from (4.1a)
- (iii) neglect of retardation, i.e. the approximation (4.4)
- (iv) replacement of \mathcal{V} by \mathcal{V}' .

These approximations are justified as follows.

- (i) In § 3 and § 4 we considered the processes that are represented diagrammatically in Fig. 4. In other words, we considered the correction to Fig. 3a,

which are of first order in H_c , and replaced $V(k)$ by the effective interaction (elimination of divergence). All other corrections may in principle be calculated by the prescription:

- (a) write down the processes in diagrams (e.g. Fig. 5)
- (b) construct the Green function G_2 which contribute to that diagram
- (c) insert it into (2.13) and perform the calculation
- (d) if divergences appear, they are always treated by the principle of elimination of divergence.

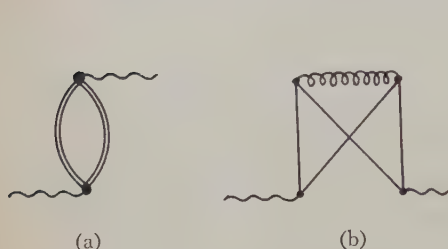


Fig. 4. Processes representing each term of (4.2)
Double line is the propagator of electron and hole including self energy.

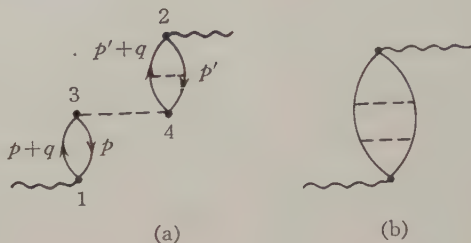


Fig. 5. Diagrams representing higher order corrections

For example, we will briefly investigate the processes represented in Fig. 5. Fig. 5a gives no contribution, just as in the case of Fig. 3b. This is the consequence of the situation that the matrix element at the point (1) or (2) in Fig. 5a is of the form $(\mathbf{p} + \frac{1}{2}\mathbf{q})\mathbf{A}(\mathbf{q})$, which changes sign by the replacement $\mathbf{p} \rightarrow -\mathbf{p} - \mathbf{q}$, and the integration variables \mathbf{p} and \mathbf{p}' are independent of each other. Thus in general diagrams of this type, i.e., diagrams which are composed of two parts each of which has one perturbation line H' and which are both connected by single interaction line H_c , give no contribution. The contribution from Fig. 5b does not vanish. However, it is of second order in H_c , and if we compare the result with that of Fig. 4b, it contains a factor e^2 , or r_s . Thus the contribution is of higher order.

(ii) Approximation (ii) is justified in the same fashion. The contribution from (4.1a) (Fig. 4a) is just (A2), except that $\omega_{p,q}$ is replaced by $W(p, q) = E(p+q) - E(p)$, where $E(p)$ is the energy of the quasi-particle

$$E(p) = \varepsilon_p^0 - \Sigma(p).$$

Our approximation is to replace

$$W(p, q)^{-1} - \omega_{p,q}^{-1} = \frac{\Sigma(p+q) - \Sigma(p)}{W(p, q)\omega_{p,q}} \quad (5.1)$$

by

$$\frac{\Sigma(p+q) - \Sigma(p)}{\omega_{p,q}^2}, \quad (5.2)$$

so the correction to our approximation is

$$(5.1) - (5.2) = \frac{[\mathcal{V}(p+q) - \mathcal{V}(p)]^2}{W(p, q)\omega_{p,q}^2},$$

which is also of the higher order in r_s .

(iii) Properties of the approximations of (iii) and (iv) are also of the same type as before, because $\mathcal{V}^0(k, \omega) - \mathcal{V}^0(k, 0)$ and $\mathcal{V}^0(k, 0) - \mathcal{V}^0'(k)$ are proportional to e^6 times integrals which do not diverge.

Finally we note that I in (4.9b) vanishes independent of ε , as is seen in (A9). This means that our original form (3.17) does not contain the constant term, although the coefficient of q^2 is infinite. Therefore, our procedure of § 4, that is, the approximation mentioned above, does not affect the vanishing of I in (4.9b).

Thus we conclude that our result (4.9) is exact up to the second term of (4.9c).

It is interesting to compare our result with that obtained by Donovan and March. They give

$$(\chi/\chi_0) = 1 + \frac{\alpha r_s}{6\pi} [\log r_s + 2(2 + \log 0.417 - \log 2) + \dots]$$

while (4.9) gives

$$(\chi/\chi_0) = 1 + \frac{\alpha r_s}{6\pi} \left[\log r_s + 4 + \log \frac{\alpha}{2\pi} \right].$$

Numerically,

$$2(2 + \log 0.417 - \log 2) = 1.12$$

$$4 + \log (\alpha/2\pi) = 1.51.$$

We find that the correction is finite but very small.

We are indebted to Prof. T. Usui, Miss E. Fujita and other members of the group of many-body problem for helpful discussions.

Appendix I

Landau diamagnetism

(3.8) can be readily integrated with respect to ε , using (3.3) :

$$F_0^{\alpha\beta}(\mathbf{q}, \omega) = -i \left(\frac{e}{m} \right)^2 \sum_p (p + \frac{1}{2}\mathbf{q})_\alpha (p + \frac{1}{2}\mathbf{q})_\beta n_p^0 (1 - n_{p+q}^0) \\ \times \left[\frac{1}{\omega_{p,q} - \omega - i\delta} + \frac{1}{\omega_{p,q} + \omega - i\delta} \right]. \quad (\text{A1})$$

(A1) is put in (3.6), where we notice that (A1) is symmetrical with respect to ω , so that

$$K_{(0)}^{\alpha\beta} = -(1/c) \text{Im} F_0^{\alpha\beta}(\mathbf{q}, 0)$$

$$= \frac{2e^2}{m^2 c} \sum_{\mathbf{p}} (p + \frac{1}{2}q)_\alpha (p + \frac{1}{2}q)_\beta \frac{n_{\mathbf{p}}^0 (1 - n_{\mathbf{p}+\mathbf{q}}^0)}{\omega_{\mathbf{p},\mathbf{q}}}. \quad (\text{A2})$$

Now we transform the coordinate system from $(\alpha\beta\gamma)$ to (xyz) , where we take z -axis parallel to \mathbf{q} . Thus

$$p_\alpha = \mathbf{p} \cdot \boldsymbol{\alpha} = \sum_i p_i \alpha_i, \text{ etc.},$$

then the integration of (A2) over \mathbf{p} is straightforward and yields

$$K_{(0)}^{\alpha\beta} = \frac{ne^2}{mc} q_\alpha q_\beta + \left(\delta_{\alpha\beta} - \frac{q_\alpha q_\beta}{q^2} \right) \frac{e^2}{4\pi^2 mc} \left\{ \frac{5}{12} k_0^3 - \frac{1}{16} q^2 k_0^2 \right. \\ \left. + \frac{1}{4q} \left(k_0^2 - \frac{q^2}{4} \right)^2 \log \left| \frac{k_0 + \frac{1}{2}q}{k_0 - \frac{1}{2}q} \right| \right\}.$$

Expanding logarithm in power series of q and adding (2.8),

$$K_L^{\alpha\beta}(\mathbf{q}) = K_1^{\alpha\beta} + K_{(0)}^{\alpha\beta} = - (q^2 \delta_{\alpha\beta} - q_\alpha q_\beta) \left[\frac{ne^2}{mc} \cdot \frac{1}{4k_0} + \dots \right].$$

So that

$$\chi_0 = - \frac{e^2 k_0}{12\pi^2 mc^2} = - \frac{1}{12\pi^2 \alpha r_s} (1/137)^2 \quad (\text{A3})$$

where we have put $n = k_0^3/3\pi^2$, taking spin into consideration.

Appendix II

Derivation of (3.14)

From (3.10) and (3.13),

$$F_2^{\alpha\beta} = i \left(\frac{e}{m} \right)^2 \sum_{\mathbf{p}\mathbf{p}'} \int \frac{d\varepsilon d\varepsilon'}{(2\pi)^2} (p + \frac{1}{2}q)_\alpha (p' + \frac{1}{2}q)_\beta V(\mathbf{p} - \mathbf{p}') \\ \times G_0(\mathbf{p}, \varepsilon) G_0(\mathbf{p} + \mathbf{q}, \varepsilon + \omega) G_0(\mathbf{p}', \varepsilon') G_0(\mathbf{p}' + \mathbf{q}, \varepsilon' + \omega) \\ = -i \left(\frac{e}{m} \right)^2 \sum_{\mathbf{p}\mathbf{p}'} (p + \frac{1}{2}q)_\alpha (p' + \frac{1}{2}q)_\beta V(\mathbf{p} - \mathbf{p}') \left[\frac{n_{\mathbf{p}+\mathbf{q}}^0 (1 - n_{\mathbf{p}}^0)}{\omega_{\mathbf{p}\mathbf{q}} - \omega + i\delta} \right. \\ \left. - \frac{n_{\mathbf{p}}^0 (1 - n_{\mathbf{p}+\mathbf{q}}^0)}{\omega_{\mathbf{p},\mathbf{q}} - \omega - i\delta} \right] \times \left[\frac{n_{\mathbf{p}'+\mathbf{q}}^0 (1 - n_{\mathbf{p}'}^0)}{\omega_{\mathbf{p}'\mathbf{q}} - \omega + i\delta} - \frac{n_{\mathbf{p}'}^0 (1 - n_{\mathbf{p}'+\mathbf{q}}^0)}{\omega_{\mathbf{p}'\mathbf{q}} - \omega - i\delta} \right]. \quad (\text{A4})$$

Here we note that $F_2^{\alpha\beta}(\mathbf{q}, -\omega) = F_2^{\alpha\beta}(\mathbf{q}, \omega)$, because in (A4) if we change the variable from \mathbf{p}, \mathbf{p}' to $-\mathbf{p} - \mathbf{q}, -\mathbf{p}' - \mathbf{q}$, nothing is changed except the sign of ω . Thus in (3.6) the first term of the integrand vanishes. So that

$$K_{(2)}^{\alpha\beta} = - \frac{1}{c} \text{Im} F_2^{\alpha\beta}(\mathbf{q}, 0), \quad (\text{A5})$$

which gives at once the result (3.14), since $\partial(\omega_{\mathbf{p},\mathbf{q}}) = 0$ everywhere in the range of integration.

Appendix III

Calculation of $K_c^{\alpha\beta}(q)$

In the calculation of (4.7) we can choose, because of gauge-invariance, a special coordinate system where the z -axis is parallel to \mathbf{q} . As was stated in § 4, we make an approximation of replacing (4.8a) by (4.8b). (4.7) then becomes

$$K_c^{\alpha\beta}(q) = q^2 K_c(q)$$

or

$$K_c(q) = \frac{1}{c} \left(\frac{e}{m} \right)^2 \frac{1}{q^2} \sum_{\mathbf{p}, \mathbf{p}'} \psi'(\mathbf{p} - \mathbf{p}') (n_{\mathbf{p}+\mathbf{q}}^0 - n_{\mathbf{p}}^0) (n_{\mathbf{p}'+\mathbf{q}}^0 - n_{\mathbf{p}'}^0) \\ \times \frac{p_z}{\omega_{\mathbf{p}, \mathbf{q}}} \left[\frac{p'_z}{\omega_{\mathbf{p}', \mathbf{q}}} - \frac{p_z}{\omega_{\mathbf{p}, \mathbf{q}}} \right]. \quad (\text{A6})$$

Since we seek for the value of $K_c(q)$ in the limit of small q , we expand $(n_{\mathbf{p}+\mathbf{q}}^0 - n_{\mathbf{p}}^0)$ and $(n_{\mathbf{p}'+\mathbf{q}}^0 - n_{\mathbf{p}'}^0)$ in power series of q . Changing the variable in (A6) from \mathbf{p}, \mathbf{p}' to $\mathbf{p} - \frac{1}{2}\mathbf{q}, \mathbf{p}' - \frac{1}{2}\mathbf{q}$ for convenience, we get

$$n_{\mathbf{p}+\mathbf{q}/2}^0 - n_{\mathbf{p}-\mathbf{q}/2}^0 = -\frac{(\mathbf{p} \cdot \mathbf{q})}{p} \delta(\mathbf{p} - k_0) + \frac{1}{8} \left\{ \left[\frac{q^2(\mathbf{p} \cdot \mathbf{q})}{p^3} - \frac{(\mathbf{p} \cdot \mathbf{q})^3}{p^5} \right] \delta(\mathbf{p} - k_0) \right. \\ \left. + \left[\frac{(\mathbf{p} \cdot \mathbf{q})^3}{p^4} - \frac{q^2(\mathbf{p} \cdot \mathbf{q})}{p^2} \right] \delta'(\mathbf{p} - k_0) - \frac{(\mathbf{p} \cdot \mathbf{q})^3}{3p^3} \delta''(\mathbf{p} - k_0) \right\} + O(q^5), \quad (\text{A7})$$

thus

$$K_c(q) = \frac{4\pi e^4}{(2\pi)^6 c q^4} \int_0^\infty dp dp' \int_{-1}^1 dx dx' \int_0^{2\pi} d\varphi d\varphi' \frac{1}{(\mathbf{p} - \mathbf{p}')^2 + 2\epsilon k_0^2} \\ \times \frac{y \cos \varphi}{x} \left[\frac{y' \cos \varphi'}{x'} - \frac{y \cos \varphi}{x} \right] \left\{ q^2 x x' \delta(\mathbf{p} - k_0) \delta(\mathbf{p}' - k_0) \right. \\ \left. - \frac{q^4}{8} x \delta(\mathbf{p} - k_0) \left[\frac{x'}{k_0^2} y'^2 \delta(\mathbf{p}' - k_0) - \frac{x'}{p'} y'^2 \delta'(\mathbf{p}' - k_0) - \frac{x'^3}{3} \delta''(\mathbf{p}' - k_0) \right] \right. \\ \left. - \frac{q^4}{8} x' \delta(\mathbf{p}' - k_0) \left[\frac{x}{k_0^2} y^2 \delta(\mathbf{p} - k_0) - \frac{x}{p} y^2 \delta'(\mathbf{p} - k_0) - \frac{x^3}{3} \delta''(\mathbf{p} - k_0) \right] \right\} \\ = (4.9b), \quad (\text{A8})$$

where

$$x = \cos \vartheta, \quad x' = \cos \vartheta',$$

$$y^2 = 1 - x^2, \quad y'^2 = 1 - x'^2;$$

$$I = \int d\mathbf{p} d\mathbf{p}' \frac{x x'}{(\mathbf{p} - \mathbf{p}')^2 + 2\epsilon k_0^2} \cdot \frac{y \cos \varphi}{x} \left[\frac{y' \cos \varphi'}{x'} - \frac{y \cos \varphi}{x} \right] \\ \times \delta(\mathbf{p} - k_0) \delta(\mathbf{p}' - k_0),$$

$$J = \int d\mathbf{p} d\mathbf{p}' \frac{1}{(\mathbf{p} - \mathbf{p}')^2 + 2\epsilon k_0^2} \cdot \frac{y \cos \varphi}{x} \left[\frac{y' \cos \varphi'}{x'} - \frac{y \cos \varphi}{x} \right]$$

$$\times \left\{ x \delta(p - k_0) \left[\frac{x'}{k_0^2} y'^3 \delta(p' - k_0) - \frac{x'}{p'} y'^2 \delta'(p' - k_0) - \frac{x'^3}{3} \delta''(p' - k_0) \right] \right. \\ \left. + x' \delta(p' - k_0) \left[\frac{x}{k_0^2} y^2 \delta(p - k_0) - \frac{x}{p} y^2 \delta'(p - k_0) - \frac{x^3}{3} \delta''(p - k_0) \right] \right\}.$$

In order to compute the integrals I and J , we use the following formulas:

$$(p - p')^2 + 2\epsilon k_0^2 = p^2 + p'^2 + 2\epsilon k_0^2 - 2pp'(xx' + yy' \cos \varphi'), \quad \varphi'' = \varphi - \varphi' \\ \int_0^{2\pi} \frac{d\varphi d\varphi'}{\alpha - \beta \cos \varphi''} xy \cos \varphi \left[\frac{y' \cos \varphi'}{x'} - \frac{y \cos \varphi}{x} \right] \\ = \pi \int_0^{2\pi} d\varphi'' \frac{xy}{\alpha - \beta \cos \varphi''} \left[\frac{y' \cos \varphi''}{x'} - \frac{y}{x} \right].$$

Then the integrations over p , p' and φ' are performed to obtain

$$I = (\pi/2) k_0^2 J_1, \\ (1/\pi) J = J_1 - \frac{1}{2} J_1' + \epsilon J_2 + \frac{1}{2} \left(\frac{1}{3} - \epsilon \right) J_2' - \frac{\epsilon^2}{3} J_3'$$

where

$$J_n = \int_{-1}^1 dx dx' \int_0^{2\pi} d\varphi \frac{yy' \cos \varphi - (x'/x) y^2}{(\alpha - \beta \cos \varphi)^n}, \\ J_n' = \int_{-1}^1 dx dx' \int_0^{2\pi} d\varphi \frac{yy' \cos \varphi - (x'/x) y^2}{(\alpha - \beta \cos \varphi)^n} (x^2 + x'^2), \\ \alpha = 1 + \epsilon - xx', \quad \beta = yy'.$$

Integrations over φ , x and x' are calculated straightforwardly. Here we write only the result:

$$J_1 = J_2 = 0 \\ J_1' = (8\pi/3) \left[-\frac{2}{3} - 2\epsilon(2 + \epsilon) + \epsilon(1 + \epsilon)(2 + \epsilon)\gamma \right] \\ J_2' = (8\pi/3) [6(1 + \epsilon) - (2 + 6\epsilon + 3\epsilon^2)\gamma]$$

where

$$\gamma = \log((2 + \epsilon)/\epsilon),$$

and

$$\epsilon^2 J_3' \rightarrow 0 \text{ as } \epsilon \rightarrow 0.$$

Thus we finally get

$$I = 0, \tag{A9} \\ J = (8\pi^2/9) [\log(\epsilon/2) + 4] + \text{terms which vanish with } \epsilon.$$

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Electron Pairs in the Theory of Superconductivity

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The detailed mathematical correspondence between the Bogoliubov theory and the quasi-chemical equilibrium theory is established by means of identities. The Bogoliubov-BCS Ansatz is a special case of the quasi-chemical equilibrium Ansatz, corresponding to perfect Bose-Einstein condensation of the electron pairs.

Some time ago it was suggested^{1),2)} that the phenomenon of superconductivity can be understood in terms of the formation of electron pairs which behave like Bose-Einstein molecules formed of Fermi-Dirac atoms, and which undergo Bose-Einstein condensation at a definite transition temperature. Since then, this approach has been refined mathematically³⁾ but no self-consistent calculation has been carried out with it.

A self-consistent theory of superconductivity has been developed by Bogoliubov⁴⁾ and others,⁵⁾ giving good agreement with most experiments. Although it has been stated⁴⁾ that the physical picture underlying these self-consistent calculations is the same as a Bose-Einstein condensation of electron pairs, the precise formal relationship has not been entirely obvious. It is the purpose of the present note to give this formal relation. Similar work has also been done by Dyson.⁶⁾ Although in most respects the formalism of Bogoliubov et al.⁴⁾ is more elegant and easier to work with, for the present purpose it turns out to be more straightforward to establish the correspondence with the formalism of reference 5). Since Yosida⁷⁾ has proved that the two formalisms are equivalent, the detailed correspondence between the quasi-chemical equilibrium approaches and the Bogoliubov theory is thus established.

We show below that the zero-temperature B.C.S. wave function is mathematically identical with a special case of the zero-temperature limit of the quasi-chemical equilibrium theory. Differences exist at non-zero temperature. Even at zero temperature, the more general wave function obtained from the quasi-chemical equilibrium picture may be of importance in understanding the Knight shift in superconductors.⁸⁾

The ground state wave function of B.C.S. is the N -particle projection of the wave function

$$\mathcal{P} = \prod_k [(1 - h_k)^{1/2} + (h_k)^{1/2} b_k^+] |0\rangle \quad (1)$$

where $|0\rangle$ is the vacuum state, h_k is a function of k to be determined by minimizing the ground state energy, and b_k^+ is the operator

$$b_k^+ = a_{k\uparrow}^+ a_{-k\downarrow}^+ \quad (2)$$

Let us assume that $h_k < 1$ for all k ; we take out a c -number factor $\prod_k (1 - h_k)^{1/2}$ and define

$$\varphi_k = [h_k / (1 - h_k)]^{1/2} \quad (3)$$

We then expand the product, noting that the operators b_k^+ commute with each other:

$$\prod_k (1 + \varphi_k b_k^+) = 1 + \sum_k \varphi_k b_k^+ + \sum_{k \neq m} \varphi_k \varphi_m b_k^+ b_m^+ + \dots \quad (4)$$

The operator

$$W^+ = \sum_k \varphi_k b_k^+ = \sum_k \varphi_k a_{k\uparrow}^+ a_{-k\downarrow}^+ \quad (5)$$

creates a pair of electrons with opposite spin, in the pair quantum state with unnormalized wave function $\varphi(k\uparrow, -k\downarrow) \equiv \varphi_k$. As a result of the operator identity $(b_k^+)^2 = 0$, we can drop the restriction $k \neq m$ in the last term of (4), and we obtain for (1)

$$\mathcal{P} = \text{constant} \left[1 + W^+ + \frac{(W^+)^2}{2!} + \dots \right] |0\rangle \quad (6)$$

The p 'th term in this series produces a wave function with exactly $2p$ electrons. Hence projection of (6) onto the N -particle space means retention of only one term in the series, namely the term of power $p = N_2 = \frac{1}{2}N$. Thus the B.C.S. ground state wave function is identically equal to

$$\mathcal{P}_{\text{B.C.S.}} = \text{constant} (W^+)^{N_2} |0\rangle \quad (N = 2N_2) \quad (7)$$

The physical interpretation of this function is simple: Electrons are created in pairs, each pair having one up-spin and one down-spin electron. There are $N_2 = \frac{1}{2}N$ pairs altogether, accounting for all N electrons, and every pair occupies the same pair quantum state $\varphi(k\uparrow, -k\downarrow)$. Since all the pairs go into the same pair quantum state, this is an extreme case of the Bose-Einstein condensation postulated in the quasi-chemical equilibrium picture.

In deriving (7) from (1), we assumed that $h_k < 1$, for all k . Suppose instead there exists a value k_0 , less than the usual Fermi momentum k_F , such that

$$\begin{aligned} h_k &= 1 & \text{for } k < k_0 \\ h_k &< 1 & \text{for } k > k_0. \end{aligned} \quad (8)$$

Since the factors in the product in Eq. (1) commute, we may consider the factors with $k < k_0$ first; we define the wave function

$$\Phi_{N_1} = \prod_{k < k_0} b_k^+ |0\rangle \quad (9)$$

where N_1 is the number of single-electron states below momentum k_0 . The wave function (9) is just a filled Fermi sea state, describing a Fermi sea of single electrons, *not* pair states. The paired electrons sit on top of this Fermi sea; that is, we now treat the factors in (1) with $k > k_0$ just as before, to obtain the wave function

$$\Psi = \text{constant } (W^+)^{N_2} \Phi_{N_1} \quad (N = N_1 + 2N_2). \quad (10)$$

The operator W^+ is defined as before, by (3) and (5), except that the values of k in the sum in (5) are now restricted to $k > k_0$.

Thus, the B.C.S. theory with the assumption (8) represents a less extreme case of the quasi-chemical equilibrium picture, in that there are now single particles (below $k = k_0$) in chemical equilibrium with electron pairs; the latter, however, are still fully Bose-condensed.

There is no compelling reason for accepting this extreme Bose condensation picture even at zero temperature. On the contrary, the repulsive pair correlations which arise from the Coulomb interaction between electrons must have an effect even in the ground state. This effect may be described qualitatively by saying that the single-particle states, out of which the pairs are made up in the B.C.S. theory, have a finite lifetime. A more satisfactory way of looking at the same effect, however, is in terms of the non-condensed pairs of the quasi-chemical equilibrium picture. Purely repulsive pair correlations appear naturally within this formalism. They can never by themselves lead to Bose-Einstein condensation, but there is no reason for them to disappear completely just because there are Bose-condensed attractive pair correlations. The non-condensed pairs provide correlations between any two electrons, not just between electrons with $\mathbf{k}\uparrow$ and $-\mathbf{k}\downarrow$.⁸⁾

The form (7) or (10) allows a qualitative understanding of the origin of the "stiffness" in the wave function which is needed for a Meissner effect. The wave function must be stiff enough to give a Meissner effect, yet must not be so stiff as to violate gauge invariance. Gauge invariance is achieved by allowing the pair state wave function $\varphi(\mathbf{k}_1, \mathbf{k}_2)$ to adjust itself to the vector potential. In particular, it is necessary to include terms with $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{q}$ and $\mathbf{k}_1 + \mathbf{k}_2 = -\mathbf{q}$, in addition to the terms with $\mathbf{k}_1 + \mathbf{k}_2 = 0$. This has been shown by Bogoliubov.⁹⁾ The wave function of the N -particle system is nevertheless stiff, simply because *all* electron pairs are put into the *same* quantum state of the pair, *i.e.*, because of the Bose-Einstein condensation. The statement that all the electron pairs are in the same pair state is manifestly gauge invariant.

From every set of pair wave functions $\varphi_\alpha(\mathbf{k}_1 s_1, \mathbf{k}_2 s_2)$ (s is spin coordinate) we can construct a set of pair creation operators W_α^+ through

$$W_\alpha^+ = \sum_{\mathbf{k}_1 s_1, \mathbf{k}_2 s_2} \varphi_\alpha(\mathbf{k}_1 s_1, \mathbf{k}_2 s_2) a_{\mathbf{k}_1 s_1}^+ a_{\mathbf{k}_2 s_2}^+ \quad (11)$$

Equation (5) is a special case of (11), namely the one in which $\varphi=0$ unless $\mathbf{k}_1=-\mathbf{k}_2$ and $s_1=-s_2$.

The operators W_α^+ do *not* obey the exact Bose commutation rules. However, it can be shown¹⁰⁾ that the phenomenon of Bose condensation still occurs in spite of the altered commutation rules.

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Bose Einstein Condensation of Correlated Pairs

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In the pair-correlation, or quasi-chemical equilibrium, approximation to statistical mechanics, there appears the possibility of a condensation phenomenon closely analogous to a Bose-Einstein condensation of correlated pairs of particles. However, the expansion used in earlier work to establish this phenomenon fails to yield an adequate approximation at and below the condensation point. In this paper, we give an alternative development, first formally for an arbitrary number of quantum states of the pair, then carry out in detail for the case of only one, or only two, quantum states of the pair. It is shown that, in spite of the Fermi-Dirac statistics of the particles making up the pairs, it is nevertheless possible to accommodate an arbitrarily large number of particles within a single pair state. Furthermore, if two quantum states of the pair are possible, and conditions are such that either the pair state would be occupied by a macroscopic number of particles if it alone were present, then the Pauli exclusion principle works in such a way that the lower one of the two pair states is occupied by a macroscopic number of particles, whereas the occupation number of the upper pair state remains independent of the volume. This result is *not* restricted to pairs whose mean separation exceeds their internal size, but applies also to pairs which completely overlap each other. The result is therefore of significance in the theory of superconductivity where the electron pairs have an internal size of the order of the Pippard coherence length, which is much larger than the mean separation of their centres of gravity.

§ 1. Introduction

In the quasi-chemical equilibrium, or pair correlation, approximation to statistical mechanics,^{1),2)} there appears the possibility of a condensation phenomenon similar to a Bose-Einstein condensation of the "molecules" which are associated with correlated pairs of particles. In reference 1) it was shown that the thermodynamic properties of the system are formally the same as those of an ideal gas of molecules, obeying Bose-Einstein statistics, in chemical equilibrium with an ideal gas of atoms obeying Fermi-Dirac statistics. The overlap between molecular wave functions gives rise to correction terms, which were shown to be unimportant not only down to the Bose-Einstein condensation temperature, but right through condensation.

In reference 2), however, it was found that these correction terms must be altered, due to a trivial error in a combinatorial factor. When this is done, the correction terms remain unimportant above the condensation temperature, but they

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become of major importance at and below condensation. Hence it has become necessary to re-investigate the nature of the condensation phenomenon, using a method of evaluating the trace of the density matrix which provides an adequate approximation not only down to condensation, but right through the transition region as well.

The physical reason for the importance of overlap terms below condensation is as follows: Above condensation, the "molecules" are well separated from each other. The overlap integrals arise from exchanges of atoms between different molecules, without exchanging the molecules as units; hence these overlap integrals give rise to small corrections only. However, if the condensation is similar to the Bose-Einstein condensation, then below the condensation temperature a finite fraction of all the molecules occupies the translational ground state $\mathbf{K}=0$. In x -space, this means that these molecules pervade all of the container; thus they must necessarily overlap with each other significantly, and the overlap integrals can no longer be ignored *a priori*.

The general formula for the partition function of the system in this approximation involves the eigenvalues v_α and eigenfunctions $\varphi_\alpha(k_1, k_2)$ of the "quenched pair correlation matrix". We shall consider in this paper two extreme special cases:

- a) The molecule has only one eigenstate,
- b) The molecule has only two eigenstates.

If the condensation actually involves a macroscopic number of molecules occupying the translational ground state $\mathbf{K}=0$, (i. e. $\mathbf{k}_1 = -\mathbf{k}_2$), then these assumptions should nevertheless prove instructive.

Section 2 contains a general transformation for the grand canonical partition function, which we then specialize to the cases (a) and (b) above. Section 3 contains an explicit and complete evaluation of the partition function for case (a), assuming a simple analytic forms for the internal eigenfunction of the molecule in its ground state. We then show that a certain approximation is sufficient not only down to the transition temperature, but sufficiently far below it to encompass the entire transition region. This approximation involves only the volume V_m of the molecule; it is independent of the detailed internal structure of the molecule, and is valid as long as the average distance between molecules exceeds their internal size.

A discussion of the relation between this new approximation and the earlier power series expansion of the partition function (Appendix II of reference 1)) is given at the end of section 3. We show that the earlier expansion allows additional reductions, i. e., some of the sums which appears can be split into products of smaller sums. The new approximation scheme is equivalent to carrying out all these reductions, and retaining only a finite number of the irreducible quantities which appear. However, *all* terms which can be built from these irreducible quantities are retained, to all orders.

Section 4 contains a first discussion of case (b). The main result is that, below condensation, the lower one of the two quantum states becomes occupied by a number of particles independent of the volume. This is of course highly reminiscent of the usual Bose-Einstein condensation.

The discussion of section 4 is based on the assumption that the average distance between molecules greatly exceeds the internal size of the molecules. This assumption is not valid in the theory of superconductivity,³⁾ where the "quasi-molecules" have internal dimensions of the order of the Pippard coherence length, about 10^{-4} cm. It is important, therefore, to establish that the essential aspects of the Bose condensation persist even if the internal size of the "molecule" greatly exceeds the average distance between "molecules". This is accomplished, for the special case (b) of only two quantum states, in section 5 of this paper.

To the extent that one can generalize from this special case to the general case of infinitely many translational quantum states of the molecule, the work reported here provides a justification for the basic assumption used by Bogoliubov³⁾ and others,⁴⁾ namely to retain only those pair correlations for which $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{K} = 0$.

It should be noted that pair correlations with $\mathbf{K} \neq 0$ need not disappear even at absolute zero; nothing in the present theory requires this, and there is good reason to believe that such pair correlations are in fact there, and are needed to explain the experimental observations on the Knight shift in superconductors.⁵⁾ All that we show here is that the pair correlations with $\mathbf{K} = 0$ dominate the picture in the sense of a Bose condensate, i.e., the number of "quasi-molecules" of this type is proportional to the volume V , below the condensation temperature. Quasi-molecules with $\mathbf{K} \neq 0$, though represented only by intensive numbers for each such quantum state, can nevertheless make a significant contribution, by virtue of the fact that there are an infinity of such quantum states. However, the method of evaluation of the theory given in this paper is not applicable to this case, since it is restricted to at most two quantum states.

Although most of the work here is formal and algebraic, we shall need occasionally to insert expressions for the eigenvalues v_α and the molecular wave functions $\varphi_\alpha(k_1, k_2)$, to estimate orders of magnitude. Whenever this is required, we shall use assumptions corresponding to real chemical equilibrium between atoms and bound molecules, rather than to unbound "quasi-molecules". In particular, we choose

$$v_\alpha = z^2 u_\alpha = \exp \left[\beta \left(2\mu - E_s - \frac{\hbar^2 \mathbf{K}^2}{4m} \right) \right]. \quad (1.1)$$

Here $z = \exp(\beta\mu) = \exp(\mu/kT)$ is the activity, E_s is the (negative) internal energy of the molecular ground state, \mathbf{K} is the wave vector of the centre-of-gravity motion, m is the atomic mass. The index α has been split into two parts, the internal quantum number s of the molecule and its centre-of-gravity momentum \mathbf{K} .

It should be noted that for the cases (a) and (b) considered here, the func-

tional dependence of (1.1) on \mathbf{K} is not important. (1.1) merely serves to provide an estimate of the ratio v_1/v_2 for the two quantum states of case (b). In the physical case, in which infinitely many values of \mathbf{K} are permitted, the functional dependence is of course significant, and it is very likely that (1.1) is qualitatively right above the transition temperature, but qualitatively wrong below it. In view of this possibility, we shall concentrate our attention primarily on features which are independent of the details of the dependence of v_α on K_α . The most important of these features is condensation of a macroscopic number of molecules into the lowest state (i.e., the state α with the highest positive value of v_α). It will be shown that such a condensation does occur, even if the "molecules" overlap each other completely.

As far as the wave functions φ_α of the "molecules" are concerned, we shall assume translational invariance, so that $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{K}$ is a constant of the motion. For the sake of simplicity, we shall also assume that the internal wave function is independent of \mathbf{K} , i. e., Galilean invariance. This latter assumption is almost surely violated in superconductors, since the solid lattice selects out a preferred frame of reference. However, as an approximation for low-lying quantum states, the assumption of an internal wave function independent of \mathbf{K} is reasonable, and that is all which concerns us in the present investigation. Thus, our assumption for $\varphi_\alpha(k_1, k_2)$ is

$$\varphi_\alpha(k_1, k_2) = \varphi_{\mathbf{K},s}(\mathbf{k}_1 s_1, \mathbf{k}_2 s_2) = \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{K}) w_s[\tfrac{1}{2}(\mathbf{k}_1 - \mathbf{k}_2), s_1, s_2], \quad (1.2)$$

where $\mathbf{k}_1, \mathbf{k}_2$ are the momenta of the two particles, s_1 and s_2 are their spin coordinates.

§ 2. General expression for the partition function in the quasi-chemical equilibrium approximation

In reference 2) we derived several equivalent expressions for the grand canonical partition function of a system in the quasi-chemical equilibrium (pair correlation) approximation. First, the partition function of the "atoms" is separated from that of the "molecules", Eq. (3.12) of reference 2). The molecular partition function is expressed as a vacuum expectation value, Eq. (3.13) of reference 2); the operators which appear in this equation are combinations of Fermi-Dirac operators which obey "not-quite-Bose" commutation rules; see Eq. (3.26) of reference 2). The result is then re-expressed in terms of operators which obey exact Bose commutation rules, Eq. (4.22) of reference 2). This equation is our starting point here, hence we repeat it for ready reference:

$$\exp(-\beta Q_M) = (0 | \exp(R+S) \exp(R^+) | 0), \quad (2.1)$$

where

$$R = \sum_{\alpha} v_{\alpha}^{1/2} A_{\alpha} B_{\alpha}, \quad (2.2)$$

$$S \equiv -\frac{1}{2} \sum_{\alpha\beta\gamma\delta} C_{\tau\delta}^{\alpha\beta} v_{\alpha}^{1/2} A_{\alpha} B_{\beta}^{+} B_{\gamma} B_{\delta}, \quad (2.3)$$

$$C_{\tau\delta}^{\alpha\beta} = 4 \sum_{k_1 k_2 k_3 k_4} \varphi_{\alpha}(k_1, k_2) \varphi_{\beta}(k_3, k_4) \varphi_{\gamma}^{*}(k_2, k_3) \varphi_{\delta}^{*}(k_4, k_1). \quad (2.4)$$

The operators A_{α} and B_{α} operate in disjoint Hilbert spaces, and both sets of operators obey Bose commutation rules:

$$[A_{\alpha}, A_{\beta}^{+}]_{-} = [B_{\alpha}, B_{\beta}^{+}]_{-} = \delta_{\alpha\beta}; \text{ all others commute.} \quad (2.5)$$

Although Eq. (2.1) involves no Fermi-Dirac operators, it suffers from the defect that the operators S , (2.3), contains creation operators B^{+} , and therefore fails to commute with R . Thus the exponential $\exp(R+S)$ becomes difficult to evaluate. Quite apart from this obvious difficulty, there is another, less apparent, restriction on the use of (2.1), which we shall discuss at the end of this section. In section 4 of reference 2) we derived an expansion of (2.1) in powers of the coefficients C , which latter are similar to overlap integrals. Unfortunately, this expansion turned out to be useless at and below condensation. The present section is devoted to deriving an alternative form for (2.1) in which all the operators in the arguments of the exponentials commute with each other. This form is *not* restricted to just one or two quantum states of the molecule, but is valid generally.

We start by rewriting the operator $\exp(R+S)$ in the form⁶⁾

$$\exp(R+S) = \exp(S) \text{EXP} \left[\int_0^1 R(t') dt' \right],$$

where EXP is the "ordered exponential function" and

$$R(t) = \exp(-tS) R \exp(+tS) = \sum_{\alpha} v_{\alpha}^{1/2} A_{\alpha} B_{\alpha}(t) \quad (2.7)$$

$$B_{\alpha}(t) = \exp(-tS) B_{\alpha} \exp(+tS). \quad (2.8)$$

Now let us introduce the overlap integral between two molecular quantum states α and β as an operator in k -space. We define:

$$\langle k | q_{\beta}^{\alpha} | k' \rangle = \sum_{k''} \varphi_{\alpha}(k, k'') \varphi_{\beta}^{*}(k'', k'). \quad (2.9)$$

we shall use the symbol "tr" for the trace over this k -space. We also use the orthonormality relation

$$\sum_{k, k'} \varphi_{\alpha}(k, k') \varphi_{\beta}^{*}(k, k') = \delta_{\alpha\beta} \quad (2.10)$$

and the antisymmetry of the φ functions to obtain the result:

$$\text{tr}(q_{\beta}^{\alpha}) = -\delta_{\alpha\beta}. \quad (2.11)$$

Furthermore, the quantity $C_{\tau\delta}^{\alpha\beta}$, Eq. (2.4), is given by

$$C_{\tau\delta}^{\alpha\beta} = 4 \text{tr}(q_{\gamma}^{\alpha} q_{\delta}^{\beta}). \quad (2.12)$$

Let us now introduce the k -space operator $L(t)$ by

$$\langle k|L(t)|k'\rangle = 2 \sum_{\alpha\beta} \langle k|q_\beta^\alpha|k'\rangle v_\alpha^{1/2} A_\alpha B_\beta(t). \quad (2.13)$$

Then it follows from (2.7) and (2.10) that

$$R(t) = -\frac{1}{2} \text{tr}[L(t)]. \quad (2.14)$$

By differentiating $R(t)$ with respect to t we get:

$$\begin{aligned} dR/dt &= \exp(-tS)(RS - SR)\exp(+tS) = R(t)S - SR(t) \\ &= -\frac{1}{2} \sum_{\alpha\beta\gamma\delta} (v_\alpha v_\beta)^{1/2} C_{\gamma\delta}^{\alpha\beta} A_\alpha A_\beta B_\gamma(t) B_\delta(t) \\ &= -\frac{1}{2} \text{tr}[L^2(t)]. \end{aligned} \quad (2.15)$$

Thus we have obtained the trace equation

$$\text{tr}[dL/dt] = \text{tr}[L^2]. \quad (2.16)$$

By using the completeness relation for the eigenfunctions $\varphi_\alpha(k, k')$:

$$\sum_\alpha \varphi_\alpha(k_1, k_2) \varphi_\alpha^*(k_1', k_2') = \delta(k_1 - k_1') \delta(k_2 - k_2') \quad (2.17)$$

it is easy to verify that the corresponding operator equation is true as well, i.e.

$$dL/dt = L^2(t) \quad (2.18)$$

holds for all matrix elements, not merely for the trace. This first order differential equation has the initial condition

$$L(0) = 2 \sum_{\alpha\beta} q_\beta^\alpha A_\alpha B_\beta \equiv M, \quad (2.19)$$

and hence the unique solution

$$L(t) = \frac{M}{1 - tM}. \quad (2.20)$$

Now $R(t)$ is given by (2.14). Since $L(t_1)$ and $L(t_2)$ commute for all t_1 and t_2 , the ordered exponential function in (2.6) becomes simply the conventional exponential function of the operator

$$\int_0^1 R(t) dt = -\frac{1}{2} \text{tr} \int_0^1 \frac{M}{1 - tM} dt = \frac{1}{2} \text{tr} \ln(1 - M). \quad (2.21)$$

We therefore have obtained the *operator identity*

$$\exp(R + S) = \exp(S) \exp\left[\frac{1}{2} \text{tr} \ln(1 - M)\right]. \quad (2.22)$$

We now make use of the fact that the left-most B -operator in S , and hence also in each term S^N in the expansion of $\exp(S)$, is a B^+ , which gives zero when operating on the vacuum state from the right. Thus we have the identity:

$$(0|\exp(S) = (0|. \quad (2.23)$$

Combination of (2.1), (2.22), and (2.23) yields

$$\exp(-\beta\Omega_M) = (0|\exp[\frac{1}{2}\text{tr}\ln(1-M)]\exp(R^+)|0). \quad (2.24)$$

This is our final form, which has the desired properties:

a) All the operators A and B in (2.2) satisfy exact Bose commutation rules; there are no Fermi-Dirac operators left.

b) Bose creation and destruction operators never appear in the argument of the same exponentials; one exponential contains only Bose destruction operators, the other only creation operators.

For later use, we can make one more, minor change: the quantity R^+ involves a factor $v_\alpha^{1/2}$ for each factor $A_\alpha^+ B_\alpha^+$; see Eq. (2.2). In the vacuum expectation value (2.24), the N 'th term in the expansion of one of the exponentials "connects" only with the N 'th term in the expansion of the other exponential. Hence it is permissible to shift these factors $v_\alpha^{1/2}$ from one of the exponentials to the other, so as to obtain the equivalent form

$$\exp(-\beta\Omega_M) = (0|\exp[\frac{1}{2}\text{tr}\ln(1-2\sum_{\alpha\beta} v_\alpha^{1/2} v_\beta^{1/2} q_\beta^\alpha A_\alpha B_\beta)]\exp(\sum_\gamma A_\gamma^+ B_\gamma^+)|0). \quad (2.25)$$

We close this section with a discussion concerning the use of our starting form (2.1). In the derivation of this form, in reference 2), use was made of the closure relation (2.17) to get a simple form for the double commutator $[[b_\alpha, b_\tau^+], b_\delta^+]$, Eq. (4.6) of ref. 2). If one expands (2.1) and makes use of the commutation rules for the B^- operators which occur in the quantity S , one encounters expressions in which closure can be used again, for example:

$$\sum_\delta C_{\tau\delta}^{\alpha\beta} C_{\sigma\delta}^{\delta\rho} = 4^2 \sum_\delta \text{tr}(q_\tau^\alpha q_\delta^\beta) \text{tr}(q_\sigma^\delta q_\delta^\rho).$$

This involves a closure sum over δ , and is equal to

$$\sum_\delta C_{\tau\delta}^{\alpha\beta} C_{\sigma\delta}^{\delta\rho} = -4^2 \text{tr}(q_\tau^\alpha q_\sigma^\beta q_\sigma^\rho). \quad (2.26)$$

In using (2.1), it is advisable to make use of these closure reductions before any other simplifications are made. In going from (2.1) to (2.24), we have made use of closure, to derive (2.18). Since all the A and B operators in $\exp[\frac{1}{2}\text{tr}\ln(1-M)]$ commute with each other, no further closure reductions are possible on the form (2.24), i.e. (2.24) is already "fully reduced".

§ 3. One quantum state of the molecule

We now make the assumption that the "molecule" has only one quantum state, i.e.,

$$v_\alpha = 0 \text{ for all } \alpha \text{ except } \alpha_0. \quad (3.1)$$

Since the sum over α, β in (2.25) contains the factor $(v_\alpha v_\beta)^{1/2}$, all terms drop out except the term $\alpha = \beta = \alpha_0$. Omitting the index α , we then get

$$\exp(-\beta\mathcal{Q}_M) = (0|\exp[\frac{1}{2}\text{tr}\ln(1-2vqAB)]\exp(A^+B^+)|0). \quad (3.2)$$

The k -space operator q is given by (2.14) as

$$\langle k|q|k'\rangle = \langle k|q_{\alpha_0}^{\alpha_0}|k'\rangle = -\sum_{k''} \varphi_{\alpha_0}(k, k'') \varphi_{\alpha_0}^*(k', k''). \quad (3.3)$$

At this stage we use the translation invariance of the theory; the centre-of-gravity momentum $K=k+k'$ is conserved, i.e., we use the delta-function factor in expression (1.2). For simplicity, we shall choose $K=0$ for the state α_0 which we retain (the subsequent evaluation is independent of this assumption). Then in (3.3) k'' must equal $-k$, from the first factor, and it must equal $-k'$, from the second factor. Thus we obtain the result

$$\langle k|q|k'\rangle = -|\varphi_{\alpha_0}(k, -k)|^2 \delta_{k,k'} \equiv -\mu_k \delta_{k,k'}. \quad (3.4)$$

That is, the k -space operator q is diagonal and negative-definite. Thus (3.2) becomes

$$\exp(-\beta\mathcal{Q}_M) = (0|\exp[\frac{1}{2}\sum_k \ln(1+2v\mu_k AB)]\exp(A^+B^+)|0). \quad (3.5)$$

Next, we observe that the operators B and B^+ serve two purposes:

a) In a power series expansion of both exponentials, only equal powers give non-zero contributions,

b) The N 'th power term contributes a factor $(0|B^N(B^+)^N|0)=N!$. The function (a) is however redundant, because the operators A and A^+ by themselves give exactly the same effect. Hence, if we can somehow supply a factor $N!$, we can omit the B and B^+ operators altogether. We use the identity

$$N! = \int_0^\infty e^{-t} t^N dt \quad (3.6)$$

in the following way: We replace the operator B in (3.5) by the c -number t , the operator B^+ by unity, and supply the integration over t as in (3.6), to get

$$\exp(-\beta\mathcal{Q}_M) = \int_0^\infty dt e^{-t} (0|\exp[\frac{1}{2}\sum_k \ln(1+2v\mu_k tA)]\exp(A^+)|0). \quad (3.7)$$

Next, we use the operator identity

$$f(A)\exp(A^+) = \exp(A^+)f(A+1) \quad (3.8)$$

where $f(A)$ is an arbitrary function of the operator A , and 1 is the unit operator. We apply this identity to the operator in (3.7), and use the fact that

$$(0|\exp(A^+) = (0| \quad (3.9a)$$

$$f(A+1)|0) = f(1)|0) \quad (3.9b)$$

to get our final c -number formula:

$$\exp(-\beta\mathcal{Q}_M) = \int_0^\infty dt e^{-t} \exp[\frac{1}{2}\sum_k \ln(1+2v\mu_k t)]. \quad (3.10)$$

Thus, in the case of only one quantum state to the molecule, the partition function calculation has been reduced to two quadratures: the integral over t , and the sum over k .

At this stage, we shall choose a particular, not unreasonable, wave function for the molecule, and evaluate the integrals. We shall also show that a certain approximation, which is independent of finer details of the molecular wave function, is valid in an interesting region.

The molecular wave function which we choose is the "zero-range approximation"

$$\psi(\mathbf{r}_1, \mathbf{r}_2) = \psi(|\mathbf{r}_1 - \mathbf{r}_2|) = N \frac{\exp(-\alpha r)}{r} \quad (3.11)$$

This function is permissible for the singlet spin state; we have omitted spin indices throughout the discussion so far. Strictly speaking, the index k contains both the momentum vector \mathbf{k} of the particle, and its spin coordinate s (which is either $+1$ or -1). It is easily seen, however, that for the case of singlet states the sum over spin indices just gives unity. Hence from now on k stands just for the momentum of the particle.

Normalization of the wave function (3.11) gives

$$N = (\alpha/2\pi V)^{1/2} \quad (3.12)$$

where V is the volume of the container. The k -space wave function corresponding to (3.11) is

$$\varphi(k_1, k_2) = \delta(k_1 + k_2) \frac{4\pi N}{\alpha^2 + k^2} \quad (3.13)$$

and the expression in the exponent of (3.10) can be transformed by an integration by parts into the integral

$$\frac{1}{2} \sum_k \ln(1 + 2v\mu_k t) = \frac{V\alpha^3}{3\pi^2} \int_0^\infty \frac{xy^4 dy}{(y^2 + 1)[x + (y^2 + 1)^2]} \quad (3.14)$$

where

$$x = \frac{16\pi vt}{V\alpha^3} \quad (3.15)$$

The integral is equal to

$$\frac{\pi}{2} \left[1 + (\sqrt{1+x} - 2) \sqrt{\frac{1}{2}(1 + \sqrt{1+x})} \right] = \frac{3\pi x}{16} \left(1 - \frac{x}{16} + \dots \right) \quad (3.16)$$

Combining (3.14) and (3.16), we get the result

$$\frac{1}{2} \sum_k \ln(1 + 2v\mu_k t) = vt - H(t) \quad (3.17)$$

where

$$H(t) = vt - \frac{V\alpha^3 v}{6\pi} \left[1 + (\sqrt{1+x} - 2) \sqrt{\frac{1}{2}(1 + \sqrt{1+x})} \right] \quad (3.18)$$

This function $H(t)$ has the following properties :

$$H(t) > 0 \quad \text{all } t > 0 \quad (3.19a)$$

$$dH/dt > 0 \quad \text{all } t > 0 \quad (3.19b)$$

$$H(t) = \frac{\pi}{V\alpha^3} (vt)^2 + \text{order } t^3 \quad \text{for small } t. \quad (3.19c)$$

Substitution of (3.17) into (3.10) gives the following integral for the partition function of the system :

$$\exp(-\beta\Omega_M) = \int_0^\infty dt \exp[(v-1)t - H(t)]. \quad (3.20)$$

This integral has a completely different behaviour for $v < 1$ than for $v > 1$. For $v < 1$, the properties (3.19) of $H(t)$ imply that $H(t)$ can be ignored completely ; for the integral converges even with $H(t) = 0$, and according to (3.19c) the corrections due to $H(t)$ vanish in the limit of infinite volume V . We thus get

$$\exp(-\beta\Omega_M) = \frac{1}{1-v} \quad \text{for } v < 1. \quad (3.21)$$

This is of course precisely the "ideal Bose gas" approximation for the case of one quantum state. However, in the ideal Bose gas, expression (3.21) is valid right up to $v=1$; hence it is possible to put an arbitrary number of particles, including a number proportional to the volume V of the container, into this one quantum state, merely by letting $v=z^2u$ come very close to unity. We shall see that this is no longer the case in the present theory.

Let us now look at the case $v > 1$. With $v > 1$, the integral (3.20) would diverge without the function $H(t)$. The integrand has its maximum value when the exponent has its maximum, i.e., when

$$dH/dt = v-1 \quad \text{for } t=t_{\max}. \quad (3.22)$$

A straightforward calculation gives

$$t_{\max} = \frac{V\alpha^3}{4\pi} v(v^2-1). \quad (3.23)$$

Note that a positive t_{\max} exists only for $v > 1$; note further that t_{\max} is itself proportional to the volume. Because of this we can use a saddle-point evaluation of the integral ; $-\beta\Omega_M$ then equals the value of the exponent at $t=t_{\max}$ (except for unimportant terms which have a much weaker volume dependence). This gives

$$-\beta\Omega_M = \frac{V\alpha^3}{12\pi} (v+2)(v-1)^2 \quad \text{for } v > 1. \quad (3.24)$$

Let us determine the equilibrium number of particles in this one quantum state, as a function of the chemical activity z , or of $v=z^2u$. We get, in general,

$$N = 2v \frac{\partial(-\beta \mathcal{Q}_M)}{\partial v} \quad (3.25)$$

and hence

$$N = \frac{2v}{1-v} \quad \text{for } v < 1 \quad (3.26a)$$

$$N = \frac{V\alpha^3}{2\pi} v(v^2 - 1) \quad \text{for } v > 1. \quad (3.26b)$$

(3.26a) is the Bose gas result: The factor $v/(1-v)$ is the expectation value of the number of "molecules" in this quantum state, and the factor 2 arises because each molecule consists of 2 particles. For $v > 1$, however, we get a number of particles proportional to the volume. There is an essential difference with respect to the ideal Bose gas of elementary bosons, however, in that the number N/V contributed by this one quantum state is finite only if v actually exceeds unity, and hence only if the pressure p (which equals \mathcal{Q}_M/V by general thermodynamic reasoning) exceeds the pressure at condensation. Thus, *we can still put a number of molecules proportional to the volume V into one molecular quantum state, but the Fermi-Dirac statistics of the elementary particles shows itself in that we must exert a finite pressure in order to achieve a finite number density N/V .* The value of the pressure depends on the internal wave function of the molecule. In effect, we are using the "tail" of the molecular wave function in k -space in order to accomodate particles without violating the exclusion principle. This can be shown very simply by repeating the above calculation with an assumed molecular wave function:

$$\begin{aligned} \varphi(k_1, k_2) &= \delta(k_1 + k_2) w \quad \text{for } |k| < k_0 \\ &= 0 \quad \text{for } |k| > k_0 \end{aligned} \quad (3.13a)$$

instead of (3.13). Straightforward calculation gives, instead of (3.26b),

$$N = \frac{V k_0^3}{6\pi^2} \left(1 - \frac{1}{v}\right) \quad \text{for } v > 1. \quad (3.26c)$$

Unlike (3.26b), (3.26c) leads to an upper limit for the number density N/V of molecules in this quantum state, no matter how large we make v . The pressure contains a term $\ln(v)$ and thus becomes infinite for $v \rightarrow \infty$. The essential difference between the wave functions (3.13) and (3.13a) is just the sharp cut-off in k -space. The wave function (3.13a) is unphysical and the result (3.26c) is so as well; we have quoted it here merely to establish that we are using the tail of the molecular wave function $\varphi(k, -k)$ to accomodate the added particles.

A useful approximation can be found by assuming that v exceeds 1 only very slightly. With this assumption, we obtain from (3.26b)

$$N/V \cong \frac{\alpha^3}{\pi} (v - 1) \quad \text{for } 0 < v - 1 \ll 1. \quad (3.27)$$

Now $\pi\alpha^3$ is just the volume of the molecule, which we shall call V_m . Solving (3.27) for the excess $(v-1)$, we get

$$v-1 \cong \frac{NV_m}{V} = \frac{\text{Volume of one molecule}}{\text{Available volume per molecule}}. \quad (3.28)$$

In the case of a gas of "dineutrons" with a density of 10^{24} dineutrons per cm^3 , the volume V_m is of the order of 10^{-37} cm^3 , and the ratio (3.28) is of the order of 10^{-13} . Thus the excess of v above unity is very small indeed, and the approximation $v-1 \ll 1$ is amply sufficient.

This suggests the use of the following *approximation* to the partition function integral (3.20): we replace the true function $H(t)$ by its lowest order term in an expansion in power of t , namely by (3.19c). This makes no difference at all for $v < 1$; for $v > 1$ we now show that we get results which are equal to the leading terms in an expansion of the exact results in powers of the excess $v-1$. Indeed, condition (3.22) now gives

$$t_{\max} \cong \frac{V\alpha^3}{2\pi} \frac{v-1}{v^2} \quad (3.23')$$

and the approximate value of the exponent at this point leads to

$$-\beta\Omega_M \cong \frac{V\alpha^3}{4\pi} \left(\frac{v-1}{v} \right)^2. \quad (2.24')$$

These differ from the exact results (3.23) and (3.24) by terms of order $(v-1)^2$ only. If $v-1$ is a small number, we have therefore found a consistent and useful approximation to the exact partition function, which is valid not only up to the point $v=1$, but is valid sufficiently far beyond this point to encompass a region of considerable physical interest.

Let us now trace back just what this approximation means in the general case. We have taken the expression (3.14) and have restricted ourselves to the first two terms (proportional to t and to t^2 , respectively) in its expansion in powers of t . This however is nothing else but the ordinary power series expansion of the logarithm; i.e., we write

$$\frac{1}{2} \sum_k \ln(1 + 2v\mu_k t) = vt \sum_k \mu_k - (vt)^2 \sum_k \mu_k^2 + \dots \quad (3.29)$$

The first sum on the right side is unity:

$$\sum_k \mu_k = \sum_k |\varphi(k, -k)|^2 = 1.$$

The second sum on the right can be used to define the volume of the molecule, V_m , by

$$\sum_k \mu_k^2 = \sum_k |\varphi(k, -k)|^4 = V_m/V. \quad (3.30)$$

What we have shown, then, is that for the case of only one quantum state of the

molecule, the statistical mechanics of the system can be described to a sufficient approximation once V_m is known; finer details of the internal structure of the molecule are of no importance as long as the ratio (3.28) is small.

This is an eminently reasonable result and could have been expected *a priori*. However, the approximation of keeping only the leading two terms in an expansion of the logarithm in (3.10) is *not* equivalent to an expansion of the grand canonical potential in powers of the ratio (3.28). This latter expansion is just the one derived in Appendix II of reference 1), and corrected in reference 2). It fails to be useful beyond the point $v=1$, whereas our present approximation method carried us beyond the point $v=1$.

Let us investigate the relation between the two methods. The earlier expansion contains coefficients $\psi_s(\alpha_1, \alpha_2, \dots, \alpha_s)$ which are, except for a factor, equal to the sum

$$\psi_s(\alpha_1, \dots, \alpha_s) \propto \sum_P' \sum_{k_1, \dots, k_s} (-1)^P \varphi_{\alpha_1}(k_1, k_2) \varphi_{\alpha_2}(k_3, k_4) \dots \varphi_{\alpha_s}(k_{2s-1}, k_{2s}) \\ \times \varphi_{\alpha_1}^*(k_1', k_2') \dots \varphi_{\alpha_s}^*(k_{2s-1}', k_{2s}'). \quad (3.31)$$

The prime on the sum over permutations indicates a restriction to "irreducible permutations which contain no natural pairs." The restriction is best explained by example: The permutation

$$\begin{pmatrix} 12 & 34 & 56 & 78 \\ 34 & 56 & 78 & 12 \end{pmatrix}$$

is "irreducible" in the sense that the sum in (3.31) does not split up into a product of two or more smaller sums. However, this permutation merely permutes the "natural pairs" (1, 2), (3, 4), (5, 6) and (7, 8) among each other, without breaking up any pair. Hence this permutation is *not* included in (3.31). If we now permute particles 4 and 5, say, in the bottom row, we obtain the permutation.

$$\begin{pmatrix} 12 & 34 & 56 & 78 \\ 35 & 46 & 78 & 12 \end{pmatrix}$$

This permutation involves the breaking up of some of the "natural pairs", for example the natural pair (1, 2) is replaced by the "unnatural pair" (3, 5). However, the natural pair (5, 6) is replaced by another natural pair, namely (7, 8), and the natural pair (7, 8) is replaced by the natural pair (1, 2). Hence this permutation still contains *some* natural pairs, and is therefore also excluded from the sum in (3.31).

We get an acceptable permutation by interchanging particles 8 and 1, say, in the bottom row of the above permutation, i.e., the permutation

$$\begin{pmatrix} 12 & 34 & 56 & 78 \\ 35 & 46 & 71 & 82 \end{pmatrix} \quad (3.32)$$

This permutation is among the ones which appear in the sum (3.31), and as long

as the s quantum states $\alpha_1, \alpha_2, \dots, \alpha_s$ ($s=4$ here) are all different from each other, nothing more can be done to reduce the sum (3.31).

However, additional reductions *are* possible if some of the α 's coincide. In our present case, of only one quantum state, all α 's are in fact equal to α_0 . The sum (3.31) for the case of the permutation (3.32) becomes

$$\sum_{k_1 \dots k_8} \varphi(k_1, k_2) \varphi(k_3, k_4) \varphi(k_5, k_6) \varphi(k_7, k_8) \varphi^*(k_3, k_5) \varphi^*(k_4, k_6) \varphi^*(k_7, k_1) \varphi^*(k_8, k_2) \\ = \left[\sum_{k_1 k_2 k_3 k_7} \varphi(k_1, k_2) \varphi^*(k_2, k_3) \varphi(k_3, k_7) \varphi^*(k_7, k_1) \right]^2 = (V_m/V)^2. \quad (3.33)$$

Thus, this particular term is decidedly reducible, even though it would have been irreducible if the four quantum states had all been different.

The procedure of expanding the logarithm in (3.10) in powers of t and keeping terms up to t^r , say, is equivalent to reducing out the earlier expansion, and summing up, exactly, all those terms arising from combinations of the first r irreducible sums over k 's which arise. It is thus not surprising that the earlier expansion fails near condensation, no matter how many terms are retained, whereas the present procedure, in which only irreducible quantities appear, gives a useful approximation with only two terms of the expansion of the logarithm.

It may be noticed that the term in $(vt)^3$ arising from the expansion (3.29) has a positive coefficient, and the same is true of all odd powers of vt . When we break off with such an odd power, and substitute into (3.10), the integral diverges for large t . Thus, if a strictly convergent approximation is desired, one should truncate at an even power of t . The difficulty with odd powers of t is more apparent than real, however. The exponent in the integral has a maximum at a value of t proportional to the volume, then drops down; only for very large t does the exponent increase again and go up to infinity. This latter behaviour is obviously spurious, arising from a region in which the power series expansion of the logarithm should not be used. Thus, when one truncates with an odd power of t , a sensible asymptotic evaluation of the integral consists in using the saddle point method for the first maximum of the exponent, and ignoring the spurious divergence at $t=\infty$. This is related to the fact that the basic Ansatz (Eqs. (2.3) or (2.26) of reference 2)) makes no allowance for the fact that the pair correlation functions become modified when more and more particles are crowded into the volume, until eventually the whole pair correlation approximation breaks down. If we truncate the expression for the grand canonical partition function as a sum over numbers of particles N (Eq. (2.1) of reference 2)) at some maximum number N_0 proportional to the volume V , we obtain natural cutoffs for the integrals which arise, and no infinities can occur.

The approximation of neglecting all the higher terms in the expansion (3.29) will be called the "quadratic approximation" henceforth. It is completely adequate for a gas of ordinary bound molecules which have an average separation larger than their internal dimensions, i.e., for which the ratio (3.28) is much less than

unity. Since this approximation is much easier to handle than the exact result, we shall start our discussion of the two-quantum-state case by using this approximation (section 4). However, in the theory of superconductivity the inequality (3.28) is badly violated. The internal size of the electron pair which forms the "quasimolecule" is of the order of 10^{-4} cm, and the average separation between centres of gravities of pairs is less than this.

It is therefore necessary to establish the Bose condensation phenomenon for the general case, and this is done in section 5. No new feature emerges from this discussion; the quadratic approximation, although not valid numerically in this region, already contains all the *qualitative* features of the general result.

§ 4. Two quantum states of the molecule. Quadratic approximation

We assume that the molecule has just two translational states, the ground state with centre-of-gravity momentum $\mathbf{K}=0$, and a translational state with \mathbf{K} different from zero. The internal wave function w , see (1.2), is assumed to be the same for both states.

As a result of conservation of centre-of-gravity momentum, the operator $\langle k|q_\beta^\alpha|k'\rangle$ has non-zero matrix elements only for

$$\mathbf{k}-\mathbf{k}'=\mathbf{K}_\alpha-\mathbf{K}_\beta. \quad (4.1)$$

As a consequence of this selection rule, we get the statement

$$\text{tr}(q_{\beta_1}^{\alpha_1} q_{\beta_2}^{\alpha_2} \cdots q_{\beta_s}^{\alpha_s}) = 0 \text{ unless } \mathbf{K}_{\alpha_1} + \mathbf{K}_{\alpha_2} + \cdots + \mathbf{K}_{\alpha_s} = \mathbf{K}_{\beta_1} + \mathbf{K}_{\beta_2} + \cdots + \mathbf{K}_{\beta_s}. \quad (4.2)$$

In the special case of only two quantum states, $\alpha=1$ and $\alpha=2$, say, the selection rule (4.2) leads to the following simple result: Of the s indices α , let n_1 be equal to 1, n_2 be equal to 2, $n_1+n_2=s$; similarly, of the s indices β , n'_1 are equal to 1, n'_2 are equal to 2; then *the trace (4.2) vanishes unless $n_1=n'_1$ and $n_2=n'_2$* . This follows directly from the observation that the condition on the \mathbf{K} vectors, (4.2), can be rewritten as

$$n_1 \mathbf{K}_1 + n_2 \mathbf{K}_2 = n'_1 \mathbf{K}_1 + n'_2 \mathbf{K}_2 \quad (4.2a)$$

and $\mathbf{K}_1, \mathbf{K}_2$ are linearly independent vectors.

We now use this to simplify the general formula (2.25). If we imagine both exponentials in (2.25), as well as the logarithm, expanded in power series, then we get expressions involving traces (4.2) and powers of these traces. Furthermore, we get vacuum expectation values of the form

$$\langle 0 | A_{\alpha_1} A_{\alpha_2} \cdots A_{\alpha_s} B_{\beta_1} B_{\beta_2} \cdots B_{\beta_s} A_{\tau_1}^+ A_{\tau_2}^+ \cdots A_{\tau_s}^+ B_{\tau_1}^+ B_{\tau_2}^+ \cdots B_{\tau_s}^+ | 0 \rangle. \quad (4.3)$$

Such a vacuum expectation value vanishes unless the sets of indices $\alpha_1 \cdots \alpha_s$, $\beta_1 \cdots \beta_s$, and $\tau_1 \cdots \tau_s$ are permutations of each other. However, the condition obtained for the trace above, that $n_1=n'_1$ and $n_2=n'_2$ for the case of two quantum states, already

implies that $\alpha_1 \cdots \alpha_s$ and $\beta_1 \cdots \beta_s$ are permutations of each other. Thus the only effect of the operators $B \cdots B^+$ in (4.3) is to supply a factor $n_1! n_2!$. We again use the identity (3.6) for the factorial and the identities (3.8), (3.9) to reduce (2.25) to the expression:

$$\exp(-\beta\Omega_M) = \int_0^\infty dt_1 \int_0^\infty dt_2 \exp(-t_1 - t_2) \exp\left[\frac{1}{2} \operatorname{tr} \ln(1 - 2 \sum_{\alpha\beta} (v_\alpha t_\alpha v_\beta t_\beta)^{1/2} q_\beta^\alpha)\right]. \quad (4.4)$$

The sums over α, β in the argument of the logarithm go over the range 1, 2 only. The form (4.4) is still exact.

We now make the "quadratic approximation" of expanding the logarithm and keeping terms up to t^2 but no higher. We use (2.11) to simplify the linear term; the quadratic terms involve the traces

$$\operatorname{tr}(q_1^1 q_1^1) = \operatorname{tr}(q_2^2 q_2^2) = V_m/V \quad (4.5)$$

$$\operatorname{tr}(q_2^1 q_1^2) = \operatorname{tr}(q_1^1 q_2^2) = \sum_k |w(k)|^2 |w(k - \frac{1}{2} K_2)|^2 = V_e/V \quad (4.6)$$

where V_m is the molecular volume given by (3.30), and V_e is interpretable as a "volume overlap" or "exchange volume"; clearly, V_e does not differ much from V_m as long as $\frac{1}{2}K_2$ is small compared to the range of the internal wave function in K -space; if we denote the "size of the molecule" by a , then

$$V_e \simeq V_m \quad \text{for } K_2 a \ll 1. \quad (4.7)$$

The quadratic approximation to the partition function (4.4) is

$$\begin{aligned} \exp(-\beta\Omega_M) = & \int_0^\infty dt_1 \int_0^\infty dt_2 \exp[(v_1 - 1)t_1 + (v_2 - 1)t_2 - (V_m/V)(v_1 t_1)^2 \\ & - (V_m/V)(v_2 t_2)^2 - (4V_e/V)v_1 t_1 v_2 t_2]. \end{aligned} \quad (4.8)$$

In spite of the simple appearance of this integral, its evaluation in terms of tabulated functions is not easy. Rather than doing that, we shall discuss the nature of the integrand and develop reasonable approximation methods depending upon the values of v_1 and v_2 , as well as on the approximate equality (4.7).

Case 1: The simplest case is the one in which both v_1 and v_2 lie below unity. Then the quadratic terms in the exponent can be ignored completely, with an error which becomes zero in the limit of infinite volume. We thus get the Bose approximation:

$$\exp(-\beta\Omega_M) = \frac{1}{(1-v_1)(1-v_2)} \quad \text{for } v_1, v_2 < 1. \quad (4.9)$$

Of the two quantities v_1, v_2 , the v associated with the lower quantum state $K=0$ is larger than the other v , according to (1.1). We shall let state 1 be the ground state hence

$$v_1 > v_2. \quad (4.10)$$

The next case to discuss, therefore, is

Case 2: $v_1 > 1$, $v_2 < 1$.

We see, by direct differentiation of the exponent in (4.8) with respect to t_2 , that the exponent decreases with increasing t_2 , for all positive values of t_1 and t_2 . Hence the contributing region lies close to the t_1 -axis, t_2 always remaining of order unity. We can therefore ignore the term $(V_m/V)(v_2 t_2)^2$ completely. With this approximation, the t_2 integral becomes the integral of a negative exponential, and we obtain

$$\exp(-\beta\Omega_M) = \int_0^\infty dt \frac{\exp[(v_1-1)t_1 - (V_m/V)(v_1 t_1)^2]}{1 - v_2 + v_2(4V_e/V)v_1 t_1}. \quad (4.11)$$

Since the exponential is a rapidly varying factor, the main contributing region is determined by the maximum of the exponent. This maximum occurs at

$$v_1 t_1 = \frac{V}{2V_m} \left(1 - \frac{1}{v_1}\right) \quad \text{Position of max. exponent.} \quad (4.12)$$

At this point, the denominator of (4.11) equals

$$1 - v_2 + v_2(2V_e/V_m) \left(1 - \frac{1}{v_1}\right) \equiv D. \quad (4.13)$$

The width of the maximum around the position (4.12) is of order $V^{1/2}$, hence the variation of the denominator of (4.11) over this width becomes zero in the limit of infinite volume. We can also extend the limit of the t_1 integration to $-\infty$. This gives

$$\exp(-\beta\Omega_M) = (V/V_m)^{1/2} v_1^{-1} \exp\left[\frac{V}{4V_m} \left(\frac{v_1-1}{v_1}\right)^2\right] \frac{1}{D}. \quad (4.14)$$

This result is as expected: The contribution from state 2, with $v_2 < 1$, is given by the factor $1/D$, which is very nearly equal to the Bose gas factor $1/(1-v_2)$, because of the inequality (3.28). State 1, on the other hand, contributes a free energy proportional to the volume, and in fact identically the same as if this were the only quantum state present, see Eq. (3.24').

Case 3: Both v_1 and v_2 greater than 1.

We must now study the quadratic form in the exponent of (4.8). We first show that the quadratic form has no true maximum anywhere. To see this, observe that we can eliminate the linear terms in t_1 and t_2 by simple displacement of the origin. A true maximum could occur only if the quadratic form

$$(V_m/V)(x_1^2 + x_2^2) + (4V_e/V)x_1 x_2 = F(x_1, x_2) \quad (4.15)$$

were positive definite. However, the eigenvalues of (4.15) are easily seen to be $(V_m + 2V_e)/V$ and $(V_m - 2V_e)/V$. The second of these is negative according to

(4.7). Hence the point $x_1=x_2=0$ is a saddle point of (4.15), not a minimum, and no true minimum exists.

Thus the exponent in (4.8) must assume its maximum value somewhere on the *boundary* of the allowed region $t_1, t_2 > 0$. There is one maximum along the t_1 -axis, at the position (4.12), and a second maximum along the t_2 -axis, at the position

$$v_2 t_2 = \frac{V}{2V_m} \left(1 - \frac{1}{v_2}\right) \quad \text{Second maximum.} \quad (4.16)$$

The value of the exponent at the maximum (4.21) is just the exponent which appears in (4.14), and the value of the exponent at the maximum (4.16) is obtained by replacing v_1 by v_2 . Hence we get

$$\frac{\text{Value of integrand at first max.}}{\text{Value of integrand at second max.}} = \exp \left\{ \frac{V}{4V_m} \left[\left(\frac{v_1-1}{v_1} \right)^2 - \left(\frac{v_2-1}{v_2} \right)^2 \right] \right\}. \quad (4.17)$$

Since v_1 exceeds v_2 , by assumption, this ratio becomes exponentially large as the volume increases.* We can therefore ignore the contribution from this second maximum in the limit of infinite volume, we are allowed to approximate to the integrand in such a way that the approximation is good in the region:

$$v_1 t_1 = \text{order } (V) \quad v_2 t_2 = \text{order } (1). \quad (4.18)$$

The simplest approximation in this region is to neglect the term $(V_m/V)(v_2 t_2)^2$ in the exponent of (4.8). This, however, is precisely the method used to obtain (4.11)** and hence (4.14). Thus (4.14) is still a correct expression for the partition function in Case 3.

This is the main result of this section: *Even though both v_1 and v_2 exceed unity, only the ground state gives rise to a free energy, and particle number, proportional to the volume.* The Pauli exclusion principle operates in such way that the number of molecules in state 2 remains intensive, even though this number would have been extensive if state 2 had been the only state present.

We may think of the diagonal quadratic terms in the exponent of (4.8) as Pauli exclusion of a quantum state against itself, whereas the term $(4V_e/V)v_1 t_1 v_2 t_2$ can be interpreted as Pauli exclusion of one quantum state against another. What our calculation has shown, then, is that the exclusion of phase space for one quantum state due to another is dominant over the exclusion of phase space by one quantum state against itself; when the system is asked to accommodate a number of particles

* This is true even if K_2 is the *first* excited translational state of the molecule, for then the ratio $(v_1 - v_2)/v_1$ is of order L^{-2} , where L is the side length of the cubical box (see (1.1)). The exponent in (4.17) becomes proportional to L .

** One caution is necessary here: The t_1 integration in (4.11) can no longer start at $t_1=0$, since the denominator would be negative there. In fact, the t_2 integral makes sense in this approximation only as long as the denominator in (4.11) is positive. However, the main contribution comes from the region near (4.12), where the denominator D , (4.13), is definitely positive.

proportional to the volume, it finds it preferable to put all the particles into the lower one of the two quantum states, rather than distributing them more evenly.

The fact that an extensive number of particles goes into the lowest state of the system is of course highly reminiscent of the Bose-Einstein condensation phenomenon for elementary bosons. The only difference is that we now need to push v_1 beyond the value $v_1=1$, in order to accommodate an extensive number of particles, whereas with elementary boson v_1 would have been less than unity, by an amount of order $1/V$.

§ 5. Two quantum states of the molecule; general case

The result of section 4 is adequate, within the limitation of only two quantum states, for ordinary molecular gases in which the molecules are on the average far apart compared to their internal sizes. However, the most interesting application of this formalism is to the theory of superconductivity, and there the electron pair correlations extend over distances of the order of 10^{-4} cm, much larger than the average distance between centres of gravities of the pairs. Hence it is necessary to extend our proof of Bose-Einstein condensation to highly overlapping "molecules."

On the other hand, we are still primarily interested in the case of small centre-of-gravity momenta K , and we shall develop, in this section, an approximation scheme based on the condition

$$K_2 a \ll 1 \quad (5.1)$$

where a denotes the linear size of the molecule or quasi-molecule. Condition (5.1) is much weaker than the condition for the applicability of the quadratic approximation:

$$a/d \ll 1 \quad (5.2)$$

where d is the average separation between centres of gravities of molecules.

We start from the exact result (4.4). The operator $\langle k|q_\alpha^a|k'\rangle$ is diagonal in k -space, but q_2^1 and q_1^2 have matrix elements connecting k to $k+K$ and $k-K$. Let us choose the vector K to lie along the z -axis. The fundamental Brillouin zone for this problem is then the two-dimensional slab:

$$-\infty < k_x < +\infty \quad -\infty < k_y < +\infty \quad -\frac{1}{2}K < k_z < +\frac{1}{2}K. \quad (5.3)$$

Let k_0 denote any vector within this region. We then define k_n by

$$k_n = k_0 + nK \quad (n \text{ integral}) \quad (5.4)$$

and obtain the result:

$$\text{tr} \ln [1 - 2 \sum_{\alpha\beta} (x_\alpha x_\beta)^{1/2} q_\beta^a] = \sum'_{k_0} \text{tr} \ln F(k_0) \quad (5.5)$$

where the prime on the sum indicates that values of k_0 outside the region (5.3) are excluded; $F(k_0)$ is an infinite matrix with elements on the diagonal and just off the diagonal, only. These elements are

$$F_{n,n} = 1 + 2x_1 w^2(k_n) + 2x_2 w^2(k_n - \frac{1}{2}K) \quad (5.6a)$$

$$F_{n,n+1} = F_{n+1,n} = 2(x_1 x_2)^{1/2} w(k_n) w(k_n - \frac{1}{2}K), \quad (5.6b)$$

where we have assumed w to be real, for simplicity. Since

$$\exp[\text{tr}(\ln F)] = \det F, \quad (5.7)$$

let us discuss the determinant of the matrix F . First of all, we show that the off-diagonal elements of F occur in the determinant only through their squares. We let $F^{(N)}$ stand for the truncated operator F , in which only N rows and columns of the matrix are kept. For $N=1$ and $N=2$ we have, respectively,

$$\det(F^{(1)}) = F_{11} \quad (5.8a)$$

$$\det(F^{(2)}) = F_{11}F_{22} - (F_{12})^2. \quad (5.8b)$$

Adding one more row and column to the matrix, we get the recursion relation:

$$\det F^{(N+1)} = F_{N+1,N+1} \det F^{(N)} - (F_{N,N+1})^2 \det F^{(N-1)}. \quad (5.9)$$

Since the off-diagonal elements of F occur only through their squares in both (5.8) and (5.9), the induction is complete.

Next, we show that *products of adjacent off-diagonal elements do not occur in $\det F$* . To see this, assume that the element $F_{n,n+1}$ actually occurs in a term of $\det F$. Then, according to the preceding discussion, the element $F_{n+1,n}$ must be a factor in the same term. Now $F_{n+1,n+2}$ lies in the same row as $F_{n+1,n}$ and $F_{n+2,n+1}$ lies in the same column as $F_{n,n+1}$. Hence neither $F_{n+1,n+2}$ nor $F_{n+2,n+1}$ can occur as factors in the term considered. The same argument excludes factors $F_{n-1,n}$ and $F_{n,n-1}$.

Since adjacent off-diagonal elements cannot occur, we are permitted to take out, as a factor, the product of all the diagonal elements, and write

$$\det F = \prod_n F_{nn} \cdot \det(1+G) \quad (5.10)$$

where G has zeros along its diagonal, and

$$G_{n,n+1} = G_{n+1,n} = \frac{F_{n,n+1}}{(F_{nn} F_{n+1,n+1})^{1/2}}. \quad (5.11)$$

The properties established so far lead to an exact expansion of $\det(1+G)$; we let $g_n = (G_{n,n+1})^2$ to get

$$\det(1+G) = 1 - \sum_n g_n + \sum_n \sum_{n' \geq n+2} g_n g_{n'} - \sum_n \sum_{n' \geq n+2} \sum_{n'' \geq n'+2} g_n g_{n'} g_{n''} + \dots \quad (5.12)$$

Although this result is exact, it is quite useless for our purposes; we require a product form, in order to be able to take the logarithm in a simple fashion.

Let μ be an eigenvalue of the linear operator G , and let u_n be the n th component of the corresponding eigenvector. We then have the recurrence relation

$$G_{n,n-1} u_{n-1} + G_{n,n+1} u_{n+1} = \mu u_n \quad (5.13)$$

which must be solved subject to the condition that u_n is normalizable, i.e., u_n must decrease to zero both for $n \rightarrow \infty$ and for $n \rightarrow -\infty$. Let μ_m be the m th eigenvalue. Then

$$\det(1+G) = \prod_m (1+\mu_m). \quad (5.14)$$

Furthermore, the eigenvalues occur in pairs; let μ be an eigenvalue, and u_n be the corresponding eigenvector. Then the vector v_n defined by

$$v_n = (-1)^n u_n$$

also satisfies (5.13), with μ replaced by $-\mu$. We therefore obtain

$$\det(1+G) = \prod'_m (1-\mu_m^2) \quad (5.15)$$

where the prime indicates that only positive values of μ_m are kept.

So far, everything has been exact. We now make an approximation based on the condition (5.1). This condition implies that $w(k_n)$, $w(k_n - \frac{1}{2}K)$, $w(k_n + \frac{1}{2}K)$, etc., are very nearly equal to each other. Thus in the difference equation (5.13) the coefficients $G_{n,n-1}$ and $G_{n,n+1}$ are nearly equal. If they were exactly equal to each other, the difference equation would have an exponential solution, $u_n = \exp(nq)$, familiar from the case of a linear chain of equal masses connected by equal springs. We now make the assumption that the ratio u_{n+1}/u_n very nearly equals the ratio u_n/u_{n-1} . This is similar to the well-known W. K. B. approximation for the corresponding differential equation. Just as in the W. K. B. approximation, there are regions in which the solution oscillates, and regions in which it behaves exponentially. The results are:

Exponential region:

$$\mu > 2(G_{n,n-1} G_{n,n+1})^{1/2} \quad (5.16)$$

$$\frac{u_{n+1}}{u_n} \simeq \frac{\mu \pm (\mu^2 - 4G_{n,n-1} G_{n,n+1})^{1/2}}{2G_{n,n+1}}. \quad (5.17)$$

Of these two roots, one is larger than 1, the other less than 1. Thus the two linearly independent approximate solutions are such that one increases exponentially with increasing n , the other decreases exponentially.

Oscillating Region:

$$\mu < 2(G_{n,n-1} G_{n,n+1})^{1/2} \quad (5.18)$$

$$\frac{u_{n+1}}{u_n} \simeq \left(\frac{G_{n,n-1}}{G_{n,n+1}} \right)^{1/2} \exp \left\{ \pm i \arccos \left[\frac{\mu}{2(G_{n,n-1} G_{n,n+1})^{1/2}} \right] \right\}. \quad (5.19)$$

Just as in the W. K. B. method, some care is required to establish the connection across the turning points.* In our case, since $G_{n,n+1}$ decreases to zero both as $n \rightarrow \infty$ and as $n \rightarrow -\infty$, there are two turning points, with an oscillating region

* A more detailed discussion of this W. K. B. method for difference equations will be published elsewhere.

between them and exponential regions outside. In order to get an eigenvector, we must start from $n = -\infty$ with a solution which increases exponentially with increasing n ; we then join to the oscillating region at the turning point $n = n_1(\mu)$; when we continue the solution to the other turning point, $n = n_2(\mu)$, and join across it, we must join correctly to the solution which decreases exponentially with increasing n . This provides a condition on the eigenvalue μ . For our purposes, we can replace sums by integrals and state the condition as

$$\int_{n_1(\mu)}^{n_2(\mu)} \arccos \left[\frac{\mu}{2G(t)} \right] dt = m\pi \quad (5.20)$$

where m is an integer and $G(t)$ is a smooth function which for integral $t = n$ assumes the values

$$G(n) = (G_{n,n-1} G_{n,n+1})^{1/2}. \quad (5.21)$$

We now use these results in formula (5.15). We take the logarithm on both sides, and replace the sum over m by an integral to get

$$\ln [\det(1+G)] = \sum'_m \ln(1-\mu_m^2) = \int_0^{\mu_0} d\mu \left(-\frac{dm}{d\mu} \right) \ln(1-\mu^2) \quad (5.22)$$

where μ_0 is the largest eigenvalue of G ,

$$\mu_0 = 2G(0) \quad (5.23)$$

and $m(\mu)$ is the function defined by (5.20). In (5.22) we integrate by parts. The integrated term vanishes at both limits, since $m(\mu_0) = 0$, and we get

$$\ln [\det(1+G)] = -\frac{2}{\pi} \int_0^{\mu_0} d\mu \int_{n_1(\mu)}^{n_2(\mu)} dt \frac{\mu \arccos[\mu/2G(t)]}{1-\mu^2}.$$

The limits become much simpler when we interchange the order of integration; furthermore, the inner integral can then be done exactly:

$$\begin{aligned} \ln [\det(1+G)] &= -\frac{2}{\pi} \int_{-\infty}^{+\infty} dt \int_0^{2G(t)} d\mu \frac{\mu \arccos[\mu/2G(t)]}{1-\mu^2} \\ &= \frac{1}{2} \int_{-\infty}^{\infty} dt \ln \left[\frac{1}{2}(1-2G^2) + \frac{1}{2}(1-4G^2)^{1/2} \right]. \end{aligned} \quad (5.24)$$

It is consistent with our approximations all along to replace this last integral by a sum. It is also consistent to replace $G^2(t)$ at $t = n$ by $G_{n,n+1}^2$ rather than by $G_{n,n-1} G_{n,n+1}$. In view of the fact that in the exact result, (5.12), the matrix elements of G occur only through their squares, this procedure provides a somewhat better approximation. Our final form is then

$$\ln [\det (1+G)]=\text{tr} [\ln (1+G)]=\frac{1}{2} \sum_{n=-\infty}^{+\infty} \ln \left[\frac{1}{2} (1-2G_{n,n+1}^2) + \frac{1}{2} (1-4G_{n,n+1}^2)^{1/2} \right]. \quad (5.25)$$

We now combine Eqs. (5.5), (5.7), (5.10) and (5.25), and we notice that the combination of the sum over n in (5.25) and the restricted sum over k_0 in (5.5) is just equivalent to an unrestricted sum over the whole k -space. This gives the result:

$$\text{tr} \{ \ln [1 - 2 \sum_{\alpha\beta} (x_\alpha x_\beta)^{1/2} q_\beta^\alpha] \} = H_1 + H_2 \quad (5.26a)$$

$$H_1 = \sum_k \ln F(k) \quad (5.26b)$$

$$H_2 = \sum_k \frac{1}{2} \ln \left[\frac{1}{2} (1 - 2G_k^2) + \frac{1}{2} (1 - 4G_k^2)^{1/2} \right] \quad (5.26c)$$

$$F(k) = 1 + 2x_1 w^2(k) + 2x_2 w^2(k - \frac{1}{2}K) \quad (5.26d)$$

$$G_k^2 = \frac{4x_1 x_2 w^2(k) w^2(k - \frac{1}{2}K)}{F(k) F(k + K)} \quad (5.26e)$$

This approximate form is valid as long as K is small compared to the internal momenta in the molecule, whether or not the molecules overlap with each other in space. It is easily seen, by expansion in power series in x_1 and x_2 , that we recover the quadratic approximation from (5.26). That is, (5.26) includes the quadratic approximation, but has a much wider range of validity.

In order to get a general result easily, we now assume that K is very small indeed, so that we can ignore the difference between $w(k)$ and $w(k - \frac{1}{2}K)$, replacing both of them by $w(k)$. This is certainly valid if the excited quantum state under consideration is a very low-lying state, since then K is of order L^{-1} , where L is the side length of the box. This replaces (5.26d) and (5.26e) by the simpler expressions

$$F(k) = 1 + 2(x_1 + x_2) w^2(k) \quad (5.26d')$$

$$G_k^2 = \frac{4x_1 x_2 w^4(k)}{[1 + 2(x_1 + x_2) w^2(k)]^2} \quad (5.26e')$$

The partition function of the system is then given by (4.4) and (5.26) as

$$\exp(-\beta\Omega_M) = (v_1 v_2)^{-1} \int_0^\infty dx_1 \int_0^\infty dx_2 \exp \left(-\frac{x_1}{v_1} - \frac{x_2}{v_2} + \frac{1}{2} H_1 + \frac{1}{2} H_2 \right). \quad (5.27)$$

In the condensation region, we can evaluate this integral by a saddle point method. Thus we need to know the location of the maximum of the exponent in (5.27), within the positive quadrant of the x_1 - x_2 plane. Rather than looking for this maximum directly, we shall do so in two steps. First, we shall consider a straight line of form

$$x_1 + x_2 = c. \quad (5.28)$$

where c is some positive constant, and we shall locate the maximum of the exponent along that line. Thereafter, we shall let c itself vary, so that we cover the whole positive quadrant, eventually.

According to (5.26b) and (5.26d') the quantity H_1 involves x_1 and x_2 only through their sum, $x_1 + x_2$. Thus H_1 is constant along the line (5.28), and we can ignore it as far as finding the maximum is concerned. The expression $-(x_1/v_1) - (x_2/v_2)$ has its maximum value on the x -axis, for $v_1 > v_2$. There remains the term $\frac{1}{2}H_2$. However, H_2 vanishes identically when $x_1x_2=0$, i.e., both on the x_1 -axis and on the x_2 -axis; and H_2 is negative definite everywhere else.* Thus, along the line (5.28), the exponent assumes its maximum value on the x_1 -axis, i.e., at $x_1=c$, $x_2=0$.

Since this is true no matter what the value of c is, we conclude that *the maximum of the exponent of (5.27) within the region of integration lies along the x_1 -axis*. This, however, is all we need for establishing the Bose-Einstein condensation; the rest of the argument is identical to that in section 4.

This proof is restricted to values of K so small that (5.26d) and (5.26e) can be replaced by (5.26d') and (5.26e'), respectively. Since this is the region of the main interest anyway, we shall not go any further here.

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* These two properties of H_2 are not merely results of our W. K. B. approximation, but are exact. The logarithm of the exact result (5.15) is non-positive. For $x_1x_2=0$, we have $G=0$ as a matrix, and hence $\ln[\det(1+G)]=0$ exactly.

On the Vibration of Disordered Linear Lattice. III

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Eigenfrequency spectrum of isotopic two-component disordered lattice has been calculated approximately by a method which requires only a comparatively small amount of numerical work. An argument based on perturbation theory shows formally that the spectrum of completely random lattice is the same as that of virtual regular lattice composed of atoms with average mass, except at the edge and outside of the band. We have first investigated how far this statement is valid and obtained the result that the smaller the concentration of lighter atoms, the larger the frequency domain in which the spectrum can be regarded as approximately the same as that of virtual regular lattice. Next, we have calculated the spectrum in the neighborhood of the edge of the band where the above statement does not hold, by applying the moment-trace method only to that region. The result is that when the concentration of lighter atoms is comparable with or larger than that of heavier atoms, there is only one presumably rounded maximum at the position of the band-edge of virtual regular lattice, whereas when the number of lighter atom becomes smaller, there appears an impurity band, its separation from the main band coming out the more distinct, as the concentration of lighter atoms gets smaller. Both results are natural provided the spectrum is to approach that of Poisson lattice as the lighter atoms become few.

§ 1. Introduction

In previous papers¹⁾ we treated the problem of eigenfrequency distribution of linear disordered lattices containing isotopic impurities by the method of transfer matrix. Eigenfrequencies were obtained from the equation $\text{Trace } \mathbf{H} = 2$, where \mathbf{H} is a product of transfer matrices. When impurities are randomly distributed, we must investigate the statistics of the quantity $\text{Trace } \mathbf{H}$. In the limiting case of infinitely small density of the impurities, i.e., in the case of "Poisson lattice", it was possible to calculate its distribution function, from which we could conclude that there appears an extremely narrow impurity band centering about the impurity frequency of the lattice containing only one impurity atom, while the distribution of in-band frequencies remains the same as that of regular lattice. Schmidt²⁾ treated the problem by another but similar method and reached the same conclusion. He moreover obtained an approximate expression for the density distribution function of the impurity band.

In the case of finite density of impurities, i.e., for "non-Poisson" lattice, however, it is impossible to obtain such a perspective result. What we could obtain was at most the average eigenfrequency equation $\langle \text{Trace } \mathbf{H} \rangle = 2$. The eigenfrequency spectrum derived from this proved to be identical with that of

"virtual regular lattice" composed of fictitious atoms with masses equal to the average mass of the atoms in the random lattice. Owing to the fact that the quantity $\text{Trace } \mathbf{H}$ is not ergodic, however, we could not conclude that this is just the frequency spectrum of the real random lattice. There remained the possibility that the true average distribution is different from the distribution of virtual regular lattice, and also the possibility that there are fortuitous fluctuations from this distribution.

The latter possibility may readily be excluded by the following argument: The squared eigenfrequencies are given by the eigenvalues of a matrix \mathbf{A} whose diagonal elements are $\alpha_{ii}=2k/m_i$, and nondiagonal ones are $\alpha_{ij}=\hat{\sigma}_{i\pm 1, j} \cdot k/(m_i m_j)^{1/2}$, where m_i is the mass of atom in the i th lattice point, and k is the force-constant which is assumed to be independent of i . If the distribution of the mass of atoms is statistically stationary, matrices corresponding to different samples of the lattice will differ from one another only in permutation of rows and columns, and the secular equation determining the eigenvalues will remain the same for every individual sample. This means that the eigenfrequency distribution itself is ergodic, contrary to the quantity $\text{Trace } \mathbf{H}$. The above argument is indeed not rigorous in the mathematical sense, but physically it can be accepted almost beyond doubt. In fact, in the papers already published on the frequency spectrum of random lattices, its ergodicity was always assumed explicitly or implicitly as an obvious fact. The argument based on perturbation theory also supports this point of view.³⁾

The former possibility, on the contrary, turns out to be highly probable. A calculation of Maradudin et al.,⁴⁾ based on the moment-trace method, shows that in the spectrum of random lattice there appears no such singular peak as in the case of regular lattice, but only a rounded finite peak. Though their calculation is only an approximation, this situation is most probable, since, as is already mentioned in their paper, the randomness will destroy any tendency for large numbers of frequencies to be localized in a small frequency range. A perturbation-theoretic argument shows, on the other hand, that this is not the case and, if the distribution of mass is completely random as in the case treated by Maradudin et al., the spectrum should be identical with that of virtual regular lattice at least within the band.⁵⁾ Since, however, this type of argument skips the problem of convergence and of certain ambiguities in the calculation which is inherent to perturbation procedure, its conclusion is not necessarily true.

Yet we cannot entirely deny the validity of the result of perturbation theory. Firstly, it was shown by Maradudin and Weiss⁵⁾ by a different method that the spectrum should coincide with that of virtual regular lattice for sufficiently low frequencies. Secondly, the result for the Poisson lattice above mentioned suggests that the width of the frequency range in which the spectrum is (approximately) identical with that of virtual regular lattice becomes larger and larger as the concentration of lighter atom becomes small. For sufficiently small concentration the spectrum will differ from that of virtual regular lattice only in a narrow

region at the edge of the band.

In the present paper we first investigate, for isotopic two-component disordered lattices with several concentration ratios, up to what extent the spectrum can be regarded as being identical with that of virtual regular lattice (§ 2 and § 3). We arrive at the result, as expected, that the smaller the concentration of lighter atom, the wider the frequency domain in which the spectrum can be regarded as approximately the same as that of virtual regular lattice.

Now, since the Poisson lattice has, as mentioned above, an impurity band well separated from the main, the non-Poisson lattice should also have an impurity band in so far as the concentration of lighter atoms is sufficiently small, for it is physically reasonable to consider that in such a situation both lattices should show the same behavior, although they are essentially different from the mathematical point of view. Further, it may be conjectured that, when the number of lighter atoms becomes large, the impurity band will gradually merge in the main band and remain merely as a tail of the latter.

It is the second purpose of this paper to investigate how this transition occurs. In principle it is possible to pursue it by the moment-trace method used by Maradudin et al. In order to obtain detailed structures of the spectrum near the edge of the band, however, it will become necessary to calculate considerable number of moments, which involves enormously complex calculations. Here we shall show that, if we utilize the result above obtained, it is possible to pursue the transition, though qualitatively, by calculating only a moderate number of moments. The method consists in replacing the spectrum in the region up to a certain frequency ω_m by that of virtual regular lattice and applying the moment-trace method to the remaining frequency region. When the concentration of lighter atoms is sufficiently small, the latter region becomes so narrow that we can obtain a considerable amount of information as to the fine structure of the spectrum by calculating only a moderate number of moments.

The calculation was carried out for the case of mass ratio 3:2 and for several concentration ratios, using the moments up to the 12th. The result shows that, when the number of lighter atoms becomes equal to that of the heavier ones, there appears an indication of an impurity band. Its separation from the main band becomes more and more distinct as the lighter atoms become less abundant, and the spectrum approaches that of Poisson lattice.

In a recent work,⁶⁾ of which we were not aware until our calculation was almost finished, Domb et al. computed the spectra of one-dimensional isotopic random lattices by the orthodox moment-trace method, and obtained similar results to ours. We shall make some remarks on their work in the concluding section.

§ 2. Moments of the squared-frequency distribution

Let M_1 and M_2 ($M_1 > M_2$) be the masses of the two components, and $p(M_1)$ and $p(M_2)$ their probabilities of occupation. We consider here only the case in

which the distribution of mass is completely random, i.e., there is no correlation between masses of atoms occupying different sites. Then the distribution can be described by these probabilities only. We denote the average mass by $\langle M \rangle$, average reciprocal mass by $\langle 1/M \rangle$, etc. The force-constant k is assumed for simplicity to be unity.

As is well known, the i th moment \mathcal{M}_i of the whole squared-frequency distribution is given by the average value of the diagonal elements of \mathbf{A}^i . The calculation is straightforward when the mass distribution is completely random. We quote here only the results up to the 6th moment:

$$\begin{aligned}
 \mathcal{M}_1 &= 2\langle 1/M \rangle, \\
 \mathcal{M}_2 &= 2\langle 1/M \rangle^2 + 4\langle 1/M^2 \rangle, \\
 \mathcal{M}_3 &= 8\langle 1/M^3 \rangle + 12\langle 1/M^2 \rangle \langle 1/M \rangle, \\
 \mathcal{M}_4 &= 4\langle 1/M \rangle^2 \langle 1/M^2 \rangle + 32\langle 1/M \rangle \langle 1/M^3 \rangle + 18\langle 1/M^2 \rangle^2 + 16\langle 1/M^4 \rangle, \\
 \mathcal{M}_5 &= 20\langle 1/M^2 \rangle^2 \langle 1/M \rangle + 20\langle 1/M^3 \rangle \langle 1/M \rangle^2 + 80\langle 1/M^4 \rangle \langle 1/M \rangle \\
 &\quad + 100\langle 1/M^3 \rangle \langle 1/M^2 \rangle + 32\langle 1/M^5 \rangle, \\
 \mathcal{M}_6 &= 156\langle 1/M \rangle \langle 1/M^2 \rangle \langle 1/M^3 \rangle + 6\langle 1/M^2 \rangle^2 \langle 1/M \rangle^2 + 72\langle 1/M^4 \rangle \langle 1/M \rangle^2 \\
 &\quad + 192\langle 1/M^5 \rangle \langle 1/M \rangle + 264\langle 1/M^4 \rangle \langle 1/M^2 \rangle + 146\langle 1/M^3 \rangle^2 \\
 &\quad + 24\langle 1/M^2 \rangle^3 + 64\langle 1/M^6 \rangle.
 \end{aligned} \tag{2.1}$$

The frequency spectrum of virtual regular lattice composed of atoms with mass $\langle M \rangle$ is known to be

$$f_0(\omega) = (2/\pi) (\omega_L^2 - \omega^2)^{-1/2}, \tag{2.2}$$

where $\omega_L = 2/\langle M \rangle^{1/2}$ is the extreme frequency at the band-edge, and consequently the squared-frequency distribution function is

$$g_0(\rho) = (1/\pi) (\rho \rho_L - \rho^2)^{-1/2}, \tag{2.3}$$

where $\rho = \omega^2$ and $\rho_L = \omega_L^2 = 4/\langle M \rangle$ (extreme squared-frequency).

Now assume that in the frequency domain $\mathcal{D} \equiv (0, \omega_m)$ the frequency- and squared-frequency-distribution functions of random lattice, $f(\omega)$ and $g(\rho)$, are identical with $f_0(\omega)$ and $g_0(\rho)$, respectively. Then the average value of ρ in \mathcal{D} is given by

$$\mathcal{N}_1 \equiv \int_0^{\rho_m} \rho g(\rho) d\rho \bigg/ \int_0^{\rho_m} g(\rho) d\rho = \frac{2}{\pi} \int_0^{\omega_m} d\omega \frac{\omega^2}{(\omega_L^2 - \omega^2)^{1/2}} \bigg/ A(\omega_m), \tag{2.4}$$

where $\rho_m = \omega_m^2$, and

$$A(\omega_m) \equiv \frac{2}{\pi} \int_0^{\omega_m} \frac{d\omega}{(\omega_L^2 - \omega^2)^{1/2}} \tag{2.5}$$

is the fraction of the frequencies in \mathcal{D} . It is clear, on the other hand, that

$$\mathcal{M}_1 = \int_0^{\rho_m} \rho g(\rho) d\rho + \int_{\rho_m}^{\rho_{\max}} \rho g(\rho) d\rho, \quad (2.6)$$

where $\rho_{\max} = \omega_{\max}^2$ is the upper limit of the squared-frequency the value of which will be given later.

From (2.4) and (2.6) we obtain

$$(\mathcal{H}_1 - \mathcal{M}_1) / [1 - A(\omega_m)] = \mathcal{H}_1 - \int_{\rho_m}^{\rho_{\max}} \rho g(\rho) d\rho / [1 - A(\omega_m)]. \quad (2.7)$$

The second term of the right-hand side is no other than the average value of ρ in the frequency domain $\mathcal{E} \equiv (\omega_m, \omega_{\max})$. Denoting this by \mathcal{L}_1 we have

$$\mathcal{L}_1 = [\mathcal{M}_1 - \mathcal{H}_1 A(\omega_m)] / [1 - A(\omega_m)], \quad (2.8)$$

from which (using (2.4)) we can calculate \mathcal{L}_1 as a function of ω_m .

The same relation is obtained also for the p th moment of ρ in \mathcal{D} and \mathcal{E} , i.e., denoting them by \mathcal{H}_p and \mathcal{L}_p , respectively, we have

$$\mathcal{L}_p = [\mathcal{M}_p - \mathcal{H}_p A(\omega_m)] / [1 - A(\omega_m)]. \quad (2.9)$$

Putting $\omega = \omega_L \sin \varphi$, $\omega_m = \omega_L \sin(\pi/2 - \theta) = \omega_L \cos \theta$, we get from (2.4) and (2.5)

$$\mathcal{H}_1 A(\omega_m) = \frac{2\rho_L}{\pi} \int_0^{\pi/2-\theta} \sin^2 \varphi d\varphi = \frac{2}{\pi \langle M \rangle} (\pi - 2\theta - \sin 2\theta) \quad (2.10)$$

and

$$A(\omega_m) = 1 - 2\theta/\pi. \quad (2.11)$$

In the same way we have also

$$\begin{aligned} \mathcal{H}_2 A &= \frac{2\rho_L^2}{\pi} \int_0^{\pi/2-\theta} \sin^4 \varphi d\varphi = \frac{3}{\langle M \rangle} \mathcal{H}_1 A - \frac{2}{\pi \langle M \rangle^2} \cdot 4 \cos^3 \theta \sin \theta, \\ \mathcal{H}_3 A &= \frac{2\rho_L^3}{\pi} \int_0^{\pi/2-\theta} \sin^6 \varphi d\varphi = \frac{10}{3 \langle M \rangle} \mathcal{H}_2 A - \frac{2}{\pi \langle M \rangle^3} \cdot \frac{32}{3} \cos^5 \theta \sin \theta, \\ \mathcal{H}_4 A &= \frac{2\rho_L^4}{\pi} \int_0^{\pi/2-\theta} \sin^8 \varphi d\varphi = \frac{7}{2 \langle M \rangle} \mathcal{H}_3 A - \frac{2}{\pi \langle M \rangle^4} \cdot 32 \cos^7 \theta \sin \theta, \\ \mathcal{H}_5 A &= \frac{2\rho_L^5}{\pi} \int_0^{\pi/2-\theta} \sin^{10} \varphi d\varphi = \frac{18}{5 \langle M \rangle} \mathcal{H}_4 A - \frac{2}{\pi \langle M \rangle^5} \cdot \frac{512}{5} \cos^9 \theta \sin \theta, \\ \mathcal{H}_6 A &= \frac{2\rho_L^6}{\pi} \int_0^{\pi/2-\theta} \sin^{12} \varphi d\varphi = \frac{11}{3 \langle M \rangle} \mathcal{H}_5 A - \frac{2}{\pi \langle M \rangle^6} \cdot \frac{1024}{3} \cos^{11} \theta \sin \theta. \end{aligned} \quad (2.12)$$

§ 3. Estimation of cutting frequency ω_m

Let us next estimate the cutting frequency ω_m up to which the assumption that the spectrum is approximately the same as that of virtual regular lattice is valid. From the theorem⁷⁾ that eigenfrequencies never exceed the maximum frequency $4/M_2$ of the regular lattice composed of lighter atoms only, we obtain a condition

$$\mathcal{L}_1 \leq 4/M_2. \quad (3.1)$$

Using (2.8), (2.11) and (2.12), (3.1) becomes

$$2\theta[2\langle M \rangle/M_2 - 1] - \sin 2\theta > \pi[\langle M \rangle \langle 1/M \rangle - 1]. \quad (3.2)$$

Since the left-hand side tends to zero as $\theta \rightarrow 0$, this inequality gives θ a lower limit θ_l , and hence gives ω_m an upper limit ω_l .

Table I

Concentration ratio	Estimate of θ_l
$p(M_1)=0.4, \quad p(M_2)=0.6$	9°
$p(M_1)=0.5, \quad p(M_2)=0.5$	7°
$p(M_1)=0.9, \quad p(M_2)=0.1$	$1^\circ 30'$

Table I shows rough estimates of θ_l for three concentration ratios in the case of $M_1=1.2$ and $M_2=0.8$. We can readily infer therefrom that we should take smaller values of ω_m as the number of lighter atoms gets larger. This is a natural consequence in view of the supposition stated in the Introduction, and will be further confirmed below. Similar conditions must hold also for higher moments:

$$\mathcal{L}_p \leq (4/M_2)^p. \quad (3.3)$$

A simple calculation shows, however, that they impose only weaker restrictions on ω_m than (3.1).

Now there is another condition to be considered: The central moments in \mathcal{C} , i.e., the moments about \mathcal{L}_1 , must have the values which are consistent with one another. For example, the moments of even order cannot be negative, and the fourth moment cannot be too large compared with the second moment (in consideration of the width of the interval \mathcal{C}), etc. We surmise that, in general, if we can select an appropriate θ -value in the region in which the central moments vary reasonably with θ , maintaining the values consistent with one another, it will give a justifiable value of ω_m . As will be seen below, this condition imposes much more severe restriction on ω_m than the condition (3.2).

Now that we need not here make distinction between absolute and ordinary moments, there are still other conditions⁸⁾ to be satisfied:

$$\mathcal{L}_p^{p+1} \leq \mathcal{L}_{p+1}^p, \quad p=2, 3, 4, \dots \quad (3.4)$$

Some calculations show, however, that these conditions are much less restrictive than the one stated above. In the following therefore we leave the conditions (3.2), (3.3) and (3.4) out of account.

The central moments μ_p can be calculated from \mathcal{L}_p by the formulas :

$$\begin{aligned}\mu_2 &= \mathcal{L}_2 - \mathcal{L}_1^2, \\ \mu_3 &= \mathcal{L}_3 - 2\mathcal{L}_1\mathcal{L}_2 + 3\mathcal{L}_1^3, \\ \mu_4 &= \mathcal{L}_4 - 4\mathcal{L}_2\mathcal{L}_1 + 6\mathcal{L}_1^2\mathcal{L}_2 - 3\mathcal{L}_1^4, \\ \mu_5 &= \mathcal{L}_5 - 5\mathcal{L}_4\mathcal{L}_1 + 10\mathcal{L}_3\mathcal{L}_1^2 - 10\mathcal{L}_2\mathcal{L}_1^3 + 4\mathcal{L}_1^5, \\ \mu_6 &= \mathcal{L}_6 - 6\mathcal{L}_5\mathcal{L}_1 + 15\mathcal{L}_4\mathcal{L}_1^2 - 20\mathcal{L}_3\mathcal{L}_1^3 + 15\mathcal{L}_2\mathcal{L}_1^4 - 5\mathcal{L}_1^6.\end{aligned}\quad (3.5)$$

Figs. 1-4 show some of the results for the case of $M_1=1.2$ and $M_2=0.8$. The curves give the values of μ_p 's as functions of $\cos \theta = \omega_m/\omega_L$, i.e., of the ratio of the width of \mathcal{D} to that of the band of virtual regular lattice. From Fig. 1, which shows the moments of even order for the case of $p(M_1)=0.4$, we see that when $\cos \theta$ and hence the width of \mathcal{D} becomes large, all moments become negative. This is the result as it should be, since, if the actual distribution has a rounded peak with a tail outside the band, instead of an infinite peak at the edge of the band, and if we take ω_m too near to the edge, the value of $A(\omega_m)$ will be estimated to be too large and hence, by (3.5) and (2.9), the value of moments of even order will be estimated to be too small. In the region about $\cos \theta \sim 0.76$, we may well suppose that the moments are varying reasonably, maintaining the values consistent with one another. This value of θ , however, gives ω_m which is far from the edge of the band.

Fig. 2 corresponds to the case $p(M_1)=0.5$. Here also the moments tend to be negative for large $\cos \theta$, but the width of the region of negative

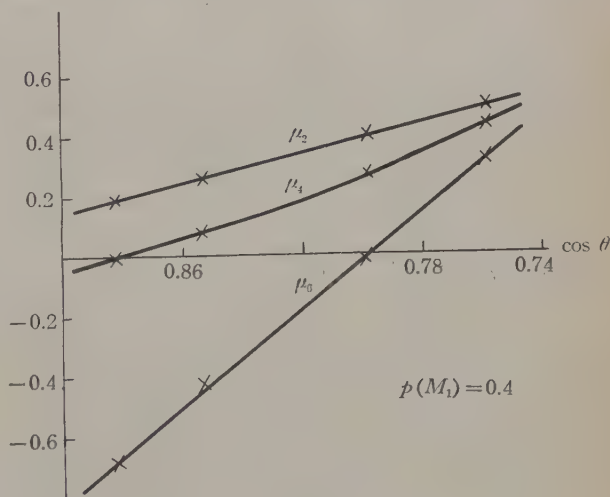


Fig. 1

moments is much smaller than in Fig. 1, and we can take ω_m much nearer to the band-edge ($\cos \theta = 0.90 \sim 0.87$). Fig. 3 shows the moments of odd order and \mathcal{L}_1-4 , i.e., the distance of \mathcal{L}_1 from the band-edge ($\rho_L=4$). We see that there occurs alteration of sign of μ_3 and μ_5 at about $\mathcal{L}_1=4$, which is naturally expected if the actual distribution is to have a rounded maximum in place of an infinite peak.

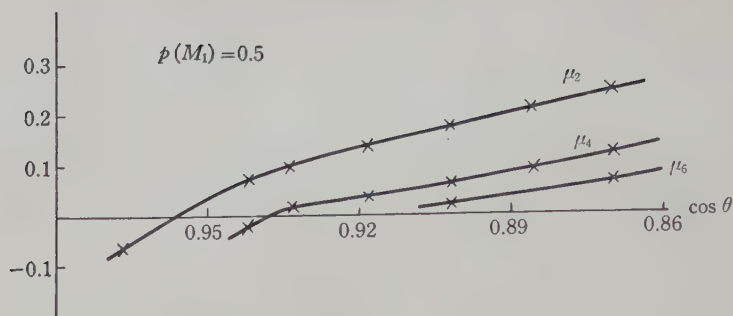


Fig. 2

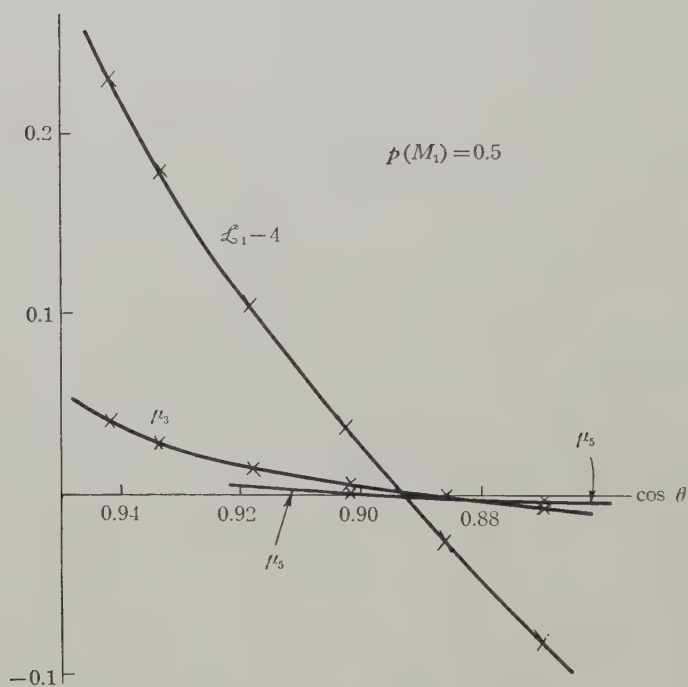


Fig. 3

Thus we are amply justified in choosing ω_m corresponding to $\cos \theta = 0.90 \sim 0.87$.

From the above results we may well expect that we can take ω_m nearer to the band-edge, as $p(M_1)$ becomes larger, i.e., as the concentration of lighter atom becomes smaller. In fact, calculation shows that, as illustrated in Fig. 4, the region in which μ_2 is negative diminishes rapidly when the concentration of lighter atom becomes small. It can be easily inferred that higher moments of even order also show the same tendency.

It is remarkable that in the case of $p(M_1) = 0.9$, μ_2 slightly increases before

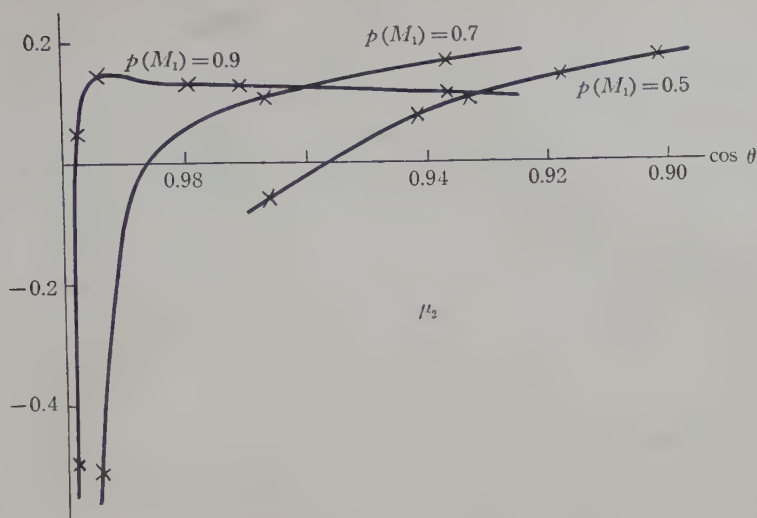


Fig. 4

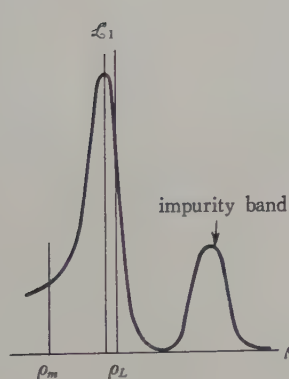


Fig. 5A

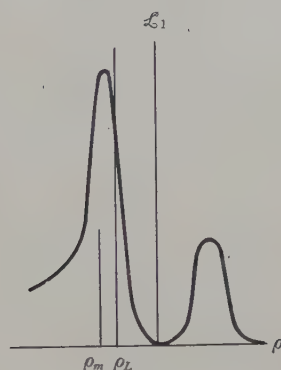


Fig. 5B

it becomes negative. This behavior can be expected from the supposition that the distribution approaches that of Poisson lattice when $p(M_1) \rightarrow 1$: Consider the distribution as shown in Fig. 5. If we take ρ_m comparatively remote from ρ_L , as in Fig. 5A, \mathcal{L}_1 will lie within the peak of the distribution and the dispersion about it will be small, whereas if we take ρ_m nearer to the edge, as in Fig. 5B, \mathcal{L}_1 will lie outside the peak so that the dispersion about it rather becomes larger. For higher moments such an effect will manifest itself more conspicuously, that is, it will become appreciable even at smaller $p(M_1)$. That this is actually the case is shown by Fig. 6, where we see that μ_6 increases, while μ_2 and μ_4 decrease, with increasing $\cos \theta$ in the frequency region indicated in the figure.

Thus all the results shown in Figs. 1-4 suggest that the actual distribution has a rounded maximum in place of an infinite peak and, moreover, is accompanied

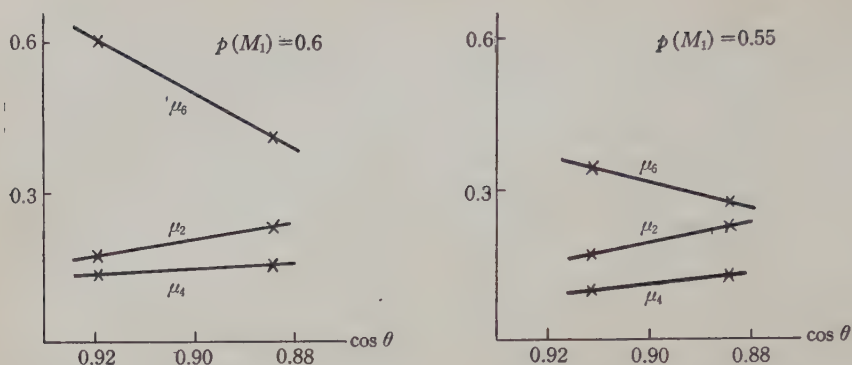


Fig. 6

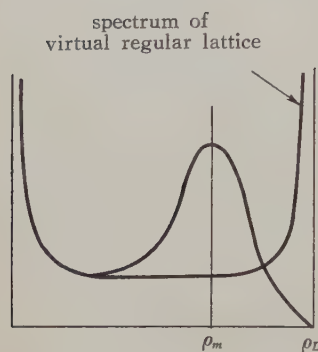


Fig. 7

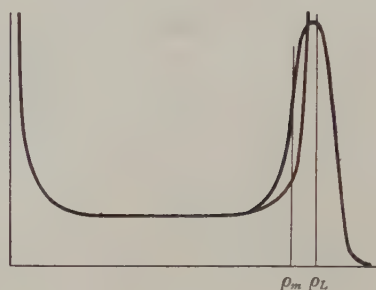


Fig. 8

by an impurity band. It is to be noted here that there is another possibility of explanation for the behavior mentioned above of the moments of even order: If the actual distribution be of the form as in Fig. 7, a situation reverse to the one mentioned in the explanation of Fig. 1 may arise, so that $A(\omega_m)$ will be estimated to be too small. Then the moments of even order can increase with $\cos \theta$. Since in the case of large $p(M_1)$ such a situation can arise near the band-edge (Fig. 8), the increase of even moments found above may partly be ascribed to this origin. To avoid this effect we should perhaps take ω_m slightly less than the value estimated in the manner explained above.

If we take into account the higher moments than the 6th, the upper limit of ω_m will come out smaller, finally tending to zero. This is natural, since the larger the number of moments we take into account, the more exactly the distribution can be determined, so that the deviation from the spectrum of virtual regular lattice which comes into question becomes smaller. As far as we use a relatively small number of moments, the upper limit of ω_m will remain finite, and it may be allowed, as long as $p(M_1)$ is sufficiently large, to replace the spectrum by that of virtual regular lattice, in the corresponding degree of approximation, up to a fairly large frequency near the band-edge.

§ 4. Calculation of the squared-frequency distribution

From the results obtained above we can draw a somewhat detailed information as to the form of frequency distribution in the neighborhood of the band-edge, especially as to the character of the impurity band, by the aid of moment-trace method involving only a moderate number of moments (here, up to the 12th moment of ω , or the 6th moment of ρ).

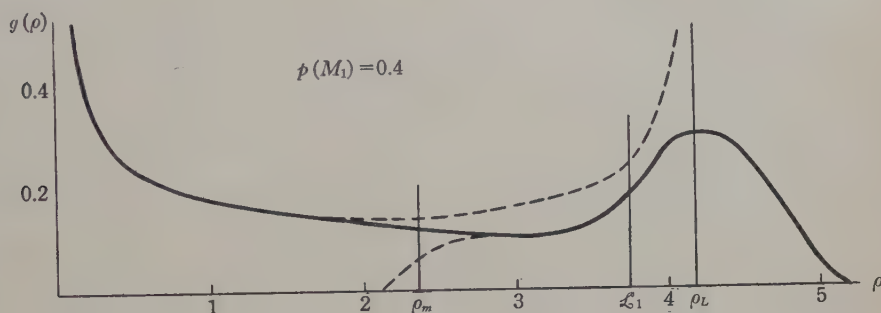


Fig. 9

The calculation proceeds as follows: Taking the results of the previous section into consideration, choose an appropriate value of ω_m , and calculate the moments

\mathcal{L}_p , then the central moments μ_p , by using (2.9), (2.11), (2.12), (2.1) and (3.4). Next, presupposing that the distribution in \mathcal{E} is not much different from the normal distribution as far as the concentration of lighter atom is not too small, use the Gram-Charlier series⁽⁸⁾ which starts from the normal distribution as the zeroth approximation. Then we have for the distribution function of the standardized variable $x = (\rho - \mathcal{L}_1)/\sigma$ ($\sigma = \mu_2^{1/2}$)

$$g(x) = \varphi(x) + c_3 \varphi^{(3)}(x)/3! + c_4 \varphi^{(4)}(x)/4! + \dots, \quad (4.1)$$

where $\varphi(x)$ is the normal distribution function and $\varphi^{(p)}(x)$ is its p th derivative, which is given in terms of Hermite polynomial $H_p(x)$ as

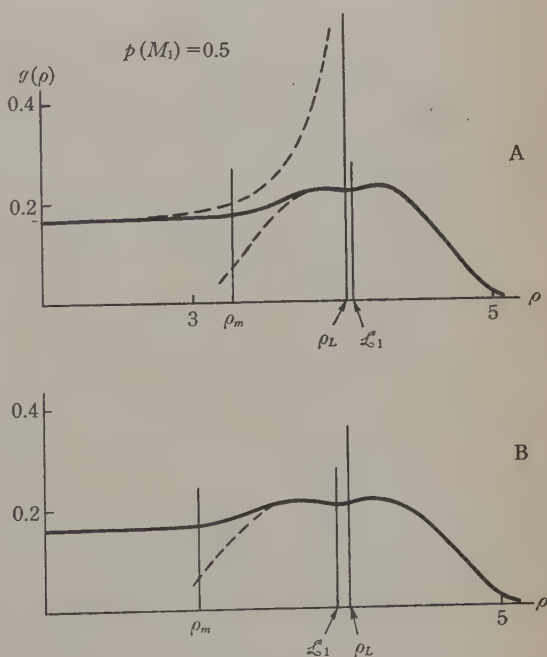


Fig. 10

$$\varphi^{(p)}(x) = (-1)^p H_p(x) \varphi(x).$$

The coefficients c_p 's are calculated from μ_p as⁸⁾

$$c_3 = -\mu_3/\sigma^3,$$

$$c_4 = \mu_4/\sigma^4 - 3,$$

$$c_5 = -\mu_5/\sigma^5 + 10\mu_3/\sigma^3,$$

$$c_6 = \mu_6/\sigma^6 - 15\mu_4/\sigma^4 + 30.$$

(4.2)

We see that c_3 is just the negative of skewness, and c_4 is the excess itself.

We choose ω_m as near as possible, but not too extremely near, to the edge of the band of virtual regular lattice, in order to reduce as much as possible the effect mentioned at the end of § 3, which may possibly come into play.

Figs. 9–13 show some results obtained for the case of $M_1=1.2$ and $M_2=0.8$. Here the upper limit of the squared frequency is $\rho_{\max}=5$. The broken line which

extends upwards from the neighborhood of $\rho=\rho_m$ indicates the spectrum of virtual regular lattice, and that which extends downwards is the tail of the distribution in \mathfrak{L} obtained by calculation. The solid curve is obtained by smoothly connecting the distribution curve of virtual regular lattice and the calculated curve in \mathfrak{L} . (Such an arbitrariness is inevitable in a rough approximation as we used.) Corresponding values of \mathcal{L}_1 , μ_p ($p>1$) and c_p ($p>2$) are tabulated in Table II.

These figures clearly illustrate how the spectrum approaches that of Poisson lattice as the concentration of lighter atom becomes small. When the concentration of lighter atoms is larger than that of heavier one, there is only one (presumably) rounded maximum at the edge of the band

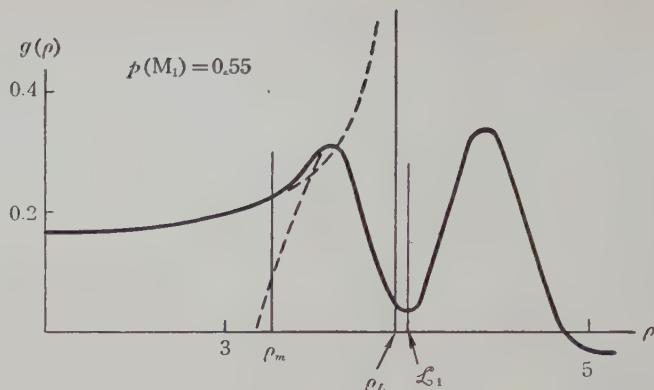


Fig. 11

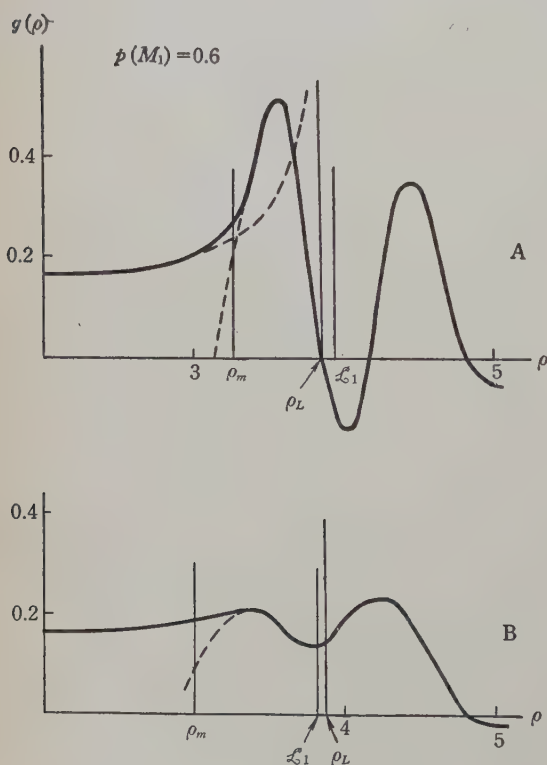


Fig. 12

(Fig. 9), whereas when they become equal, there appears an indication of an impurity band (Fig. 10), and its separation from the main band becomes rapidly more and more distinct as the lighter atoms become less abundant (Figs. 11–13). These results are rather remarkable, in view of the degree of approximation here employed, although perhaps the curves represent the actual form of distribution only qualitatively.

In Figs. 10 and 12, A and B show the results for two somewhat different choices of ρ_m . In Fig. 10 they differ only slightly, whereas in Fig. 12 there appears a marked difference, which is due to the large differences of higher coefficients c_p (Table II). This phenomenon can be explained by the fact that in Fig. 12A \mathcal{L}_1 lies outside the band of virtual regular lattice, while in Fig. 12B it lies inside the band.

A similar situation as stated in the explanation of Fig. 5 can then be brought about, so that the values of these coefficients may come out to be considerably different. This does not mean that our procedure is useless, but means only that in our approximation, which uses only six first moments, the form of distribution can be reproduced only very roughly when \mathcal{L}_1 comes to such a position as in Fig. 5A. The sensitivity of the form of calculated curves to the choice of ω_m rather enables one to infer with better reliability the structure of the spectrum in the domain \mathcal{E} . (From this it may also be inferred that if we had used the moment-trace method for the whole frequency region, we could not have obtained such an amount of information as above about the detailed structure of the distribution in the neighborhood of band-edge, unless we computed enormous number of moments.) It is supposed that, in the case of Fig. 10, we have no such situation owing to the absence of impurity band. Consequently the form of calculated curve depends so little on the position of \mathcal{L}_1 , that the actual distribution is reproduced pretty well irrespective of the choice of ω_m . For the same reason the curve in Fig. 9 may also be regarded as a fairly good representation of the actual distribution, although here \mathcal{L}_1 is much smaller than ρ_L .

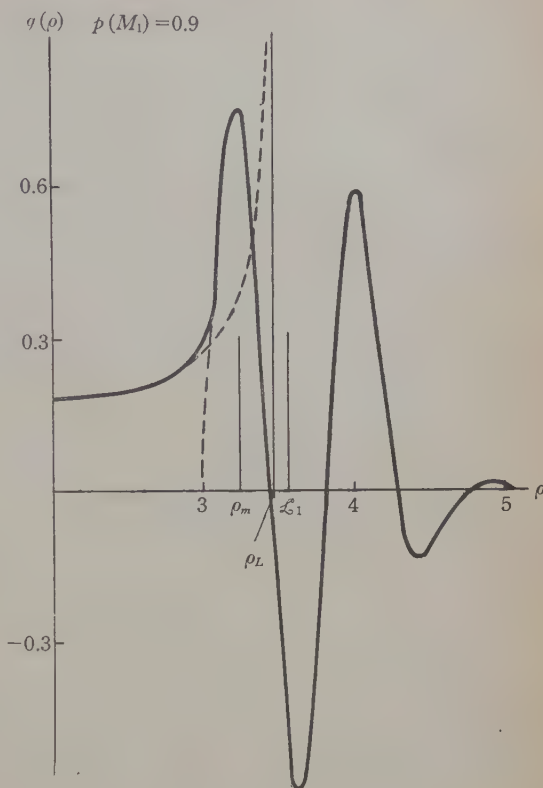


Fig. 13

Table II

	$p(M_1)=0.4$	$p(M_1)=0.5$ A	$p(M_1)=0.5$ B	$p(M_1)=0.55$	$p(M_1)=0.6$ A	$p(M_1)=0.6$ B	$p(M_1)=0.9$
\mathcal{L}_1	3.72295	4.03824	3.91805	3.99280	3.94413	3.80487	3.55320
μ_2	0.49576	0.17503	0.24521	0.17235	0.17033	0.23566	0.12080
μ_3	-0.04905	0.00567	-0.00274	-0.00776	-0.02143	-0.01054	-0.03409
μ_4	0.43482	0.06251	0.12026	0.10161	0.13663	0.15755	0.12111
μ_5	-0.04360	0.00258	-0.00212	-0.10749	-0.20126	-0.11382	-0.21154
μ_6	0.30780	0.02365	0.06715	0.34124	0.60197	0.41089	0.57357
c_3	0.14051	-0.07741	0.02255	0.10852	0.2048	0.09211	0.81190
c_4	-1.23088	-0.95955	-0.99996	0.42058	1.7093	-0.16300	5.29856
c_5	-1.15310	-0.57250	-0.15448	-0.97769	14.7600	-0.53026	33.58566
c_6	5.98924	3.80402	4.55397	45.33970	81.1665	18.84190	230.865

This is the reason why we adopted such values of ω_m that \mathcal{L}_1 becomes outside the band in all cases but in Fig. 9.

§ 5. Concluding Remarks

It may be said that the qualitative behavior of the spectrum of one-dimensional isotopic lattice has thus been revealed to a considerable extent. Now some discussions about the nature of approximation will be in order.

The convergence of Gram-Charlier series becomes worse when the difference between the distribution to be inferred and the normal one comes out large. In Fig. 12A there appear regions in which the value of distribution density is negative. In Fig. 13 this becomes much more drastic, and moreover the curve exhibits a violent oscillation. This means that the convergence of the series rapidly becomes bad when the concentration of lighter atoms becomes small. This situation naturally is expected, since the spectrum approaches that of Poisson lattice as $p(M_2) \rightarrow 0$, which is highly singular. Strictly speaking, therefore, we cannot use the Gram-Charlier series in these cases, and must have recourse to some other method. In the case of Fig. 12B, however, we may still consider that the curve reproduces the actual form of the spectrum qualitatively. Even in the case of Fig. 13, the curve may be deemed to suggest that the actual distribution becomes quite singular and markedly different from the normal one, although the form of the curve itself is almost meaningless. Anyhow, it would perhaps be difficult to reproduce such a singular distribution with a good approximation, in so far as we follow the moment-trace method.*

Our results cannot be compared directly with those of Domb et al., owing to

* Domb et al.⁶⁾ obtained the curves similar to Fig. 13 for certain cases, and tried to explain the appearance of "two impurity bands" by a physical argument. In connection with the above remark, however, we cannot put too much confidence in such an argument.

the difference in the adopted values of constants. Inspection of the curves shows, however, that their general qualitative conclusions agree with ours. For example, it can be concluded also from our results that, as the concentration of light masses increases, the subsidiary peak (impurity band) becomes more and more dominant, before it merges into the main band. It seems that their results are in general not so excellent compared with ours, as expected from the large number of moments they used. This is probably because they used the moment-trace method for the whole frequency region. The "delta-function method", which they used at the same time, does not play such an essential role as our procedure of band-cutting, but only a subsidiary role of correcting the spectrum in the low frequency region. The result of this correction merely indicates that it is a good approximation to regard the spectrum as approximately the same as that of virtual regular lattice up to fairly high frequencies, the situation which makes the starting point of our procedure.

It may thus be concluded that although our method contains some ambiguities in the estimation of the cutting frequency ω_m and, as a result, in the interpretation of the resulting curves, this deficiency can be compensated to a large extent by the remarkable simplicity of calculation, provided sufficient care is taken in dealing with the ambiguities. Conversely, if we do not mind carrying out a complex calculation, it will be possible to obtain much more information than hitherto obtained as to the detailed structure of the spectrum, by formulating our method more rigorously and using it more systematically.

The author finally wishes to express his sincere thanks to Prof. T. Tanaka of Kyushu University for his continual encouragement, and to Mr. Asahi for several helpful discussions. He is also indebted to the Ministry of Education for the Scientific Research Fund.

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S-Wave Pion-Nucleon Scattering

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Low energy S -wave pion-nucleon scattering is re-investigated based on the following two assumptions:

I) Dilute nucleon-antinucleon pair extends to a range about $r_0 \sim (2\mu)^{-1}$; μ being a pion mass.

II) Pion-pion interaction with attractive force contributes to the S -wave pion-nucleon scattering.

Main characteristic features of S -wave pion-nucleon scattering are shown to be reproduced under these assumptions and at the same time reason of difficulties inherent in S -wave scattering is made clear in terms of potential scattering.

§ 1. Introduction

Small magnitudes of the phase shifts and their relatively large isotopic spin dependence, as is well known, are the characteristic features of low energy S -wave pion-nucleon scattering, for which no satisfactory explanation has yet been given from pion field theory.¹⁾

Two different approaches may be possible to this problem; the first is to notice that dispersion theory yields the relation²⁾

$$\Delta(\mu) = \frac{4}{\mu^3} \left(\frac{f^2}{4\pi} \right) + \frac{1}{2\pi^2} \int_{\mu}^{\infty} \frac{d\omega}{k} [\sigma_-(\omega) - \sigma_+(\omega)], \quad (1.1)$$

where

$$\Delta(\mu) = (a_- - a_+)/\mu = 2(a_1 - a_3)/3\mu.$$

The first term of the right-hand side of Eq. (1.1) represents the second order perturbation result with renormalized coupling constant and the contribution from the second term may be interpreted as a recoil correction which somewhat suppresses the contribution from the first term. We put numerically

$$\text{the right-hand side of Eq. (1.1)} = 0.32 \frac{1}{\mu^2} - 0.14 \frac{1}{\mu^2}. \quad (1.2)$$

Eq. (1.2) quite well reproduces Orear's experimental formula.³⁾ This means that at least the difference of scattering length is calculable within the present quantum field theory. To obtain their phase shifts separately, we must know their average, $\frac{1}{2}(a_- + a_+) = (a_1 + 2a_3)/3$. Analogous formula to Eq. (1.1), however, does not

seem to hold since the corresponding integral for this case does not converge. In this respect, it is very interesting to speculate that both scattering lengths a_1 and a_3 cannot in fact be calculated, only their difference $a_1 - a_3$ being calculable within the framework of the present quantum field theory. From this dispersion theoretic point of view, at least an additional parameter $(a_1 + 2a_3)/3$ should be added to the fundamental quantities M , μ , f^2 and λ (pion-pion coupling constant) to determine dynamical properties of pion-nucleon system. Similar speculation has been made by K. Symaznik⁴⁾ in connection with the renormalization problem in β -decay interaction.

Another approach, which is based upon the usual Hamiltonian formalism, is to suppose that difficulties inherent in the S -wave problems do not lie in our insufficient technique in solving the equation but in lacking our knowledge about the nature of the interaction. That is, if a suitable form of the interaction Hamiltonian is taken, both phase shifts δ_1 and δ_3 should be calculated to reproduce experimental data at least qualitatively.

In this paper, we shall discuss, standing on the latter point of view, that main features of the S -wave phase shifts can be reproduced under the following assumptions.

(I) The source of S -wave pions is extended to a range $r_0 \approx (2\mu)^{-1}$, μ being the pion mass.

(II) Attractive pion-pion interaction ($\lambda(\varphi \cdot \varphi)^2$; $\lambda < 0$) is important for the S -wave pion-nucleon scattering.

Assumption (I) means that the core of the nucleon which is probably made up of nucleon-antinucleon pairs is a very dilute quantity which spreads out to a large extent. In this paper we shall not touch the question why the nucleon core is so much extended. We merely point out that such an extended core is consistent with the conclusion from the electromagnetic structure of the nucleon⁵⁾ and also from the analysis of the nucleon-antinucleon interaction.⁶⁾ It will be noticed that the extended nucleon core ($\approx 1/2\mu$) does not necessarily contradict the momentum dependence of the phase shift.

As for assumption (II), the contribution from pion-pion interaction to the S -wave pion-nucleon scattering has already been discussed by several authors⁷⁾ and the effect of this interaction has been shown to be small for appropriate magnitude of the coupling constant, the reason of which will be made clear later. This interaction, however, gives considerable contribution to the S -wave pion-nucleon scattering when combined with assumption (I).

In § 2, we shall calculate qualitative behaviour of the phase shifts based on assumptions (I) and (II), and § 3 is devoted to the discussion of the result thus obtained.

§ 2: Calculation of the phase shift

In this section, we calculate phase shifts of the S -wave pion-nucleon scattering

based on assumptions (I) and (II). Under these assumptions, a perturbation treatment is expected to become a good approximation on account of a long range and shallow potential as we shall see in § 3. So we make use of the usual Tamm-Dancoff method, neglecting higher configurations to see a qualitative behaviour of the phase shifts.

Effective Hamiltonian can be expressed as

$$H = \frac{g^2}{2M} \left(\int \rho(r) \boldsymbol{\varphi}(r) d^3r \right)^2 + \left(\frac{g}{2M} \right)^2 \boldsymbol{\tau} \left[\int \rho(r) \boldsymbol{\varphi}(r) d^3r \times \int \rho(r) \boldsymbol{\pi}(r) d^3r \right] + \int \boldsymbol{\varphi}(r) V(r) \boldsymbol{\varphi}(r) d^3r \quad (2.1)$$

where $V(r)$ means "potential" induced by pion-pion interaction (Fig. 1).

In the static approximation this can be written as⁹⁾

$$\hat{V}(q^2) = \int V(r) e^{-i\mathbf{q} \cdot \mathbf{r}} d^3r = \frac{-i5\lambda}{2\pi^3} \int \frac{[A(k_0) + 2B(k_0) + C(k_0)] k^4 dk}{(k_0^2 - \omega_k^2)(k_0^2 - \omega_{k-q}^2)} dk_0 \dots, \quad (2.2)$$

$$\hat{V}(0) \cong \left(\frac{\lambda}{4\pi} \right) \left(\frac{f^2}{4\pi} \right) \frac{80}{\mu^2} \int \frac{k^4}{\omega^4} dk.$$

The range of this potential, being of a two-pion exchange type, is $\sim 1/2\mu$.

The first and second terms of the expression (2.1) are the usual Foldy term of the S -wave interaction; the first term contributes as an isotopic spin independent repulsive potential, while the second term contributes as an isotopic spin dependent potential. We take the sign of λ so that the third term is attractive.

In the interest of simplicity of calculation and without losing the essential feature of the problem, we replace the third term of the expression (2.1) by a separable source version

$$H' = \frac{g^2}{2M} \bar{\boldsymbol{\varphi}} \cdot \bar{\boldsymbol{\varphi}} + \left(\frac{g}{2M} \right)^2 \boldsymbol{\tau} [\bar{\boldsymbol{\varphi}} \times \bar{\boldsymbol{\pi}}] + \hat{V}(0) \bar{\boldsymbol{\varphi}} \cdot \bar{\boldsymbol{\varphi}} \quad (2.3)$$

with

$$\bar{\boldsymbol{\varphi}} = \int \rho(r) \boldsymbol{\varphi}(r) d^3r \quad \text{and} \quad V(r) = \hat{V}(0) \rho(r).$$

Here the same function $\rho(r)$ is used for the third term since the range of $V(r)$ is about the same as that of $\rho(r)$.

In this case, the third term induced by the pion-pion interaction amalgamates into the first term of the expression (2.1) to alter the strength of a repulsive potential.

Applying usual Tamm-Dancoff method, we obtain the following result.

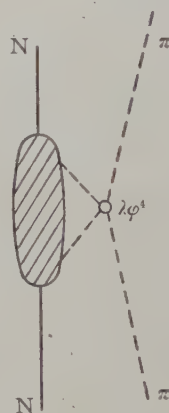


Fig. 1. Contribution of pion-pion interaction to the S -wave pion-nucleon scattering

$$\tan \delta_i = -\frac{k\omega_k}{2\pi} \langle k|T|k \rangle \quad (i=1, 3),$$

$$\langle k|T|k \rangle = \frac{\langle k|t|k \rangle}{1 - \langle k|t|k \rangle^{-1} \frac{P}{2\pi^2} \int dk' \hat{\rho}(k') k'^2 \frac{|\langle k'|t|k' \rangle|^2}{\omega_k - \omega_{k'}}}, \quad (2.4)$$

$$\langle k|t|k \rangle = \frac{g^2}{2M} \frac{1}{\omega_k} \left[1 + \frac{\omega_k}{2M} (\boldsymbol{\tau} \cdot \boldsymbol{\omega}) \right] + \frac{1}{\omega_k} \hat{V}(0),$$

$$(\boldsymbol{\tau} \cdot \boldsymbol{\omega}) = \begin{cases} -2 & i=1 \\ 1 & i=3, \end{cases}$$

$$\hat{\rho}(k) = \int e^{-ikr} \rho(r) d^3r = \begin{cases} 1 & k < A_s \\ 0 & k > A_s \end{cases}$$

where k means incident pion momentum.

Cut off momentum A_s is taken to be 2μ , corresponding to assumption (I). The phase shifts $\delta_i (i=1, 3)$ given by the expression (2.4) are plotted as the

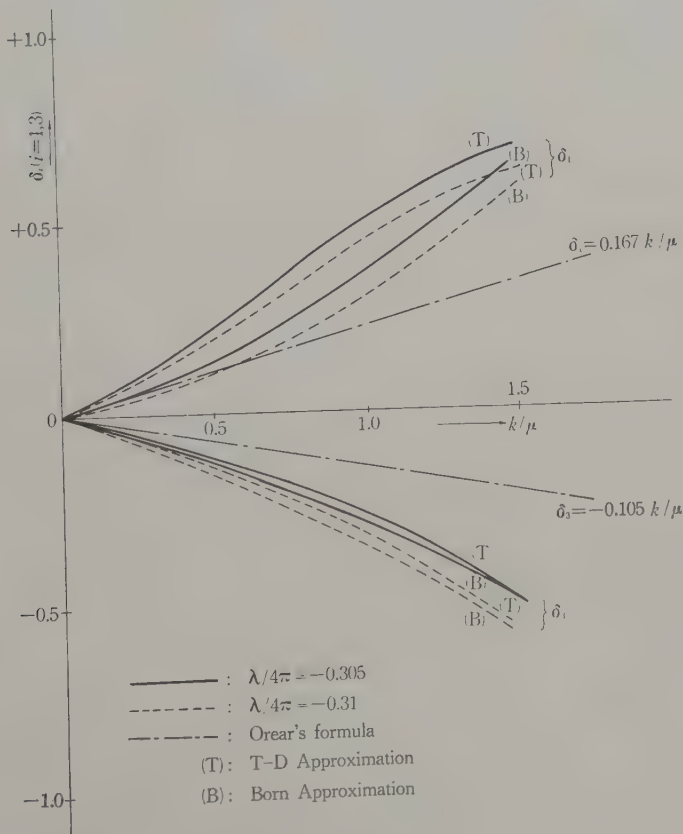


Fig. 2

function of the incident pion energy for appropriate values of λ in Fig. 2. As is seen, the qualitative features of the experimental curves are well reproduced for the value of $\lambda/4\pi \cong -0.3$.* The remaining difference between theory and experiment may be attributed to the recoil effect and could be explained as follows.** The nucleon, due to the emission of virtual P -wave pions, is at some time in the relative P -state with the incoming "S-wave" pion and gives rise to a P -wave interaction. This process in fact gives a negative contribution and this is the physical meaning of the second term of Eq. (1.1) or (1.2).

§ 3. Concluding discussion

In § 2 and § 3, we have investigated the S -wave pion-nucleon scattering based on assumptions (I) and (II). Qualitative agreement with experiment will be rather remarkable, considering the present rough calculation. This circumstance will be understood easily in terms of potential scattering as follows. Usual Hamiltonian for S -wave scattering (first and second terms of the expression (2.1)) corresponds to a repulsive short range potential which leads us to the fact that effect of an isotopic spin dependence becomes necessarily small. This is because the difference in height of the potentials is not much felt by the incident wave on account of a large damping inside the repulsive potential (Fig. 3, (A)). The situation remains

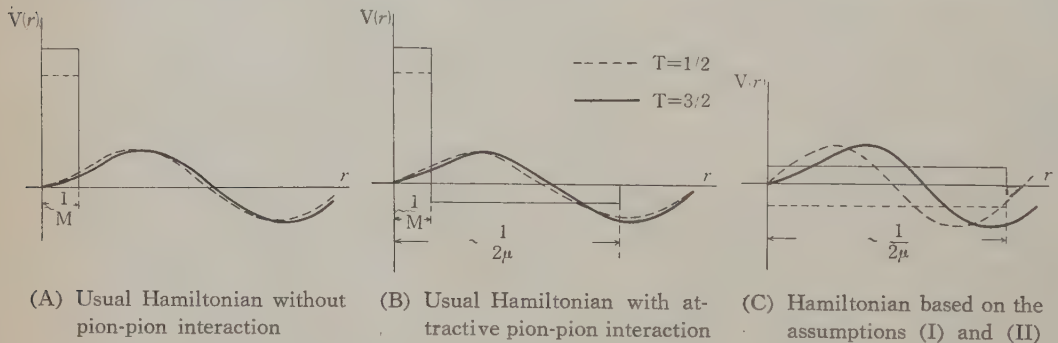


Fig. 3

unaltered also in the presence of the pion-pion interaction because the range of the potential induced by pion-pion interaction is much larger than that of the repulsive potential (Fig. 3, (B)). Assumption (I), however, alters the potential to a

* We have used the expression (2.2), neglecting rescattering correction. Rescattering correction, however, gives a positive contribution to the Born term.⁸⁾ So, numerical value of λ given here should be regarded as a lower limit.

** The authors are indebted to professor G. Takeda for his suggestion of this explanation.

Dr. K. Ishida has pointed out that this difference is due to the τ -spin precession by the virtual P -wave pions.

long range ($r_0 \approx (2\mu)^{-1}$) and shallow shape but their isotopic spin dependence remains the same as before. In addition, on account of assumption (II), the potential for the $T=1/2$ state changes to attractive but remains repulsive for the $T=3/2$ state with an appropriate magnitude of the coupling constant (Fig. 3, (C)). In this case, the Born approximation will become a good approximation because the potential is shallow and long range so that the condition of a validity of Born approximation is fulfilled. In fact, the damping effect is shown to be small as is seen from Fig. 2. These situations are graphically shown in Fig. 3, (A), (B), and (C).

From the treatment of P -wave pion-nucleon interaction we know that cutoff for P -wave pions should be taken as $A_p \approx M$ in contrast to our S -wave cutoff $A_s \approx 2\mu$. This situation may be visualized by a core structure with the extended (to $\approx 1/2\mu$) outer region and a very small (with radius $\approx 1/M$) spinning part. The former is supposed to be the source for S -wave pions while the latter is responsible for the emission or absorption of P -wave pions. This structure also explains that the charge radius of the core is rather large ($\approx 1/2\mu$) while its magnetic moment is not so large ($\approx e/M$).

It is expected that the higher order effect in g^2 is not important since the S -wave cutoff is very small ($A_s \ll M$). Usually radiative corrections due to P -wave pions are not large because of the smallness of P -wave coupling ($(\mu g/2M)^2/4\pi \approx 0.08$). Therefore it is all right to neglect terms of g^4 or higher in Eq. (2.1).

It is clear that sign and magnitude of λ is essential to the present treatment of S -wave problem. Recent investigation⁹⁾ on the pion-pion interaction, based on the dispersion relation approach, will soon provide us a definite information on this point. If λ turns out to be positive, the first standpoint will be more promising.

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Relativistic Rotators and Bilocal Theory

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The notion of relativistic, or "particle" rotator, which is the system of four "beingrößen" centered on a moving point in Minkowski space, has recently been introduced to describe kinematically the average motion of extended particles in space time. In this paper, we further study this system as such and show its similarity with the bilocal theory introduced by Yukawa. The special example of the hyper-spherical rotator is treated in detail by replacing the original beingrößen variables with complex triad variables and relativistic Euler angles.

Introduction

By definition let us call relativistic or "particle" rotator the system kinematically defined,

- a. by the coordinates $x_\mu(\tau)$ of its origin (x) , and
- b. by the set of four orthogonal and unitary four vectors centered on (x) , or, in other words, by a moving tetrad in Minkowski space.¹⁾

The corresponding set of parameters can be used to describe several types of physical problems which, very characteristically, cannot be described on the basis of the point particle model. We shall mention only two.

In a series of recent papers²⁾ it has been shown that the hydrodynamical description of the spinor wave equations of quantum mechanics could not be carried out on the basis of point-like elements. Indeed, in order to describe the existence of an angular momentum density in such waves one has to introduce, alongside the coordinates $x_\mu(\tau)$ of the lines of flow and the invariant conserved density ρ supplementary "beingrößen" kinematical variables b_μ^ξ describing the local "spin".

On the other hand, on the basis of a general model of extended particles proposed by Møller³⁾ and developed by two of us, (D. B. and J. P. V.)⁴⁾ one can show that the motion of average variables can be developed on the basis of the rotator kinematical variables. The motion of the origin of the tetrad just corresponds to the behaviour of a central geometrical point (the so-called "center of matter density") while the space and time like part of the tetrad's instantaneous

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rotation $\omega_{\mu\nu}$ describes the rotation and acceleration of matter in its neighbourhood.

Furthermore, de Broglie and two of us (P. H. and J. P. V.) have proposed the idea that the time has come to substitute relativistic rotators for point particles as classical foundation of quantum mechanics.

In the present paper, we shall make no specific physical assumption about the possible physical signification of the rotator variables and study directly relativistic rotators with the help of the usual Lagrangian and Hamiltonian method.

In section 1 we shall discuss the behaviour of isolated rotators and show that the conservation equations resulting from general Lagrangian imply the existence of a second remarkable point center of a particular inertial frame, showing the similarity of our point of view with Yukawa's bilocal theory⁵⁾ in Minkowski space.

In section 2 we shall discuss a special type of Lagrangian and show in particular that this allows a simple classical interpretation of de Broglie's relation $E = h\nu$.

Finally, in section 3 we shall introduce relativistic Euler angles as internal variables and show that these new variables greatly simplify subsequent quantization of the theory.

§ 1

According to our program let us first recall certain general results on the Lagrangian and Hamiltonian method. Let us make two basic assumptions:

A. That our rotator can be described by:

a) The coordinates $x_\mu(\tau)$ of the origin of the tetrad, where τ represents the proper time of the world line l followed by this point (x).

b) The components b_μ^ξ of the four vectors constituting the tetrad. Let us adopt the usual symbolic conventions. The index μ represents tensor components, varying from one to four (Latin indices representing space components vary only from one to three), and its repetition implies the usual summation. Since we calculate in Minkowski space all fourth components are purely imaginary for real vectors. In b_μ^ξ the index ξ (which also varies from one to four) is not a usual tensor component but rather differentiates the vectors themselves. We thus have three space-like vectors b_μ^ξ and one time-like vector which we write ib_μ^4 . Their orthogonal and unitary character is represented by the relations:

$$b_\mu^\xi b_\nu^\xi = \delta_{\mu\nu}, \quad b_\mu^\xi b_\mu^\eta = \delta^{\xi\eta}. \quad (1a)$$

From b_μ^ξ 's which are functions of the proper time τ of their origin, the instantaneous rotational velocity of the tetrad is defined by the skew tensor:

$$\omega_{\mu\nu} = b_\mu^\xi \dot{b}_\nu^\xi.$$

As we have to assume¹⁾ that ib_μ^4 points along the four velocity of the origin such that

$$\dot{x}_\mu = ic b_\mu^4, \quad (1b)$$

we see that the acceleration of x_μ and the instantaneous rotation of the tetrad in any rest frame Σ (satisfying the relations $b_k^4 = b_k^4 = 0$) are determined by the vectors:

$$\alpha_\mu = \omega_{\mu\nu} b_\nu^4$$

$$\omega_\mu = \tilde{\omega}_{\mu\nu} b_\nu^4 = \frac{i}{2} \varepsilon_{\mu\nu\alpha\beta} \dot{b}_\alpha^\xi b_\beta^\xi b_\nu^4$$

where $\varepsilon_{\mu\nu\alpha\beta}$ is the completely antisymmetrical Ricci-Levi-Civita tensor and the symbol \dot{A} represents the derivative $dA/d\tau$.

B. The second assumption is that its laws of motion can be deduced from a variation principle with a scalar Lagrangian $L(\dot{x}_\alpha, \ddot{x}_\alpha, b_\alpha^\xi, \dot{b}_\alpha^\xi)$ where all variables depend on τ only. This implies that the physical motion between two points M_1 and M_2 will correspond to the minimum of the world-line integral $\int_{M_1}^{M_2} L d\tau$, that is,

$$\delta \int_{M_1}^{M_2} L(\dot{x}_\alpha, \ddot{x}_\alpha, b_\alpha^\xi, \dot{b}_\alpha^\xi) d\tau = 0 \quad (2)$$

for arbitrary variations $\delta x_\mu, \delta b_\alpha^\xi$ which vanish at M_1 and M_2 .

Following Noether⁽⁶⁾ we then see that the corresponding Euler equations:

$$\frac{d}{d\tau} G_\mu = 0 \quad \text{with} \quad G_\mu = \frac{\partial L}{\partial \dot{x}_\mu} - \frac{d}{d\tau} \frac{\partial L}{\partial \ddot{x}_\mu} \quad (3a)$$

and

$$\frac{d}{d\tau} \frac{\partial L}{\partial \dot{b}_\mu^\xi} = \frac{\partial L}{\partial b_\mu^\xi}, \quad (3b)$$

which determine the equations of motion, imply two conservation equations. Indeed (2) must remain unchanged under any variation resulting from ∂x_μ 's and ∂b_μ^ξ 's determined by an arbitrary infinitesimal Lorentz transform $\partial \varepsilon_{\alpha\beta} = -\partial \varepsilon_{\beta\alpha}$. For such a transformation we have

$$\delta x_\mu = \partial \varepsilon_{\mu\nu} x_\nu, \quad \delta b_\mu^\xi = \partial \varepsilon_{\mu\nu} b_\nu^\xi. \quad (4)$$

From the invariance of L with respect to this infinitesimal Lorentz transformation we obtain the relation

$$\dot{M}_{\alpha\beta} = G_\alpha \dot{x}_\beta - G_\beta \dot{x}_\alpha, \quad (5)$$

where

$$M_{\alpha\beta} = \left(b_\alpha^\xi \frac{\partial L}{\partial \dot{b}_\beta^\xi} + \dot{x}_\alpha \frac{\partial L}{\partial \dot{x}_\beta} \right) - \left(b_\beta^\xi \frac{\partial L}{\partial \dot{b}_\alpha^\xi} + \dot{x}_\beta \frac{\partial L}{\partial \dot{x}_\alpha} \right).$$

As is known⁽⁷⁾ relations (3) and (5) represent the two essential relations of conservation of momentum and angular momentum of particles with extended structures.

Starting from A. and B., the next step is to pass to Hamiltonian formalism. If L depends in general on a certain set of kinematical variables q_μ^ξ (where α denotes the usual components and ξ differentiates the variables) so that $L=L(q_\alpha^\xi, \dot{q}_\alpha^\xi)$ we can introduce the set of canonical variables Π_α^ξ by the relations

$$\Pi_\alpha^\xi = \partial L / \partial \dot{q}_\alpha^\xi$$

and define a relativistic Hamiltonian H by the expression

$$H(\Pi_\alpha^\xi, q_\alpha^\xi) = \Pi_\alpha^\xi \dot{q}_\alpha^\xi - L(q_\alpha^\xi, \Pi_\alpha^\xi) \quad (6)$$

where we have expressed in $L(q_\alpha^\xi, \dot{q}_\alpha^\xi)$ \dot{q}_α^ξ in terms of Π_α^ξ . The equations of motion can then be written in the well-known form:

$$\dot{\Pi}_\alpha^\xi = -\partial H / \partial q_\alpha^\xi \quad \text{and} \quad \dot{q}_\alpha^\xi = \partial H / \partial \Pi_\alpha^\xi. \quad (7)$$

H is evidently a constant of the motion since

$$\dot{H} = \frac{\partial H}{\partial \Pi_\alpha^\xi} \dot{\Pi}_\alpha^\xi + \frac{\partial H}{\partial q_\alpha^\xi} \dot{q}_\alpha^\xi = 0$$

in view of (7). We shall see that H is just proportional to the rest mass term as in the point particle case. From relations (7) we deduce moreover that the proper time derivation of any function f along the world line followed by x_μ is given by

$$\dot{f} = df/d\tau = [f, H] \quad (8)$$

where $[\quad]$ denotes the usual Poisson bracket with respect to the variables q_μ^ξ and their canonical momenta.

As an example, let us describe in this way the usual point particle. We write $L = \frac{1}{2} m \dot{x}_\mu \dot{x}_\mu$ and obtain immediately

$$G_\mu = m \dot{x}_\mu,$$

$$H = \frac{1}{2m} G_\mu G_\mu. \quad (9)$$

The canonical equations become $\dot{G}_\mu = 0$, so that

$$H = -\frac{1}{2} mc^2 \quad \text{since} \quad \dot{x}_\mu \dot{x}_\mu = -c^2.$$

The Hamiltonian formalism for the case where

$$L = L(x_\mu, \dot{x}_\mu, \ddot{x}_\mu, q_\mu, \dot{q}_\mu)$$

is performed by the Ostrogradski's method.⁸⁾ Introducing the new variables

$$\dot{x}_\mu = v_\mu, \quad n_\mu = \frac{\partial L}{\partial \ddot{x}_\mu}$$

and putting

$$G_\mu = \frac{\partial L}{\partial \dot{x}_\mu} - \dot{n}_\mu,$$

and

$$H = G_\mu v_\mu + n_\mu \ddot{x}_\mu - L(x_\mu, v_\mu, \ddot{x}_\mu),$$

we get

$$\begin{aligned} \dot{x}_\mu &= \partial H / \partial G_\mu, & \dot{G}_\mu &= -\partial H / \partial x_\mu, \\ \dot{v}_\mu &= \partial H / \partial n_\mu, & \dot{n}_\mu &= -\partial H / \partial v_\mu, \end{aligned}$$

which determine the evolution of the canonically conjugated variables (x_μ, G_μ) and (v_μ, n_μ) . This form will be employed later.

If we now depart from the point particle model and introduce new parameters such as b_μ^ξ into the Lagrangian, we can show immediately that such a step implies to pass from local to bilocal theory.

Indeed, let us call r_μ the four vector defined by

$$M^2 r_\mu = M_{\mu\nu} G_\nu, \quad (10)$$

where $-M^2 = G_\mu G_\mu$ is evidently a constant of the motion, and we assume $M > 0$.

If we introduce the world point y with coordinates defined by

$$y_\mu(\tau) = x_\mu - r_\mu = x_\mu(\tau) - M_{\mu\nu}(\tau) G_\nu(\tau) / M^2, \quad (11)$$

we can show that this point (which corresponds in the extended droplet theory to Møller's so-called "center of gravity") moves, in the absence of exterior forces, along a straight world line l_0 with a four velocity $u_\mu = G_\mu / M$ where M is the preceding constant of the motion.

To show this, let us differentiate (11) with respect to τ . We get

$$dy_\mu / d\tau = \dot{x}_\mu - (G_\mu \dot{x}_\nu - G_\nu \dot{x}_\mu) G_\nu / M^2 = \dot{x}_\mu + G_\mu \frac{m}{M^2} - \dot{x}_\mu = \frac{m}{M^2} G_\mu, \quad (12)$$

where we have written $m = -G_\mu \dot{x}_\mu$. This proves that y_μ moves with a four velocity parallel to G_μ , the relation between the proper time $d\tau'$ of l_0 and $d\tau$ being $d\tau' = (m/M) d\tau$.

Thus the introduction of new "line" variables alongside the coordinate x_μ determines a second point y_μ moving in a straight line l_0 around which l spirals in a more or less complex way according to the exact form of L . Moreover, if we introduce an inertial frame Π (in which $G_i = 0$) we see that r_μ is purely space-like in that frame since $r_\mu G_\mu = 0$.

Both points are in a sense canonically associated, x_μ being connected with kinematical and y_μ with dynamical variables. We notice immediately that our model is quite similar to Yukawa's classical model of bilocal theory⁵⁾ and thus the introduction of new "line" variables $b_\mu^\xi(\tau)$ necessarily implies a passage from "local" to "bilocal" theory. This is not very astonishing since Yukawa's model was precisely proposed as the simplest possible extension of the point particle idea. Our model, however, implies more degrees of freedom than the simple bilocal model. As was said before, the b_μ^ξ variables associated with the relativistic rotator can be

understood as describing "internal" average motions of matter in the immediate neighbourhood of x_μ in the case of extended relativistic particles. In this light Yukawa's model appears as a very simplified schematization of extended particle models.

On the other hand the new variables may be thought as a possible classical example of the "hidden" variables introduced *a priori* in the causal interpretation of quantum mechanics. Naturally, they are only "hidden" in the sense that we usually neglect them when we suppress particle extension in space time, reduce world tubes to world lines and leave aside internal motions (rotations, etc.) of matter around typical average points such as the center of matter density.

§ 2

We shall now study as a typical example of the preceding formalism the relativistic rotator described by the Lagrangian:

$$L = -\frac{1}{4} I \omega_{\alpha\beta} \omega_{\alpha\beta} + \lambda_{\mu\nu} \left(b_\mu^r b_\nu^r - \frac{1}{c^2} \dot{x}_\mu \dot{x}_\nu - \delta_{\mu\nu} \right) - \frac{\lambda}{4} (\omega_{\alpha\beta} \omega_{\alpha\beta} - K^2) \quad (13)$$

with $r \sim 1, 2, 3$ and

$$\omega_{\alpha\beta} = \dot{b}_\alpha^r b_\beta^r - \frac{1}{c^2} \ddot{x}_\alpha \dot{x}_\beta.$$

We shall assume moreover that the four vectors \dot{x}_α and b_α^r are physically initially well determined but that b_α^1 and b_α^2 can be arbitrarily rotated around the hyperplane containing b_α^3 and $\dot{x}_\alpha = ic b_\alpha^4 = v_\alpha$ without changing the motion. This corresponds, as we shall see, to gauge invariance.

In (13) the first term

$$T = -\frac{1}{4} I \omega_{\alpha\beta} \omega_{\alpha\beta} = -\frac{1}{4} I \dot{b}_\mu^{\xi} b_\mu^{\xi} \quad (14)$$

$$(\dot{x}_\alpha = ic b_\alpha^4), \quad \xi, \eta \sim 1, 2, 3, 4$$

is a formal relativistic generalization of the rotation energy of the three-dimensional spherical rigid body, the term $I + \lambda$ playing the part of the non-relativistic moment of inertia. This results from the fact that the corresponding angular momentum tensor is

$$M_{\alpha\beta} = \dot{x}_\alpha \frac{\partial L}{\partial \dot{x}_\beta} - \dot{x}_\beta \frac{\partial L}{\partial \dot{x}_\alpha} + b_\alpha^r \frac{\partial L}{\partial \dot{b}_\beta^r} - b_\beta^r \frac{\partial L}{\partial \dot{b}_\alpha^r} = (I + \lambda) \omega_{\alpha\beta}, \quad (15)$$

if we take into account the relations

$$b_\mu^r b_\nu^r - \frac{1}{c^2} \dot{x}_\mu \dot{x}_\nu = \delta_{\mu\nu}.$$

This results immediately from the second term $\lambda_{\mu\nu} (b_\mu^r b_\nu^r - 1/c^2 \cdot \dot{x}_\mu \dot{x}_\nu - \delta_{\mu\nu})$ in (13), with the symmetrical Lagrange multipliers $\lambda_{\mu\nu}$. The third term is also a Lagrange condition which means the constancy of $\omega_{\alpha\beta} \omega_{\alpha\beta}$.

We here note that our rotator is apparently similar to the one considered by Nakano⁹⁾. However, our forth axis plays a distinguished rôle through the identification (1b), while Nakano does not make such identification because his beingrößen parameters are associated to an Euclidian space in contrast to our b_μ^ξ which defines an internal Lorentz space. On the other hand he imposes the dynamical subsidiary condition which is not assumed in our case, i. e., $M_{\mu\nu}G_\nu=0$. This condition means that the point y coincides with x and thus the bilocal feature degenerates.

Since (13) is symmetrical in the internal Lorentz space (i. e., for $\xi=1, 2, 3, 4$) we shall call the present model "hyper-spherical" rotator, which is also a quite special example of possible relativistic rotators.

Now the equations of motion result immediately by applying the formulas given in § 1. We get with $n_\alpha=\partial L/\partial \ddot{x}_\alpha$

$$\dot{G}_\alpha = \frac{d}{d\tau} \left(\frac{\partial L}{\partial \dot{x}_\alpha} - \dot{n}_\alpha \right) = \frac{d}{d\tau} \left(\frac{\partial L}{\partial \dot{x}_\alpha} - \frac{d}{d\tau} \frac{\partial L}{\partial \ddot{x}_\alpha} \right) = 0 \quad (16a)$$

$$\dot{M}_{\alpha\beta} = G_\alpha \dot{x}_\beta - G_\beta \dot{x}_\alpha \quad (16b)$$

$$\text{and} \quad M_{\alpha\beta} \dot{x}_\beta = (I + \lambda) \ddot{x}_\alpha \quad (16c)$$

which constitute a simple generalization of Weyssenhoff's usual relations.⁷⁾

To solve the equations of motion we first change variables and write:

$$\dot{x}_\alpha = i c b_\alpha^4 = v_\alpha. \quad (17)$$

This is legitimate since L does not depend directly on x_α . Starting then from (16c), we get

$$\begin{aligned} M_{\alpha\beta} G_\alpha v_\beta &= \frac{1}{2} M_{\alpha\beta} (G_\alpha v_\beta - G_\beta v_\alpha) = \frac{1}{2} M_{\alpha\beta} \dot{M}_{\alpha\beta} \\ &= (I + \lambda) G_\alpha \ddot{x}_\alpha = - (I + \lambda) \dot{m} c^2. \end{aligned} \quad (18)$$

This is an important relation which shows that $M_{\alpha\beta}$, $M_{\alpha\beta}$ and m are simultaneous constants of the motion, since we shall show that λ is a constant of the motion. To prove this we first note that if we insert the expression (15) into (18) we get

$$\frac{1}{2} M_{\alpha\beta} \dot{M}_{\alpha\beta} = - \frac{1}{2} (I + \lambda) \dot{\lambda} K^2,$$

so that the relation (18) is written as

$$\dot{m} c^2 = - \frac{1}{2} \dot{\lambda} K^2. \quad (19)$$

Next we apply the Hamiltonian formalism stated in the preceding section to the form (13) of the Lagrangian.

We get

$$\begin{cases} \Pi_{\alpha}^r = \frac{\partial L}{\partial \dot{b}_{\alpha}^r} = -\frac{1}{2}(I+\lambda)\dot{b}_{\alpha}^r, \\ n_{\alpha} = \frac{\partial L}{\partial \ddot{x}_{\alpha}} = \frac{1}{2c^2}(I+\lambda)\ddot{x}_{\alpha}, \end{cases} \quad (20)$$

so that

$$\Pi_{\alpha}^r \dot{b}_{\alpha}^r + n_{\alpha} \ddot{x}_{\alpha} = \frac{1}{2}(I+\lambda)\omega_{\alpha\beta}\omega_{\alpha\beta},$$

and

$$\begin{aligned} H &= G_{\alpha} v_{\alpha} + \Pi_{\alpha}^r \dot{b}_{\alpha}^r + n_{\alpha} \ddot{x}_{\alpha} - L \\ &= -mc^2 - \frac{1}{4}(I+\lambda)\omega_{\alpha\beta}\omega_{\alpha\beta} - \lambda_{\alpha\beta} \left(b_{\alpha}^r b_{\beta}^r - \frac{1}{c^2} \dot{x}_{\alpha} \dot{x}_{\beta} - \delta_{\alpha\beta} \right) + \frac{\lambda}{4}(\omega_{\alpha\beta}\omega_{\alpha\beta} - K^2). \end{aligned} \quad (21)$$

As we know that H and $\omega_{\alpha\beta}\omega_{\alpha\beta}$ are constants of the motion, the derivation of (21) yields that

$$\dot{m}c^2 = -\frac{1}{4}\dot{\lambda}\omega_{\alpha\beta}\omega_{\alpha\beta} = -\frac{1}{4}\dot{\lambda}K^2. \quad (22)$$

The relation (22) compared with (19) proves the fact we anticipated $\dot{\lambda}=0$. We thus conclude therefore that m and $M_{\alpha\beta}M_{\alpha\beta}$ are also constants of the motion.

On the basis of this result,* the complete integration of the motion can be carried out explicitly as has been developed by Halbwachs.¹⁰⁾ Starting from the basic relations (16) and the constancy of m and $M_{\alpha\beta}M_{\alpha\beta}$, he has obtained the following results:

a. The "spin" $S_{\mu} = \tilde{M}_{\mu\nu} \dot{x}_{\nu}$ is a constant of the motion ($\dot{S}_{\mu}=0$) with a constant length, $S^2 = S_{\mu}S_{\mu}$

where

$$\tilde{M}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}M_{\alpha\beta}.$$

b. The acceleration \ddot{x}_{μ} has also a constant length and is always parallel to

$$r_{\mu} = \frac{1}{M^2 c^2} M_{\mu\nu} G_{\nu}.$$

c. The "radius" r_{μ} has also a constant length ($r_{\mu}r_{\mu}=\text{constant}$) and remains perpendicular in the frame H (for which $G_t=0$) to the spin. As a consequence, in that frame, the motion of x_{μ} around y_{μ} reduces to a circular motion with constant angular velocity Ω with

$$\Omega^2 \simeq M^2 c^4 / S_{\mu}S_{\mu}.$$

* More general analysis of the motion (without necessarily requiring the constancy of $M_{\alpha\beta}^2$) was performed by one of the authors (T.T.). This reveals the relations between six conserved quantities of internal motion which are identified as the known intrinsic properties of elementary particles (to be published).

This, as Möller³ remarked, is just the classical analogue of Schrödinger's Zitterbewegung.

d. The rotation of the frame defined by b_μ^r is necessarily constant. This can be shown in the following way: we first notice that the quantities

$$\omega^i = \frac{1}{2} \epsilon^{ijk} \dot{b}_\mu^j b_\mu^k$$

are constants of the motion ($\dot{\omega}^i = 0$) and represent the projections of the instantaneous rotation velocity on the moving frame (b_μ^r). According to our preceding results the "spin" has as projections on the axes of the moving tetrad constant components $I\omega^1, I\omega^2, I\omega^3$ and 0. If it points initially along b_μ^3 (that is, if we start with $\omega^1 = \omega^2 = 0$), it will remain so and b_μ^1 and b_μ^2 will rotate around b_μ^3 with a constant angular velocity $\omega = \dot{b}_\mu^1 b_\mu^2$, so that $\dot{b}_\mu^1 = \omega b_\mu^2$. Naturally in that case b_μ^1 and b_μ^2 are not physically completely determined since we can always rotate them by a constant angle around b_μ^3 .

The preceding motion can also be understood as the classical analogue of De Broglie's relation $E = h\nu$. Indeed if we start with the initial relation,

$$mc^2 = -G_\alpha \dot{x}_\alpha = S_\mu \omega_\mu = (I\omega_\mu) \omega_\mu, \quad (23)$$

this relation will be conserved by the motion so that if we suppose that $I\omega = S = \hbar$ initially we get

$$E = h\nu = mc^2$$

in the rest frame. The total motion then appears as a spiral motion of the origin x combined with a space rotation of the tetrad on itself, a behaviour already strangely similar to quantum theoretical motions.

§ 3

We shall now introduce a new set of kinematical parameters: the relativistic Euler angles. This step is justified by the fact that it allows simple comparison with the non-relativistic theory of rotating spherical rigid bodies and also, as we shall see in a subsequent paper, facilitates the subsequent quantization of the theory.

The introduction of the new variables rests on the remark that the b_μ^ξ parameters define only six independent quantities taking relations (1) into account. This means that the orientation of the b_μ^ξ frame Σ_0 with respect to any fixed laboratory frame Π_0 (defined by the tetrad a_μ^ξ also satisfying (1)) is determined (except for an arbitrary constant rotation) by the six parameters¹¹⁾ of the homogeneous Lorentz transformation which transforms Π_0 into Σ_0 .

As one knows¹²⁾ these relativistic Euler angles determines the Lorentz transformation from a_μ^ξ to b_μ^ξ by the matrix relation:

$$\begin{aligned}
& \begin{pmatrix} b_\mu^1 \\ b_\mu^2 \\ b_\mu^3 \\ b_\mu^4 \end{pmatrix} = \begin{pmatrix} \cos(\varphi^+/2) & \sin(\varphi^+/2) & 0 & 0 \\ -\sin(\varphi^+/2) & \cos(\varphi^+/2) & 0 & 0 \\ 0 & 0 & \cos(\varphi^+/2) & \sin(\varphi^+/2) \\ 0 & 0 & -\sin(\varphi^+/2) & \cos(\varphi^+/2) \end{pmatrix} \\
& \times \begin{pmatrix} \cos(\varphi^-/2) & \sin(\varphi^-/2) & 0 & 0 \\ -\sin(\varphi^-/2) & \cos(\varphi^-/2) & 0 & 0 \\ 0 & 0 & \cos(\varphi^-/2) & -\sin(\varphi^-/2) \\ 0 & 0 & \sin(\varphi^-/2) & \cos(\varphi^-/2) \end{pmatrix} \begin{pmatrix} \cos(\theta^+/2) & 0 & -\sin(\theta^+/2) & 0 \\ 0 & \cos(\theta^+/2) & 0 & -\sin(\theta^+/2) \\ \sin(\theta^+/2) & 0 & \cos(\theta^+/2) & 0 \\ 0 & \sin(\theta^+/2) & 0 & \cos(\theta^+/2) \end{pmatrix} \\
& \times \begin{pmatrix} \cos(\theta^-/2) & 0 & -\sin(\theta^-/2) & 0 \\ 0 & \cos(\theta^-/2) & 0 & \sin(\theta^-/2) \\ \sin(\theta^-/2) & 0 & \cos(\theta^-/2) & 0 \\ 0 & -\sin(\theta^-/2) & 0 & \cos(\theta^-/2) \end{pmatrix} \begin{pmatrix} \cos(\psi^+/2) & \sin(\psi^+/2) & 0 & 0 \\ -\sin(\psi^+/2) & \cos(\psi^+/2) & 0 & 0 \\ 0 & 0 & \cos(\psi^+/2) & \sin(\psi^+/2) \\ 0 & 0 & -\sin(\psi^+/2) & \cos(\psi^+/2) \end{pmatrix} \\
& \times \begin{pmatrix} \cos(\psi^-/2) & \sin(\psi^-/2) & 0 & 0 \\ -\sin(\psi^-/2) & \cos(\psi^-/2) & 0 & 0 \\ 0 & 0 & \cos(\psi^-/2) & -\sin(\psi^-/2) \\ 0 & 0 & \sin(\psi^-/2) & \cos(\psi^-/2) \end{pmatrix} \begin{pmatrix} a_\mu^1 \\ a_\mu^2 \\ a_\mu^3 \\ a_\mu^4 \end{pmatrix}. \quad (24)
\end{aligned}$$

The complex angles $\omega^+ = \{\theta^+, \varphi^+, \psi^+\}$ and $\omega^- = \{\theta^-, \varphi^-, \psi^-\}$ correspond to the relativistic generalization of the three-dimensional Euler angles. They are defined by the relations:

$$\varphi^\pm = \varphi_1 \pm i\varphi_2, \quad \theta^\pm = \theta_1 \pm i\theta_2, \quad \psi^\pm = \psi_1 \pm i\psi_2 \quad (25)$$

where the real quantities $\varphi_1, \theta_1, \psi_1$ correspond to the usual space Euler angles, while $\varphi_2, \theta_2, \psi_2$ represent hyperbolic angles (varying from $-\infty$ to $+\infty$) expressing pure Lorentz transforms. The two sets ω^+ and ω^- are thus complex conjugates of each other.

Before we express the Hamiltonian as a function of these new variables we shall briefly discuss their geometrical meaning.

As already implied in the work of Einstein and Mayer,¹³⁾ we construct from b_μ^\pm a set of complex self-dual antisymmetrical tensors:

$$F_{\mu\nu}^{r\pm} = \varepsilon^{rst} b_\mu^s b_\nu^t \pm (b_\mu^r b_\nu^4 - b_\nu^r b_\mu^4),$$

of which independent components are

$$B_k^{r\pm} = b_k^r b_4^4 - b_k^4 b_4^r \pm \varepsilon_{ijk} b_i^r b_j^4. \quad (26)$$

Then it can be shown that each set of $\{B_k^{r+}\}$ and $\{B_k^{r-}\}$ behaves as a set of three complex vectors spanning a complex three-dimensional orthonormal frame of axes:

$$B_i^{r\pm} B_j^{r\pm} = \delta_{ij}, \quad B_k^{r\pm} B_k^{s\pm} = \delta^{rs}. \quad (27)$$

In exactly the same way one can construct from a_μ^\pm the corresponding quantities by

$$A_{\mu\nu}{}^{r\pm} = \varepsilon^{rst} a_{\mu}{}^s a_{\nu}{}^t \pm (a_{\mu}{}^r a_{\nu}{}^4 - a_{\nu}{}^r a_{\mu}{}^4),$$

$$A_k{}^{r\pm} = a_i{}^r a_k{}^4 - a_k{}^r a_i{}^4 \pm \varepsilon_{ijk} a_i{}^r a_j{}^4, \quad (26')$$

with

$$A_i{}^{r\pm} A_j{}^{r\pm} = \delta_{ij}, \quad A_k{}^{s\pm} A_k{}^{s\pm} = \delta^{rs}. \quad (27')$$

Now, utilizing definition (26) one can demonstrate after a short calculation that

$$\begin{pmatrix} B_k{}^{1\pm} \\ B_k{}^{2\pm} \\ B_k{}^{3\pm} \end{pmatrix} = \begin{pmatrix} \cos\varphi^{\pm} \cos\theta^{\pm} \cos\phi^{\pm} - \sin\varphi^{\pm} \sin\phi^{\pm}, & \cos\varphi^{\pm} \cos\theta^{\pm} \sin\phi^{\pm} + \sin\varphi^{\pm} \cos\phi^{\pm}, & -\cos\varphi^{\pm} \sin\theta^{\pm} \\ -\sin\varphi^{\pm} \cos\theta^{\pm} \cos\phi^{\pm} - \cos\varphi^{\pm} \sin\phi^{\pm}, & -\sin\varphi^{\pm} \cos\theta^{\pm} \sin\phi^{\pm} + \cos\varphi^{\pm} \cos\phi^{\pm}, & -\sin\varphi^{\pm} \sin\theta^{\pm} \\ \sin\theta^{\pm} \cos\phi^{\pm}, & \sin\theta^{\pm} \sin\phi^{\pm}, & \cos\theta^{\pm} \end{pmatrix} \times \begin{pmatrix} A_k{}^{1\pm} \\ A_k{}^{2\pm} \\ A_k{}^{3\pm} \end{pmatrix}. \quad (28)$$

This shows that defining four complex three-dimensional frames A^+ with $A_k{}^{r+}$, A^- with $A_k{}^{r-}$, B^+ with $B_k{}^{r+}$, and B^- with $B_k{}^{r-}$, we can pass from A^+ to B^+ and from A^- to B^- by two complex three-dimensional rotations defined by the complex Euler angles ω^+ and ω^- , since relations (28) are identical (except for the complex character of all quantities) with the classical expression for real three-dimensional ordinary rotations (in terms of real Euler angles) in ordinary space.

As the set $B_k{}^{r+}$, $A_k{}^{r+}$ (or $B_k{}^{r-}$, $A_k{}^{r-}$) are equivalent to $b_{\mu}{}^{\xi}$ and $a_{\mu}{}^{\xi}$ this demonstrates in a very simple way the well-known isomorphism established by Einstein⁽¹⁴⁾ and Cartan⁽¹⁵⁾ between Lorentz transforms and three-dimensional complex rotation. These complex rotations can be represented in two complex conjugated three-dimensional spaces, E^+ and E^- . If we choose $a_{\mu}{}^{\xi}$ and $b_{\mu}{}^{\xi}$ as rest frames, we see that $A_k{}^{r\pm} = a_k{}^r$, and $B_k{}^{r\pm} = b_k{}^r$ so that the complex rotations ω^+ and ω^- can be understood as taking place in the three-dimensional ordinary space.

If we now return to Minkowski space we must remember that any rotation (defined by a skew symmetrical tensor) takes place around a bivector which glides on itself under the corresponding matrix transformation. The meaning of relations (24) and (28) then becomes clear. Relations (24) define a set of six successive rotations around six bivectors in a certain determined order (right to left) which bring H_0 into Σ_0 . Relation (28) defines the corresponding transformation which brings the system of bivectors $A_k{}^{r\pm}$ on $B_k{}^{r\pm}$. One notes also that any self-dual skew tensor of the (+) type (such as $m_{\mu\nu} + i\tilde{m}_{\mu\nu}$) is orthogonal to the (-) type ($n_{\mu\nu} - i\tilde{n}_{\mu\nu}$), so that we have identically $(m_{\mu\nu} + i\tilde{m}_{\mu\nu})(n_{\mu\nu} - i\tilde{n}_{\mu\nu}) = 0$.

For more details on relativistic Euler angles the reader may consult specialized mathematical papers.^{(12), (16)}

Let us now return to the hyper-spherical rotator. Let us call T (kinematical energy) the first term in the Lagrangian (13).

In order to prepare the transition to the use of relativistic Euler angles we can first express T as a function of B_k^{\pm} . A rather long but simple calculation gives

$$T = \frac{I}{2} \varepsilon_{ijk} \varepsilon_{ilm} (\dot{B}_j^{r+} B_k^{r+} \dot{B}_l^{s+} B_m^{s+} + \dot{B}_j^{r-} B_k^{r-} \dot{B}_l^{s-} B_m^{s-}) = \frac{I}{2} (\dot{B}_k^{r+} \dot{B}_k^{r+} + \dot{B}_k^{r-} \dot{B}_k^{r-}). \quad (29)$$

We see here appearing for the first time an essential property of the hyper-spherical rotator: its Hamiltonian can be split into two complex conjugated parts the first depending on ω^+ , the second on ω^- .

This is very natural if one remarks that the skew tensor can be split into a sum of two self-dual tensors of the types $(+)$ and $(-)$. We have

$$\omega_{\mu\nu} = \frac{1}{2} (\omega_{\mu\nu} + i\tilde{\omega}_{\mu\nu}) + \frac{1}{2} (\omega_{\mu\nu} - i\tilde{\omega}_{\mu\nu}) = \frac{1}{2} \omega_{\mu\nu}^+ + \frac{1}{2} \omega_{\mu\nu}^-, \quad (30)$$

so that

$$\omega_{\mu\nu} \omega_{\mu\nu} = \frac{1}{4} \omega_{\mu\nu}^+ \omega_{\mu\nu}^+ + \frac{1}{4} \omega_{\mu\nu}^- \omega_{\mu\nu}^- = \frac{1}{2} \omega_{ik}^+ \omega_{ik}^+ + \frac{1}{2} \omega_{ik}^- \omega_{ik}^-, \quad (31)$$

taking into account the self-dual character of $\omega_{\mu\nu}^+$ and $\omega_{\mu\nu}^-$. Relation (31) multiplied by $I/2$ is equivalent to (29).

Introducing the expressions

$$\omega_k^{\pm} = \varepsilon_{ijk} \dot{B}_i^{\pm} B_j^{\pm} = \frac{1}{2} \varepsilon_{ijk} \omega_{ij}^{\pm},$$

we see that the hyper-spherical Hamiltonian can be written in the new form.

$$H = mc^2 + \frac{1}{2} \mathcal{G} \omega_k^+ \omega_k^+ + \frac{1}{2} \mathcal{G} \omega_k^- \omega_k^- + \sum_{\pm} \lambda_{ij}^{\pm} (B_i^{\pm} B_j^{\pm} - \delta_{ij}) + \lambda/2 (\omega_k^+ \omega_k^+ + \omega_k^- \omega_k^- - K^2) \quad (32)$$

where

$$\mathcal{G} = I + \lambda.$$

That is exactly⁽¹²⁾ the sum of two complex conjugated three-dimensional spherical rigid rotators (the proper time τ playing the part of ordinary time). This simplifies everything.

The angular momenta associated to ω_k^{\pm} are just

$$S_k^{\pm} = \mathcal{G} \omega_k^{\pm} \quad (33)$$

and correspond to the splitting of the angular momentum $M_{\alpha 3}$ into two self-dual parts:

$$M_{\alpha\beta} = \frac{1}{2}(M_{\alpha\beta} + i\widetilde{M}_{\alpha\beta}) + \frac{1}{2}(M_{\alpha\beta} - i\widetilde{M}_{\alpha\beta}) = \frac{1}{2}M_{\alpha\beta}^+ + \frac{1}{2}M_{\alpha\beta}^-, \quad (34)$$

and we wrote the space components as

$$S_k^\pm = \frac{1}{2}\varepsilon_{ijk}M_{ij}^\pm, \quad (35)$$

while the space-time parts are just proportional to them (since $M_{\alpha\beta}^\pm = \pm i\widetilde{M}_{\alpha\beta}^\pm$), so that we can write:

$$T = \frac{1}{4\mathcal{G}}S_k^+S_k^+ + \frac{1}{4\mathcal{G}}S_k^-S_k^-. \quad (36)$$

The physical meaning of expressions like S_k^\pm is clear. These quantities represent the projections of the internal angular momentum $M_{\alpha\beta}$ on the fixed self-dual bivectors (or rather their space-like parts) out of which we have built the fixed frames A^+ and A^- . For example, we have

$$S_3^+ = M_{\alpha\beta}A_{\alpha\beta}^{3+} = M_{\alpha\beta}^+A_{\alpha\beta}^{3+} \quad (37)$$

and similar expressions for all such quantities.

Our last step is to make a new change of variables and express the Lagrangian and Hamiltonian in terms of relativistic Euler angles. This, with the help of expressions (28), can be carried out exactly like in the three-dimensional case, so we shall just recall the results.

We first calculate the projections of the angular velocities $\vec{\omega}^\pm$ (with components ω_k^\pm) on the axes of the moving frames B^+ and B^- .

We get (denoting by primed quantities such projections in order to differentiate them from unprimed quantities which represent projections on the fixed frames A^\pm):

$$\begin{cases} \omega'^{1\pm} = \cos\varphi^\pm \sin\theta^\pm \dot{\phi}^\pm + \sin\varphi^\pm \dot{\theta}^\pm \\ \omega'^{2\pm} = -\sin\varphi^\pm \sin\theta^\pm \dot{\phi}^\pm + \cos\varphi^\pm \dot{\theta}^\pm \\ \omega'^{3\pm} = \dot{\phi}^\pm + \cos\theta^\pm \dot{\psi}^\pm \end{cases} \quad (38)$$

and the Lagrangian becomes with the help of (38):

$$\begin{aligned} L = & \mathcal{G}/2(\dot{\varphi}^{+2} + \dot{\psi}^{+2} + \dot{\theta}^{+2} + 2\dot{\varphi}^+\dot{\psi}^+\cos\theta^+) \\ & + \mathcal{G}/2(\dot{\varphi}^{-2} + \dot{\psi}^{-2} + \dot{\theta}^{-2} + 2\dot{\varphi}^-\dot{\psi}^-\cos\theta^-) + \lambda K^2 = L^+ + L^- + \lambda K^2 \end{aligned} \quad (39)$$

where as we know λ and K^2 are constants.

This is exactly the sum $L^+ + L^-$ of the Lagrangians of two complex conjugate three-dimensional spherical rigid bodies. Naturally the Lagrange condition terms with the multipliers λ_{ij}^\pm have disappeared since the orthogonality conditions are automatically satisfied by our six Euler angles.

We can now calculate the canonical momenta associated with the new variables. Naturally, we could vary separately in the Lagrangian L^+ (and in L^-) $\varphi_1, \varphi_2, \theta_1, \theta_2, \psi_1$, and ψ_2 , (that is the real and imaginary part of ω^+ and ω^-), and consider L^+ as a function of these variables; the equations of motion being defined by:

$$\delta \int L^+(\varphi_1, \varphi_2, \theta_1, \theta_2, \psi_1, \psi_2, \dot{\varphi}_1, \dot{\varphi}_2, \dot{\theta}_1, \dot{\theta}_2, \dot{\psi}_1, \dot{\psi}_2) d\tau = 0 \quad (40)$$

and the same variational equation with L^- . However, we know that one obtains exactly the same equations of motion if one considers L^+ as a function of $\omega^+ = (\theta^+, \varphi^+, \psi^+)$ and L^- as a function of $\omega^- = (\theta^-, \varphi^-, \psi^-)$ and vary these quantities independently. This can be shown directly. We get for the corresponding momenta:

$$\begin{cases} p_{\varphi^\pm} = \partial L^\pm / \partial \dot{\varphi}^\pm = \mathcal{G}(\dot{\varphi}^\pm + \dot{\psi}^\pm \cos \theta^\pm) \\ p_{\psi^\pm} = \partial L^\pm / \partial \dot{\psi}^\pm = \mathcal{G}(\dot{\psi}^\pm + \dot{\varphi}^\pm \cos \theta^\pm) \\ p_{\theta^\pm} = \partial L^\pm / \partial \dot{\theta}^\pm = \mathcal{G} \dot{\theta}^\pm \end{cases} \quad (41)$$

so that the Hamiltonian $H = p_i \dot{q}_i - L$ becomes

$$H = \frac{1}{2} (p_{\varphi^+} \dot{\varphi}^+ + p_{\psi^+} \dot{\psi}^+ + p_{\theta^+} \dot{\theta}^+) + \frac{1}{2} (p_{\varphi^-} \dot{\varphi}^- + p_{\psi^-} \dot{\psi}^- + p_{\theta^-} \dot{\theta}^-), \quad (42)$$

if we neglect constant terms. Namely,

$$\begin{aligned} H &= \frac{1}{2\mathcal{G}} \left[p_{\theta^+}^2 + \frac{1}{\sin^2 \theta^+} (p_{\varphi^+}^2 + p_{\psi^+}^2 - 2p_{\varphi^+} p_{\psi^+} \cos \theta^+) \right] \\ &\quad + \frac{1}{2\mathcal{G}} \left[p_{\theta^-}^2 + \frac{1}{\sin^2 \theta^-} (p_{\varphi^-}^2 + p_{\psi^-}^2 - 2p_{\varphi^-} p_{\psi^-} \cos \theta^-) \right] \\ &= \frac{1}{2} (s_i'^+ \omega_i'^+ + s_i'^- \omega_i'^-) \\ &= H^+ + H^-, \end{aligned} \quad (43)$$

or

$$H = \frac{1}{2\mathcal{G}} (S_k^+ S_k^+ + S_k^- S_k^-) \quad (44a)$$

$$= \frac{1}{2\mathcal{G}} (S_k'^+ S_k'^+ + S_k'^- S_k'^-). \quad (44b)$$

One then checks immediately with the help of the usual Poisson brackets that $[S_k^+, S_j^-] = 0$, $[S_k'^+, S_j'^-] = 0$.

Comparing (44a) and (44b), we obtain the connection between our angular momenta and the projections of the momentum S_k^\pm in the form

$$\begin{cases} p_{\varphi\pm} = S_3'^{\pm} \\ p_{\theta\pm} = S_1'^{\pm} \sin \varphi^{\pm} + S_2'^{\pm} \cos \varphi^{\pm} \\ p_{\psi\pm} = -S_1'^{\pm} \sin \theta^{\pm} \cos \varphi^{\pm} + S_2'^{\pm} \sin \theta^{\pm} \sin \varphi^{\pm} + S_3'^{\pm} \cos \theta^{\pm} \end{cases} \quad (45)$$

so that the projections of the momentum on the frames B^{\pm} and A^{\pm} take the familiar form :

$$\begin{cases} S_1'^{\pm} = p_{\theta\pm} \sin \varphi^{\pm} + p_{\varphi\pm} \cot \theta^{\pm} \cos \varphi^{\pm} - p_{\psi\pm} \frac{\cos \varphi^{\pm}}{\sin \theta^{\pm}} \\ S_2'^{\pm} = p_{\theta\pm} \cos \varphi^{\pm} - p_{\varphi\pm} \cot \theta^{\pm} \sin \varphi^{\pm} + p_{\psi\pm} \frac{\sin \varphi^{\pm}}{\sin \theta^{\pm}} \\ S_3'^{\pm} = p_{\varphi\pm} \end{cases} \quad (46)$$

on B^{\pm} , and

$$\begin{cases} S_1^{\pm} = -p_{\theta\pm} \sin \psi^{\pm} - p_{\psi\pm} \cot \theta^{\pm} \cos \psi^{\pm} + p_{\varphi\pm} \frac{\cos \psi^{\pm}}{\sin \theta^{\pm}} \\ S_2^{\pm} = p_{\theta\pm} \cos \psi^{\pm} - p_{\psi\pm} \cot \theta^{\pm} \sin \psi^{\pm} + p_{\varphi\pm} \frac{\sin \psi^{\pm}}{\sin \theta^{\pm}} \\ S_3^{\pm} = p_{\psi\pm} \end{cases} \quad (47)$$

on A^{\pm} .

H^{+} and H^{-} are then evidently constants of the motion and we get, writing $(S^{\pm})^2 = S_k'^{\pm} S_k'^{\pm} = S_k^{\pm} S_k^{\pm}$, the relations

$$[H, (S^{\pm})^2] = 0, \quad [H, p_{\psi\pm}] = 0, \quad [H, p_{\varphi\pm}] = 0, \quad (48)$$

which correspond to classical three-dimensional properties of the non-relativistic rotators.

Concluding remarks

In conclusion we want to add a few remarks.

First we note that if we want to connect such a rotator with real physical movements inside relativistic fluid masses, it must be understood that its behaviour constitutes only an average and very crude abstraction of real internal motions since we then necessarily neglect an infinite number of possible degrees of freedom.

Secondly the introduction of this model as possible starting point for quantum theory (or elements of a new sub-quantum-mechanical level) raises many difficult but interesting problems. It is closely connected, for example, with very recent researches and ideas in quantum field theories, such as indefinite metric and non-linear waves, and we intend to discuss them in subsequent papers. Indeed, as three of us have shown (D. B., P. H. and J. P. V.), the quantization of the above rotator leads to the introduction of quantum numbers and energy levels which can be classified according to the well-known Nishijima-Gell-Mann scheme of elementary particles.

Evidently the present model is a quite special one of relativistic rotators. One of us (T. T.) has classified possible relativistic rotators from wider point of view which makes the physical meanings of various quantities and conditions clearer. Indeed, we only need the three-dimensional symmetry in the internal Lorentz space (i.e., the rotator be *spherical*) so as to obtain the conserved iso-spin components, and it is then shown that only its third component remains conserved when electromagnetic interaction is introduced. General theory of such spherical rotators includes, as its special examples, various models: the hyper-spherical rotator treated in this paper, Matthison-Weyssenhoff's particle,⁷⁾ Nakano's rigid body, Hönl-Papapetrou's particle, and others. These will also be given in separate papers.

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Stopping Power of High Temperature Plasma

—Effects of Ionic Collective Motion—

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The problem of energy loss of a charged particle travelling through a fully ionized gas has been studied by taking account of effects of ionic motion. Contribution due to an ionic collective motion turns out to be smaller than that due to an electronic collective motion by order of m/M , where m is the mass of electron, M that of ion. The present investigation shows that the ionic collective motion cannot be excited by a charged interloper unless one takes into account effects of thermal motion of electrons.

§ 1. Introduction

Stopping power of an ionized gas for a charged particle has been investigated by Hayakawa and other authors in relation to the problems of astrophysics.¹⁾ Here, we will examine the stopping power of a fully ionized gas at high temperature with particular interest in exploring effects of ionic collective motion.

It is well known that the energy loss due to far distance collisions between the charged interloper and electrons of the ionized gas can be calculated by treating a system of electrons as a polarizable medium with a dielectric constant

$$\varepsilon(\omega) = 1 - \omega_p^2 (\omega^2 - i\omega\nu)^{-1} \quad (1)$$

where $\omega_p^2 = 4\pi e^2 n/m$ and ν is a collision frequency. However, as it has been discussed by Akhiezer,²⁾ one has to remember that only at low electronic temperature it is allowed to regard the electron plasma as polarizable medium with the dielectric constant $\varepsilon(\omega)$ given by Eq. (1). Therefore, although the magneto-hydrodynamical approaches are widely used for investigating physical behaviours of an ionized gas, it is very essential to formulate the present problem on such a fundamental basis that has been presented by Bohm and Pines.³⁾

Now, at extremely high temperature, the thermal motion of ions may not be disregarded at all. Mutual correlation between the ions may give rise to an acoustic collective motion at low frequency region, yet so far there seems to be no investigation which has examined effects of the ionic collective motions on the stopping power of the fully ionized gas at high temperature. In what follows,

following Yamada-Nakajima's treatment of the collective description of the fully ionized gas,⁴⁾ we will examine the interaction between the incident charged particle and the fully ionized gas, and then solving a set of equations of motion we will calculate the stopping power of the fully ionized gas.

§ 2. Interaction between a charged particle and the plasma

Let us consider a unit volume of the plasma, which is composed of n electrons and N ions. Let ze be the charge of ion, then because of the neutrality of the whole system, we have a relation $n=Nz$. When a particle with electric charge Ze and mass μ falls into a plasma, the Hamiltonian of the particle is given as

$$H = \mathbf{p}_0^2 / (2\mu) - 4\pi Ze^2 \sum_{i, k} k^{-2} \exp(i\mathbf{k} \cdot (\mathbf{x}_i - \mathbf{x}_0)) \\ + 4\pi z Ze^2 \sum_{j, k} k^{-2} \exp(i\mathbf{k} \cdot (\mathbf{x}_j - \mathbf{x}_0)) \quad (2)$$

where \mathbf{p}_0 and \mathbf{x}_0 are momentum and coordinate of the incident particle, i and j are the suffices for electron and ion. According to Tomonaga's theory of the collective motion,⁵⁾ we have the collective coordinates of electrons with mass m and that of ions with mass M as follows,

$$q_k' = (m/n)^{1/2} \sum_i k^{-1} \exp(i\mathbf{k} \cdot \mathbf{x}_i), \quad (3a)$$

$$Q_k' = (M/N)^{1/2} \sum_j k^{-1} \exp(i\mathbf{k} \cdot \mathbf{x}_j). \quad (3b)$$

Using the above collective coordinates, we can separate the interaction term into short range interaction parts and long range interaction parts as follows.

$$H_{int} = -4\pi Ze^2 (n/m)^{1/2} \sum_{|\mathbf{k}| < k_c} k^{-1} e^{-i\mathbf{k} \cdot \mathbf{x}_0} q_k' + 4\pi z Ze^2 (N/M)^{1/2} \sum_{|\mathbf{k}| < k_c'} k^{-1} e^{-i\mathbf{k} \cdot \mathbf{x}_0} Q_k' \\ - 4\pi Ze^2 \sum_{|\mathbf{k}| > k_c} \sum_i k^{-2} e^{i\mathbf{k} \cdot (\mathbf{x}_i - \mathbf{x}_0)} + 4\pi z Ze^2 \sum_{|\mathbf{k}| > k_c'} \sum_j k^{-2} e^{i\mathbf{k} \cdot (\mathbf{x}_j - \mathbf{x}_0)}, \quad (4)$$

where k_c and k_c' are the critical momentum of the electronic collective motion and of the ionic collective motion, respectively.

Now, Yamada and Nakajima⁴⁾ have shown that in a Hamiltonian of the plasma a coupling term between the electronic collective motion and the ionic collective motion can be eliminated by the following orthogonal transformation,

$$q_k' = q_k + (\omega_p \mathcal{Q}_p / \omega(k)^2) Q_k, \quad (5a)$$

$$Q_k' = -(\omega_p \mathcal{Q}_p / \omega(k)^2) q_k + Q_k \quad (5b)$$

where $\omega_p^2 = 4\pi e^2 n/m$, $\mathcal{Q}_p^2 = 4\pi z^2 e^2 N/M$. After the transformation (5a, b), collective parts of the Hamiltonian of plasma is reduced to

$$H_c = \frac{1}{2} \sum_{|\mathbf{k}| < k_c} (p_k p_{-k} + \omega(k)^2 q_k q_{-k}) + \frac{1}{2} \sum_{|\mathbf{k}| < k_c'} (P_k P_{-k} + \mathcal{Q}(k)^2 Q_k Q_{-k}) \quad (6)$$

where p_{-k} , P_{-k} are the canonical momentum of q_k and Q_k , respectively. The eigen-frequencies $\omega(k)$, $\Omega(k)$ are given as follows,

$$\omega(k)^2 = \omega_p^2 + (3\kappa T_e/m)k^2 \quad k < k_c \quad (6a)$$

$$\begin{aligned} \Omega(k)^2 &= \Omega_p^2 \cdot k^2 (k^2 + k_d^2)^{-1} + (3\kappa T_i/M)k^2 \quad k < k_c \\ &= \Omega_p^2 + (3\kappa T_i/M)k^2 \quad k_c < k < k_c' \end{aligned} \quad (6b)$$

where T_e is the electronic temperature, T_i is the ionic temperature, $k_d^2 = (4\pi e^2/\kappa T_e)$ and κ is Boltzmann's constant.

Applying the transformation (5a, b) to the H_{int} (4), we get

$$\begin{aligned} H_{int} &= -4\pi Ze^2 (n/m)^{1/2} \sum_{|k| < k_c} k^{-1} e^{-ikx_0} q_k \\ &\quad + 4\pi z Ze^2 (N/M)^{1/2} \sum_{|k| < k_c'} k^{-1} (1 - (\omega_p/\omega(k))^2) e^{-ikx_0} Q_k \\ &\quad + \text{short range part.} \end{aligned} \quad (7)$$

Since the second term vanishes for the limit of $T_e \rightarrow 0$, we can conclude that when a charged interloper runs through the plasma the ionic collective motion can be excited only through the thermal motion of electrons in the plasma.

§ 3. Equation of motion of the system and stopping power

Since the short range part of the H_{int} (7) describes two-body collisions between the interloper and the particles in the plasma, its effect can be examined by usual treatment. Therefore, in the following, we will examine effects of the long range parts of the interaction between the interloper and the collective motions in the plasma. Then we may take a following total Hamiltonian for the system,

$$\begin{aligned} H_{total} &= H_c + \dot{p}_0^2/(2\mu) - 4\pi Ze^2 (n/m)^{1/2} \sum_{|k| < k_c} k^{-1} e^{-ikx_0} q_k \\ &\quad + 4\pi z Ze^2 (N/M)^{1/2} \sum_{|k| < k_c'} k^{-1} (1 - (\omega_p/\omega(k))^2) e^{-ikx_0} Q_k \end{aligned} \quad (8)$$

where H_c is given by Eq. (6).

The motion of the interloper can be described by the following equation of motion

$$\begin{aligned} \dot{p}_0 &= -4\pi Ze^2 (n/m)^{1/2} \sum_{|k| < k_c} i(\mathbf{k}/k) e^{-ikx_0} q_k \\ &\quad + 4\pi z Ze^2 (N/M)^{1/2} \sum_{|k| < k_c'} i(\mathbf{k}/k) \{1 - (\omega_p/\omega(k))^2\} e^{-ikx_0} Q_k \end{aligned} \quad (9)$$

and the equations of the electronic and ionic collective motion are given as follows,

$$\ddot{q}_k + \omega(k)^2 q_k = 4\pi Ze^2 (n/m)^{1/2} k^{-1} e^{ikx_0}, \quad (10)$$

$$\ddot{Q}_k + \Omega(k)^2 Q_k = -4\pi z Ze^2 (N/M)^{1/2} k^{-1} \{1 - (\omega_p/\omega(k))^2\} e^{ikx_0}. \quad (11)$$

When the incident particle runs through the plasma with a constant velocity v_0 , solutions of the induced forced oscillation are obtained from Eqs. (10) and (11) as

$$q_k = 4\pi Ze^2 (n/m)^{1/2} k^{-1} e^{ikv_0 t} \{2\omega(k)\}^{-1} \{(\omega(k) - k\mathbf{v}_0)^{-1} + (\omega(k) + k\mathbf{v}_0)^{-1} + i\pi\delta(\omega(k) - k\mathbf{v}_0) - i\pi\delta(\omega(k) + k\mathbf{v}_0)\}, \quad (12)$$

$$Q_k = -4\pi Ze^2 (N/M)^{1/2} k^{-1} e^{ikv_0 t} \{1 - (\omega_p/\omega(k))^2\} \times \{2\Omega(k)\}^{-1} \{(\Omega(k) - k\mathbf{v}_0)^{-1} + (\Omega(k) + k\mathbf{v}_0)^{-1} + i\pi\delta(\Omega(k) - k\mathbf{v}_0) - i\pi\delta(\Omega(k) + k\mathbf{v}_0)\} \quad (13)$$

which satisfy the proper initial conditions that at $t = -\infty$, $q_k = \dot{q}_k = 0$ and $Q_k = \dot{Q}_k = 0$. In Eqs. (12) and (13), the terms of the δ -function represent resonance excitation of the collective motions, and the other terms stand for the stationary state solution.

Now, Bohm and Pines³⁾ have calculated contributions of the far distant collisions to the stopping power by considering excitation of the collective motion in the electron plasma. Their calculation has been carried through by obtaining an electric field due to the steady state solution of the collective density fluctuation q_k under the boundary condition that the excitation of the collective motion takes place behind the incident particle.

To derive the energy loss, however, it is sufficient to find a force F acting on the incident particle, since the energy loss dE of the particle running over the distance dx is equal to the work done by the particle, $-F \cdot dx$. Thus, we may simply find the stopping power, dE/dx . Now, the force acting on the incident particle is just equal to the right-hand side of Eq. (9). Substituting the solutions (12) and (13) into the right-hand side of Eq. (9), we can obtain the force acting on the particle while the collective motions are excited in the plasma as follows,

$$F = - (4\pi Ze^2)^2 \frac{n}{m} \sum_k \frac{i\mathbf{k}}{k^2} \frac{i\pi}{2\omega(k)} \{ \delta(\omega(k) - k\mathbf{v}_0) - \delta(\omega(k) + k\mathbf{v}_0) \} \\ - (4\pi Ze^2)^2 \frac{N}{M} \sum_k \frac{i\mathbf{k}}{k^2} \frac{i\pi}{2\Omega(k)} \left\{ 1 - \left(\frac{\omega_p}{\omega(k)} \right)^2 \right\} \\ \times \{ \delta(\Omega(k) - k\mathbf{v}_0) - \delta(\Omega(k) + k\mathbf{v}_0) \}. \quad (14)$$

If z -axis is chosen to be in the incident direction, x - and y -component of the force vanish because of the axial symmetry of the summands of Eq. (14). So finally, we get

$$F_z = \frac{(Ze)^2}{2\pi} \frac{\omega_p^2}{v_0^2} \int dk_x dk_y \frac{1}{k_x^2 + k_y^2 + k_z^2} \Big|_{k_z = \omega(k)/v_0} \\ + \frac{(Ze)^2}{2\pi} \frac{\Omega_p^2}{v_0^2} \int dk_x dk_y \frac{(3\kappa T_e/m)^2 k^2}{[\omega_p^2 + (3\kappa T_e/m)k^2]^2} \Big|_{k_z = \Omega(k)/v_0}. \quad (15)$$

Here, it will be worth remarking that if we take ω_p in the place of $\omega(k)$ in the first term of Eq. (15) it is reduced to Eq. (56) of reference 3). As it is discussed in section 2, however, the second term of Eq. (15) vanishes identically if we neglect the dispersion term of the frequency $\omega(k)$. Hence, if we want to investigate the contributions of the ionic collective motion, we have to take into account the dispersion term of $\omega(k)$ in the first term of Eq. (15).

In the following, let us take $k_e = k_e'$ for convenience, and let us approximate the $\Omega(k)$ by the following expression,

$$\Omega(k)^2 = (3\kappa T^*/M) k^2, \quad T^* = T_i + T_e. \quad (16)$$

Then, the condition $k_z = \omega(k)/v_0$ determines the value of k_z in terms of k_x, k_y as follows,

$$k_z^2 = \omega_p^2 / (v_0^2 - v_T^2) + (v_T^2 / (v_0^2 - v_T^2)) (k_x^2 + k_y^2), \quad (17)$$

provided that $v_0^2 \neq v_T^2 = 3\kappa T_e/m$. The condition $k_z = \Omega(k)/v_0$ gives

$$k_z^2 = (v^{*2} / (v_0^2 - v^{*2})) (k_x^2 + k_y^2), \quad (18)$$

provided that $v_0^2 \neq v^{*2} = 3\kappa T^*/M$. Substituting Eqs. (17) and (18) into Eq. (15), we get

$$\begin{aligned} F_z = & \frac{(Ze)^2}{2\pi} \frac{\omega_p^2}{v_0^2} \cdot \frac{v_0^2 - v_T^2}{v_0^2} \int dk_x dk_y \left[\frac{\omega_p^2}{v_0^2} + (k_x^2 + k_y^2) \right]^{-1} \\ & + \frac{(Ze)^2}{2\pi} \frac{\Omega_p^2}{v_0^2} \cdot \frac{v_0^2 - v^{*2}}{v_0^2} \int dk_x dk_y (k_x^2 + k_y^2) \left[\frac{\omega_p^2}{v_T^2} \cdot \frac{v_0^2 - v^{*2}}{v_0^2 - v_T^2} + (k_x^2 + k_y^2) \right]^{-1}. \end{aligned} \quad (19)$$

On performing the integration, using polar coordinates, the upper limits are obtained as

$$(k_x^2 + k_y^2)_{max} = ((v_0^2 - v_T^2)/v_0^2) k_e^2 - (\omega_p^2/v_0^2) \quad (20)$$

for the first integral and as

$$(k_x^2 + k_y^2)_{max} = ((v_0^2 - v^{*2})/v_0^2) k_e^2 \quad (21)$$

for the second integral, respectively. After the integration, we obtain the following result,

$$\begin{aligned} F_z = & \frac{1}{2} (Ze)^2 \left(\frac{\omega_p}{v_0} \right)^2 ((v_0^2 - v_T^2)/v_0^2) \log_e \left| \frac{v_0^2 - v_T^2}{\omega_p^2} k_e^2 \right| \\ & + \frac{1}{2} (Ze)^2 \left(\frac{\Omega_p}{v_0} \right)^2 ((v_0^2 - v^{*2})/v_0^2) \{ \log_e |1 + 3(k_e/k_d)^2| - k_e^2 / (3k_d^2 + k_e^2) \}. \end{aligned} \quad (22)$$

This is equal to the rate of energy loss per unit distance $dE/dz|_{far}$. The argument of the logarithm in the first term agrees that of Pines' formula (Eq. (48) of reference 6)), but differs in two respects from his formula. Our formula has a

factor $(v_0^2 - v_T^2/v_0^2)$ for the logarithmic term and misses the term $(v_0^2 - v_T^2) \times (v_T^2/v_0^2) k_c^2$ of Pines' formula. Since our expression Eq. (15) is reduced to Bohm-Pines' formula (Eq. (56) of reference 3)) when $T_e \rightarrow 0$, our result seems to be the most consistent generalization in taking into account the effect of the dispersive nature of the electronic plasma oscillation.

§ 4. Concluding discussions

In this elementary investigation of the problem, we have shown clearly that the electronic thermal motion has an important effect on the ionic collective motion. When the charged particle runs through a fully ionized plasma, the ionic collective motion can be excited only through the dispersive effect of the electronic collective motion. In general, we may conclude that it is essential to take into account the effect of the electronic temperature in a study of cooperative phenomena of the ions in an ionized gas.

The effect of the ionic collective motion on the energy loss is found to be smaller than that of the electronic collective motion by the order of (m/M) . Although this is an expected consequence, when the velocity of the incident particle is slower than a critical velocity v_c ,

$$v_c = (v_T^2 + \omega_p^2/k_c^2)^{1/2}, \quad (23)$$

which is derived from the conditions $k_z = \omega(k)/v_0$ and $|\mathbf{k}| < k_c$, the energy loss due to the far distance collision reflects the effects of the excitation of the ionic collective motion exclusively. Taking $k_c = k_d$ for the definiteness, we get

$$-\left. \frac{dE}{dz} \right|_{far} = 0.6 (Ze)^2 (\Omega_p^2/v_0^2) \quad (24)$$

when $v_0 < v_c$. For the specified value of $k_c = k_d$, $(\omega_p/k_z)^2$ is equal to $(v_T^2/3)$. Therefore, the expression of the energy loss, Eq. (24), seems to have a rather wide margin to be significant. Since the effect of the far distance collisions for the energy loss is known to be of the same order of magnitude as that of the short distance collision effect, we may confirm experimentally a discontinuous drop of the energy loss at the critical velocity v_c .

Another interesting phenomenon seems to occur when the incident particle has such a velocity that

$$v_0 > v_0 \gg (3\kappa T_e/M)^{1/2}, (3\kappa T_i/M)^{1/2}. \quad (25)$$

In such a case, the incident particle gains energy from the electronic collective motion and transmit the energy to the ionic collective motion. Hence, when a charged particle beam is shot into the high temperature plasma, there may arise a short circuit effect for the energy transfer between the electronic motion and the ionic motion.

We conclude this section by discussing that when one employs an electromagnetic effect to diagnose the high temperature plasma, it is very difficult to get direct information about behaviour of the ions since the electrons respond so easily to the external disturbance that masks the response of the ion. Hence, it seems to be very interesting to examine possibilities of the use of non-electromagnetic effect such as a neutron beam or an ultra-sonic sound beam for the high temperature plasma diagnosis.

The author is much obliged to Prof. N. Fukuda for his critical discussions and constant encouragement. It is also his great pleasure to thank Associate Prof. M. Yokota and Dr. M. Sato for valuable discussions.

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Letters to the Editor

The opinions expressed in these columns do not necessarily reflect those of the Board of Editors. Communications should be submitted in duplicate and should be held to within 100 lines (pica type) on standard size letter paper (approx. 21×30 cm.), so that each letter may be arranged into two pages when printed. Do not forget to count in enough space for formulas, figures or tables.

S-Wave Pion-Nucleon Interaction and Nucleon Core

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December 7, 1959

Although many remarkable results for pion-nucleon interaction have already been reported, no satisfactory explanation for *s*-wave interaction has yet been given from pion field theory. This situation may be due to the fact that *s*-wave interaction is strongly affected by the nucleon structure. To our regret, we have not a reliable theory by which the phenomena in the neighborhood of a nucleon core may be explained. The purpose of this paper is to examine how the current pion field theory should be modified in such a range as $1/M$ and to discuss the assumption which has been introduced in the previous paper.¹⁾ In order to study these problems we think it appropriate to take into consideration the *s*-wave pion-nucleon scattering.

When the real parts of the forward scattering amplitudes for π^+p and π^-p scattering are denoted by $D_+(\omega)$ and $D_-(\omega)$ respectively, the expressions for them can be written down by using the dispersion relation²⁾.

$$D^{(1)}(\omega) = [D_-(\omega) + D_+(\omega)]/2 \\ = F_1(\omega) + G_1(\omega),^* \quad (1)$$

$$D^{(2)}(\omega) = [D_-(\omega) - D_+(\omega)]/2 \\ = F_2(\omega) + G_2(\omega),^* \quad (2)$$

$$F_1(\omega) = 2f^2 \frac{(\mu^2/2M)^2 - \mu^2}{\omega^2 - (\mu^2/2M)^2} \cdot \frac{1}{2M}, \quad (3)$$

$$F_2(\omega) = -2f^2 \frac{(\mu^2/2M)^2 - \mu^2}{\omega^2 - (\mu^2/2M)^2} \cdot \frac{\omega}{\mu^2}. \quad (4)$$

In these equations, k is the wave number of pion and ω its total energy, μ and M are the masses of pion and nucleon respectively, and f is equal to $(\mu g/2M)$, where g is the renormalized pseudo-scalar coupling constant. $F_1(\omega)$ and $F_2(\omega)$ stand for the contributions from the bound state and ought to represent the second order perturbation results with renormalized coupling constant.

First of all we concentrate our attention on $F_1(\mu)$ and $F_2(\mu)$ in the case of pion momentum $k=0$. From the expressions of $D^{(2)}(\omega)$ in (2) and $D^{(1)}(\omega)$ in (1), it may be said that the effects due to the internal structure of nucleon are almost cancelled out in the former, while these effects play an important

* With regard to the expressions for $G_1(\omega)$ and $G_2(\omega)$, see reference 1).

role in the latter. Then $F_1(\mu)$ and $F_2(\mu)$ will mainly represent the contributions from the nucleon core (inner region $\sim 1/M$) and those from the pion cloud (outer region $\sim 1/\mu$) respectively. This can also be seen through the following expressions,

$$F_1(\mu) = -f^2/M, \quad F_2(\mu) = 2f^2/\mu. \quad (5)$$

Moreover, it will be possible to regard these $F_1(\mu)$ and $F_2(\mu)$ as the perturbation results in the true pion field theory.*

On the other hand, the practical straightforward calculation based on the Lorentz-covariant perturbation (the second order) gives the following results,

$$D_{\pm}(\mu) = -\frac{g^2}{M(1+\mu/M)(1\mp\mu/2M)}. \quad (6)$$

Therefore

$$\begin{aligned} D^{(1)}(\mu) &\cong -\frac{g^2}{M(1+\mu/M)} \\ &\cong -g^2/M, \\ D^{(2)}(\mu) &\cong -\frac{g^2(\mu/2M)}{M(1+\mu/M)} \cong 2f^2/\mu. \end{aligned} \quad (7)$$

$D^{(2)}(\mu)$ in Eq. (7) is almost identical with $F_2(\mu)$, while $D^{(1)}(\mu)$ in Eq. (7) differs from $F_1(\mu)$. This discrepancy may be due to the incorrect application of the theory to the region of nucleon core which has been established in the region of pion cloud, and may correspond to the fact that the usual renormalized coupling constant g differs from

the g_s which has been defined by Deser, Thirring and Goldberger.³⁾ At any rate, comparing Eq. (5) with Eq. (7), we can see that the phenomena in the neighborhood of pion cloud can be explained by the current pion field theory, while those in the neighborhood of a nucleon core cannot be explained. This defect can, however, be amended if only g^2 in the expression of $D^{(1)}(\mu)$ in Eq. (7) is replaced by f^2 . From this result we may conclude as follows. *The correct dynamics should have such a property as the effective coupling constant g is reduced to $(\mu/2M)g$ in appearance when the dynamics is applied to the problems in the region of a nucleon core.* When pv -coupling theory is adopted, we obtain the similar one with the expression in (6) except the difference in coupling constant between f^2 and g^2 . Thus it may be said that *pv -coupling theory is promising in describing the behavior of the interaction in a nucleon core.*

In the previous paper,¹⁾ we have considered the π - N scattering in the limiting case of $\mu=0$ and $\omega \rightarrow 0$ in order to derive a significant quantity from $G_1(\omega)$. In the same limiting case Deser, Thirring and Goldberger³⁾ have shown the following result:

$$D_+(0) = \bar{D}_-(0) = -g_s^2/M, \quad (8)$$

where g_s^2 differs from the usual renormalized one and turns out to be very small. But it seems to be difficult to obtain the correct value of g_s^2 . We think their result is very natural, because it will be difficult to estimate the contribution from a nucleon core which may play the most important role in

* It must be noted that the values of s -wave phase shifts obtained from the relation of (5) qualitatively agree fairly well with the experimental ones.

the scattering in the limiting case of $\mu=0$. But, if the practical effect brought by the application of the correct dynamics to the phenomena in the region of a nucleon core is expressed in terms of the change of the value of coupling constant as was mentioned above, the unknown value of g_s^2 should be taken equal to f^2 . This is a basis of the assumption $g_s^2=f^2$ which has been introduced in the previous paper.¹⁾ Finally we should like to emphasize that, as was shown previously,¹⁾ the experimental result for s -wave phase shifts $\alpha_3=-0.11\eta$, $\alpha_1=0.16\eta$ ⁴⁾ can be explained by the calculation based on this assumption, where η is the pion momentum in the center of mass system in units of μ .

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On the Eigenvalue Problem Associated with the Solution of Generalized Diffusion Equation

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The solution f of the generalized diffusion equation under no external forces,

$$\frac{\partial f}{\partial t} = \frac{1}{\sqrt{g}} \sum_{\alpha} \frac{\partial}{\partial q^{\alpha}} \left\{ \sqrt{g} \sum_{\beta} D^{\alpha\beta} \right. \\ \left. \times \left(\frac{\partial f}{\partial q^{\beta}} + \frac{f}{kT} \frac{\partial V}{\partial q^{\beta}} \right) \right\}, \quad (1)$$

must satisfy the associated equation:¹⁾

$$\frac{\partial}{\partial t} \{ \exp(V/2kT) f \} \\ = L \{ \exp(V/2kT) f \}, \quad (1')$$

where L is a self-adjoint operator:

$$L = \frac{1}{\sqrt{g}} \sum_{\alpha} \frac{\partial}{\partial q^{\alpha}} \left(\sqrt{g} \sum_{\beta} D^{\alpha\beta} \frac{\partial}{\partial q^{\beta}} \right) \\ + \frac{1}{2kT} \frac{1}{\sqrt{g}} \sum_{\alpha} \left\{ \frac{\partial}{\partial q^{\alpha}} \left(\sqrt{g} \right. \right. \\ \left. \left. \times \sum_{\beta} D^{\alpha\beta} \frac{\partial V}{\partial q^{\beta}} \right) \right\} \\ - \frac{1}{(2kT)^2} \sum_{\alpha} \sum_{\beta} D^{\alpha\beta} \frac{\partial V}{\partial q^{\alpha}} \frac{\partial V}{\partial q^{\beta}}, \quad (2)$$

g is the metric determinant in the space of generalized co-ordinates q^{α} 's, $D^{\alpha\beta}$ the diffusion tensor, and V the internal potential energy. Expanding f in terms of ϕ_{λ} , the orthonormal eigenfunction of L :

$$L\phi_{\lambda} + \lambda\phi_{\lambda} = 0, \\ \int \phi_{\lambda}^* \phi_{\mu} \sqrt{g} \prod_{\alpha} dq^{\alpha} = \delta_{\lambda\mu}, \quad (3)$$

we may have

$$f = \exp(-V/2kT) \\ \times \sum_{\lambda} c_{\lambda}(0) \exp(-\lambda t) \phi_{\lambda}, \quad (4)$$

where the summation is expectantly taken over $\lambda \geq 0$.²⁾ In fact, based on Eq. (2), we can verify that the eigenvalue of L is limited to

$$\lambda \geq 0, \quad (5)$$

and so the terms of $\lambda < 0$ never appear in Eq. (4). The proof is as follows.

Under no external forces, the irreversible entropy production of this system at the constant temperature T is given by²⁾

$$\begin{aligned} T \left(\frac{dS}{dt} \right)_{irr.} &= - \frac{d}{dt} \int (V + kT \log f) \\ &\quad \times f \sqrt{g} \prod_{\alpha} dq^{\alpha} \\ &= kT \int \sum_{\alpha} \sum_{\beta} D^{\alpha\beta} \left(\frac{\partial \log f}{\partial q^{\alpha}} + \frac{1}{kT} \frac{\partial V}{\partial q^{\alpha}} \right) \\ &\quad \times \left(\frac{\partial \log f}{\partial q^{\beta}} + \frac{1}{kT} \frac{\partial V}{\partial q^{\beta}} \right) f \sqrt{g} \prod_{\alpha} dq^{\alpha} \\ &\geq 0, \end{aligned} \quad (6)$$

where the inequality is valid at any time t for any function $f(q, t) \geq 0$ satisfying Eq. (1) and also for any function $f(q, 0) \geq 0$ satisfying the normalizing condition

$$\int f(q, 0) \sqrt{g} \prod_{\alpha} dq^{\alpha} = 1, \quad (7)$$

q denoting the abbreviation of q^{α} 's.

On the other hand, integrating by parts, we can have for any complex

$$\begin{aligned} \phi(q) &= \varphi_1(q) + \varphi_2(q) \\ \int \phi^* L \phi \sqrt{g} \prod_{\alpha} dq^{\alpha} \\ &= \int (\varphi_1 L \varphi_1 + \varphi_2 L \varphi_2) \sqrt{g} \prod_{\alpha} dq^{\alpha} \\ &= - \int \sum_{\alpha} \sum_{\beta} D^{\alpha\beta} \left\{ \left(\frac{\partial \log \varphi_1}{\partial q^{\alpha}} + \frac{1}{2kT} \frac{\partial V}{\partial q^{\alpha}} \right) \right. \\ &\quad \times \left(\frac{\partial \log \varphi_1}{\partial q^{\beta}} + \frac{1}{2kT} \frac{\partial V}{\partial q^{\beta}} \right) \varphi_1^2 \\ &\quad + \left(\frac{\partial \log \varphi_2}{\partial q^{\alpha}} + \frac{1}{2kT} \frac{\partial V}{\partial q^{\alpha}} \right) \\ &\quad \times \left. \left(\frac{\partial \log \varphi_2}{\partial q^{\beta}} + \frac{1}{2kT} \frac{\partial V}{\partial q^{\beta}} \right) \varphi_2^2 \right\} \\ &\quad \times \sqrt{g} \prod_{\alpha} dq^{\alpha}, \end{aligned} \quad (8)$$

where φ_1 and φ_2 are arbitrary real functions of q . Choosing φ_1 and φ_2 proportional to $\{f(q, 0)\}^{1/2}$ and using Eq. (6), we can easily see non-negativeness of Eq. (8), and then

$$\int \phi^* L \phi \sqrt{g} \prod_{\alpha} dq^{\alpha} / \int \phi^* \phi \sqrt{g} \prod_{\alpha} dq^{\alpha} \leq 0 \quad (8')$$

leads to Eq. (5).

Thus we can have²⁾

$$\sum_{\lambda \geq 0} \phi_{\lambda}^*(q_0) \phi_{\lambda}(q) = \frac{1}{\sqrt{g(q_0)}} \prod_{\alpha} \delta(q^{\alpha} - q_0^{\alpha}), \quad (9)$$

where the terms of $\lambda < 0$ never appear.

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On the Test of Global Symmetry*

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In this note, we consider the absorption process of K^- -meson by proton at low energy and give a method to test the global symmetry under the following assumptions:

(I) Intermediate \bar{K} - N states and processes $\pi + \Sigma \rightleftharpoons \pi + \Lambda^{**}$ are neglected.

* This work has already been published in the *Soryûshiron Kenkyû* **20** (1959), 197.

(Mimeographed informal circular in Japanese).

** S -wave scattering for this process does not occur in the order of $1/M$.

(II) Only S -wave is taken into consideration.*

(III) Global symmetry is valid.

$$\begin{aligned} K^- + P &\rightarrow \Sigma^{\pm} + \pi^{\mp} \\ K^- + P &\rightarrow \Lambda^0 + \pi^0. \end{aligned} \quad (1)$$

Transition matrix for the process (1) may be written as

$$\begin{aligned} \langle \pi Y | T | K^- P \rangle &= -\langle \pi Y | S | \pi Y \rangle \\ &\times \langle \pi Y | T^+ | K^- P \rangle \end{aligned} \quad (2)$$

where unitarity of S -matrix and assumption (I) are used. Using the assumption (II) and the invariance under the Wigner time reversal, we obtain the following relation for the phase in eigenchannel of Eq. (3),

$$e^{i\Delta_I} = -e^{2i\delta_I} e^{-i\Delta_I}$$

$$\text{or } \Delta_I = \delta_I + (n+1/2)\pi \quad (3)$$

where I is total isospin, Δ_I and δ_I represent the absorption phase and the π - Y scattering phase shift in the I -state, respectively. If we consider the π - Y scattering in the final state of the process (1), π interacts with isodoublet $Y = (\Sigma^+, Y^0)$ and $Z = (Z^0, \Sigma^-)$ in the same form as π interacts with nucleon by the assumption (III), where $Y^0 = (\Sigma^0 - \Lambda^0)/\sqrt{2}$ and $Z^0 = (\Sigma^0 + \Lambda^0)/\sqrt{2}$.

On the other hand, π - Σ scattering occurs in the $I=0, 1, 2$, states and π - Λ scattering does in the $I=1$ state, so we can obtain the relation between these two cases. We denote the π - Σ and π - Λ scattering phase shifts as δ_0 ,

δ_1 , δ_2 and δ_1' respectively. π - Y and π - Z scattering phase shifts are denoted by $\alpha_{1/2}$ and $\alpha_{3/2}$. We can expand the scattering S -matrix by the projection operator of the $I=1/2$ and $3/2$ states.

$$S = P_{1/2} e^{2i\alpha_{1/2}} + P_{3/2} e^{2i\alpha_{3/2}}. \quad (4)$$

Isospin states of $(\pi\Sigma)$ and $(\pi\Lambda)$ states are denoted by

$$\begin{aligned} |\pi\Sigma; I=0\rangle &= |\Sigma 0\rangle \\ |\pi\Sigma; I=1\rangle &= |\Sigma 1\rangle, \text{ etc.} \end{aligned} \quad (5)$$

Using (5), we can evaluate the S -matrix element for the process (1),

$$e^{2i\delta_0} = \langle \Sigma 0 | S | \Sigma 0 \rangle = e^{2i\alpha_{1/2}} \quad (6a)$$

$$\begin{aligned} e^{2i\delta_1} &= \langle \Sigma 1 | S | \Sigma 1 \rangle \\ &= (2e^{2i\alpha_{1/2}} + e^{2i\alpha_{3/2}})/3 \text{ etc.} \end{aligned} \quad (6b)$$

From (6a) or (6b), we obtain

$$\cos 2(\alpha_{1/2} - \alpha_{3/2}) = 1. \quad (7)$$

From (6a) and (6b),

$$\begin{aligned} \cos 2(\delta_1 - \delta_0) &= 5/9 + (4/9) \cos 2(\alpha_{1/2} - \alpha_{3/2}). \end{aligned} \quad (8)$$

From (7) and (8), we get

$$\cos 2(\delta_1 - \delta_0) = 1,$$

$$\text{so } |\delta_1 - \delta_0| = n\pi \quad n=0, 1, 2, \dots$$

If we take $n=0$,

$$|\delta_1 - \delta_0| = 0. \quad (9)$$

Even if we take $n \neq 0$, the following results would not be altered. (3) and (9) lead to

$$|\delta_1 - \delta_0| = |\Delta_1 - \Delta_0| = 0. \quad (10)$$

After all, we have shown that the difference of absorption phases between $I=0$ and 1 is equal to that of scattering phase shifts and that the difference vanishes exactly.

* Present experimental results in the low energy region show that the S -wave is predominant.

On the other hand, the most recent experimental data¹⁾ have shown that

$$|\Delta_1 - \Delta_0| \simeq 62^\circ \pm 4^\circ. \quad (11)$$

This result is quite inconsistent with the relation (10). In Fig. 1, we have

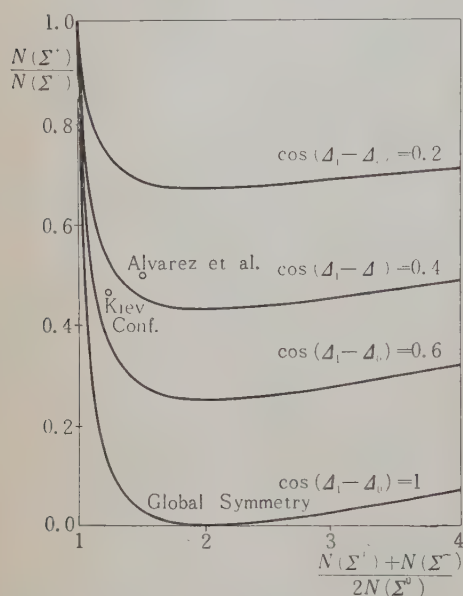


Fig. 1. $N(\Sigma^+)/N(\Sigma^-)$ and $\{N(\Sigma^+) + N(\Sigma^-)\}/2N(\Sigma^0)$ taking $\cos(\Delta_1 - \Delta_0)$ as parameter.

shown the relation between $N(\Sigma^+)/N(\Sigma^-)$ and $\{N(\Sigma^+) + N(\Sigma^-)\}/2N(\Sigma^0)$, taking $\cos(\Delta_1 - \Delta_0)$ as parameter. Circles are the experimental result by Alvarez et al.²⁾ and that reported at Kiev Conference¹⁾ (1959). If global symmetry is valid, the experimental data must be on the curve of $\cos(\Delta_1 - \Delta_0) = 1$. However, the experimental data¹⁾²⁾ are quite far from the curve of $\cos(\Delta_1 - \Delta_0) = 1$.

It will be quite difficult to understand the very large phase difference (11) with the errors due to the assumptions (I) and (II). Therefore, we may obtain rather conclusive evidence

for inconsistency of global symmetry with the present experiments. It should be noted that above conclusion was derived only by kinematical considerations and not by details of the theory. For the assumption (I), we have investigated the contribution of the intermediate $\bar{K}-N$ states of Eq. (2) and obtained the preliminary result that we would be unable to get the consistent result with experiment¹⁾²⁾ so far as we admitted the assumption (III). Details will be published in a later issue of this journal.

The authors wish to express their sincere thanks to Professor D. Ito for his kind guidance and encouragement.

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- 2) L. W. Alvarez et al., *Nuovo Cimento* **5** (1957), 1026.

Nonlocal Interaction, Causality and Integrability Condition

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December 14, 1959

Nonlocal interaction formalisms have been thought not to satisfy the microcausality condition,¹⁾ and also not to satisfy the integrability conditions when one wants to formulate them in the Lorentz-invariant formalisms.²⁾ These both facts have a close connection to each other and are one of the most formidable obstacles for the construction of S -matrix.

Then, how can we regard them to have self-consistency? Do they not satisfy any integrability condition?

It is a well-known fact that under the assumptions of usual relation between spin and statistics and of local interactions, the Lorentz invariant theories satisfy the CTP-invariance automatically and the CTP-invariance is equivalent to the condition of weak microcausality.³⁾ Does this fact not hold in the cases of nonlocal interactions, too?

As an example, we take up interaction Hamiltonian between the electromagnetic field $A_\mu(x)$ and the spinor field $\psi(x)$,

$$H_{int} = -i(e/2) \int \{ \bar{\psi}(x') \gamma_\mu A_\mu(x'') \psi(x'') \\ - \psi(x'') A_\mu(x''') \gamma_\mu^T \bar{\psi}(x') \} \\ \times d^4 x' d^4 x'' d^4 x''' \\ \times F((x' - x'')^2, (x' - x''')^2), \quad (1)$$

where F is a c -number factor commutable with all other factors and depends only on $(x' - x'')^2$ and $(x' - x''')^2$ because of Lorentz invariance. Then we have a charge-current four vector

$$j_\mu(x) = i(e/2) \int \{ \bar{\psi}(x') \gamma_\mu \psi(x'') \\ - \psi(x'') \gamma_\mu^T \bar{\psi}(x') \} d^4 x' d^4 x'' \\ \times F((x' - x'')^2, (x' - x'')^2). \quad (2)$$

Now, if we adopt the same transformation properties of field operators under the CTP-inversion as in the case of local interaction, $j_\mu(x)$ changes under the CTP-inversion as usual:

$$j_\mu'(x) = i(e/2) \int \{ \psi'(x'') \gamma_\mu^T \bar{\psi}'(x') \\ - \bar{\psi}'(x') \gamma_\mu \psi'(x'') \} d^4 x' d^4 x'' \\ \times F((x' - x'')^2, (x' - x'')^2)$$

$$= -i(e/2) \int \{ e^{i\alpha} \gamma_5 \psi(-x'') \gamma_\mu^T e^{-i\alpha} \\ \times \bar{\psi}(-x') \gamma_5 - e^{-i\alpha} \bar{\psi}(-x') \gamma_5 \gamma_\mu e^{i\alpha} \\ \times \gamma_5 \psi(-x'') \} d^4 x' d^4 x'' \\ \times F((-x' + x'')^2, (-x' + x'')^2) \\ = -ie/2 \int \{ \bar{\psi}(x') \gamma_\mu \psi(x'') \\ - \psi(x'') \gamma_\mu^T \bar{\psi}(x') \} d^4 x' d^4 x'' \\ \times F((x' - x'')^2, (x' + x'')^2) \\ = -j_\mu(-x). \quad (3)$$

The vacuum expectation values of products of field operators are the same as in the local case and the vertex $\gamma_\mu F((x' - x'')^2, (x' - x''')^2)$ is CTP-invariant.

Consequently, in this case the CTP-theorem and, therefore, the weak microcausality hold as usual, i.e., for self adjoint, real fields $A(x)$ and $B(x')$,

$$\langle [A(x), B(x')] \rangle_0 = 0$$

$$\text{for } (x - x')^2 > 0. \quad (4)$$

So, in this case, the commutators of observable quantities are found to be zero for space-like points if we take the vacuum expectation values of them even though the original commutators are not zero.

From this fact we may conjecture that in the cases of nonlocal interaction formalisms, even though the integrability conditions are not satisfied, when we take the vacuum expectation values of them, they are satisfied and we may have a hint for the construction of S -matrix. The details of this problem will be discussed elsewhere.

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On the Extremum Property in the Variation Principle in the Theory of Transport Processes

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December 26, 1959

We have recently introduced a variation principle in the theory of transport processes¹⁾ and also in that of electric or magnetic susceptibility or relaxation phenomena.²⁾ We can develop the problem a little further. We shall here investigate on the transport phenomena and discuss on the latter problem in the next short note. On the problem of transport phenomena the variation principle can be reduced to the form quite analogous to the Kohler-Sondheimer principle³⁾ concerning the Boltzmann equation.

As we have shown,¹⁾ any transport coefficient σ_{AA} and the internal entropy production θ in the system are obtained respectively as the extremum value of the functionals of state operators $\Psi^{(+)}$ and $\Psi^{(-)}$,

$$\begin{aligned} &\sigma_{AA}(\Psi^{(+)}, \Psi^{(-)}) \text{ or } \theta(\Psi^{(+)}, \Psi^{(-)}) \\ &= \int_0^\beta d\lambda \operatorname{Tr} \{ \Psi^{(-)} \rho_c e^{\lambda H} (s \Psi^{(+)} \\ &\quad + i[H, \Psi^{(+)}]) e^{-\lambda H} \\ &\quad + (\Psi^{(+)} - \Psi^{(-)}) \rho_c e^{\lambda H} Q e^{-\lambda H} \}, \end{aligned} \quad (1)$$

with

$$Q = J_A \text{ for } \sigma_{AA}(\Psi^{(+)}, \Psi^{(-)}),$$

or

$$Q = \sum_A X_A J_A \text{ for } \theta(\Psi^{(+)}, \Psi^{(-)}), \quad (1a)$$

where H is the Hamiltonian of the isolated system, the operator J_A represents the flow, e.g., the electric (or the matter) current \mathbf{J}_e (or $\mathbf{J}_m = \mathbf{J}_e/e$) or the energy current \mathbf{J}_w , X_A is the generalized force conjugate to \mathbf{J}_A : $\mathbf{J}_e \leftrightarrow \mathbf{E} - (T/e) \operatorname{grad}(\zeta/T)$, $\mathbf{J}_w \leftrightarrow T \operatorname{grad}(1/T)$, and ρ_0 represents the equilibrium or the grand canonical distribution. By making use of the substitution

$$\begin{aligned} \Psi^{(\pm)} &= \pm \frac{1}{2\pi} \int_{-\infty}^{\infty} dE R^{(\pm)}(E) \Psi R^{(\mp)}(E), \\ R^{(\pm)}(E) &= \frac{1}{E - H \pm i(s/2)}, \end{aligned} \quad (2)$$

we can rewrite the expression (1) as

$$\begin{aligned} &\sigma_{AA}(\Psi) \text{ or } \theta(\Psi) = -(\Psi, L\Psi) \\ &\quad + 2(\Psi, U), \end{aligned} \quad (3)$$

where we have defined the inner product

$$\begin{aligned} &(\Phi, \Psi) = (\Psi, \Phi) \\ &\equiv \int_0^\beta d\lambda \operatorname{Tr} \{ \Phi \rho_c e^{\lambda H} \Psi e^{-\lambda H} \}, \end{aligned} \quad (4)$$

and the operation

$$L\Psi \equiv \frac{1}{4\pi} \int_{-\infty}^{\infty} dE (R^{(+)} \Psi R^{(-)} + R^{(-)} \Psi R^{(+)}), \quad (5)$$

and have made use of the abbreviation $U \equiv LQ$. By substituting (5) into (4), we get in the limit $s \rightarrow +0$,

$$(\Phi, L\Psi) = \frac{1}{4\pi} \int_{-\infty}^{\infty} dE \int_0^\beta d\lambda \operatorname{Tr} \{ \Phi \rho_c e^{\lambda H}$$

$$\begin{aligned} & \times (R^{(+)}\mathcal{T}R^{(-)} + R^{(-)}\mathcal{T}R^{(+)})e^{-\lambda H}\} \\ &= \frac{1}{2} \int_0^\infty dt \int_0^\beta d\lambda \operatorname{Tr} [\rho_0 \Phi \{ \mathcal{T}(t+i\lambda) \\ &+ \mathcal{T}(-t+i\lambda) \}] = (\mathcal{T}, L\Phi), \end{aligned} \quad (6)$$

which shows the self-adjoint property of the operation L . We can see also the non-negative property

$$(\mathcal{T}, L\mathcal{T}) \geq 0 \quad (7)$$

for any hermitian operator \mathcal{T} . Based upon the properties (6) and (7) of the operation L , we can now prove that the extremum of (3) is nothing but the maximum. Thus the formal feature of our principle is quite analogous to the Kohler-Sondheimer principle, although there are essentially remarkable differences between them. It is possible to suggest the variation principles in our case in the same way as in the Kohler-Sondheimer principle. We suggest them here in the following forms:

- (I) $(\mathcal{T}, L\mathcal{T}) = \text{maximum}$, under the condition $(\mathcal{T}, L\mathcal{T}) = (\mathcal{T}, U)$.
- (II) $-(\mathcal{T}, L\mathcal{T}) + 2(\mathcal{T}, U) = \text{maximum}$.
- (III) $(\mathcal{T}, L\mathcal{T})(\mathcal{T}, U)^{-2} = \text{minimum}$.

The maximum values in (I) and (II) give the transport coefficient σ_{AA} (conductivity) or the internal entropy production θ , and the minimum in (III) gives their reciprocals (resistance). If we take $s = +0 + i\omega$ (ω real), we can make the same argument concerning the real and imaginary parts of (6). We shall soon write its detailed report in this journal.

I should like to express my cordial thanks to Professors K. Tomita and T. Matsubara and other members of Rese-

arch Institute for Fundamental Physics, Kyoto University, for their kind and helpful discussions. I also wish to express my heartfelt thanks to Professor T. Matsubara who had advised me to visit his Institute.

- 1) H. Nakano, *Prog. Theor. Phys.* **22** (1959), 453; **23** (1960), 180.
- 2) H. Nakano, *Prog. Theor. Phys.* **23** (1960), 182.
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On the Extremum Problem in the Variation Principle in the Theory of Susceptibility or Relaxation Phenomena

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December 28, 1959

In the previous note¹⁾ we developed a step further the variation principle in the theory of transport processes we had suggested before in this journal.²⁾ We shall here make the same investigation on the problem of susceptibility or the relaxation phenomena. For the sake of definiteness we shall discuss the magnetic problem.

As we have shown,³⁾ the magnetic susceptibility $\chi_{\nu\nu}$ and also the increase of internal energy U_H due to the application of magnetic field can be obtained respectively as the extremum values of functionals of state operators $\Phi^{(+)}$ and $\Phi^{(-)}$,

$$\begin{aligned} \chi_{\nu\nu}(\Phi^{(+)}, \Phi^{(-)}) \text{ or } U_H(\Phi^{(+)}, \Phi^{(-)}) \\ = \text{Tr}\{\Phi^{(-)}, [H, \Phi^{(+)}] - i s \Phi^{(+)}\} \\ + i(\Phi^{(+)} + \Phi^{(-)})[\rho_c, Q]\}, \end{aligned} \quad (1)$$

where Q represents the operator

$$Q \equiv \begin{cases} M_\nu & \text{for } \chi_{\nu\nu}(\Phi^{(+)}, \Phi^{(-)}), \\ - \sum_{\nu=x,y,z} H_\nu M_\nu & \text{for } U_H(\Phi^{(+)}, \Phi^{(-)}). \end{cases} \quad (1a)$$

$$(1b)$$

Here the operator M_ν stands for the ν -component ($\nu=x, y, z$) of magnetization per unit volume of the system and H_ν for the applied magnetic field. H and ρ_c represent respectively the Hamiltonian and the equilibrium or grand canonical distribution of the system in the absence of the external field. By making the substitution

$$\begin{aligned} R^{(\pm)} = \pm \frac{1}{2\pi} \int_{-\infty}^{\infty} R^{(\pm)}(E) \Phi R^{(\mp)}(E) dE, \\ R^{(\pm)}(E) = \frac{1}{E - H \pm i(s/2)}, \end{aligned} \quad (2)$$

we can rewrite the expression (1) as

$$\begin{aligned} \chi_{\nu\nu}(\Phi) \text{ or } U_H(\Phi) = - \frac{1}{4\pi} \int_{-\infty}^{\infty} dE \\ \times \text{Tr}\{[\rho_c, \Phi](R^{(+)}\Phi R^{(-)} - R^{(-)}\Phi R^{(+)}) \\ + 2[\rho_c, \Phi](R^{(+)}Q R^{(-)} - R^{(-)}Q R^{(+)})\} \\ = \frac{i}{2} \int_{-\infty}^{\infty} dt e^{-st} \text{Tr}\{-[\rho_c, \Phi] \\ \times \{\Phi(t) - \Phi(-t)\} \\ + 2[\rho_c, \Phi]\{Q(t) - Q(-t)\}\}. \end{aligned} \quad (3)$$

We now define the inner product of any two operators Φ and Ψ

$$(\Psi, \Phi) = (\Phi, \Psi) \equiv \text{Tr}(\Psi\Phi) \quad (4)$$

and the operation

$$\begin{aligned} \chi\Phi \equiv \frac{1}{4\pi} \int_{-\infty}^{\infty} [R^{(+)}\Phi R^{(-)} \\ - R^{(-)}\Phi R^{(+)}, \rho_c] dE. \end{aligned} \quad (5)$$

Then we get

$$\begin{aligned} (\Psi, \chi\Phi) = \frac{1}{4\pi} \int_{-\infty}^{\infty} dE \text{Tr}\{[\rho_c, \Psi] \\ \times (R^{(+)}\Phi R^{(-)} - R^{(-)}\Phi R^{(+)})\} \end{aligned} \quad (6)$$

with regard to which we can easily prove the self-adjoint and non-negative properties,

$$(\Phi, \chi\Psi) = (\Psi, \chi\Phi), \quad (\Phi, \chi\Phi) \geq 0 \quad (7)$$

for any hermitian operators Φ and Ψ . The expression (3) can now be rewritten as

$$\begin{aligned} \chi_{\nu\nu}(\Phi) \text{ or } U_H(\Phi) = -(\Phi, \chi\Phi) \\ + 2(\Phi, X) \end{aligned} \quad (8)$$

where $X \equiv \chi Q$. In the limit $s \rightarrow +0$, the expression (8) or (3) takes the maximum value when the state operator is the solution or the Schrödinger equation of the system in the presence of the applied magnetic field. In the same way as in the previous note³⁾ we can express the variation principle in different forms,

(I) $(\Phi, \chi\Phi) = \text{maximum}$ under the condition $(\Phi, \chi\Phi) = (\Phi, X)$,

(II) $-(\Phi, \chi\Phi) + 2(\Phi, X) = \text{maximum}$,

(III) $(\Phi, \chi\Phi)(\Phi, X)^{-2} = \text{minimum}$,

where the maximum values in (I) and (II) are the susceptibility $\chi_{\nu\nu}$ or the internal energy increase U_H and the minimum values in (III) are their reciprocals. In the case $s = +0 + i\omega$ (ω real), where the absorption and relaxation phenomena appear, we have to

investigate the real and positive parts and consider the variation principle with regard to these parts. The detailed report will soon be published in this Journal.

I should like to express my sincere thanks to Professor T. Matsubara who gave me the occasion to visit the Re-

search Institute for Fundamental Physics of Kyoto University.

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Formulas in the Fermi Theory of Beta Decay. II*—On the Beta Ray Angular Correlation—*

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Our previous work on beta spectrum, which has taken into account the finite charge distribution of the nucleus and the screening effect, is extended to the beta ray angular correlation. Formulas are given for the allowed and the first forbidden beta-gamma angular correlations including polarization of the gamma ray, and the second forbidden beta-gamma directional correlation, where we assume the mixed interaction of vector and axial vector types (VA) and the non-conservation of parity in beta decay. Since these formulas for the allowed and the first forbidden transitions have already been given by Morita (M.) and Morita (R. S.) for the case of the point nuclear charge, we only rewrite them in this paper. If we use the effective radii theory again, our final formulas can be expressed in forms similar to the familiar expressions for light nuclei in the usual formulations. The deviations of the coefficients from the usual ones are graphically shown as functions of beta ray energies for $Z=\pm 10, \pm 30, \pm 50, \pm 70$ and $Z=\pm 90$ (for β^\mp -decay), and significant deviations are found for heavy nuclei except for the allowed transition. Those are especially significant for the non-unique first forbidden, and the unique first and second forbidden transitions (several ten percents for $Z\sim 90$). One of the important results is that our theory is again insensitive to the nuclear charge distributions.

§ 1. Introduction

Since the discovery of non-conservation of parity in beta decay,¹⁾ much experimental data on beta ray angular correlations, which include the measurement of the polarization of both gamma ray and nuclear spin, have been accumulated. As the results, the validity of the two-components theory of neutrino²⁾ and the universal Fermi interaction ($V-A$ type³⁾) are now decisive.⁴⁾ Furthermore, recent analyses⁵⁾ and experiments⁶⁾ show that the interaction Hamiltonian for beta decay is probably invariant under time reversal,⁷⁾ and now most of the fundamental features of beta decay can be explained.

On the other hand, there is another important problem, that is, the finiteness of the nuclear charge distribution in a nucleus has very serious effect⁸⁾ on the β -decay of heavy nuclei especially in forbidden transitions. Many authors have investigated this effect with appropriate assumptions for charge distribution,⁹⁾ but most of their treatments require very laborious calculations. The accuracy of the result is, however, not so improved as one may expect, because of the uncertainties of the nuclear matrix elements and the associated approximations.

Yamada and the author¹⁰⁾ have obtained formulas for beta spectrum, which include most of the important corrections in the theory of beta decay synthetically. The corrections are (1) the finite nuclear size effect in a narrow sense,⁸⁾ (2) the finite de Broglie wave length effect,¹¹⁾ (3) the effect of the decay in the internal region of the nucleus¹²⁾ and (4) the screening effect of atomic electrons (though their values are not given explicitly). The result is expressed in a very practical form similar to those in Konopinski and Uhlenbeck's low- Z approximations. The errors involved were investigated in detail and for almost all nuclei we can use the formulas within errors of several percent. One of the important characteristics of our theory is that our formulas do not depend explicitly on the nuclear charge distribution.

The aim of this paper is to extend our previous work on beta spectrum to the case of beta-ray angular correlations. In their recent works Kotani and Ross⁹⁾ treated the finite nuclear size effect in a similar way to ours, and their treatment in their second paper on β - γ angular correlation corresponds to the content of the present paper. Many authors have already derived formulas for the allowed and forbidden transitions,^{13),14)} but most expressions were given on the assumption that the nucleus can be treated as a point charge and with the associated approximations. We can see that the deviations from these simple formulas become considerably large for heavy nuclei, and, for instance, the coefficient of $P_2(\cos \theta)$ terms is reduced about 50 percent in the case of Tm^{170} as was shown by Dolginov,¹⁵⁾ where θ is the angle between the beta and gamma rays and we took the same magnitudes of the nuclear matrix elements. The present analysis also shows the large deviations from their simplest formulas clearly, especially for the non-unique first forbidden transitions. In § 2 we investigate the asymptotic behaviour of the electron wave functions at infinity. We find that we can approximately describe the Coulomb phase shifts by those in the case of point nuclear charge, for the effect of the finite nuclear charge distribution of the nucleus to it is very small. In § 3 we obtain formulas for the first forbidden beta transitions, by assuming VA interaction and non-conservation of parity. Formulas for the allowed and the second forbidden transitions are given in the Appendix. We follow the notation and formulation by Yamada and Morita,¹³⁾ and Morita and Morita¹⁴⁾ mainly. All the effects of the finite nuclear size and the screening are included in the particle parameters, $b_{LL}^{(n)}$'s, and special attention is paid to the effect (3). The final formulas are expressed in the familiar forms similar to those for light nuclei of the usual theories (e.g., Ref. 14)), and some factors which represent the deviations from it are shown graphically in Figs. 1a-5c for β^\mp -decay. It will be understood that the errors in the usual approximations, except that of the allowed transition, are quite large for heavy nuclei. Especially for the non-unique first forbidden transition and the unique first and second forbidden transitions the deviations amount to a factor 1/2 or so. One of the interesting features of our theory is again the insensitivity of the nuclear charge distribution. § 4 is prepared for persons not interested in the derivation of formulas but only in their application. Some concluding remarks are given in § 5.

§ 2. Asymptotic form of electron wave function at infinity

In the theory of angular correlations, the effect of the finite charge distribution of the nucleus is included in modification of electron wave function inside the nucleus and of Coulomb phase shift of the electron at infinity. The former effect has already been given by Eqs. (13) and (14) of Ref. 10) for arbitrary spherically symmetric charge distribution, and the large differences between the wave functions for the uniform and surface charge distributions and those for the point nuclear charge have been shown graphically in Figs. 3a–6b.

Now we have to derive the asymptotic form of electron wave function at infinity, by taking into account the finite nuclear size effect. The wave function outside the nucleus is expressed as a linear combination of the regular and irregular wave functions; $f_{\kappa}^{reg}(r)$, $g_{\kappa}^{reg}(r)$, $f_{\kappa}^{irr}(r)$ and $g_{\kappa}^{irr}(r)$

$$\begin{aligned} f_{\kappa}(r) &= a_{\kappa}[1 + \Delta_{\kappa}(r)]f_{\kappa}^{reg}(r) + b_{\kappa}[1 + \Delta_{\kappa}^{irr}(r)]f_{\kappa}^{irr}(r), \\ g_{\kappa}(r) &= a_{\kappa}[1 + \Delta_{\kappa}(r)]g_{\kappa}^{reg}(r) + b_{\kappa}[1 + \Delta_{\kappa}^{irr}(r)]g_{\kappa}^{irr}(r), \end{aligned} \quad : r < \rho \quad (1)$$

where r is distance between the electron and nuclear center and ρ is defined as it means that there is no charge at $r > \rho$, and $f_{\kappa}(r)$ and $g_{\kappa}(r)$ are the small and large components of the radial wave functions, respectively. The quantum number κ is defined as follows; $\kappa = \mp(j+1/2)$ according to $j = l \pm 1/2$, where j is the magnitude of total angular momentum and l is that of orbital angular momentum of the large component. The radial wave functions are normalized so that there are R electrons in a large sphere of radius R . $\Delta_{\kappa}(r)$ and $\Delta_{\kappa}^{irr}(r)$ are correction terms due to the screening effect (see Eq. (7) of Ref. 10)) and if this effect can be safely omitted (i.e., except for the region of small beta-ray energies) we can put $\Delta_{\kappa} = \Delta_{\kappa}^{irr} = 0$. The coefficients, a_{κ} and b_{κ} , are easily obtained by connecting (1) with inner wave functions (Eq. (9) or (14) of Ref. 10)) smoothly at the boundary of the charge distribution and by using the normalization condition at infinity.

Asymptotic forms of $f_{\kappa}^{reg}(r)$ and $g_{\kappa}^{reg}(r)$ at infinity was derived by Rose as follows :

$$\begin{aligned} r f_{\kappa}^{reg}(r) &\cong -\sqrt{\frac{W-1}{W}} \sin(pr + \delta_{\kappa}), \\ r g_{\kappa}^{reg}(r) &\cong \sqrt{\frac{W+1}{W}} \cos(pr + \delta_{\kappa}), \end{aligned} \quad : r \rightarrow \infty \quad (2)$$

$$\begin{aligned} \delta_{\kappa} &= -\arg \Gamma(\gamma_{\kappa} + i\alpha ZW/p) + \eta_{\kappa} - \frac{\pi}{2} \gamma_{\kappa} + \alpha ZW/p \cdot \ln 2pr, \\ e^{+i\eta_{\kappa}} &= -(k + i\alpha Z/p)^{1/2} / (\gamma_{\kappa} + i\alpha ZW/p)^{1/2}, \\ e^{i\eta_{\kappa}} &= -i(k - i\alpha Z/p)^{1/2} / (\gamma_{\kappa} + i\alpha ZW/p)^{1/2}, \end{aligned} \quad (3)$$

where W and p are the energy and momentum of electron in natural unit ($\hbar = c = 1$, and electron mass is equal to unity), and α is the fine structure constant, Z the atomic number of the daughter nucleus, and $\gamma_{\kappa} = (k^2 - \alpha^2 Z^2)^{1/2}$. $k = |\kappa|$ is the positive integer.

$\arg \Gamma(\gamma_k + i\alpha ZW/p)$ means the argument of complex gamma function. The expression for $e^{i\eta_k}$ was already used in the previous paper in the process of the derivation of the asymptotic expansions near the nucleus (Eq. (4) of Ref. 10)) from the original expression by Rose.¹⁶⁾ Since uncertainty of the signs in $e^{i\eta_k}$ causes no trouble after all, we shall use the above choice hereafter.

Asymptotic forms of $f_\kappa^{irr}(r)$ and $g_\kappa^{irr}(r)$ are similarly given as follows:

$$\begin{aligned} rf_\kappa^{irr}(r) &\cong -\sqrt{\frac{W-1}{W}} \sin(pr + \bar{\delta}_\kappa), \\ rg_\kappa^{irr}(r) &\cong \sqrt{\frac{W+1}{W}} \cos(pr + \bar{\delta}_\kappa). \end{aligned} \quad : r \rightarrow \infty \quad (4)$$

$$\begin{aligned} \bar{\delta}_\kappa &= -\arg \Gamma(-\gamma_k + i\alpha ZW/p) + \bar{\eta}_\kappa + \frac{\pi}{2} \gamma_k + \alpha ZW/p \cdot \ln 2pr, \\ e^{i\bar{\eta}-k} &= -i(k + i\alpha Z/p)^{1/2} / (\gamma_k - i\alpha ZW/p)^{1/2}, \\ e^{i\bar{\eta}k} &= (k - i\alpha Z/p)^{1/2} / (\gamma_k - i\alpha ZW/p)^{1/2}. \end{aligned} \quad (5)$$

These functions correspond to Eq. (5) of Ref. 10). If we use (3) and (5), the asymptotic forms of (1) at infinity are written as

$$\begin{aligned} rf_\kappa(r) &\cong -\sqrt{\frac{W-1}{W}} \sin(pr + \zeta_\kappa), \\ rg_\kappa(r) &\cong \sqrt{\frac{W+1}{W}} \cos(pr + \zeta_\kappa). \end{aligned} \quad : r \rightarrow \infty \quad (6)$$

$$\begin{aligned} \tan \zeta_\kappa &= \frac{\sin \bar{\delta}_\kappa + H_\kappa \sin \bar{\delta}_\kappa}{\cos \bar{\delta}_\kappa + H_\kappa \cos \bar{\delta}_\kappa}, \\ H_\kappa &= \frac{b_\kappa [1 + \mathcal{A}_\kappa^{irr}(\rho)]}{a_\kappa [1 + \mathcal{A}_\kappa(\rho)]} = -\frac{f_\kappa^{in}(\rho) g_\kappa^{reg}(\rho) - g_\kappa^{in}(\rho) f_\kappa^{reg}(\rho)}{f_\kappa^{in}(\rho) g_\kappa^{irr}(\rho) - g_\kappa^{in}(\rho) f_\kappa^{irr}(\rho)}, \end{aligned} \quad (7)^*$$

* H_κ is written explicitly as follows (see Ref. 10) for details):

$$H_{\mp k} = -\sqrt{F_{k-1}(\rho)/F_{k-1}^{irr}(\rho)} \left(\frac{kW \pm \gamma_k}{kW \mp \gamma_k} \right)^{1/2} \frac{E_{k^\mp}}{D_{k^\mp}},$$

D_{k^\mp} was given in Ref. 10), p. 298 and

$$\begin{aligned} E_{k^\mp} &= \frac{\alpha Z}{\sqrt{2k(k+\gamma_k)}} J_k(\rho) + \sqrt{\frac{k+\gamma_k}{2k}} \left(\rho^{-2k} \int_0^\rho dr \cdot r^{2k} V(r) I_k(r) \right) \\ &\quad - \frac{2(2kW \pm 1)\rho}{(2k+1)(2k-1)} \left\{ \frac{\alpha Z}{1+2\gamma_k} \left[(2k-1) \frac{\alpha Z}{\sqrt{2k(k+\gamma_k)}} J_k(\rho) \right. \right. \\ &\quad \left. \left. + (2k+1) \sqrt{\frac{k+\gamma_k}{2k}} \left(\rho^{-2k} \int_0^\rho dr \cdot r^{2k} V(r) I_k(r) \right) \right. \right. \\ &\quad \left. \left. + \sqrt{\frac{k+\gamma_k}{2k}} K_k^{(1)}(\rho) - \frac{\alpha Z}{\sqrt{2k(k+\gamma_k)}} \rho^{-1} K_k^{(2)}(\rho) \right\}, \end{aligned}$$

where $I_k(r)$, $K_k^{(1)}(r)$ and $K_k^{(2)}(r)$ are some functions which depend on the nuclear charge distribution (i.e., potential $V(r)$).

where $f_{\kappa}^{in}(r)$ and $g_{\kappa}^{in}(r)$ are the unnormalized inner wave functions of the electron given in Eq. (2) of Ref. 10). Here the expression (7) is the same as that given by Dolginov.¹⁵⁾ For the purpose of estimating the deviation of the phase shift ζ_{κ} from δ_{κ} , it is convenient to use the following equation, since, as we shall see in § 3, the phase shifts are always included in a form of subtraction between two of them.

$$\tan(\zeta_{\kappa} - \zeta_{\kappa'}) = \frac{\sin(\delta_{\kappa} - \delta_{\kappa'}) + H_{\kappa} \sin(\bar{\delta}_{\kappa} - \delta_{\kappa'}) + H_{\kappa'} \sin(\delta_{\kappa} - \bar{\delta}_{\kappa'}) + H_{\kappa} H_{\kappa'} \sin(\bar{\delta}_{\kappa} - \bar{\delta}_{\kappa'})}{\cos(\delta_{\kappa} - \delta_{\kappa'}) + H_{\kappa} \cos(\bar{\delta}_{\kappa} - \delta_{\kappa'}) + H_{\kappa'} \cos(\delta_{\kappa} - \bar{\delta}_{\kappa'}) + H_{\kappa} H_{\kappa'} \cos(\bar{\delta}_{\kappa} - \bar{\delta}_{\kappa'})}.$$

The values of H_{κ} for $Z = \pm 90$ (β^{\mp} -decay), $W=5$ and $k=1-4$ are evaluated in Table I, where we assumed the uniform charge distribution and used $\rho = 1.2 \times 10^{-13} \times A^{1/3}$ cm, and which indicates that the deviations of $\zeta_{\kappa} - \zeta_{\kappa'}$ from $\delta_{\kappa} - \delta_{\kappa'}$ are usually quite negligible, except for $\kappa = \pm 1$. For example, for $Z = \pm 90$ and $W=5$ the deviation of $\sin(\zeta_{-1} - \zeta_1)/\sin(\delta_{-1} - \delta_1)$ from unity is about 0.3 percent and for $\cos(\zeta_1 - \zeta_{-2})/\cos(\delta_1 - \delta_{-2})$ it is smaller than 0.1 percent. If time reversal invariance in beta decay is violated, such a term as $\cos(\zeta_{-1} - \zeta_1)$ appears and $\cos(\zeta_{-1} - \zeta_1)/\cos(\delta_{-1} - \delta_1)$ deviates about 20 percent from unity. For smaller Z and W , H_{κ} will decrease rapidly, since it is nearly proportional to $\alpha Z(p\rho)^{2\gamma_k}$. These facts lead to the conclusion in the next section, that if time reversal invariance is valid in beta decay, our theory is again very insensitive to the nuclear charge distribution.

Table I. Mixing ratios of irregular to regular wave functions of electron
The values are evaluated for $Z = \pm 90$ (β^{\mp} -decay), $W=5$ and $k(=|\kappa|)=1-4$, where we assume the uniform charge distribution inside the nucleus and use $\rho = 1.2 A^{1/3} \times 10^{-13}$ cm (A : mass number).

k		1	2	3	4
β^{-}	H_{-k}	-1.84×10^{-2}	-3.24×10^{-6}	-5.55×10^{-10}	-5.67×10^{-14}
	H_k	-1.40 //	-2.23 //	-3.75 //	-3.81 //
β^{+}	H_{-k}	2.46 //	3.67 //	5.96 //	5.84 //
	H_k	1.77 //	2.49 //	4.01 //	3.91 //

§ 3. Formulas for the first forbidden beta-gamma angular correlations

In the first subsection, we shall rewrite the formulas for the first forbidden beta-gamma angular correlations, taking into account the finite nuclear size effect and the screening effect. In this paper, we mainly follow the notations and formulations in the previous works by Yamada and the author,¹⁰⁾ Yamada and Morita,¹³⁾ and Morita (M.) and Morita (R.S.)¹⁴⁾ (abbreviated as (M, M) hereafter). As we already mentioned in § 2, the effects are mostly included only in the radial part of the electron wave function inside the nucleus and the Coulomb phase

shift at infinity, where the latter effect is shown to be generally negligible, that is, $\xi_\kappa - \xi_{\kappa'} \cong \delta_\kappa - \delta_{\kappa'}$. Thus in the formulas given by (M, M) only particle parameters $b_{LL'}^{(n)}$ must be rewritten with careful consideration on the finite nuclear size effect (especially the effect (3) in § 1).

The other important problem is the treatment of the phase shift, which always brings about some difficulties and cumbersome calculations into the theory of angular correlations. For a point nuclear charge, simple expressions in a convenient form are derived with some approximations (e.g., Eqs. (36)–(52) of (M, M)), but their validity will be satisfactory only for light nuclei (less than $Z \sim 30$ at most). However, it is still a good approximation for the allowed transition. They are, of course, calculable up to the desired accuracy, so we expect an exact estimation to be made.

In the following treatment, we shall show a method, by which the phase shift can be separated into two parts, and our final formulas can be given in a form which is very similar to those by Eqs. (36)–(52) of (M, M). The deviations in coefficients are expressed by several multiplication factors, which are graphically shown in the figures. In the second subsection, further parametrization is performed and then a number of nuclear matrix elements are restricted to a few adjustable parameters. Formulas for the allowed and second forbidden transitions are given in the Appendix.

3A. Reformulation of angular correlation function

The complete formulas for the first forbidden transition, including the non-conservation of parity and polarizations of related quantum and nuclei, are given by (M, M).¹⁴⁾ They are expressed by parameters $b_{LL'}^{(n)}$ defined in Eqs. (3) and (4) of Ref. 17), which include all radial wave functions and phase shifts of electron. If we use them, the angular correlation between beta rays and circularly polarized gamma rays was given in Eq. (3) of Ref. 14) as follows:

$$\begin{aligned} \mathcal{W}(\theta, p; \beta - \gamma) = \sum_n \left\{ \sum_{L \leq L'} (-1)^{j_1 - j} b_{LL'}^{(n)} W(j_1 j_1 LL'; nj) (2j_1 + 1)^{1/2} \right. \\ \left. \times \left[\sum_{L_1, L_1'} (-1)^{L_1 + L_1'} p^{\delta + \delta' + L_1 + L_1' + n} (j_1 || L_1 || j_2) (j_1 || L_1' || j_2) F_n(L_1 L_1' j_2 j_1) \right] \right\} \times P_n(\cos \theta), \end{aligned} \quad (8)$$

with

$$\begin{aligned} F_n(LL' j_a j_b) &= F_n(L' L j_a j_b) \\ &= (-1)^{j_a - j_b - 1} \{ (2j_b + 1) (2L + 1) (2L' + 1) \}^{1/2} \\ &\times (LL' 1 - 1 | n 0) W(j_a j_b LL'; nj_a), \end{aligned}$$

where θ is the angle between beta rays and circularly polarized gamma rays, and j, j_1 and j_2 are spins of successive nuclear levels for the decay scheme, $j(\beta) j_1(\gamma) j_2$. L means the ranks of the gamma rays. δ and δ' are equal to 0(+1) for magnetic (electric) radiation, and p is +1(−1) for the left (right) circularly polarized

gamma rays. $W(abcd; ef)$ and $(j_1 j_2 m_1 m_2 | jm)$ are the Racah and the Clebsch-Gordan coefficients. If we observe only the directions of the beta and gamma rays, only the terms involving $P_n(\cos \theta)$ with even n remain.

Similarly, the beta-ray angular distribution from oriented nuclei is expressed as

$$\mathcal{W}(\theta; \beta) = \sum_{n, L \leq L'} F_n(j) (-1)^{j_1 - j + L + L' + n} W(jjLL'; nj_1) b_{LL'}^{(n)} P_n(\cos \theta), \quad (9)$$

with

$$F_n(j) = \sum_m (-1)^{j-m} (jjm-m|n0) a_m,$$

where θ is the angle between beta rays and nuclear orientation and m is the magnetic quantum number of j . a_m is the relative population of the initial magnetic substates. The correction factor for the first forbidden transition is similarly expressed as:

$$C_1(W) = b_{00}^{(0)} - (1/\sqrt{3}) b_{11}^{(0)} + (1/\sqrt{5}) b_{22}^{(0)}. \quad (10)$$

(M, M) also gives the expressions for triple cascade transition.

Here $b_{LL'}^{(n)}$ given by (M, M) must be rewritten in the following forms, if we take into account the finite nuclear size effects (especially the effect of the decay in the internal region of the nucleus). We here assume only VA interaction among $SVTAP$, and take the Hamiltonian density given by Eq. (15') of Ref. 10), where, if G_i is the coupling constant of parity conserving parts, G_i' is that of parity non-conserving parts, or vice versa. For β^- -decay,

$$\begin{aligned} b_{00}^{(0)} &= (|G_A|^2 + |G_A'|^2) [(q^2/9) L_0^+(\boldsymbol{\sigma} \cdot \mathbf{r}, \boldsymbol{\sigma} \cdot \mathbf{r}) + M_0^+(\boldsymbol{\sigma} \cdot \mathbf{r}, \boldsymbol{\sigma} \cdot \mathbf{r}) \\ &\quad - (2q/3) N_0^+(\boldsymbol{\sigma} \cdot \mathbf{r}, \boldsymbol{\sigma} \cdot \mathbf{r}) + L_0^+(\gamma_5, \gamma_5) + 2i\{ (q/3) L_0^+(\boldsymbol{\sigma} \cdot \mathbf{r}, \gamma_5) \\ &\quad - N_0^+(\boldsymbol{\sigma} \cdot \mathbf{r}, \gamma_5) \}], \\ b_{01}^{(1)} &= 2\text{Re}(G_A^* G_V' + G_A'^* G_V) [- (q^2/9) \{ A_1(\boldsymbol{\sigma} \cdot \mathbf{r}, \mathbf{r}) + A_1(\mathbf{r}, \boldsymbol{\sigma} \cdot \mathbf{r}) \} \\ &\quad + (q/3) \{ N_{11}(\boldsymbol{\sigma} \cdot \mathbf{r}, \mathbf{r}) + N_{11}(\mathbf{r}, \boldsymbol{\sigma} \cdot \mathbf{r}) \} + m_1(\boldsymbol{\sigma} \cdot \mathbf{r}, \mathbf{r}) \\ &\quad + m_1(\mathbf{r}, \boldsymbol{\sigma} \cdot \mathbf{r}) - (2q/3) L_{12}(\boldsymbol{\sigma} \cdot \mathbf{r}, \mathbf{r}) + 2N_{12}(\boldsymbol{\sigma} \cdot \mathbf{r}, \mathbf{r}) \\ &\quad - i(q/3) \{ A_1(\boldsymbol{\sigma} \cdot \mathbf{r}, \boldsymbol{\alpha}) - A_1(\boldsymbol{\alpha}, \boldsymbol{\sigma} \cdot \mathbf{r}) \} + iN_{11}(\boldsymbol{\sigma} \cdot \mathbf{r}, \boldsymbol{\alpha}) \\ &\quad + i(q/3) \{ A_1(\gamma_5, \mathbf{r}) - A_1(\mathbf{r}, \gamma_5) \} + iN_{11}(\mathbf{r}, \gamma_5) \\ &\quad + 2iL_{12}(\gamma_5, \mathbf{r}) - \{ A_1(\gamma_5, \boldsymbol{\alpha}) - A_1(\boldsymbol{\alpha}, \gamma_5) \} \\ &\quad + 2\text{Im}(G_A^* G_V' + G_A'^* G_V) [(q^2/9) \{ A_1'(\boldsymbol{\sigma} \cdot \mathbf{r}, \mathbf{r}) - A_1'(\mathbf{r}, \boldsymbol{\sigma} \cdot \mathbf{r}) \} \\ &\quad - (q/3) \{ J_{11}(\boldsymbol{\sigma} \cdot \mathbf{r}, \mathbf{r}) - J_{11}(\mathbf{r}, \boldsymbol{\sigma} \cdot \mathbf{r}) \} - \{ m_1'(\boldsymbol{\sigma} \cdot \mathbf{r}, \mathbf{r}) \\ &\quad - m_1'(\mathbf{r}, \boldsymbol{\sigma} \cdot \mathbf{r}) \} + (2q/3) H_{12}(\boldsymbol{\sigma} \cdot \mathbf{r}, \mathbf{r}) - 2J_{12}(\boldsymbol{\sigma} \cdot \mathbf{r}, \mathbf{r}) \\ &\quad + i(q/3) \{ A_1'(\boldsymbol{\sigma} \cdot \mathbf{r}, \boldsymbol{\alpha}) + A_1'(\boldsymbol{\alpha}, \boldsymbol{\sigma} \cdot \mathbf{r}) \} - iJ_{11}(\boldsymbol{\sigma} \cdot \mathbf{r}, \boldsymbol{\alpha}) \\ &\quad - i(q/3) \{ A_1'(\gamma_5, \mathbf{r}) + A_1'(\mathbf{r}, \gamma_5) \} + iJ_{11}(\mathbf{r}, \gamma_5) \end{aligned}$$

$$\begin{aligned}
& -2i\mathbf{H}_{12}(\gamma_5, \mathbf{r}) + A_1'(\gamma_5, \boldsymbol{\alpha}) - A_1'(\boldsymbol{\alpha}, \gamma_5)] \\
& + 4\text{Re}(G_A^* G_A') [-i(q^2/9) \{A_1(\boldsymbol{\sigma} \cdot \mathbf{r}, \boldsymbol{\sigma} \times \mathbf{r}) - A_1(\boldsymbol{\sigma} \times \mathbf{r}, \boldsymbol{\sigma} \cdot \mathbf{r})\} \\
& + i(q/3) \{N_{11}(\boldsymbol{\sigma} \cdot \mathbf{r}, \boldsymbol{\sigma} \times \mathbf{r}) + N_{11}(\boldsymbol{\sigma} \times \mathbf{r}, \boldsymbol{\sigma} \cdot \mathbf{r})\} - i\{m_1(\boldsymbol{\sigma} \cdot \mathbf{r}, \boldsymbol{\sigma} \times \mathbf{r}) \\
& - m_1(\boldsymbol{\sigma} \times \mathbf{r}, \boldsymbol{\sigma} \cdot \mathbf{r})\} - i(q/3) L_{12}(\boldsymbol{\sigma} \cdot \mathbf{r}, \boldsymbol{\sigma} \times \mathbf{r}) + iN_{12}(\boldsymbol{\sigma} \cdot \mathbf{r}, \boldsymbol{\sigma} \times \mathbf{r}) \\
& - (q/3) \{A_1(\gamma_5, \boldsymbol{\sigma} \times \mathbf{r}) + A_1(\boldsymbol{\sigma} \times \mathbf{r}, \gamma_5)\} - N_{11}(\boldsymbol{\sigma} \times \mathbf{r}, \gamma_5) - L_{12}(\gamma_5, \boldsymbol{\sigma} \times \mathbf{r})], \\
b_{11}^{(0)} = & -\sqrt{3} \left\{ (|G_V|^2 + |G_V'|^2) [(q^2/3) L_0^+(\mathbf{r}, \mathbf{r}) + 2L_1^+(\mathbf{r}, \mathbf{r}) \right. \\
& + M_0^+(\mathbf{r}, \mathbf{r}) + (2q/3) N_0^+(\mathbf{r}, \mathbf{r}) + L_0^+(\boldsymbol{\alpha}, \boldsymbol{\alpha}) \\
& + i(2q/3) \{L_0^+(\mathbf{r}, \boldsymbol{\alpha}) - N_0^+(\mathbf{r}, \boldsymbol{\alpha})\}] + (|G_A|^2 + |G_A'|^2) \\
& \times [(q^2/6) L_0^+(\boldsymbol{\sigma} \times \mathbf{r}, \boldsymbol{\sigma} \times \mathbf{r}) + (2q/3) N_0^+(\boldsymbol{\sigma} \times \mathbf{r}, \boldsymbol{\sigma} \times \mathbf{r}) \\
& + \frac{1}{2} L_1^+(\boldsymbol{\sigma} \times \mathbf{r}, \boldsymbol{\sigma} \times \mathbf{r}) + M_0^+(\boldsymbol{\sigma} \times \mathbf{r}, \boldsymbol{\sigma} \times \mathbf{r})] \\
& + 2\text{Re}(G_V^* G_A + G_V'^* G_A') [iL_1^+(\mathbf{r}, \boldsymbol{\sigma} \times \mathbf{r}) - iM_0^+(\mathbf{r}, \boldsymbol{\sigma} \times \mathbf{r}) \\
& - i(q/3) \{N_0^+(\mathbf{r}, \boldsymbol{\sigma} \times \mathbf{r}) - N_0^+(\boldsymbol{\sigma} \times \mathbf{r}, \mathbf{r})\} \\
& \left. + (q/3) L_0^+(\boldsymbol{\sigma} \times \mathbf{r}, \boldsymbol{\alpha}) + N_0^+(\boldsymbol{\sigma} \times \mathbf{r}, \boldsymbol{\alpha}) \right\}, \\
b_{11}^{(1)} = & \sqrt{2} \left\{ 4\text{Re}(G_V^* G_V') [A_2(\mathbf{r}, \mathbf{r}) - (q/3) N_{11}(\mathbf{r}, \mathbf{r}) - m_1(\mathbf{r}, \mathbf{r}) \right. \\
& - (q/3) L_{12}(\mathbf{r}, \mathbf{r}) + N_{12}(\mathbf{r}, \mathbf{r}) + A_1(\boldsymbol{\alpha}, \boldsymbol{\alpha}) \\
& + i(q/3) \{A_1(\mathbf{r}, \boldsymbol{\alpha}) - A_1(\boldsymbol{\alpha}, \mathbf{r})\} - iN_{11}(\mathbf{r}, \boldsymbol{\alpha}) + iL_{12}(\boldsymbol{\alpha}, \mathbf{r})] \\
& + 2\text{Re}(G_A^* G_A') [(q^2/6) A_1(\boldsymbol{\sigma} \times \mathbf{r}, \boldsymbol{\sigma} \times \mathbf{r}) + \frac{1}{2} A_2(\boldsymbol{\sigma} \times \mathbf{r}, \boldsymbol{\sigma} \times \mathbf{r}) \\
& + (2q/3) N_{11}(\boldsymbol{\sigma} \times \mathbf{r}, \boldsymbol{\sigma} \times \mathbf{r}) - 2m_1(\boldsymbol{\sigma} \times \mathbf{r}, \boldsymbol{\sigma} \times \mathbf{r}) \\
& - (q/3) L_{12}(\boldsymbol{\sigma} \times \mathbf{r}, \boldsymbol{\sigma} \times \mathbf{r}) - N_{12}(\boldsymbol{\sigma} \times \mathbf{r}, \boldsymbol{\sigma} \times \mathbf{r})] \\
& + 2\text{Re}(G_V^* G_A' + G_V'^* G_A) [(q/3) \{A_1(\boldsymbol{\alpha}, \boldsymbol{\sigma} \times \mathbf{r}) + A_1(\boldsymbol{\sigma} \times \mathbf{r}, \boldsymbol{\alpha})\} \\
& + N_{11}(\boldsymbol{\sigma} \times \mathbf{r}, \boldsymbol{\alpha}) - (1/2) L_{12}(\boldsymbol{\alpha}, \boldsymbol{\sigma} \times \mathbf{r}) \\
& + i(q^2/6) \{A_1(\mathbf{r}, \boldsymbol{\sigma} \times \mathbf{r}) - A_1(\boldsymbol{\sigma} \times \mathbf{r}, \mathbf{r})\} \\
& - i(1/2) \{A_2(\boldsymbol{\sigma} \times \mathbf{r}, \mathbf{r}) - A_2(\mathbf{r}, \boldsymbol{\sigma} \times \mathbf{r})\} - i(q/3) \{N_{11}(\boldsymbol{\sigma} \times \mathbf{r}, \mathbf{r}) \\
& + N_{11}(\mathbf{r}, \boldsymbol{\sigma} \times \mathbf{r})\} + i\{m_1(\mathbf{r}, \boldsymbol{\sigma} \times \mathbf{r}) - m_1(\boldsymbol{\sigma} \times \mathbf{r}, \mathbf{r})\} \\
& + i(q/3) L_{12}(\boldsymbol{\sigma} \times \mathbf{r}, \mathbf{r}) - i(q/6) L_{12}(\mathbf{r}, \boldsymbol{\sigma} \times \mathbf{r}) + i(1/2) N_{12}(\mathbf{r}, \boldsymbol{\sigma} \times \mathbf{r}) \\
& + iN_{12}(\boldsymbol{\sigma} \times \mathbf{r}, \mathbf{r})] + 2\text{Im}(G_V^* G_A' + G_V'^* G_A) [- (q/3) \{A_1'(\boldsymbol{\alpha}, \boldsymbol{\sigma} \times \mathbf{r}) \\
& - A_1'(\boldsymbol{\sigma} \times \mathbf{r}, \boldsymbol{\alpha})\} + J_{11}(\boldsymbol{\sigma} \times \mathbf{r}, \boldsymbol{\alpha}) + \frac{1}{2} H_{12}(\boldsymbol{\alpha}, \boldsymbol{\sigma} \times \mathbf{r}) \\
& - i(q^2/6) \{A_1'(\boldsymbol{\sigma} \times \mathbf{r}, \mathbf{r}) + A_1'(\mathbf{r}, \boldsymbol{\sigma} \times \mathbf{r})\} - i(1/2) \{A_2'(\boldsymbol{\sigma} \times \mathbf{r}, \mathbf{r}) \\
& + A_2'(\mathbf{r}, \boldsymbol{\sigma} \times \mathbf{r})\} + i(q/3) \{J_{11}(\boldsymbol{\sigma} \times \mathbf{r}, \mathbf{r}) + J_{11}(\mathbf{r}, \boldsymbol{\sigma} \times \mathbf{r})\} \\
& \left. - i\{m_1'(\boldsymbol{\sigma} \times \mathbf{r}, \mathbf{r}) + m_1'(\mathbf{r}, \boldsymbol{\sigma} \times \mathbf{r})\} + i(q/3) H_{12}(\boldsymbol{\sigma} \times \mathbf{r}, \mathbf{r}) \right\}
\end{aligned}$$

$$\begin{aligned}
& +i(q/6)\mathbf{H}_{12}(\mathbf{r}, \boldsymbol{\sigma} \times \mathbf{r}) + i\mathbf{J}_{12}(\boldsymbol{\sigma} \times \mathbf{r}, \mathbf{r}) - i(1/2)\mathbf{J}_{12}(\mathbf{r}, \boldsymbol{\sigma} \times \mathbf{r}) \Big\}, \\
b_{11}^{(2)} = & -\sqrt{6} \Big\{ (|G_V|^2 + |G_V'|^2) [(2q/3)L_{12}(\mathbf{r}, \mathbf{r}) - L_1^+(\mathbf{r}, \mathbf{r}) \\
& - 2N_{12}(\mathbf{r}, \mathbf{r}) - 2iL_{12}(\boldsymbol{\alpha}, \mathbf{r})] + (|G_A|^2 + |G_A'|^2) \\
& \times [(q/3)L_{12}(\boldsymbol{\sigma} \times \mathbf{r}, \boldsymbol{\sigma} \times \mathbf{r}) - (1/4)L_1^+(\boldsymbol{\sigma} \times \mathbf{r}, \boldsymbol{\sigma} \times \mathbf{r}) \\
& + N_{12}(\boldsymbol{\sigma} \times \mathbf{r}, \boldsymbol{\sigma} \times \mathbf{r})] + 2\text{Re}(G_V^*G_A + G_V'^*G_A') [(1/2)L_{12}(\boldsymbol{\alpha}, \boldsymbol{\sigma} \times \mathbf{r}) \\
& + i(q/6)L_{12}(\mathbf{r}, \boldsymbol{\sigma} \times \mathbf{r}) - i(q/3)L_{12}(\boldsymbol{\sigma} \times \mathbf{r}, \mathbf{r}) \\
& - i(1/2)L_1^+(\mathbf{r}, \boldsymbol{\sigma} \times \mathbf{r}) - i(1/2)N_{12}(\mathbf{r}, \boldsymbol{\sigma} \times \mathbf{r}) - iN_{12}(\boldsymbol{\sigma} \times \mathbf{r}, \mathbf{r}) \\
& + 2\text{Im}(G_V^*G_A + G_V'^*G_A') [(1/2)H_{12}(\boldsymbol{\alpha}, \boldsymbol{\sigma} \times \mathbf{r}) + i(q/6)H_{12}(\mathbf{r}, \boldsymbol{\sigma} \times \mathbf{r}) \\
& + i(q/3)H_{12}(\boldsymbol{\sigma} \times \mathbf{r}, \mathbf{r}) - i(1/2)J_{12}(\mathbf{r}, \boldsymbol{\sigma} \times \mathbf{r}) + iJ_{12}(\boldsymbol{\sigma} \times \mathbf{r}, \mathbf{r})] \Big\}, \\
b_{02}^{(0)} = & (1/\sqrt{6}) (|G_A|^2 + |G_A'|^2) [6iL_{12}(\gamma_5, B_{ij}) - 2qL_{12}(\boldsymbol{\sigma} \cdot \mathbf{r}, B_{ij}) \\
& + 6N_{12}(\boldsymbol{\sigma} \cdot \mathbf{r}, B_{ij})], \\
b_{12}^{(1)} = & -(\sqrt{5}/2\sqrt{3}) \Big\{ 2\text{Re}(G_V^*G_A' + G_V'^*G_A) [- (q^2/3) \{A_1(\mathbf{r}, B_{ij}) \\
& + A_1(B_{ij}, \mathbf{r})\} - (3/5) \{A_2(\mathbf{r}, B_{ij}) + A_2(B_{ij}, \mathbf{r})\} \\
& - qL_{12}(\mathbf{r}, B_{ij}) + 3N_{12}(\mathbf{r}, B_{ij}) + 3iL_{12}(\boldsymbol{\alpha}, B_{ij})] \\
& + 2\text{Im}(G_V^*G_A' + G_V'^*G_A) [(q^2/3) \{A_1'(\mathbf{r}, B_{ij}) - A_1'(B_{ij}, \mathbf{r})\} \\
& + (3/5) \{A_2'(\mathbf{r}, B_{ij}) - A_2'(B_{ij}, \mathbf{r})\} + qH_{12}(\mathbf{r}, B_{ij}) \\
& - 3J_{12}(\mathbf{r}, B_{ij}) - 3iH_{12}(\boldsymbol{\alpha}, B_{ij})] \\
& + 2\text{Re}(G_A^*G_A') [-i(q^2/3) \{A_1(\boldsymbol{\sigma} \times \mathbf{r}, B_{ij}) - A_1(B_{ij}, \boldsymbol{\sigma} \times \mathbf{r})\} \\
& + i(3/5) \{A_2(\boldsymbol{\sigma} \times \mathbf{r}, B_{ij}) - A_2(B_{ij}, \boldsymbol{\sigma} \times \mathbf{r})\} \\
& + 2iqL_{12}(\boldsymbol{\sigma} \times \mathbf{r}, B_{ij}) + 6iN_{12}(\boldsymbol{\sigma} \times \mathbf{r}, B_{ij})] \Big\}, \\
b_{12}^{(2)} = & - (1/2) \Big\{ 2\text{Re}(G_V^*G_A + G_V'^*G_A') [-qL_{12}(\mathbf{r}, B_{ij}) - 3L_1^+(\mathbf{r}, B_{ij}) \\
& + 3N_{12}(\mathbf{r}, B_{ij}) + 3iL_{12}(\boldsymbol{\alpha}, B_{ij})] \\
& + 2\text{Im}(G_V^*G_A + G_V'^*G_A') [-qH_{12}(\mathbf{r}, B_{ij}) + 3J_{12}(\mathbf{r}, B_{ij}) \\
& + 3iH_{12}(\boldsymbol{\alpha}, B_{ij})] + 2(|G_A|^2 + |G_A'|^2) [iqL_{12}(\boldsymbol{\sigma} \times \mathbf{r}, B_{ij}) \\
& + i\frac{3}{2}L_1^+(\boldsymbol{\sigma} \times \mathbf{r}, B_{ij}) + 3iN_{12}(\boldsymbol{\sigma} \times \mathbf{r}, B_{ij})] \Big\}, \\
b_{12}^{(3)} = & (9/\sqrt{10}) \Big\{ -2\text{Re}(G_V^*G_A' + G_V'^*G_A) [A_2(\mathbf{r}, B_{ij}) + A_2(B_{ij}, \mathbf{r})] \\
& + 2\text{Im}(G_V^*G_A' + G_V'^*G_A) [A_2'(\mathbf{r}, B_{ij}) - A_2'(B_{ij}, \mathbf{r})] \\
& + 2i\text{Re}(G_A^*G_A') [A_2(\boldsymbol{\sigma} \times \mathbf{r}, B_{ij}) - A_2(B_{ij}, \boldsymbol{\sigma} \times \mathbf{r})] \Big\}, \\
b_{22}^{(0)} = & \sqrt{5} (|G_A|^2 + |G_A'|^2) [(q^2/12)L_0^+(B_{ij}, B_{ij}) + (3/4)L_1^+(B_{ij}, B_{ij})], \\
b_{22}^{(1)} = & -\sqrt{10} \cdot 2\text{Re}(G_A^*G_A') [(q^2/12)A_1(B_{ij}, B_{ij}) + (9/20)A_2(B_{ij}, B_{ij})],
\end{aligned}$$

$$b_{22}^{(2)} = -(3\sqrt{7}/4\sqrt{2}) (|G_A|^2 + |G_A'|^2) L_1^+(B_{ij}, B_{ij}),$$

$$b_{22}^{(3)} = (9/\sqrt{10}) 2\text{Re}(G_A^* G_A') \cdot A_2(B_{ij}, B_{ij}). \quad (11)$$

Here q is momentum of the neutrino, and $L_{k-1}^+(X, Y)$, $M_{k-1}^+(X, Y)$ and $N_{k-1}(X, Y)$ have been defined in Ref. 10) (see Eq. (17) on p. 309), but we have used here the following definitions for nucleon operator $\mathfrak{f}X$ as "the reduced matrix element", a quantity similar to that denoted by Racah's ($\| \|$) symbol or Yamada and Morita's \mathfrak{M} symbol.

$$L_{k-1}^+(X, Y) = (2F_0(\rho)p^2)^{-1} \left[\left(\int \mathfrak{M}(X) \frac{g_{-k}(r)}{r^{k-1}} \right)^* \left(\int \mathfrak{M}(Y) \frac{g_{-k}(r)}{r^{k-1}} \right) \right. \\ \left. + \left(\int \mathfrak{M}(X) \frac{f_k(r)}{r^{k-1}} \right) \left(\int \mathfrak{M}(Y) \frac{f_k(r)}{r^{k-1}} \right) \right]. \quad (12)$$

$$M_{k-1}^+(X, Y) = (2F_0(\rho)p^2)^{-1} \left[\left(\int \mathfrak{M}(X) \frac{f_{-k}(r)}{r^k} \right)^* \left(\int \mathfrak{M}(Y) \frac{f_{-k}(r)}{r^k} \right) \right. \\ \left. + \left(\int \mathfrak{M}(X) \frac{g_k(r)}{r^k} \right)^* \left(\int \mathfrak{M}(Y) \frac{g_k(r)}{r^k} \right) \right]. \quad (13)$$

$$N_{k-1}^+(X, Y) = (2F_0(\rho)p^2)^{-1} \left[\left(\int \mathfrak{M}(X) \frac{f_{-k}(r)}{r^k} \right)^* \left(\int \mathfrak{M}(Y) \frac{g_{-k}(r)}{r^{k-1}} \right) \right. \\ \left. - \left(\int \mathfrak{M}(X) \frac{g_k(r)}{r^k} \right) \left(\int \mathfrak{M}(Y) \frac{f_k(r)}{r^{k-1}} \right) \right]. \quad (14)$$

Similarly, the other terms are defined as follows (if we use the notation of (M, M)) :

$$L_{k,k+1}(X, Y) = (2F_0(\rho)p^2)^{-1} \\ \times \left[\left(\int \mathfrak{M}(X) \frac{g_{-k}(r)}{r^{k-1}} \right)^* \left(\int \mathfrak{M}(Y) \frac{f_{k+1}(r)}{r^k} \right) \cos(\zeta_{-k} - \zeta_{k+1}) \right. \\ \left. - \left(\int \mathfrak{M}(X) \frac{f_k(r)}{r^{k-1}} \right)^* \left(\int \mathfrak{M}(Y) \frac{g_{-(k+1)}(r)}{r^k} \right) \cos(\zeta_k - \zeta_{-(k+1)}) \right],$$

$$H_{k,k+1}(X, Y) = (2F_0(\rho)p^2)^{-1} \\ \times \left[\left(\int \mathfrak{M}(X) \frac{g_{-k}(r)}{r^{k-1}} \right)^* \left(\int \mathfrak{M}(Y) \frac{f_{k+1}(r)}{r^k} \right) \sin(\zeta_{-k} - \zeta_{k+1}) \right. \\ \left. - \left(\int \mathfrak{M}(X) \frac{f_k(r)}{r^{k-1}} \right)^* \left(\int \mathfrak{M}(Y) \frac{g_{-(k+1)}(r)}{r^k} \right) \sin(\zeta_k - \zeta_{-(k+1)}) \right]. \quad (15)$$

$$N_{k,k+1}(X, Y) = (2F_0(\rho)p^2)^{-1} \\ \times \left[\left(\int \mathfrak{M}(X) \frac{f_{-k}(r)}{r^k} \right)^* \left(\int \mathfrak{M}(Y) \frac{f_{k+1}(r)}{r^k} \right) \cos(\zeta_{-k} - \zeta_{k+1}) \right. \\ \left. + \left(\int \mathfrak{M}(X) \frac{g_k(r)}{r^k} \right)^* \left(\int \mathfrak{M}(Y) \frac{g_{-(k+1)}(r)}{r^k} \right) \cos(\zeta_k - \zeta_{-(k+1)}) \right],$$

$$\begin{aligned}
 J_{k,k+1}(X, Y) &= (2F_0(\rho) p^2)^{-1} \\
 &\times \left[\left(\int \mathfrak{M}(X) \frac{f_{-k}(r)}{r^k} \right)^* \left(\int \mathfrak{M}(Y) \frac{f_{k+1}(r)}{r^k} \right) \sin(\zeta_{-k} - \zeta_{k+1}) \right. \\
 &\left. + \left(\int \mathfrak{M}(X) \frac{g_k(r)}{r^k} \right)^* \left(\int \mathfrak{M}(Y) \frac{g_{-(k+1)}(r)}{r^k} \right) \sin(\zeta_k - \zeta_{-(k+1)}) \right]. \quad (16)
 \end{aligned}$$

$$\begin{aligned}
 L_{k,k+1}(X, Y) &= (2F_0(\rho) p^2)^{-1} \\
 &\times \left[\left(\int \mathfrak{M}(X) \frac{f_k(r)}{r^{k-1}} \right)^* \left(\int \mathfrak{M}(Y) \frac{f_{k+1}(r)}{r^k} \right) \sin(\zeta_k - \zeta_{k+1}) \right. \\
 &\left. + \left(\int \mathfrak{M}(X) \frac{g_{-k}(r)}{r^{k-1}} \right)^* \left(\int \mathfrak{M}(Y) \frac{g_{-(k+1)}(r)}{r^k} \right) \sin(\zeta_{-k} - \zeta_{-(k+1)}) \right],
 \end{aligned}$$

$$\begin{aligned}
 H_{k,k+1}(X, Y) &= (2F_0(\rho) p^2)^{-1} \\
 &\times \left[\left(\int \mathfrak{M}(X) \frac{f_k(r)}{r^{k-1}} \right)^* \left(\int \mathfrak{M}(Y) \frac{f_{k+1}(r)}{r^k} \right) \cos(\zeta_k - \zeta_{k+1}) \right. \\
 &\left. + \left(\int \mathfrak{M}(X) \frac{g_{-k}(r)}{r^{k-1}} \right)^* \left(\int \mathfrak{M}(Y) \frac{g_{-(k+1)}(r)}{r^k} \right) \cos(\zeta_{-k} - \zeta_{-(k+1)}) \right]. \quad (17)
 \end{aligned}$$

$$\begin{aligned}
 N_{k,k}(X, Y) &= (2F_0(\rho) p^2)^{-1} \left[\left(\int \mathfrak{M}(X) \frac{f_{-k}(r)}{r} \right)^* \left(\int \mathfrak{M}(Y) \frac{f_k(r)}{r^{k-1}} \right) \right. \\
 &\left. - \left(\int \mathfrak{M}(X) \frac{g_k(r)}{r^k} \right)^* \left(\int \mathfrak{M}(Y) \frac{g_{-k}(r)}{r^{k-1}} \right) \right] \sin(\zeta_{-k} - \zeta_k),
 \end{aligned}$$

$$\begin{aligned}
 J_{k,k}(X, Y) &= (2F_0(\rho) p^2)^{-1} \\
 &\times \left[\left(\int \mathfrak{M}(X) \frac{f_{-k}(r)}{r^k} \right)^* \left(\int \mathfrak{M}(Y) \frac{f_k(r)}{r^{k-1}} \right) \right. \\
 &\left. + \left(\int \mathfrak{M}(X) \frac{g_k(r)}{r^k} \right)^* \left(\int \mathfrak{M}(Y) \frac{g_{-k}(r)}{r^{k-1}} \right) \right] \cos(\zeta_{-k} - \zeta_k). \quad (18)
 \end{aligned}$$

$$\begin{aligned}
 N_{k,k+1}(X, Y) &= (2F_0(\rho) p^2)^{-1} \\
 &\times \left[\left(\int \mathfrak{M}(X) \frac{f_{-k}(r)}{r^k} \right)^* \left(\int \mathfrak{M}(Y) \frac{g_{-(k+1)}(r)}{r^k} \right) \sin(\zeta_{-k} - \zeta_{-(k+1)}) \right. \\
 &\left. - \left(\int \mathfrak{M}(X) \frac{g_k(r)}{r^k} \right) \left(\int \mathfrak{M}(Y) \frac{f_{k+1}(r)}{r^k} \right) \sin(\zeta_k - \zeta_{k+1}) \right],
 \end{aligned}$$

$$\begin{aligned}
 J_{k,k+1}(X, Y) &= (2F_0(\rho) p^2)^{-1} \\
 &\times \left[\left(\int \mathfrak{M}(X) \frac{f_{-k}(r)}{r^k} \right)^* \left(\int \mathfrak{M}(Y) \frac{g_{-(k+1)}(r)}{r^k} \right) \cos(\zeta_{-k} - \zeta_{-(k+1)}) \right. \\
 &\left. - \left(\int \mathfrak{M}(X) \frac{g_k(r)}{r^k} \right)^* \left(\int \mathfrak{M}(Y) \frac{f_{k+1}(r)}{r^k} \right) \cos(\zeta_k - \zeta_{k+1}) \right]. \quad (19)
 \end{aligned}$$

$$\begin{aligned}
 A_k(X, Y) &= (2F_0(\rho) p^2)^{-1} \left(\int \mathfrak{M}(X) \frac{g_{-k}(r)}{r^{k-1}} \right)^* \\
 &\times \left(\int \mathfrak{M}(Y) \frac{f_k(r)}{r^{k-1}} \right) \sin(\zeta_{-k} - \zeta_k),
 \end{aligned}$$

$$A'_k(X, Y) = (2F_0(\rho) p^2)^{-1} \left(\int \mathfrak{M}(X) \frac{g_{-k}(r)}{r^{k-1}} \right)^* \left(\int \mathfrak{M}(Y) \frac{f_k(r)}{r^{k-1}} \right) \cos(\zeta_{-k} - \zeta_k). \quad (20)$$

$$m_k(X, Y) = (2F_0(\rho) p^2)^{-1} \left(\int \mathfrak{M}(X) \frac{f_{-k}(r)}{r^k} \right)^* \left(\int \mathfrak{M}(Y) \frac{g_k(r)}{r^k} \right) \sin(\zeta_{-k} - \zeta_k),$$

$$m'_k(X, Y) = (2F_0(\rho) p^2)^{-1} \left(\int \mathfrak{M}(X) \frac{f_{-k}(r)}{r^k} \right)^* \left(\int \mathfrak{M}(Y) \frac{g_k(r)}{r^k} \right) \cos(\zeta_{-k} - \zeta_k). \quad (21)$$

In the derivation of (11)–(21), we have assumed that the strong interaction is invariant under time-reversal, i.e., this means that the nuclear matrix elements $\mathfrak{M}(\boldsymbol{\sigma} \times \mathbf{r})$, $\mathfrak{M}(\boldsymbol{\alpha})$, $\mathfrak{M}(\gamma_5)$, $i\mathfrak{M}(\mathbf{r})$, $i\mathfrak{M}(\boldsymbol{\sigma} \cdot \mathbf{r})$ and $i\mathfrak{M}(B_{ij})$ must have the same phase¹⁸⁾ except for the relative sign. If this assumption is violated, the following replacement $\zeta_\kappa - \zeta_{\kappa'} \rightarrow \zeta_\kappa - \zeta_{\kappa'} + \theta$ is required for each pair of matrix elements, where θ is the relative sign between them. It causes serious complexity in our theory, because we may not estimate them in this case at the present stage. However, the validity of the time-reversal invariance is now very reliable. The second functions of Eqs. (11)–(21) appear when the time-reversal invariance for beta-interaction⁷⁾ is violated, but now the validity of the invariance is also very reliable.^{5), 6)} $f_\kappa(r)$ and $g_\kappa(r)$ are the inner wave functions of the electron, which have approximately been given by Eq. (14) of Ref. 10). If we replace here $f_{-k}(r)/r^k$, $g_{-k}(r)/r^{k-1}$, etc., by their values at $r=\rho$, they tend to Eqs. (21)–(34) and (36)–(52) of (M, M), where A'_k and m'_k were cancelled out of the expressions.

Here we shall evaluate Eqs. (27)–(37) more explicitly by introducing effective nuclear radii defined in Eq. (21) of Ref. 10). Now the phase shift can be separated into two parts as

$$\partial_\kappa - \partial_{\kappa'} = \eta_\kappa - \eta_{\kappa'} + (\pi/2) (|\kappa'| - |\kappa|) + \varepsilon_{k, k'},$$

$$\varepsilon_{k, k'} = -\arg \Gamma(\gamma_k + i\alpha ZW/p) + \arg \Gamma(\gamma_{k'} + i\alpha ZW/p) + (\pi/2) (\gamma_{k'} - \gamma_k + |\kappa| - |\kappa'|),$$

and, for example, $\cos(\zeta_\kappa - \zeta_{\kappa'})$ is separated as

$$\begin{aligned} \cos(\zeta_\kappa - \zeta_{\kappa'}) &\cong \cos\{\eta_\kappa - \eta_{\kappa'} + (\pi/2) (|\kappa'| - |\kappa|)\} [\cos \varepsilon_{k, k'} \\ &\quad - \tan\{\eta_\kappa - \eta_{\kappa'} + (\pi/2) (|\kappa'| - |\kappa|)\} \sin \varepsilon_{k, k'}]. \end{aligned}$$

Here, using Eq. (3), we calculate only $\cos\{\eta_\kappa - \eta_{\kappa'} - (\pi/2) (|\kappa'| - |\kappa|)\}$ in an analytical form. $\varepsilon_{k, k'}$ is not so small except for light nuclei, and in fact for $Z=90$, ε_{12} approaches to $\pi/3$ for the low energy region. However, this treatment is still now performed for the later purpose. After the calculation, the first functions of Eqs. (15)–(21) are expressed in the following.

$$\begin{aligned} L_{k, k+1}(X, Y) &= \left(\int \mathfrak{M}(X) I_k(r) \right)^* \left(\int \mathfrak{M}(Y) I_{k+1}(r) \right) \frac{\sqrt{F_{k-1}(\rho) F_k(\rho)}}{F_0(\rho)} \\ &\quad \times \left[\frac{2^k \cdot k!}{(2k)!} p^{k-1} \right]^2 \left(-\frac{p^2}{(2k+1)W} \right) \sigma_{k, k+1}. \end{aligned} \quad (22)$$

$$N_{k,k+1}(X, Y) = \left(\int \mathfrak{M}(X) I_k(r) \right)^* \left(\int \mathfrak{M}(Y) I_{k+1}(r) \right) \frac{\sqrt{F_{k-1}(\rho) F_k(\rho)}}{F_0(\rho)} \\ \times \left[\frac{2^k \cdot k!}{(2k)!} p^{k-1} \right]^2 \frac{p^2}{(2k+1)W} \left[\frac{1}{k} \frac{\alpha Z}{2\rho_k(X)} \sigma_{k,k+1} + \frac{W}{(2k+1)} \sigma_{k,k+1}^- \right]. \quad (23)$$

$$L_{k,k+1}(X, Y) = \left(\int \mathfrak{M}(X) I_k(r) \right)^* \left(\int \mathfrak{M}(Y) I_{k+1}(r) \right) \frac{\sqrt{F_{k-1}(\rho) F_k(\rho)}}{F_0(\rho)} \\ \times \left[\frac{2^k \cdot k!}{(2k)!} p^{k-1} \right]^2 \frac{p}{(2k+1)} \chi_{k,k+1}. \quad (24)$$

$$N_{k,k}(X, Y) = - \left(\int \mathfrak{M}(X) I_k(r) \right)^* \left(\int \mathfrak{M}(Y) I_k(r) \right) \frac{F_{k-1}(\rho)}{F_0(\rho)} \left[\frac{2^k \cdot k!}{(2k)!} p^{k-1} \right]^2 \\ \times \left[\frac{p}{kW} \cdot \frac{\alpha Z}{2\rho_k(X)} + \frac{p}{2k+1} \right] \Omega_k. \quad (25)$$

$$N_{k,k+1}(X, Y) = - \left(\int \mathfrak{M}(X) I_k(r) \right) \left(\int \mathfrak{M}(Y) I_{k+1}(r) \right) \frac{\sqrt{F_{k-1}(\rho) F_k(\rho)}}{F_0(\rho)} \\ \times \left[\frac{2^k \cdot k!}{(2k)!} p^{k-1} \right]^2 \frac{p}{2k+1} \left[\frac{1}{k} \frac{\alpha Z}{2\rho_k(X)} \chi_{k,k+1} + \frac{p^2}{(2k+1)W} \chi_{k,k+1}^+ \right]. \quad (26)$$

$$A_k(X, Y) = \left(\int \mathfrak{M}(X) I_k(r) \right) \left(\int \mathfrak{M}(Y) I_k(r) \right) \frac{F_{k-1}(\rho)}{F_0(\rho)} \\ \times \left[\frac{2^k \cdot k!}{(2k)!} p^{k-1} \right]^2 \frac{p}{2W} \Omega_k. \quad (27)$$

$$m_k(X, Y) = - \left(\int \mathfrak{M}(X) I_k(r) \right)^* \left(\int \mathfrak{M}(Y) I_k(r) \right) \frac{F_{k-1}(\rho)}{F_0(\rho)} \\ \times \left[\frac{2^k \cdot k!}{(2k)!} p^{k-1} \right]^2 \frac{p}{2W} \left[\frac{1}{k^2} \left(\frac{\alpha Z}{2\rho_k(X)} \right) \left(\frac{\alpha Z}{2\rho_k(Y)} \right) \right. \\ \left. + \frac{W+1}{k(2k+1)} \frac{\alpha Z}{2\rho_k(X)} + \frac{W-1}{k(2k+1)} \frac{\alpha Z}{2\rho_k(Y)} + \frac{p^2}{(2k+1)^2} \right] \Omega_k. \quad (28)$$

The multiplication factors are defined as follows:

$$\sigma_{k,k+1} = a_{k,k+1}^+ g_{k,k+1}^+, \quad \sigma_{k,k+1}^- = \{a_{k,k+1}^+ - a_{k,k+1}^-/W\} g_{k,k+1}^+, \\ \chi_{k,k+1} = b_{k,k+1}^+ g_{k,k+1} + \frac{1}{W} b_{k,k+1}^- g_{k,k+1}^-, \\ \chi_{k,k+1}^+ = b_{k,k+1}^+ g_{k,k+1}^+ + \frac{1}{p^2} (b_{k,k+1}^+ - W b_{k,k+1}^+) (g_{k,k+1}^+ - g_{k,k+1}^-), \\ \Omega_k = (1 + \mathcal{A}_{-k}) (1 + \mathcal{A}_k), \quad (29)$$

where

$$\begin{aligned}
 a_{k,k+1}^{\pm} = & \frac{1}{2} \left[(1 + \mathcal{A}_{-k}) (1 + \mathcal{A}_{k+1}) \{ \cos \varepsilon_{k,k+1} - \tan(\eta_{-k} - \eta_{k+1} + \pi/2) \sin \varepsilon_{k,k+1} \} \right. \\
 & \left. \pm (1 + \mathcal{A}_k) (1 + \mathcal{A}_{-(k+1)}) \{ \cos \varepsilon_{k,k+1} - \tan(\eta_k - \eta_{-(k+1)} + \pi/2) \sin \varepsilon_{k,k+1} \} \right], \\
 b_{k,k+1}^{\pm} = & \frac{1}{2} \left[(1 + \mathcal{A}_{-k}) (1 + \mathcal{A}_{-(k+1)}) \{ \cos \varepsilon_{k,k+1} + \cot(\eta_{-k} - \eta_{-(k+1)} + \pi/2) \sin \varepsilon_{k,k+1} \} \right. \\
 & \left. + (1 + \mathcal{A}_k) (1 + \mathcal{A}_{k+1}) \{ \cos \varepsilon_{k,k+1} + \cot(\eta_k - \eta_{k+1} + \pi/2) \sin \varepsilon_{k,k+1} \} \right], \quad (30)
 \end{aligned}$$

$$g_{k,k'}^{\pm} = \left\{ \left(\frac{k + \gamma_k}{2k} \right) \left(\frac{k' + \gamma_{k'}}{2k'} \right) \right\}^{1/2} \pm \left\{ \left(\frac{k - \gamma_k}{2k} \right) \left(\frac{k' - \gamma_{k'}}{2k'} \right) \right\}^{1/2}. \quad (31)$$

$I_k(r)$'s involved are defined in Eq. (3) of Ref. 10), and are nearly equal to unity. They are given in Figs. 3-6a. of Ref. 10) for the cases of uniform and surface charge distributions of the nucleus. $\rho_k(X)$ and $\rho_k(Y)$ are the "effective nuclear radii", which were defined by Eq. (21) of Ref. 10) as the ratio of two nuclear matrix elements. They correspond to the ordinary nuclear radius ρ appearing in the Konopinski-Uhlenbeck low- Z approximation, and depend on k , Z and also the types of the related nuclear matrix element. The deviations from ρ will be considerably large (order of ten percent for heavy nuclei). As we here assume the invariance under time reversal for strong interactions, we can treat them as real quantities.

In the above treatment the expressions were given for β^- -decay. In order to obtain the corresponding $b_{LL'}^{(n)}$ for β^+ -decay the following substitution should be performed:

$$Z \rightarrow -Z, G_{\bar{\nu}} \rightarrow G_{\bar{\nu}}^*, G_{\nu'} \rightarrow -G_{\nu'}^*, G_A \rightarrow -G_A^* \text{ and } G_A' \rightarrow G_A'^*.$$

Now if we compare the above formulas with those given by (M, M) or Yamada and Morita,¹³⁾ we can easily find a close correspondence between them. In fact, if we put $\sigma_{12} = \sigma_{12}^- = \chi_{12} = \chi_{12}^+ = \Omega_k = 1$, $\sqrt{F_k(\rho)/F_0(\rho)} = 1$, $I_k(r) = 1$ and $\rho_k(X) = \rho_k(Y) = \rho$, then the expressions (22)-(28) tend to Eqs. (40)-(47) of (M, M), e.g.,

$$\begin{aligned}
 N_{12}(X, Y) = & \left\{ \frac{p^2}{3W} - \frac{\alpha Z}{2\rho_1(X)} \sigma_{12} + \frac{p^2}{9} \sigma_{12}^- \right\} \sqrt{\frac{F_1(\rho)}{F_0(\rho)}} \left(\int \mathfrak{M}(X) I_1(r) \right)^* \\
 & \times \left(\int \mathfrak{M}(Y) I_2(r) \right) \longrightarrow \left(\int \mathfrak{M}(X) \right)^* \left(\int \mathfrak{M}(Y) \right) N_{12},
 \end{aligned}$$

where N_{12} is defined by Eq. (41) of (M, M). This approximation is the "low- Z approximation",¹⁰⁾ and it is in fact correct for $Z \rightarrow 0$, but considerably wrong for heavy nuclei.

Here we plot $\sqrt{F_{k-1}(\rho)/F_0(\rho)}$ as function of beta-ray energy, W , in Figs. 1a-c for $k=2, 3$ and 4, where the values of W are taken from 1.2 to 5.0 (mc^2) and the solid lines corresponds to $Z = \pm 10, \pm 30, \pm 50, \pm 70$ and ± 90 for β^- -decay, respectively. We here used the assumption that the radius of nuclear charge distribution is given by $\rho = 1.2 A^{1/3} \times 10^{-13}$ cm, but it must be noted that they depend

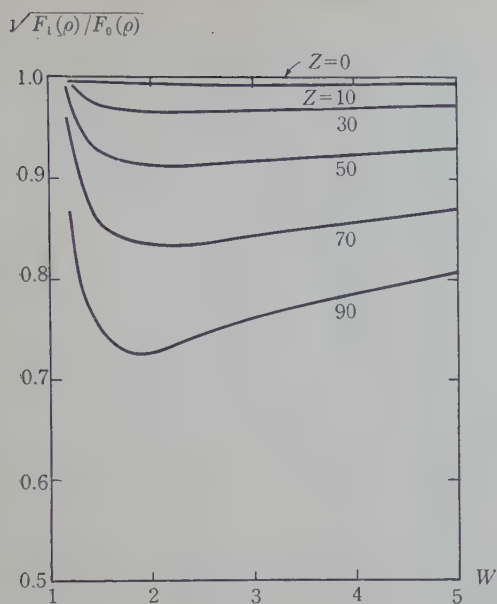


Fig. 1a. $\sqrt{F_1(\rho)/F_0(\rho)}$ as function of beta ray energies, W , and atomic number, Z

In Figs. 1a-c we estimate them from $W=1.2$ (mc^2) to 5.0, and the solid curves express them for the nuclei with $|Z|=10, 30, 50, 70$ and 90, respectively. Here we used the value $\rho=1.2 A^{1/3} \times 10^{-13}$ cm.

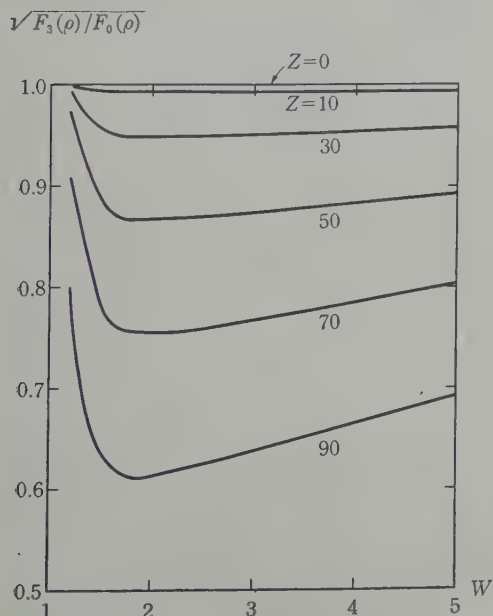


Fig. 1c. $\sqrt{F_3(\rho)/F_0(\rho)}$

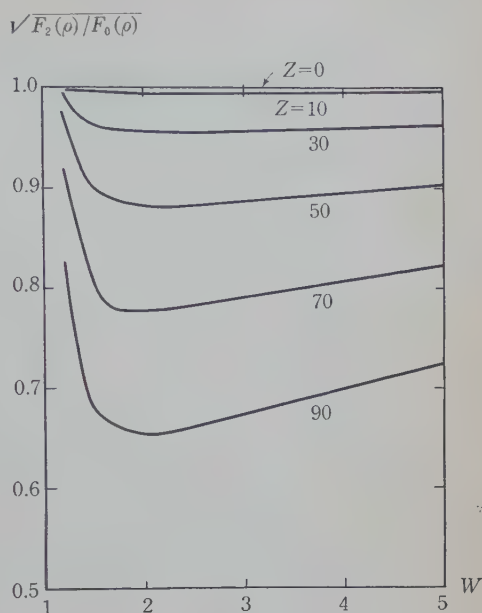
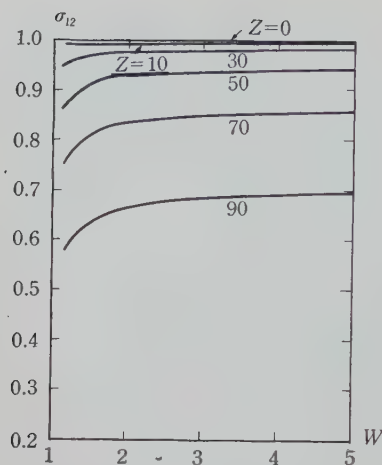
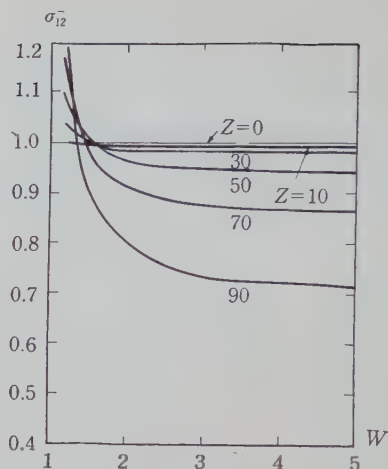


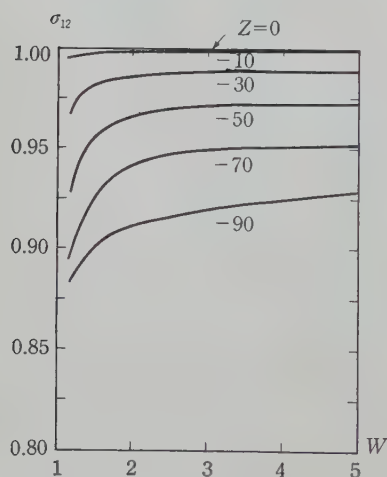
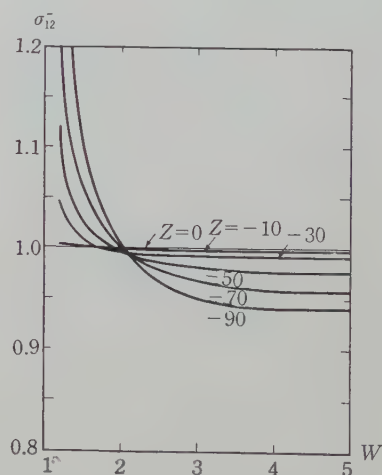
Fig. 1b. $\sqrt{F_2(\rho)/F_0(\rho)}$

very sensitively on the choice of ρ ($\propto \rho^{\gamma_k - \gamma_1 - k + 1}$). Similarly, σ_{12} and σ_{12}^- for β^\mp -decay are shown in Figs. 2a-d, and χ_{12} and χ_{12}^+ in Figs. 3a-d, where we neglected the screening. For $Z=0$, they are equal to unity in all regions, but for heavy nuclei the deviations are considerably significant. As will be seen from (29)–(31), σ and χ depend only on phase shifts of electron wave function at infinite distance (only on Z and W), and are independent of ρ . Thus they are calculable to a desired accuracy. We will discuss these effects in § 4, after getting some more information. We also give here σ_{23} and σ_{23}^- in Figs. 4a-d, which will be needed for the case of the second forbidden transition (see the Appendix).

Now, if time reversal invariance is violated in the beta-decay interaction, further terms defined in the second

Fig. 2a. σ_{12} for β^- -decayFig. 2b. σ_{12}^- for β^- -decay

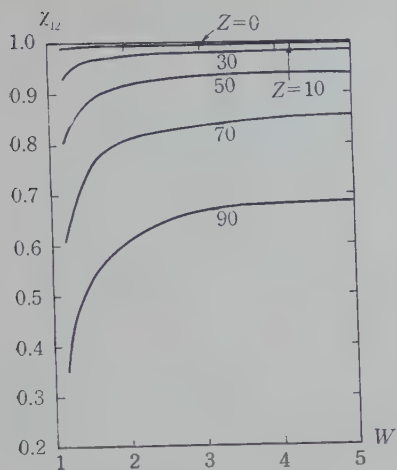
The solid curves correspond to the nuclei with $Z=10, 30, 50, 70$ and 90 , respectively.

Fig. 2c. σ_{12} for β^+ -decayFig. 2d. σ_{12}^- for β^+ -decay

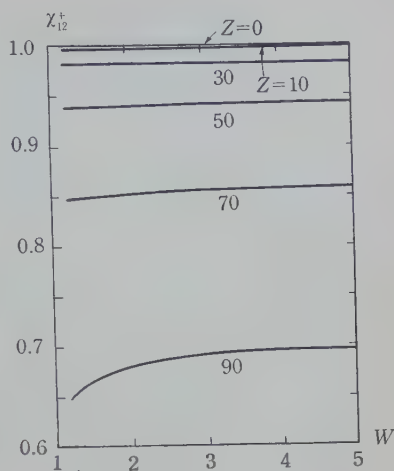
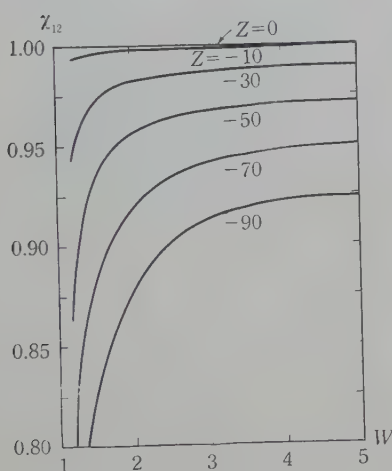
The solid curves correspond to the nuclei with $Z=-10, -30, -50, -70$ and -90 , respectively.

equations of (15)–(21) will appear.

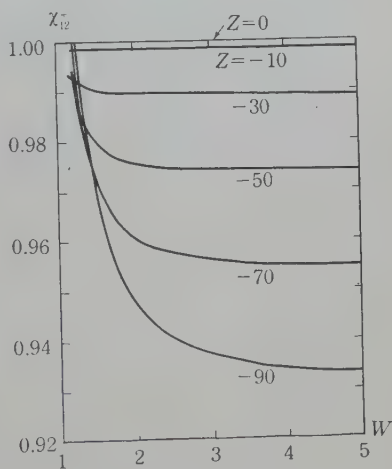
They can be easily obtained in a more explicit form with the similar treatment, but the correspondence to (M, M)'s formula is not obvious. Furthermore, following the remarks towards the end of § 2, we must perform more careful estimation for the approximation, $\zeta_\kappa - \zeta_{\kappa'} \cong \partial_\kappa - \partial_{\kappa'}$. As the validity of time reversal invariance is now very reliable by recent experimental information, we will not estimate their


 Fig. 3a. x_{12} for β^- -decay

The solid curves correspond to the nuclei with $Z=10, 30, 50, 70$ and 90 , respectively.


 Fig. 3b. x_{12}^+ for β^- -decay

 Fig. 3c. x_{12} for β^+ -decay

The solid curves correspond to the nuclei with $Z=-10, -30, -50, -70$ and -90 , respectively.


 Fig. 3d. x_{12}^+ for β^+ -decay

magnitude, but rewrite them in simple forms as the following, where we use the explicit forms of the electron wave functions inside the nucleus.

$$\begin{aligned}
 H_{k,k+1}(X, Y) = & \left(\int \mathfrak{M}(X) I_k(r) \right) * \left(\int \mathfrak{M}(Y) I_{k+1}(r) \right) \frac{V' \bar{F}_{k-1}(\rho) F_k(\rho)}{F_0(\rho)} \\
 & \times \left[\frac{2^k \cdot k!}{(2k)!} p^{k-1} \right]^2 \frac{p}{2k+1} \sigma'_{k,k+1}.
 \end{aligned} \tag{22a}$$

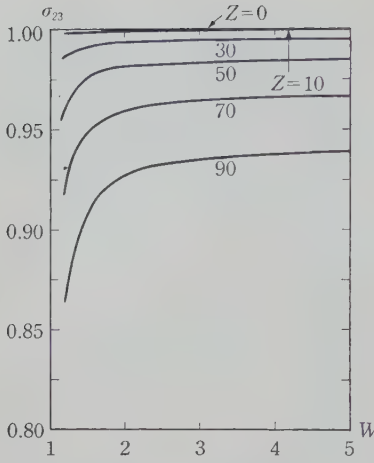


Fig. 4a. σ_{23} for β^- -decay

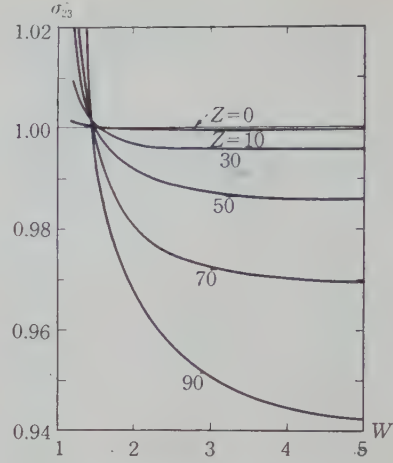


Fig. 4b. $\bar{\sigma}_{23}$ for β^- -decay

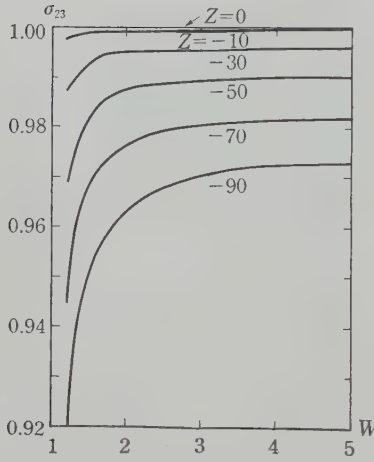


Fig. 4c. σ_{23} for β^+ -decay

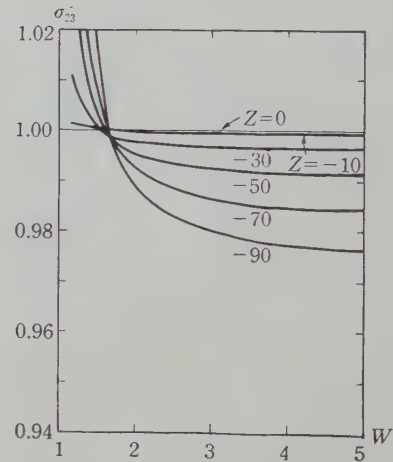


Fig. 4d. $\bar{\sigma}_{23}$ for β^+ -decay

$$J_{k,k+1}(X, Y) = \left(\int \mathfrak{M}(X) I_k(r) \right) * \left(\int \mathfrak{M}(Y) I_{k+1}(r) \right) \frac{\sqrt{F_{k-1}(\rho) F_k(\rho)}}{F_0(\rho)} \\ \times \left[\frac{2^k \cdot k!}{(2k)!} p^{k-1} \right]^2 \left(-\frac{p}{2k+1} \right) \left[\frac{1}{k} - \frac{\alpha Z}{2\rho_k(X)} \sigma'_{k,k+1} + \frac{p^2}{(2k+1)W} \sigma'_{k,k+1} \right] \quad (23a)$$

$$H_{k,k+1}(X, Y) = \left(\int \mathfrak{M}(X) I_k(r) \right) * \left(\int \mathfrak{M}(Y) I_{k+1}(r) \right) \frac{\sqrt{F_{k-1}(\rho) F_k(\rho)}}{F_0(\rho)} \\ \times \left[\frac{2^k \cdot k!}{(2k)!} p^{k-1} \right]^2 \frac{p^2}{(2k+1)W} \chi'_{k,k+1} \quad (24a)$$

$$\begin{aligned} J_{k,k}(X, Y) &= \left(\int \mathfrak{M}(X) I_k(r) \right) * \left(\int \mathfrak{M}(Y) I_k(r) \right) \frac{F_{k-1}(\rho)}{F_0(\rho)} \\ &\times \left[\frac{2^k \cdot k!}{(2k)!} p^{k-1} \right]^2 \frac{\alpha Z}{k(2k+1)W} \Omega'_k. \end{aligned} \quad (25a)$$

$$\begin{aligned} J_{k,k+1}(X, Y) &= \left(\int \mathfrak{M}(X) I_k(r) \right) * \left(\int \mathfrak{M}(Y) I_{k+1}(r) \right) \frac{\sqrt{F_{k-1}(\rho) F_k(\rho)}}{F_0(\rho)} \\ &\times \left[\frac{2^k \cdot k!}{(2k)!} p^{k-1} \right]^2 \left(-\frac{p^2}{(2k+1)W} \right) \left[\frac{1}{k} \frac{\alpha Z}{2\rho_k(X)} \chi'_{k,k+1} + \frac{W}{2k+1} \chi'^{-}_{k,k+1} \right]. \end{aligned} \quad (26a)$$

$$\begin{aligned} A'_k(X, Y) &= \left(\int \mathfrak{M}(X) I_k(r) \right) * \left(\int \mathfrak{M}(Y) I_k(r) \right) \frac{F_{k-1}(\rho)}{F_0(\rho)} \\ &\times \left[\frac{2^k \cdot k!}{(2k)!} p^{k-1} \right]^2 \left(-\frac{\alpha Z}{2kW} \right) \Omega'_k. \end{aligned} \quad (27a)$$

$$\begin{aligned} m'_k(X, Y) &= \left(\int \mathfrak{M}(X) I_k(r) \right) * \left(\int \mathfrak{M}(Y) I_k(r) \right) \frac{F_{k-1}(\rho)}{F_0(\rho)} \left[\frac{2^k \cdot k!}{(2k)!} p^{k-1} \right]^2 \\ &\times \frac{\alpha Z}{2kW} \left[\frac{1}{k^2} \left(-\frac{\alpha Z}{2\rho_k(X)} \right) \left(-\frac{\alpha Z}{2\rho_k(Y)} \right) + \frac{W+1}{k(2k+1)} \frac{\alpha Z}{2\rho_k(X)} \right. \\ &\left. + \frac{W-1}{k(2k+1)} \frac{\alpha Z}{2\rho_k(Y)} + \frac{p^2}{(2k+1)^2} \right] \Omega'_k. \end{aligned} \quad (28a)$$

$\sigma'_{k,k+1}$ et al. are defined as follows:

$$\begin{aligned} \sigma'_{k,k+1} &= d_{k,k+1}^+ f_{k,k+1}^+ - \frac{1}{W} d_{k,k+1}^- f_{k,k+1}^-, \\ \sigma'^{\pm}_{k,k+1} &= d_{k,k+1}^+ f_{k,k+1}^+ + \frac{1}{p^2} (d_{k,k+1}^+ \mp W d_{k,k+1}^-) (f_{k,k+1}^+ \pm f_{k,k+1}^-), \\ \chi'_{k,k+1} &= h_{k,k+1}^+ f_{k,k+1}^+, \\ \chi'^{-}_{k,k+1} &= \left(h_{k,k+1}^+ - \frac{1}{W} h_{k,k+1}^- \right) f_{k,k+1}^+, \\ \Omega'_k &= (1 + \mathcal{A}_{-k}) (1 + \mathcal{A}_k) \{ \cos \hat{\xi}_{-k,k} - \tan(\delta_{-k} - \delta_k) \sin \hat{\xi}_{-k,k} \}, \end{aligned} \quad (29a)$$

with

$$\begin{aligned} d_{k,k+1}^+ &= \frac{1}{2} \left[(1 + \mathcal{A}_{-k}) (1 + \mathcal{A}_{k+1}) \{ \cos \hat{\xi}_{-k,k+1} + \cot(\gamma_{-k} - \gamma_{k+1} + \pi/2) \sin \hat{\xi}_{-k,k+1} \} \right. \\ &\quad \left. \pm (1 + \mathcal{A}_k) (1 + \mathcal{A}_{-(k+1)}) \{ \cos \hat{\xi}_{k,-(k+1)} + \cot(\gamma_k - \gamma_{-(k+1)} + \pi/2) \sin \hat{\xi}_{k,-(k+1)} \} \right], \\ h_{k,k+1}^{\pm} &= \frac{1}{2} \left[(1 + \mathcal{A}_{-k}) (1 + \mathcal{A}_{-(k+1)}) \{ \cos \hat{\xi}_{-k,-(k+1)} \right. \\ &\quad \left. - \tan(\gamma_{-k} - \gamma_{-(k+1)} + \pi/2) \sin \hat{\xi}_{-k,-(k+1)} \} \right. \\ &\quad \left. \pm (1 + \mathcal{A}_k) (1 + \mathcal{A}_{k+1}) \{ \cos \hat{\xi}_{k,k+1} - \tan(\gamma_k - \gamma_{k+1} + \pi/2) \sin \hat{\xi}_{k,k+1} \} \right], \end{aligned} \quad (30a)$$

$$f_{k,k'}^{\pm} = \frac{\alpha Z}{2\sqrt{k}k'} \left\{ \left(\frac{k' + \gamma_{k'}}{k + \gamma_k} \right)^{1/2} \pm \left(\frac{k + \gamma_k}{k' + \gamma_{k'}} \right)^{1/2} \right\}, \quad (31a)$$

$$\tilde{\varepsilon}_{k,k'} = \varepsilon_{k,k'} + (\zeta_k - \zeta_{k'}) - (\partial_k - \partial_{k'}). \quad (32)$$

3B. Further parametrization of $b_{LL'}^{(n)}$

In this subsection, the parameters, $b_{LL'}^{(n)}$, are parametrized by using the following parameters, which are independent of W and ρ .

$$\begin{aligned} x_1^{(0)} &= \left[i \frac{\alpha Z}{2\rho_1(\boldsymbol{\sigma} \cdot \mathbf{r})} \left(\int \mathfrak{M}(\boldsymbol{\sigma} \cdot \mathbf{r}) I_1(r) - \left(\int \mathfrak{M}(\gamma_5) I_1(r) \right) \right) \right. \\ &\quad \times \left. \left(\int \mathfrak{M}(\boldsymbol{\sigma} \times \mathbf{r}) I_1(r) \right) \right]^{-1}, \\ x_1^{(1)} &= \left[G_V \left(\int \mathfrak{M}(\boldsymbol{\alpha}) I_1(r) - \frac{\alpha Z}{2\rho_1(\boldsymbol{\sigma} \times \mathbf{r})} G_A \left(\int \mathfrak{M}(\boldsymbol{\sigma} \times \mathbf{r}) I_1(r) \right) \right. \right. \\ &\quad \left. \left. - i \frac{\alpha Z}{2\rho_1(\mathbf{r})} G_V \left(\int \mathfrak{M}(\mathbf{r}) I_1(r) \right) \right] \left[G_A \left(\int \mathfrak{M}(\boldsymbol{\sigma} \times \mathbf{r}) I_1(r) \right) \right]^{-1}, \quad (33) \\ u_1^{(0)} &= \frac{i \left(\int \mathfrak{M}(\boldsymbol{\sigma} \cdot \mathbf{r}) I_1(r) \right)}{\left(\int \mathfrak{M}(\boldsymbol{\sigma} \times \mathbf{r}) I_1(r) \right)}, \quad u_k^{(2)} = \frac{i \left(\int \mathfrak{M}(B_{ij}) I_k(r) \right)}{\left(\int \mathfrak{M}(\boldsymbol{\sigma} \times \mathbf{r}) I_1(r) \right)}, \\ y_k^{(1)} &= \frac{i G_V \left(\int \mathfrak{M}(\mathbf{r}) I_k(r) \right)}{G_A \left(\int \mathfrak{M}(\boldsymbol{\sigma} \times \mathbf{r}) I_1(r) \right)}, \quad z_2^{(1)} = \frac{\left(\int \mathfrak{M}(\boldsymbol{\sigma} \times \mathbf{r}) I_2(r) \right)}{\left(\int \mathfrak{M}(\boldsymbol{\sigma} \times \mathbf{r}) I_1(r) \right)}. \end{aligned}$$

The definitions of $x_1^{(0)}$, $u_1^{(0)}$ and $u_k^{(2)}$ are slightly different from those of the earlier paper (Eq. (29) of Ref. 10)), and the parameters for the part which involves the primed coupling constants are

$$\begin{aligned} x_1'^{(0)} &= G_A' x_1^{(0)} / G_A, \\ x_1'^{(1)} &= \left[G_V' \left(\int \mathfrak{M}(\boldsymbol{\alpha}) I_1(r) \right) - \frac{\alpha Z}{2\rho_1(\boldsymbol{\sigma} \times \mathbf{r})} G_A' \left(\int \mathfrak{M}(\boldsymbol{\sigma} \times \mathbf{r}) I_1(r) \right) \right. \\ &\quad \left. - i \frac{\alpha Z}{2\rho_1(\mathbf{r})} G_V' \left(\int \mathfrak{M}(\mathbf{r}) I_1(r) \right) \right] \left[G_A \left(\int \mathfrak{M}(\boldsymbol{\sigma} \times \mathbf{r}) I_1(r) \right) \right]^{-1}, \\ u_1'^{(0)} &= G_A' u_1^{(0)} / G_A, \quad u_k'^{(2)} = G_A' u_k^{(2)} / G_A, \\ y_k'^{(1)} &= G_V' y_k^{(1)} / G_V, \quad z_k'^{(1)} = G_A' z_k^{(1)} / G_A. \end{aligned} \quad (34)$$

Each index of the parameters in (33) and (34) indicates magnitude of a (vector) sum of the momenta of the emitted electron and neutrino. If we use them and Eqs. (26)–(36), $b_{LL'}^{(n)}$'s are expressed as follows:

$$b_{00}^{(0)} = \lambda_1 (|x_1^{(0)}|^2 + |x_1'^{(0)}|^2) + \frac{2}{3} \left(q\lambda_1^+ + \frac{p^2}{W} \nu_1^+ \right) \operatorname{Re}(x_1^{(0)*} u_1^{(0)} + x_1'^{(0)*} u_1'^{(0)}) \\ + \left(\frac{q^2}{9} \lambda_1^+ + \frac{p^2}{9} \mu_1^+ + \frac{2qp^2}{9W} \nu_1^+ \right) (|u_1^{(0)}|^2 + |u_1'^{(0)}|^2). \quad (35)$$

$$b_{01}^{(1)} = \frac{2p}{W} \Omega_1 \cdot \operatorname{Re} \left[(x_1^{(0)*} x_1'^{(1)} + x_1'^{(0)*} x_1^{(1)}) - \frac{1}{3} (q+W) (x_1^{(0)*} y_1'^{(1)} + x_1'^{(0)*} y_1^{(1)}) \right. \\ + \frac{1}{3} (q-W) (x_1^{(0)*} z_1'^{(1)} + x_1'^{(0)*} z_1^{(1)}) + \frac{1}{3} (q+W) (u_1^{(0)*} x_1'^{(1)} + u_1'^{(0)*} x_1^{(1)}) \\ - \frac{1}{9} (q^2 + qW + p^2) (u_1^{(0)*} y_1'^{(1)} + u_1'^{(0)*} y_1^{(1)}) \\ - \frac{p^2}{9} (u_1^{(0)*} z_1'^{(1)} + u_1'^{(0)*} z_1^{(1)}) \left. \right] - \frac{2p}{3} \sqrt{\frac{F_1(\rho)}{F_0(\rho)}} \operatorname{Re} \left[\chi_{12} \{ x_1^{(0)*} (2y_2'^{(1)} - z_2'^{(1)}) \right. \\ + x_1'^{(0)*} (2y_2^{(1)} - z_2^{(1)}) \} \\ + \frac{1}{3} \left(q\chi_{12} + \frac{p^2}{W} \chi_{12}^+ \right) \{ u_1^{(0)*} (2y_2'^{(1)} - z_2'^{(1)}) + u_1'^{(0)*} (2y_2^{(1)} - z_2^{(1)}) \} \left. \right] \\ - \frac{2\alpha Z}{3W} \Omega_1' \operatorname{Im} \left[(x_1^{(0)*} y_1'^{(1)} + x_1'^{(0)*} y_1^{(1)}) + (u_1^{(0)*} x_1'^{(1)} + u_1'^{(0)*} x_1^{(1)}) \right] \\ + \frac{2p^2}{3W} \sqrt{\frac{F_1(\rho)}{F_0(\rho)}} \operatorname{Im} \left[\chi_{12}' \{ x_1^{(0)*} (2y_2'^{(1)} - z_2'^{(1)}) + x_1'^{(0)*} (2y_2^{(1)} - z_2^{(1)}) \} \right. \\ + \frac{1}{3} (q\chi_{12}' + W\chi_{12}^-) \{ u_1^{(0)*} (2y_2'^{(1)} - z_2'^{(1)}) + u_1'^{(0)*} (2y_2^{(1)} - z_2^{(1)}) \} \left. \right]. \quad (36)$$

$$b_{11}^{(0)} = -\nu \sqrt{3} \left\{ \lambda_1 (|x_1^{(1)}|^2 + |x_1'^{(1)}|^2) - \frac{2}{3} \left(q\lambda_1^+ + \frac{p^2}{W} \nu_1^+ \right) \operatorname{Re}(x_1^{(1)*} y_1^{(1)} + x_1'^{(1)*} y_1'^{(1)}) \right. \\ + \frac{2}{3} \left(q\lambda_1^+ - \frac{p^2}{W} \nu_1^+ \right) \operatorname{Re}(x_1^{(1)} + x_1'^{(1)*} z_1'^{(1)}) \\ + \frac{1}{3} \left(q^2 \lambda_1^+ + \frac{p^2}{3} \mu_1^+ + \frac{2qp^2}{3W} \nu_1^+ \right) (|y_1^{(1)}|^2 + |y_1'^{(1)}|^2) \\ + \frac{2p^2}{9} \mu_1^+ \operatorname{Re}(y_1^{(1)} + y_1'^{(1)*} z_1'^{(1)}) + \frac{1}{3} \left(\frac{q^2}{2} \lambda_1^+ + \frac{p^2}{3} \mu_1^+ - \frac{2qp^2}{3W} \nu_1^+ \right) \\ \times (1 + |z_1'^{(1)}|^2) \\ + \frac{p^2}{18} \lambda_2^+ \frac{F_1(\rho)}{F_0(\rho)} [|2y_2^{(1)} - z_2^{(1)}|^2 + |2y_2'^{(1)} - z_2'^{(1)}|^2] \left. \right\}. \quad (37)$$

$$b_{11}^{(1)} = \nu \sqrt{2} \left\{ \frac{2p}{W} \Omega_1 \operatorname{Re} \left[x_1^{(1)*} x_1'^{(1)} - \frac{1}{3} (q+W) (x_1^{(1)*} y_1'^{(1)} + x_1'^{(1)*} y_1^{(1)}) \right. \right.$$

$$\begin{aligned}
& + \frac{1}{3} (q - W) (x_1^{(1)*} z_1'^{(1)} + x_1'^{(1)}) + \frac{1}{9} (p^2 + 2qW) y_1^{(1)*} y_1'^{(1)} \\
& + \frac{1}{3} \left(\frac{p^2}{3} - \frac{q^2}{2} \right) (y_1'^{(1)} + y_1^{(1)*} z_1'^{(1)}) + \frac{1}{3} \left(\frac{p^2}{3} + \frac{q^2}{4} - \frac{2qW}{3} \right) z_1'^{(1)} \Big] \\
& + \frac{p^3}{18W} \frac{F_1(\rho)}{F_0(\rho)} \operatorname{Re} [(2y_2^{(1)} - z_2^{(1)}) * (2y_2'^{(1)} - z_2'^{(1)})] \\
& + \frac{p}{3} \sqrt{\frac{F_1(\rho)}{F_0(\rho)}} \operatorname{Re} \left[\chi_{12} \{ x_1^{(1)*} (2y_2'^{(1)} - z_2'^{(1)}) + x_1'^{(1)*} (2y_2^{(1)} - z_2^{(1)}) \} \right. \\
& - \frac{1}{3} \left(q\chi_{12} + \frac{p^2}{W} \chi_{12}^+ \right) \{ y_1^{(1)*} (2y_2'^{(1)} - z_2'^{(1)}) + y_1'^{(1)*} (2y_2^{(1)} - z_2^{(1)}) \} \\
& + \frac{1}{3} \left(q\chi_{12} - \frac{p^2}{W} \chi_{12}^+ \right) \{ (2y_2'^{(1)} - z_2'^{(1)}) + z_1'^{(1)*} (2y_2^{(1)} - z_2^{(1)}) \} \\
& - \frac{2\alpha Z}{3W} \mathcal{Q}_1' \operatorname{Im} [(x_1^{(1)*} y_1'^{(1)} + x_1'^{(1)*} y_1^{(1)}) + (x_1'^{(1)} + x_1^{(1)*} z_1'^{(1)})] \\
& - \frac{p^2}{3W} \sqrt{\frac{F_1(\rho)}{F_0(\rho)}} \operatorname{Im} \left[\chi_{12}' \{ x_1^{(1)*} (2y_2'^{(1)} - z_2'^{(1)}) + x_1'^{(1)*} (2y_2^{(1)} - z_2^{(1)}) \} \right. \\
& + \frac{1}{3} (q\chi_{12}' + W\chi_{12}'^-) \{ y_1^{(1)*} (2y_2'^{(1)} - z_2'^{(1)}) + y_1'^{(1)*} (2y_2^{(1)} - z_2^{(1)}) \} \\
& \left. + \frac{1}{3} (q\chi_{12}' - W\chi_{12}'^-) \{ (2y_2'^{(1)} - z_2'^{(1)}) + z_1'^{(1)*} (2y_2^{(1)} - z_2^{(1)}) \} \right] \Big\}. \quad (38)
\end{aligned}$$

$$\begin{aligned}
b_{11}^{(2)} = & -\sqrt{6} \left\{ \frac{p^2}{3W} \sqrt{\frac{F_1(\rho)}{F_0(\rho)}} \operatorname{Re} \left[\sigma_{12} \{ x_1^{(1)*} (2y_2^{(1)} - z_2^{(1)}) + x_1'^{(1)*} (2y_2'^{(1)} - z_2'^{(1)}) \} \right. \right. \\
& - \frac{1}{3} (q\sigma_{12} + W\sigma_{12}^-) \{ y_1^{(1)*} (2y_2^{(1)} - z_2^{(1)}) + y_1'^{(1)*} (2y_2'^{(1)} - z_2'^{(1)}) \} \\
& + \frac{1}{3} (q\sigma_{12} - W\sigma_{12}^-) \{ (2y_2^{(1)} - z_2^{(1)}) + z_1'^{(1)*} (2y_2'^{(1)} - z_2'^{(1)}) \} \\
& - \frac{p^2}{36} \lambda_2^+ \frac{F_1(\rho)}{F_0(\rho)} [|2y_2^{(1)} - z_2^{(1)}|^2 + |2y_2'^{(1)} - z_2'^{(1)}|^2] \\
& + \frac{p}{3} \sqrt{\frac{F_1(\rho)}{F_0(\rho)}} \operatorname{Im} \left[\sigma_{12}' \{ x_1^{(1)*} (2y_2^{(1)} - z_2^{(1)}) + x_1'^{(1)*} (2y_2'^{(1)} - z_2'^{(1)}) \} \right. \\
& - \frac{1}{3} \left(q\sigma_{12}' + \frac{p^2}{W} \sigma_{12}'^+ \right) \{ y_1^{(1)*} (2y_2^{(1)} - z_2^{(1)}) + y_1'^{(1)*} (2y_2'^{(1)} - z_2'^{(1)}) \} \\
& \left. \left. + \frac{1}{3} \left(q\sigma_{12}' - \frac{p^2}{W} \sigma_{12}'^+ \right) \{ (2y_2^{(1)} - z_2^{(1)}) + z_1'^{(1)*} (2y_2'^{(1)} - z_2'^{(1)}) \} \right] \right\}. \quad (39)
\end{aligned}$$

$$\begin{aligned}
b_{02}^{(2)} = & \sqrt{\frac{2}{3}} \frac{p^2}{W} \sqrt{\frac{F_1(\rho)}{F_0(\rho)}} \operatorname{Re} [\sigma_{12} (x_1^{(0)*} u_2^{(2)} + x_1'^{(0)*} u_2'^{(2)}) \\
& + \frac{1}{3} (q\sigma_{12} + W\sigma_{12}^-) (u_1^{(0)*} u_2^{(2)} + u_1'^{(0)*} u_2'^{(2)})]. \quad (40)
\end{aligned}$$

$$\begin{aligned}
b_{12}^{(1)} = & -(\sqrt{5}/2\sqrt{3}) \left\{ 2p \sqrt{\frac{F_1(\rho)}{F_0(\rho)}} \operatorname{Re} \left[\chi_{12} (x_1^{(1)*} u_2'^{(2)} + x_1'^{(1)*} u_2^{(2)}) \right. \right. \\
& + \frac{1}{3} \left(q\chi_{12} + \frac{p^2}{W} \chi_{12}^+ \right) (y_1^{(1)*} u_2'^{(2)} + y_1'^{(1)*} u_2^{(2)}) \\
& + \frac{1}{3} \left(q\chi_{12} - \frac{p^2}{W} \chi_{12}^+ \right) (u_2'^{(2)} + z_1'^{(1)*} u_2^{(2)}) \\
& - \frac{q^2 p}{3W} \mathcal{Q}_1 \operatorname{Re} [(2y_1^{(1)} + z_1^{(1)}) * u_1'^{(2)} + (2y_1'^{(1)} + z_1'^{(1)}) * u_1^{(2)}] \\
& - \frac{p^3}{15W} \mathcal{Q}_2 \frac{F_1(\rho)}{F_0(\rho)} \operatorname{Re} [(2y_2^{(1)} - z_2^{(1)}) * u_2'^{(2)} + (2y_2'^{(1)} - z_2'^{(1)}) * u_2^{(2)}] \\
& - \frac{2p^2}{W} \sqrt{\frac{F_1(\rho)}{F_0(\rho)}} \operatorname{Im} \left[\chi_{12}' (x_1^{(1)*} u_2'^{(2)} + x_1'^{(1)*} u_2^{(2)}) \right. \\
& \left. \left. - \frac{1}{3} (q\chi_{12}' + W\chi_{12}^-) (y_1^{(1)*} u_2'^{(2)} + y_1'^{(1)*} u_2^{(2)}) \right] \right\}. \quad (41)
\end{aligned}$$

$$\begin{aligned}
b_{12}^{(2)} = & \frac{p^2}{W} \sqrt{\frac{F_1(\rho)}{F_0(\rho)}} \operatorname{Re} \left[\sigma_{12} (x_1^{(1)*} u_2^{(2)} + x_1'^{(1)*} u_2'^{(2)}) \right. \\
& - \frac{1}{3} (q\sigma_{12} + W\sigma_{12}^-) (y_1^{(1)*} u_2^{(2)} + y_1'^{(1)*} u_2'^{(2)}) \\
& \left. + \frac{1}{3} (q\sigma_{12} - W\sigma_{12}^-) (u_2^{(2)} + z_1'^{(1)*} u_2'^{(2)}) \right] \\
& + \frac{p^2}{6} \lambda_2^+ \frac{F_1(\rho)}{F_0(\rho)} \operatorname{Re} [(2y_2^{(1)} - z_2^{(1)}) * u_2^{(2)} + (2y_2'^{(1)} - z_2'^{(1)}) * u_2'^{(2)}] \\
& + p \sqrt{\frac{F_1(\rho)}{F_0(\rho)}} \operatorname{Im} \left[\sigma_{12}' (x_1^{(1)*} u_2^{(2)} + x_1'^{(1)*} u_2'^{(2)}) \right. \\
& \left. - \frac{1}{3} \left(q\sigma_{12}' + \frac{p^2}{W} \sigma_{12}^+ \right) (y_1^{(1)*} u_2^{(2)} + y_1'^{(1)*} u_2'^{(2)}) \right]. \quad (42)
\end{aligned}$$

$$b_{12}^{(3)} = - (1/2\sqrt{10}) \frac{p^3}{W} \mathcal{Q}_2 \frac{F_1(\rho)}{F_0(\rho)} \operatorname{Re} \{ (2y_2^{(1)} - z_2^{(1)}) * u_2'^{(2)} + (2y_2'^{(1)} - z_2'^{(1)}) * u_2^{(2)} \}. \quad (43)$$

$$b_{22}^{(0)} = (\sqrt{5}/12) \left\{ q^2 \lambda_1^+ (|u_1^{(2)}|^2 + |u_1'^{(2)}|^2) + p^2 \lambda_2^+ \frac{F_1(\rho)}{F_0(\rho)} (|u_2^{(2)}|^2 + |u_2'^{(2)}|^2) \right\}. \quad (44)$$

$$b_{22}^{(1)} = - (\sqrt{5}/2\sqrt{2}) \frac{p}{W} \operatorname{Re} \left\{ \frac{q^2}{3} \mathcal{Q}_1 (u_1^{(2)*} u_1'^{(2)}) + \frac{p^2}{5} \mathcal{Q}_2 \frac{F_1(\rho)}{F_0(\rho)} (u_2^{(2)*} u_2'^{(2)}) \right\}. \quad (45)$$

$$b_{22}^{(2)} = - (3\sqrt{7}/4\sqrt{2}) \lambda_2^+ \frac{F_1(\rho)}{F_0(\rho)} \{ |u_2^{(2)}|^2 + |u_2'^{(2)}|^2 \}. \quad (46)$$

$$b_{22}^{(3)} = (1/\sqrt{90}) \frac{p^3}{W} \mathcal{Q}_2 \frac{F_1(\rho)}{F_0(\rho)} \operatorname{Re} (u_2^{(2)*} u_2'^{(2)}). \quad (47)$$

Here it should be noted that $y_2^{(1)}$ and $z_2^{(1)}$ are always combined in a form $2y_2^{(1)} - z_2^{(1)}$, and the same situation holds for $y_2'^{(1)}$ and $z_2'^{(1)}$, too. In order to show these relations explicitly, we use the following equations, $\text{Im}(G_r^* G_r' + G_r G_r'^*) = 0 = \text{Im}(G_A^* G_A' + G_A G_A'^*)$ or $\text{Im}(u_1'^{(0)} z_2^{(1)}) = \text{Im}(x_1^{(0)} z_2'^{(1)}) = 0$, etc.

§ 4. Summary

This section is prepared for persons not interested in the derivation of the formulas but only in their application.

For β -ray angular correlation we must consider not only the behavior of wave functions of electron inside the nucleus, but also the Coulomb phase shifts at infinity. The radial wave functions of the electron have already been obtained in Eq. (14) of Ref. 10) by Yamada and the author, taking into account the finite nuclear size and the screening effects. For their explanations, refer to § 4 of Ref. 10). The corresponding expressions for their asymptotic forms at infinity have been given in Eq. (6) of § 2 of this paper, where $f_\kappa(r)$ and $g_\kappa(r)$ are small and large components in the non-relativistic limit, respectively, and are normalized so that there are R electrons in a large sphere of radius R . The conventional notation and units are used as far as possible. The phase shifts, ξ_κ , were obtained in Eq. (7) as function of the phase shifts of the regular and irregular wave functions, (i.e., δ_κ and $\bar{\delta}_\kappa$, respectively) and H_κ . H_κ represents the degree of mixture of the irregular wave function to the regular one, and gives the measure for the deviation of ξ_κ from δ_κ . It has been shown for $|\kappa|=1-4$ and for β^\mp -decay in Table I, where we assumed the uniform charge distribution of the nucleus. They are generally quite small and in actual cases we can safely use the approximation, $\xi_\kappa \simeq \delta_\kappa$, for the terms remaining when time reversal invariance is valid in beta-interaction. Therefore, if it is the case, our theory is again insensitive to the nuclear charge distributions as far as we treat the nuclear matrix elements as adjustable parameters.

In § 3, we derived the formulas of β -ray angular correlation for the first forbidden transitions. Using the expressions of Ref. 14), we have given in Eq. (8) the formulas for the angular correlation between the beta rays and circularly polarized gamma rays. If we observe only their direction the terms involving $P_n(\cos \theta)$ with even n remain. Similarly, the beta-ray angular distribution from oriented nuclei and the beta spectrum have been given in Eq. (9) and Eq. (10), respectively. When we took into account the finite size of the nucleus (especially the effect (3) in § 1) and the screening, the particle parameters $b_{LL'}^{(n)}$'s were expressed in (11) for VA interaction by using combinations of electron wave functions and phase shifts as Eqs. (12)–(21). X and Y represent arbitrary nucleon operators, and $\mathfrak{M}(X)$ and $\mathfrak{M}(Y)$ represent the “reduced nuclear matrix elements” by Yamada and Morita¹⁴⁾ or those denoted by Racah's ($\parallel \parallel$) symbol. For the Coulomb phase shift we have usually used the following two approximations: One is $\xi_\kappa - \xi_{\kappa'} \simeq \delta_\kappa - \delta_{\kappa'}$, which is permitted by the discussion in § 2, and another is the “low- Z approximation” which is valid for light nuclei

(ex. Eqs. (36)–(52) of Ref. 14)). The “low- Z approximation” has been required for the reason that one has scarcely been able to make an analytical simplification of the final expressions owing to the difficulty of the exact treatment of the Coulomb phase shift. Thus we have separated the phase shift into two parts; One is the part which is calculable in analytical forms and another is the remaining one. The latter part is usually less than the former part, but not always so especially for heavy nuclei. By making the simplification of the former part of the phase shift and using the explicit forms of electron wave functions (Eq. (14) of Ref. 10)), we could arrange the first equations of (15)–(21) in forms of Eqs. (22)–(28), which had the obvious correspondence to those in the usual theories. $I_k(r)$ is an energy independent function shown in Figs. 3a–6d for a few particular cases in Ref. 10). $I_k(r)$ is not much different from unity and $I_k(r) \rightarrow 1$ for $Z \rightarrow 0$. $\rho_k(X)$ is the “effective nuclear radius” defined by Eq. (21) of Ref. 10), and λ_k^+ , μ_k^+ , and ν_k^+ are the factors defined in Eq. (19) of Ref. 10) which are nearly equal to unity when we omit the screening effect. σ ’s, χ ’s and ϱ ’s have been defined in Eqs. (29)–(31), where J_k is the screening correction term ($J_k=0$ without screening). Those multiplication factors depend only on W and Z , but do not depend on ρ and the nuclear charge distribution. If we put $\sigma_{12}=\sigma_{12}^-=\chi_{12}=\chi_{12}^+=\varrho=1$, $\sqrt{F_k(\rho)/F_0(\rho)}=1$, $I_k(r)=1$ and $\rho_k(X)=\rho_k(Y)=\rho$, then the expressions (22)–(28) tend to Eqs. (40)–(47) of Ref. 14). σ ’s and χ ’s have been shown in Figs. 2a–4d for $Z=\pm 10, \pm 30, \pm 50, \pm 70$ and ± 90 (for β^\mp -decay) as function of W , and for heavy nuclei considerably large deviations from unity have been found. It means that the approximation for the phase shift in the usual theories is quite wrong for heavy nuclei. Similarly, we obtained in Eqs. (22a)–(28a) the expressions for the terms which should vanish when time reversal invariance is valid in beta decay. σ ’s χ ’s and ϱ ’s have been given in Eqs. (29a)–(31a) and (32), but we did not estimate them, because the validity of time reversal invariance is now reliable.

In § 3B the particle parameters $b_{LL'}^{(v)}$ ’s have finally been represented in Eqs. (35)–(41) by the parameters x ’s, y ’s, z ’s and u ’s defined in (33) and (34). The x ’s et al. in our theories are adjustable parameters, which include the nuclear matrix elements.

The explanation for them has already been given in p. 319 of Ref. 10), though their definition is slightly different from those of Ref. 10). If time-reversal invariance is valid in beta decay, all the primed parameters vanish in Eqs. (35)–(41). See the following section (§ 5) for more detailed discussions of the results. The formulas for the allowed and the second forbidden transitions are given in the Appendix.

§ 5. Concluding remarks

The expressions in the previous section and the Appendix give us information on the magnitude of the deviations from those with the usual low- Z approximation. Those are evaluated by the quantities (multiplication factor) $F_{k-1}(\rho)/F_0(\rho)$, ϱ_k ,

$\sqrt{F_1(\rho)/F_0(\rho)} \cdot \sigma_{12}$, $\sqrt{F_1(\rho)/F_0(\rho)} \cdot \sigma_{12}^-$, $\sqrt{F_1(\rho)/F_0(\rho)} \cdot \chi_{12}$, $\sqrt{F_1(\rho)/F_0(\rho)} \cdot \chi_{12}^-$, $(\sqrt{F_1(\rho)F_2(\rho)}/F_0(\rho))\sigma_{23}$ and $(\sqrt{F_1(\rho)F_2(\rho)}/F_0(\rho))\sigma_{23}^-$ and these quantities are easily evaluated from Fig. 1a-4b. The most significant deviation occurs for the non-unique first forbidden beta-gamma directional correlation. We show $\sqrt{F_1(\rho)/F_0(\rho)} \cdot \sigma_{12}$ in Fig. 5a. As is seen from the figure, the angular dependent coefficient, $b_{1,1}^{(2)}$, which is the coefficient of $P_2(\cos \theta)$ term, decreases to about 70 percent for $Z=70$ and, moreover, about 50 percent for $Z=90$ by the factor $\sqrt{F_1(\rho)/F_0(\rho)} \cdot \sigma_{12}$ in the main terms (linear in x), while the angular independent term, $b_{1,1}^{(0)}$, changes little. For the β^- -decay the situation is less significant, but $b_{1,1}^{(2)}$ decreases about 80 percent

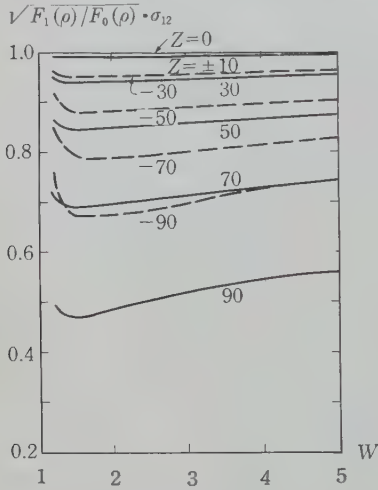


Fig. 5a. $\sqrt{F_1(\rho)/F_0(\rho)} \cdot \sigma_{12}$
for β^+ -decay

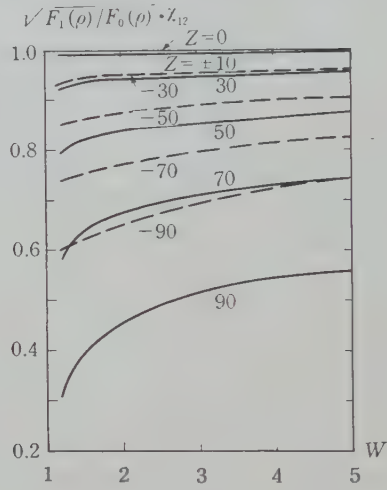


Fig. 5b. $\sqrt{F_1(\rho)/F_0(\rho)} \cdot \chi_{12}$
for β^+ -decay

In Figs. 5a-c, the solid curves correspond to the nuclei with $Z=10, 30, 50, 70$ and 90 (for β^- -decay), and the dashed curves correspond to the nuclei with $Z=-10, -30, -50, -70$ and -90 (for β^+ -decay), respectively.

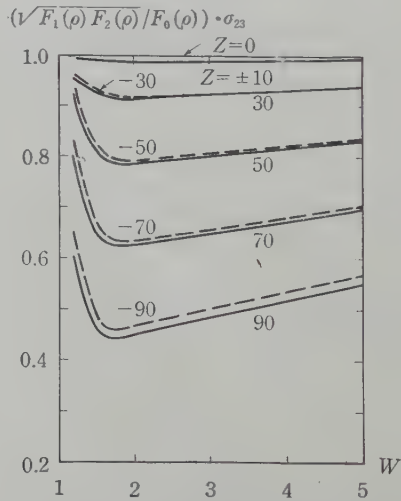


Fig. 5c. $(\sqrt{F_1(\rho)F_2(\rho)}/F_0(\rho)) \cdot \sigma_{23}$
for β^+ -decay

for $Z = -70$ and about 70 percent for $Z = -90$. The deviation is about five percent even for $Z = 30$ or smaller nuclei. When we observe polarization we must investigate $b_{LL'}^{(n)}$ with odd n . In this case the deviation is comparatively small, because the main term which is proportional to x^2 depends only on Ω_1 . The next terms linear in x depend on both Ω_1 and $\sqrt{F_1(\rho)/F_0(\rho)} \cdot \chi_{12}$, and $F_1(\rho)/F_0(\rho) \Omega_2$ deviates significantly for heavy nuclei. This is shown in Fig. 5b. For the allowed transition there is no question, because they depend only on λ_1^+ and Ω_1 . But for the unique first forbidden transition, $b_{22}^{(2)}$ and $b_{22}^{(3)}$ change considerably by factor $F_1(\rho)/F_0(\rho)$, while $b_{22}^{(1)}$ is less than those.

For the non-unique second forbidden beta-gamma directional correlation, the main term in $b_{LL'}^{(2)}$ depends only on λ_2^+ , thus the situation is not so significant as to the non-unique first forbidden transition. But the next term depends on λ_2^+ , Ω_2^+ , $\sqrt{F_1(\rho)/F_0(\rho)} \cdot \sigma_{12}$ and $(\sqrt{F_1(\rho)F_2(\rho)}/F_0(\rho))\sigma_{23}$. $(\sqrt{F_1(\rho)F_2(\rho)}/F_0(\rho))\sigma_{23}$ is shown in Fig. 5c and large deviations are found. The coefficient of $P_4(\cos \theta)$ term deviates significantly by this factor. Thus, we must not neglect these large deviations, and these quantities are calculable to the desired accuracy. Accurate tables for them are very desirable. Here we must be careful to treat the ρ -dependence. If we take these effects into account thoroughly, the error involved in our theory becomes equal to that for the case of beta spectrum in our previous paper.¹⁰⁾ In conclusion, we wish to emphasize here again that more careful treatment is needed for the theory of the β -ray angular correlation, and in addition the difference in the charge distribution of the nucleus is not so effective when we use the nuclear matrix element as adjustable parameters and the time reversal invariance is valid in beta-decay.

For electron-neutrino angular correlation the same conclusion holds in the allowed and first forbidden transitions.

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Appendix

$b_{LL'}^{(n)}$ for allowed and second forbidden transitions in the VA interaction

Following the procedure in § 3, we here give the particle parameters $b_{LL'}^{(n)}$, after the parametrization for the allowed and second forbidden transitions. For the latter transition only the expressions in the case of directional correlation (i.e., $b_{LL'}^{(n)}$ with even n) are derived, because the experiments which measure the polarization are probably not available for such a transition at the present stage.

For allowed transition

$$\begin{aligned}
 b_{00}^{(0)} &= \lambda_1^+ (|G_V|^2 + |G_V'|^2) \left| \int \mathfrak{M}(1) I_1(r) \right|^2, \\
 b_{11}^{(0)} &= -\sqrt{3} \lambda_1^+ (|G_A|^2 + |G_A'|^2) \left| \int \mathfrak{M}(\sigma) I_1(r) \right|^2, \\
 b_{01}^{(1)} &= -2(p/W) \mathcal{Q}_1 \operatorname{Re}(G_V^* G_A' + G_V'^* G_A) \left(\int \mathfrak{M}(1) I_1(r) \right)^* \left(\int \mathfrak{M}(\sigma) I_1(r) \right) \\
 &\quad - 2(\alpha Z/W) \mathcal{Q}_1' \operatorname{Im}(G_V^* G_A' + G_V'^* G_A) \left(\int \mathfrak{M}(1) I_1(r) \right)^* \left(\int \mathfrak{M}(\sigma) I_1(r) \right), \\
 b_{11}^{(1)} &= \pm 2\sqrt{2} (p/W) \mathcal{Q}_1 \operatorname{Re}(G_A^* G_A') \left| \int \mathfrak{M}(\sigma) I_1(r) \right|^2.
 \end{aligned} \tag{A1}$$

Here the plus and minus signs in the right-hand side of $b_{11}^{(1)}$ corresponds to ρ^{\pm} -decay, respectively.

For the second forbidden transition

Parameters :

$$\begin{aligned}
 x_1^{(2)} &= \left[G_V \left(\int \mathfrak{M}(A_{ij}) I_1(r) \right) - \frac{\alpha Z}{2\rho_1(T_{ij})} G_A \left(\int \mathfrak{M}(T_{ij}) I_1(r) \right) \right. \\
 &\quad \left. - 2i \frac{\alpha Z}{2\rho_1(R_{ij})} G_V \left(\int \mathfrak{M}(R_{ij}) I_1(r) \right) \right] \cdot \left[G_A \left(\int \mathfrak{M}(T_{ij}) I_1(r) \right) \right]^{-1}, \\
 x_2^{(2)} &= \left[G_V \left(\int \mathfrak{M}(A_{ij}) I_2(r) \right) - \frac{1}{2} \frac{\alpha Z}{2\rho_2(T_{ij})} G_A \left(\int \mathfrak{M}(T_{ij}) I_2(r) \right) \right. \\
 &\quad \left. - i \frac{\alpha Z}{2\rho_2(R_{ij})} G_V \left(\int \mathfrak{M}(R_{ij}) I_2(r) \right) \right] \cdot \left[G_A \left(\int \mathfrak{M}(T_{ij}) I_1(r) \right) \right]^{-1}, \\
 y_k^{(2)} &= 2i G_V \left(\int \mathfrak{M}(R_{ij}) I_k(r) \right) / G_A \left(\int \mathfrak{M}(T_{ij}) I_1(r) \right), \\
 z_k^{(2)} &= \left(\int \mathfrak{M}(T_{ij}) I_k(r) \right) / \left(\int \mathfrak{M}(T_{ij}) I_1(r) \right), \\
 u_k^{(3)} &= i \left(\int \mathfrak{M}(S_{ij\alpha}) I_k(r) \right) / \left(\int \mathfrak{M}(T_{ij}) I_1(r) \right).
 \end{aligned} \tag{A2}$$

Particle parameters :

$$\begin{aligned}
 b_{22}^{(0)} &= \sqrt{5} \left\{ \frac{q^2}{6} \left[\frac{1}{2} \lambda_1^+ |x_1^{(2)}|^2 - \left(\frac{9}{5} \lambda_1^+ + \frac{p^2}{3W} \nu_1^+ \right) \operatorname{Re}(x_1^{(2)*} y_1^{(2)}) \right. \right. \\
 &\quad \left. \left. + \left(\frac{q}{5} \lambda_1^+ - \frac{p^2}{3W} \nu_1^+ \right) \operatorname{Re}(x_1^{(2)}) + \frac{1}{2} \left(\frac{q^2}{10} \lambda_1^+ + \frac{p^2}{9} \mu_1^+ + \frac{2qp^2}{15W} \nu_1^+ \right) |y_1^{(2)}|^2 \right. \right. \\
 &\quad \left. \left. + \frac{p^2}{9} \mu_1^+ \operatorname{Re}(y_1^{(2)}) + \frac{1}{3} \left(\frac{q^2}{10} \lambda_1^+ + \frac{p^2}{6} \mu_1^+ - \frac{qp^2}{5W} \nu_1^+ \right) \right\}
 \end{aligned}$$

$$\begin{aligned}
& + \frac{p^2}{6} \frac{F_1(\rho)}{F_0(\rho)} \left[\frac{1}{2} \lambda_2^+ |x_2^{(2)}|^2 - \left(\frac{q}{3} \lambda_2^+ + \frac{p^2}{5W} \nu_2^+ \right) \operatorname{Re}(x_2^{(2)*} y_2^{(2)}) \right. \\
& + \left(\frac{q}{3} \lambda_2^+ - \frac{p^2}{5W} \nu_2^+ \right) \operatorname{Re}(x_2^{(2)*} z_2^{(2)}) \\
& + \left(\frac{q^2}{9} \lambda_2^+ + \frac{p^2}{50} \mu_2^+ + \frac{qp^2}{15W} \nu_2^+ \right) |y_2^{(2)}|^2 \\
& - \left(\frac{q^2}{9} \lambda_2^+ - \frac{p^2}{25} \mu_2^+ \right) \operatorname{Re}(y_2^{(2)*} z_2^{(2)}) \\
& + \left. \left(\frac{q^2}{18} \lambda_2^+ + \frac{p^2}{50} \mu_2^+ - \frac{qp^2}{15W} \nu_2^+ \right) |z_2^{(2)}|^2 \right] \\
& + \frac{p^4}{1800} \lambda_3^+ \frac{F_2(\rho)}{F_0(\rho)} |3y_3^{(2)} - 2z_3^{(2)}|^2 \}. \tag{A3}
\end{aligned}$$

$$\begin{aligned}
b_{22}^{(2)} = & -\sqrt{7/2} \left\{ \frac{p^2}{6} \frac{F_1(\rho)}{F_0(\rho)} \left[\frac{1}{2} \lambda_2^+ |x_2^{(2)}|^2 - \left(\frac{q}{3} \lambda_2^+ + \frac{p^2}{5W} \nu_2^+ \right) \operatorname{Re}(x_2^{(2)*} y_2^{(2)}) \right. \right. \\
& + \left(\frac{q}{3} \lambda_2^+ - \frac{p^2}{5W} \nu_2^+ \right) \operatorname{Re}(x_2^{(2)*} z_2^{(2)}) + \left(\frac{q^2}{18} \lambda_2^+ + \frac{p^2}{50} \mu_2^+ + \frac{qp^2}{15W} \nu_2^+ \right) |y_2^{(2)}|^2 \\
& - \left(\frac{q^2}{9} \lambda_2^+ - \frac{p^2}{25} \mu_2^+ \right) \operatorname{Re}(y_2^{(2)*} z_2^{(2)}) + \left. \left(\frac{q^2}{18} \lambda_2^+ + \frac{p^2}{50} \mu_2^+ - \frac{qp^2}{15W} \nu_2^+ \right) |z_2^{(2)}|^2 \right] \\
& + \frac{p^4}{1575} \lambda_3^+ \frac{F_2(\rho)}{F_0(\rho)} |3y_3^{(2)} - 2z_3^{(2)}|^2 - \frac{q^2 p^2}{18W} \sqrt{\frac{F_1(\rho)}{F_0(\rho)}} \left[\sigma_{12} \operatorname{Re}(x_1^{(2)*} y_2^{(2)}) \right. \\
& - \left(\frac{q}{5} \sigma_{12} + \frac{W}{3} \sigma_{12}^- \right) (y_1^{(2)*} y_2^{(2)}) + \left(\frac{q}{5} \sigma_{12} - \frac{W}{3} \sigma_{12}^- \right) \operatorname{Re}(y_2^{(2)}) \Big] \\
& - \frac{p^4}{210W} \frac{\sqrt{F_1(\rho) F_2(\rho)}}{F_0(\rho)} \left[\sigma_{23} \operatorname{Re}\{x_2^{(2)*} (3y_3^{(2)} - 2z_3^{(2)})\} \right. \\
& - \left(\frac{q}{3} \sigma_{23} + \frac{W}{5} \sigma_{23}^- \right) \operatorname{Re}\{y_2^{(2)*} (3y_3^{(2)} - 2z_3^{(2)})\} \\
& + \left(\frac{q}{3} \sigma_{23} - \frac{W}{5} \sigma_{23}^- \right) \operatorname{Re}\{z_2^{(2)*} (3y_3^{(2)} - 2z_3^{(2)})\} \Big] \\
& + \frac{q^2 p}{18} \sqrt{\frac{F_1(\rho)}{F_0(\rho)}} \operatorname{Im} \left[\sigma_{12}' (x_1^{(2)*} y_2^{(2)}) + \left(\frac{q}{5} \sigma_{12}' - \frac{p^2}{3W} \sigma_{12}^{\prime+} \right) y_2^{(2)} \right] \\
& - \frac{p^3}{210} \frac{\sqrt{F_1(\rho) F_2(\rho)}}{F_0(\rho)} \operatorname{Im} \left[\sigma_{23}' \{x_2^{(2)*} (3y_3^{(2)} - 2z_3^{(2)})\} \right. \\
& + \left. \left(\frac{q}{3} \sigma_{23}' - \frac{p^2}{5W} \sigma_{23}^{\prime+} \right) \{ (y_2^{(2)} + z_2^{(2)}) * (3y_3^{(2)} - 2z_3^{(2)}) \} \right] \}. \tag{A4}
\end{aligned}$$

$$b_{22}^{(4)} = (1/3\sqrt{70}) \left\{ -\frac{p^4}{W} \frac{\sqrt{F_1(\rho) F_2(\rho)}}{F_0(\rho)} \operatorname{Re} \left[\sigma_{23} \{x_2^{(2)*} (3y_3^{(2)} - 2z_3^{(2)})\} \right. \right.$$

$$\begin{aligned}
& - \left(\frac{q}{3} \sigma_{23} + \frac{W}{5} \sigma_{23}^- \right) \{ y_2^{(2)*} (3y_3^{(2)} - 2z_3^{(2)}) \} \\
& + \left(\frac{q}{3} \sigma_{23} - \frac{W}{5} \sigma_{23}^- \right) \{ z_2^{(2)*} (3y_3^{(2)} - 2z_3^{(2)}) \} \Big] + \frac{p^4}{60} \lambda_3 + \frac{F_2(\rho)}{F_0(\rho)} |3y_3^{(2)} - 2z_3^{(2)}|^2 \\
& + p^3 \frac{\sqrt{F_1(\rho) F_2(\rho)}}{F_0(\rho)} \operatorname{Im} \left[\sigma_{23}' \{ x_2^{(2)*} (3y_3^{(2)} - 2z_3^{(2)}) \} \right. \\
& \left. + \left(\frac{q}{3} \sigma_{23}' - \frac{p^2}{5W} \sigma_{23}^{\prime+} \right) \{ (y_2^{(2)} + z_2^{(2)}) * (3y_3^{(2)} - 2z_3^{(2)}) \} \right] \}. \quad (A5)
\end{aligned}$$

$$\begin{aligned}
b_{23}^{(2)} = & - (\sqrt{7}/3\sqrt{30}) \left\{ \frac{q^2 p^2}{3W} \sqrt{\frac{F_1(\rho)}{F_0(\rho)}} \operatorname{Re} \left[\sigma_{12} (x_1^{(2)*} u_2^{(3)}) \right. \right. \\
& - \left(\frac{q}{5} \sigma_{12} + \frac{W}{3} \sigma_{12}^- \right) (y_1^{(2)*} u_2^{(3)}) + \left(\frac{q}{5} \sigma_{12} - \frac{W}{3} \sigma_{12}^- \right) (z_1^{(2)*} u_2^{(3)}) \\
& + \frac{p^4}{7W} \frac{\sqrt{F_1(\rho) F_2(\rho)}}{F_0(\rho)} \operatorname{Re} \left[\sigma_{23} (x_2^{(2)*} u_3^{(3)}) \right. \\
& - \left(\frac{q}{3} \sigma_{23} + \frac{W}{5} \sigma_{23}^- \right) (y_2^{(2)*} u_3^{(3)}) + \left(\frac{q}{3} \sigma_{23} - \frac{W}{5} \sigma_{23}^- \right) (z_2^{(2)*} u_3^{(3)}) \Big] \\
& + \frac{q^2 p^2}{9} \lambda_2 + \frac{F_1(\rho)}{F_0(\rho)} \operatorname{Re} (y_2^{(2)*} u_2^{(3)}) + \frac{p^4}{105} \lambda_3 + \frac{F_2(\rho)}{F_0(\rho)} \operatorname{Re} \{ (3y_3^{(2)} - 2z_3^{(2)}) * u_3^{(3)} \} \\
& - \frac{q^2 p}{3} \sqrt{\frac{F_1(\rho)}{F_0(\rho)}} \operatorname{Im} \left[\sigma_{12}' (x_1^{(2)*} u_2^{(3)}) - \left(\frac{q}{5} \sigma_{12}' + \frac{p^2}{3W} \sigma_{12}^{\prime+} \right) (y_1^{(2)*} u_2^{(3)}) \right] \\
& - \frac{p^3}{7} \frac{\sqrt{F_1(\rho) F_2(\rho)}}{F_0(\rho)} \operatorname{Im} \left[\sigma_{23}' (x_2^{(2)*} u_3^{(3)}) \right. \\
& \left. - \left(\frac{q}{3} \sigma_{23}' + \frac{p^2}{5W} \sigma_{23}^{\prime+} \right) (y_2^{(2)*} u_3^{(3)}) \right] \}. \quad (A6)
\end{aligned}$$

$$\begin{aligned}
b_{23}^{(4)} = & (\sqrt{7}/42\sqrt{3}) \left\{ \frac{p^4}{W} \frac{\sqrt{F_1(\rho) F_2(\rho)}}{F_0(\rho)} \operatorname{Re} \left[\sigma_{23} (x_2^{(2)*} u_3^{(3)}) \right. \right. \\
& - \left(\frac{q}{3} \sigma_{23} + \frac{W}{5} \sigma_{23}^- \right) (y_2^{(2)*} u_3^{(3)}) \\
& + \left(\frac{q}{3} \sigma_{23} - \frac{W}{5} \sigma_{23}^- \right) (z_2^{(2)*} u_3^{(3)}) \Big] + \frac{p^4}{15} \lambda_3 + \frac{F_2(\rho)}{F_0(\rho)} \operatorname{Re} \{ (3y_3^{(2)} - 2z_3^{(2)}) * u_3^{(3)} \} \\
& - p^3 \frac{\sqrt{F_1(\rho) F_2(\rho)}}{F_0(\rho)} \operatorname{Im} \left[\sigma_{23}' (x_2^{(2)*} u_3^{(3)}) - \left(\frac{q}{3} \sigma_{23}' + \frac{p^2}{5W} \sigma_{23}^{\prime+} \right) (y_2^{(2)*} u_3^{(3)}) \right] \}. \quad (A7)
\end{aligned}$$

$$\begin{aligned}
b_{33}^{(0)} = & - (\sqrt{7}/108) \left\{ \frac{q^4}{10} \lambda_1 + |u_1^{(3)}|^2 + \frac{q^2 p^2}{3} \lambda_2 + \frac{F_1(\rho)}{F_0(\rho)} |u_2^{(3)}|^2 \right. \\
& \left. + \frac{p^4}{10} \lambda_3 + \frac{F_2(\rho)}{F_0(\rho)} |u_3^{(3)}|^2 \right\}. \quad (A8)
\end{aligned}$$

$$b_{33}^{(2)} = (\sqrt{7}/90\sqrt{3}) \left\{ \frac{q^2 p^2}{3} \lambda_2 + \frac{F_1(\rho)}{F_0(\rho)} |u_2^{(3)}|^2 + \frac{p^4}{7} \lambda_3 + \frac{F_2(\rho)}{F_0(\rho)} |u_3^{(3)}|^2 \right\}. \quad (\text{A8})$$

$$b_{33}^{(4)} = (\sqrt{11}/540\sqrt{14}) p^4 \lambda_3 + \frac{F_2(\rho)}{F_0(\rho)} |u_3^{(3)}|^2. \quad (\text{A9})$$

In the above expressions, we omitted for brevity the factor $|G_A \mathfrak{M}(T_{ij}) I_1(r)|^2$ which might be multiplied in each $b_{lL}^{(n)}$, and also the terms which involve the primed parameters.

In order to include the latter, we have only to add the parameters with the primed coupling constants to those with the non-primed coupling constants. The correction factor of β -spectrum for the second forbidden transition is

$$C_2(W) = (1/\sqrt{5}) b_{22}^{(0)} - (1/\sqrt{7}) b_{00}^{(0)}.$$

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A Note on the First Born Approximation in Collisions of Electron with Helium

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Dependences of the results of the first Born approximation on the choice of the approximate wave functions of helium are examined in collision processes of electron with helium. So far the wave function

$$\Psi \sim \exp\{-(27/16a_0)(r_1+r_2)\}$$

has been used almost exclusively as the analytical wave function of the ground state of helium. In this paper

$$\Psi \sim \exp\{-(\zeta/a_0)r_1 - (\zeta'/a_0)r_2\} + \exp\{-(\zeta'/a_0)r_1 - (\zeta/a_0)r_2\}$$

is used. It is found that the theoretical results are quite sensitive to the values of the parameters ζ and ζ' and with suitable choices of ζ and ζ' we obtain considerable improvements within the first Born approximation.

§ 1. Introduction

The first approximation in Born's theory of collisions is worked out for the elastic scattering of electron by helium and for the excitation of the 2^1P state of helium from the ground 1^1S state by electron impact. Our main concern is to examine the dependence of the results on the approximate wave functions of the ground state of helium. It seems that so far the following function has been used almost exclusively as the analytical wave function:

$$\Psi_0(\mathbf{r}_1, \mathbf{r}_2) = \psi(\zeta|\mathbf{r}_1)\psi(\zeta|\mathbf{r}_2) \quad (1)$$

where

$$\psi(\zeta|\mathbf{r}_i) = (\zeta^3/\pi a_0^3)^{1/2} \exp\{-(\zeta/a_0)r_i\},$$

$$(I) \quad \zeta = 27/16 = 1.6875.$$

This function, hereafter called the function (I), gives the value $E_0 = -2.8477$ a.u. as the ground state energy. In this paper we employ the so-called open-shell-type wave function:

$$\Psi_0(\mathbf{r}_1, \mathbf{r}_2) = (N/\sqrt{2}) \{ \psi(\zeta|\mathbf{r}_1)\psi(\zeta'|\mathbf{r}_2) + \psi(\zeta'|\mathbf{r}_1)\psi(\zeta|\mathbf{r}_2) \} \quad (2)$$

where

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$$N = \left[1 + \frac{64(\zeta\zeta')^3}{(\zeta + \zeta')^6} \right]^{-1/2}$$

is the normalization factor. The function (2) includes the function (1) as a special case with $\zeta = \zeta'$. We choose two sets of the parameters ζ and ζ' . The first choice is

$$(II) \quad \zeta = 1.19, \quad \zeta' = 2.184.$$

The function (2) with these parameter values will be called the function (II). According to Taylor and Parr,¹⁾ this choice gives the best value of the ground state energy $E_0 = -2.8756$ a.u. under the functional form (2). The second choice is

$$(III) \quad \zeta = 1, \quad \zeta' = 2.1.$$

The function with these parameter values, called the function (III), is designed specially for the present purpose. $\zeta = 1$ means that the screening of the nuclear Coulomb field by the inner electron is supposed to be complete for the outer electron and the wave function (III) has longer tail than (I) or (II). The choice of $\zeta' = 2.1$ for the inner electron comes from the fact that $|E_0(\zeta = 1, \zeta' = 2.1)| > |E_0(\zeta = \zeta' = 1.6875)| > |E_0(\zeta = 1, \zeta' = 2)|$. In fact, the function (III) gives the energy value $E_0 = -2.8554$ a.u. which is definitely better than that of (I), -2.8477 a.u.

The differential cross section for the elastic scattering of fast electron by helium is treated in § 2. For smaller angles serious discrepancies are found between the experiment and the result of the first Born approximation based on the function (I). Massey and Mohr²⁾ ascribed the discrepancies to the neglect of the polarization effect of helium in the first Born approximation. In fact, they obtained excellent agreement between theory and experiment by working out the second Born approximation for this problem. Their interpretation is believed to be essentially correct but we shall show that by using the functions (II) and (III) we can obtain considerable improvements even within the first Born approximation. Then it will be recognized that the discrepancies are partly due to the poorness of function (I) as the ground state wave function of helium.

In § 3, we discuss the energy dependence of the total cross section of the excitation of the 1^1S - 2^1P transition in helium. Rothenstein³⁾ attempted to improve the result of the first Born approximation based on the function (I) by extending the method used by Massey and Mohr (above) to calculate the second Born approximation for the excitation cross section. The method requires tedious and lengthy calculations and is not applicable at energies close to the threshold. Again here, we shall show that we can obtain qualitatively the same improvement by replacing the function (I) with (II) or (III) within the first Born approximation.

§ 2. Elastic scattering of fast electron by helium

The differential cross section $I(\theta)d\omega$ for an elastic collision of an electron of velocity v with a spherically symmetrical field of force of potential $V(r)$ is given

within the range of validity of Born's first approximation by

$$I(\theta)d\omega = \left| \frac{2m}{\hbar} \int_0^\infty \frac{\sin Kr}{Kr} V(r) r^2 dr \right|^2 d\omega, \quad K = 2(mv/\hbar) \sin(\theta/2). \quad (3)$$

If $\Psi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_z)$ is the wave function of the atom (atomic number z), we have

$$V(r) = -e^2 \int \left(\frac{z}{r} - \sum_{n=1}^z \frac{1}{|\mathbf{r} - \mathbf{r}_n|} \right) |\Psi(\mathbf{r}_1, \dots)|^2 d\tau_1 \dots d\tau_z. \quad (4)$$

In our present case $z=2$ and $\Psi_0(\mathbf{r}_1, \mathbf{r}_2)$ in (2) is to be inserted to the above formula. A straightforward calculation yields that

$$V(r) = -e^2 \left[1 + \frac{64(\zeta\zeta')^3}{(\zeta + \zeta')^6} \right]^{-1} \left\{ \left(\frac{1}{r} + \frac{\zeta}{a_0} \right) e^{-(2\zeta/a_0)r} + \left(\frac{1}{r} + \frac{\zeta'}{a_0} \right) e^{-(2\zeta'/a_0)r} + 128 \frac{(\zeta\zeta')^3}{(\zeta + \zeta')^6} \left(\frac{1}{r} + \frac{\zeta + \zeta'}{2a_0} \right) e^{-(\zeta + \zeta'/a_0)r} \right\}. \quad (5)$$

Then $I(\theta)$ can be calculated to be

$$I(\theta) = (4a_0^2) \left[1 + \frac{64(\zeta\zeta')^3}{(\zeta + \zeta')^6} \right]^{-2} \left[\frac{2(2\zeta)^2 + (a_0 K)^2}{\{(2\zeta)^2 + (a_0 K)^2\}^2} + \frac{2(2\zeta')^2 + (a_0 K)^2}{\{(2\zeta')^2 + (a_0 K)^2\}^2} + 128 \frac{(\zeta\zeta')^3}{(\zeta + \zeta')^6} \frac{2(\zeta + \zeta')^2 + (a_0 K)^2}{\{(\zeta + \zeta')^2 + (a_0 K)^2\}^2} \right]. \quad (6)$$

As is seen in the above formula, $I(\theta)$ remains finite at $\theta=0$ in the first Born approximation:

$$I(0) = (4a_0^2) \left[1 + \frac{64(\zeta\zeta')^3}{(\zeta + \zeta')^6} \right]^{-2} \left[\frac{1}{2\zeta^2} + \frac{1}{2\zeta'^2} + \frac{256(\zeta\zeta')^3}{(\zeta + \zeta')^8} \right]. \quad (7)$$

Then we find that

$$I_{II}(0)/I_I(0) = 1.376, \quad I_{III}(0)/I_I(0) = 2.317.$$

Table I. The scattered intensity $I(\theta)$ for an electron of incident energy 100 ev in units of πa_0^2

θ	(I) $\zeta = \zeta' = 1.6875$	(II) $\zeta = 1.19 \quad \zeta' = 2.184$	(III) $\zeta = 1.0 \quad \zeta' = 2.1$
0°	0.157	0.216	0.364
10°	0.148	0.200	0.325
20°	0.126	0.159	0.238
30°	0.0979	0.115	0.158
60°	0.0375	0.0384	0.0442
90°	0.0154	0.0153	0.0164
120°	0.00818	0.00805	0.00842
150°	0.00565	0.00556	0.00576
180°	0.00501	0.00493	0.00509

Here I_I , I_{II} and I_{III} correspond to the cases (I) $\zeta=\zeta'=27/16$, (II) $\zeta=1.19$, $\zeta'=2.184$ and (III) $\zeta=1.0$, $\zeta'=2.1$ respectively. These ratios are, of course, independent of energy. We see that the cross section for smaller angle is quite sensitive to the choice of the values of ζ and ζ' . If we make reference to Fig. 22 of the text of Mott and Massey,⁴⁾ we recognize that the above ratio, 1.376 or 2.317, means a considerable improvement for smaller angles and thus the first Born approximation seems to acquire wider validity than hitherto supposed. In Table I the calculated values over the whole angles are shown for 100ev electron. Large angle cross sections are insensitive to the choice of ζ and ζ' .

§ 3. 1¹S-2¹P transition in helium by electron impact

Let us consider an inelastic collision of an electron with a normal helium atom in which the atom is raised from the ground state 1¹S of energy E_0 to the 2¹P excited state with energy E_1 . According to the first Born approximation the differential cross section is given by⁵⁾

$$I_{01}(K) dK = (8\pi m^2 e^4 / \hbar^4 k_0^2) (dK/K^3) |\varepsilon_{01}(K)|^2, \quad (8)$$

$$\varepsilon_{01}(K) = \int \Psi_0(\mathbf{r}_1, \mathbf{r}_2) (e^{iKz_1} + e^{iKz_2}) \Psi_1^*(\mathbf{r}_1, \mathbf{r}_2) d\tau_1 d\tau_2, \quad (9)$$

$$K^2 = k_0^2 + k_1^2 - 2k_0 k_1 \cos \alpha, \quad (10)$$

where $k_0\hbar$ and $k_1\hbar$ are respectively the initial and final momentum vectors of the colliding electron and α is the scattering angle between k_0 and k_1 . The conservation of energy requires that

$$(\hbar^2/2m) (k_0^2 - k_1^2) = E_1 - E_0. \quad (11)$$

In our present treatment the wave functions are

$$\Psi_0(\mathbf{r}_1, \mathbf{r}_2) = (N/\sqrt{2}) \{ \phi(\zeta|\mathbf{r}_1) \phi(\zeta'|\mathbf{r}_2) + \phi(\zeta'|\mathbf{r}_1) \phi(\zeta|\mathbf{r}_2) \} \quad (2)$$

$$\Psi_1(\mathbf{r}_1, \mathbf{r}_2) = (1/\sqrt{2}) \{ \phi(\delta|\mathbf{r}_1) \varphi(\gamma|\mathbf{r}_2) + \varphi(\gamma|\mathbf{r}_1) \phi(\delta|\mathbf{r}_2) \} \quad (12)$$

where

$$\begin{aligned} \phi(\delta|\mathbf{r}_i) &= (\delta^3/\pi a_0^3)^{1/2} e^{-(\delta/a_0)r_i}, \\ \varphi(\gamma|\mathbf{r}_i) &= (\gamma^5/32\pi a_0^5)^{1/2} r_i \cos \theta_i e^{-(\gamma/2a_0)r_i}. \end{aligned}$$

We take that $\delta=2.003$ and $\gamma=0.965$ according to the variational calculation of Eckart⁶⁾ and the function $\Psi_1(\mathbf{r}_1, \mathbf{r}_2)$ gives $E_1=-2.1224$ a.u. With these functions we can obtain the analytical expression of $I_{01}(K)$ without difficulty. Then the corresponding total cross section is given by

$$Q_{01} = \int_{K_{\min}}^{K_{\max}} I_{01}(K) dK, \quad (13)$$

where

$$K_{\min} = k_0 - k_1, \quad K_{\max} = k_0 + k_1. \quad (14)$$

The explicit expression is

$$\begin{aligned} Q_{01} = & 128(\pi a_0^3) \left[1 + \frac{64(\zeta\zeta')^3}{(\zeta + \zeta')^6} \right]^{-1} \gamma^5 \\ & \times \left[S(\zeta', \delta)^2 \frac{\zeta^3}{\sigma^{10}} \frac{1}{(k_0 a_0)^2} \{G(x_+) - G(x_-)\} \right. \\ & + S(\zeta, \delta)^2 \frac{\zeta'^3}{\sigma'^{10}} \frac{1}{(k_0 a_0)^2} \{G(x_+) - G(x'_-)\} \\ & + 2S(\zeta', \delta)S(\zeta, \delta) (\zeta\zeta')^{3/2} \frac{\sigma\sigma'}{(\sigma'^2 - \sigma^2)^5} \frac{1}{(k_0 a_0)^2} \\ & \times \left\{ \frac{6}{\sigma^2} \left[\left(\frac{1}{2x_+^2} - \frac{1}{x_+} \right) - \left(\frac{1}{2x_-^2} - \frac{1}{x_-} \right) \right] \right. \\ & - \frac{6}{\sigma'^2} \left[\left(\frac{1}{2x'^2_+} - \frac{1}{x'_+} \right) - \left(\frac{1}{2x'^2_-} - \frac{1}{x'_-} \right) \right] \\ & + \frac{3(5\sigma^2 - \sigma'^2)}{\sigma^4} \left(\frac{1}{2x_-^2} - \frac{1}{2x_+^2} \right) - \frac{3(5\sigma'^2 - \sigma^2)}{\sigma'^4} \left(\frac{1}{2x'^2_-} - \frac{1}{2x'^2_+} \right) \\ & + \frac{(10\sigma^4 - 5\sigma^2\sigma'^2 + \sigma'^4)}{\sigma^6} [H(x_+) - H(x_-)] \\ & \left. \left. + \frac{(10\sigma'^4 - 5\sigma'^2\sigma^2 + \sigma^4)}{\sigma'^6} [H(x'_+) - H(x'_-)] \right\} \right]. \quad (15) \end{aligned}$$

Here

$$S(\zeta, \delta) = \int \psi(\zeta|\mathbf{r}) \psi(\delta|\mathbf{r}) d\tau = 8(\zeta\delta)^{3/2}/(\zeta + \delta)^3,$$

$$\sigma = \zeta + (\gamma/2), \quad \sigma' = \zeta' + (\gamma/2),$$

$$G(x) = \log \frac{x-1}{x} + \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{3x^3} + \frac{1}{4x^4} + \frac{1}{5x^5},$$

$$H(x) = \log \frac{x-1}{x} + \frac{1}{x} + \frac{1}{2x^2},$$

$$\begin{cases} x_+ = 1 + (1/\sigma^2)(k_0 + k_1)^2 a_0^2, & x'_+ = 1 + (1/\sigma'^2)(k_0 + k_1)^2 a_0^2, \\ x_- = 1 + (1/\sigma^2)(k_0 - k_1)^2 a_0^2, & x'_- = 1 + (1/\sigma'^2)(k_0 - k_1)^2 a_0^2. \end{cases}$$

As in the preceding section we have examined the three cases, (I) $\zeta = \zeta' = 27/16$, (II) $\zeta = 1.19$, $\zeta' = 2.184$ and (III) $\zeta = 1.0$, $\zeta' = 2.1$. Numerical values of total cross sections given by the above formula (15) for three cases are included in Table II while in Fig.1 the variations of the cross sections with electron energy are illustrated. It should be noticed that in Fig.1 the curves (II) and (III) are adjusted so that

three curves coincide at the electron energy of 300ev, in order to make it easy to compare these curves with other theoretical and experimental ones. Fig. 1 is to be compared with Figure 4 of Rothenstein's paper³⁾ and we see that the "shape" of the excitation function is changed to favorable direction by replacing the function (I) with (II) or (III) which is better than (I), at least, from the energetic point of view. According to Rothenstein, the second Born approximation introduces the correction which "reduces" the intensity of the small angle inelastic scattering and the total inelastic cross section. In our calculations the replacement of the function (I) with (II) or (III) gives rise to marked increases of cross sections as is seen in Table II.

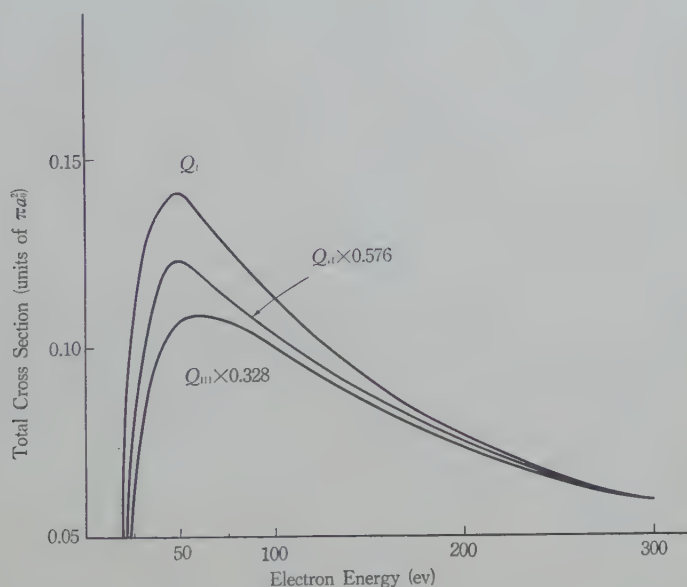


Fig. 1. Total cross sections for 1^1S-2^1P excitation of helium. Scales have been adjusted so that the three curves agree at the electron energy of 300 ev.

Table II. Total cross sections for 1^1S-2^1P excitation of helium by electron impact in units of πa_0^2

Electron energy in ev	(I) $\zeta=\zeta'=1.6875$	(II) $\zeta=1.19 \ \zeta'=2.184$	(III) $\zeta=1.0 \ \zeta'=2.1$
25	0.106	0.117	0.173
50	0.141	0.214	0.327
75	0.127	0.196	0.324
100	0.112	0.180	0.305
200	0.0767	0.130	0.226
300	0.0592	0.103	0.180

§ 4. Conclusion

It has been shown that the results of the first Born approximation for the electron scattering processes by helium are rather sensitive to the form of the approximate wave function for the ground state of helium. For example, the functions (I) and (III) are not very different energetically but yield significantly divergent results for the collision cross sections. Probably the collision cross section would be much more sensitive to the details of the approximate charge distribution of the ground state of helium than the ground state energy. Tentatively we have calculated the diamagnetic susceptibility of helium which gives a measure of the shape of the wave functions. The result is that the functions (I), (II) and (III) give -1.67×10^{-6} , -1.96×10^{-6} and -2.54×10^{-6} , respectively, while the experimental value is -1.9×10^{-6} in units of erg per gauss per mole. This implies that the electron cloud described by the function (I) is contracted too tightly and on the other hand the function (III) has a too loosely extended tail.

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On High Energy Limit of Fermion-Fermion Interaction

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It is investigated what asymptotic behaviour of the vertex part of the fermion-fermion interaction is compatible with Lehmann's spectral representation for the one-body propagator. In contrast to the results of Abrikosov et al., it is shown that no term may be neglected in the equation for the vertex part.

§ 1. Introduction

Last year Abrikosov et al.¹⁾ have proposed an asymptotic theory of the fermion-fermion interaction by applying the asymptotic theory of Landau et al.²⁾ developed for the cases of the interactions of the first kind. They adopted a system of approximate integral equations for the one-body propagator and the vertex part collecting terms which might give the main contributions, and estimated these quantities using a cut-off technique. The fundamental assumption on which the asymptotic theory of Landau et al. is based is that the so-called first approximations of the one-body propagator and the vertex part are well determined by solving a system of finite number of integral equations. This assumption is inherited by the theory of fermion-fermion interaction of Abrikosov et al. Even in the cases of interactions of the first kind it is doubtful that the sums over all terms in the perturbation expansion are logarithmic, though each term is logarithmic.^{3,4)} In the case of the fermion-fermion interaction, straightforward application of the perturbation theory gives the so-called unrenormalizable divergences, then it is meaningless to collect several classes of diagrams in conformity to the perturbation theoretical estimations.

Our purpose is to show that solution of the equation for Γ proposed by Abrikosov et al. and of any system of finite number of approximate integral equations does not have asymptotic behaviour required by the spectral condition of Lehmann⁵⁾ with use of Schwinger's⁶⁾ theory of Green functions without cut-off.

§ 2. Derivation of equations

In order to avoid unnecessary complications, we consider the self interactions of a fermion field and take the following Lagrangean density:

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$$\begin{aligned}
L(x) = & -\frac{1}{4} \left[\bar{\psi}(x), \left(\gamma_\mu \frac{\partial}{\partial x_\mu} + m_0 \right) \psi(x) \right] + \frac{1}{4} \left[\left(\frac{\partial \bar{\psi}(x)}{\partial x_\mu} \gamma_\mu - m_0 \bar{\psi}(x) \right), \psi(x) \right] \\
& + \sum_{i=S,V,T,A,I} \frac{g_i}{4} [\bar{\psi}(x), O_i \psi(x)] [\bar{\eta}(x), O_i \psi(x)] \\
& + \frac{1}{2} [\bar{\psi}(x), \eta(x)] + \frac{1}{2} [\bar{\eta}(x), \psi(x)], \quad (1)
\end{aligned}$$

where η and $\bar{\eta}$ are fictitious fermion sources. Applying Schwinger's procedure of defining the Green function, we obtain the following functional differential equation:

$$\begin{aligned}
\left[\gamma_\mu \frac{\partial}{\partial x_\mu} + m_0 + \sum g_i O_i \cdot O_i \left\{ \frac{\delta^2}{\delta \bar{\eta}(x) \delta \eta(x)} + \langle 0 | \bar{\psi}(x) | 0 \rangle \langle 0 | \psi(x) | 0 \rangle \right. \right. \\
\left. \left. - \langle 0 | \psi(x) | 0 \rangle \frac{\delta}{\delta \eta(x)} - \langle 0 | \bar{\psi}(x) | 0 \rangle \frac{\delta}{\delta \bar{\eta}(x)} \right\} \right] G(x, x'; \eta, \bar{\eta}) = \delta(x - x'). \quad (2)
\end{aligned}$$

In the theory of Green function, G or $\delta^2 G / \delta \bar{\eta} \delta \eta|_{\eta=\bar{\eta}=0}$ should be given as boundary condition (but not sufficient) for the functional differential equation if perturbation theory is avoided, and we can introduce constants with dimension of length as parameters on which G and $\delta^2 G / \delta \bar{\eta} \delta \eta|_{\eta=\bar{\eta}=0}$ depend. Then the dimensional analysis is not applicable. In this note we shall consider boundary conditions different from either that corresponding to the perturbation theory or the so-called asymptotic condition.

We define the one-body propagator and the vertex parts:

$$G(x - x') = G(x, x'; 0, 0), \quad (3)$$

$$\begin{aligned}
\Gamma(x_1, x_2; x_3, x_4) = & \int dx'_1 dx'_2 dx'_3 dx'_4 G^{-1}(x_1 - x'_1) G^{-1}(x_2 - x'_2) \\
& \times \frac{\delta^2 G(x'_1, x'_3; \eta, \bar{\eta})}{\delta \bar{\eta}(x'_2) \delta \eta(x'_4)} \Big|_{\eta=\bar{\eta}=0} G^{-1}(x'_3 - x_3) G^{-1}(x'_4 - x_4), \quad (4)
\end{aligned}$$

$$\begin{aligned}
& \Xi(x_1, x_2, x_3; x_4, x_5, x_6) \\
& = \int dx'_1 dx'_6 G^{-1}(x_3 - x'_3) \frac{\delta^2 \Gamma(x_1, x_2; x_4, x_5; \eta, \bar{\eta})}{\delta \bar{\eta}(x'_3) \delta \eta(x'_6)} \Big|_{\eta=\bar{\eta}=0} G^{-1}(x'_6 - x_6). \quad (5)
\end{aligned}$$

In the momentum representation we can write the integral equations for G and Γ as following:

$$(i\gamma_\mu p_\mu + m_0 + \Sigma^*(p)) G(p) = 1; \quad (6)$$

$$\begin{aligned}
\Sigma^*(p) = & \int dq_1 dq_2 \sum_i g_i O_i G(q_1) \cdot G(q_1 + q_2 - p) O_i G(q_2) \\
& \times \Gamma(q_1, q_2; p, q_1 + q_2 - p) + \mathfrak{M}, \quad (7)
\end{aligned}$$

$$\mathfrak{M} = \sum_i g_i \left\{ O_i \int dq G(q) O_i + S_p [O_i \int G(q) dq] O_i \right\}, \quad (8)$$

$$\Gamma = \sum_i g_i O_i \cdot O_i + \Gamma_a + \Gamma_b + \Gamma_c. \quad (9)$$

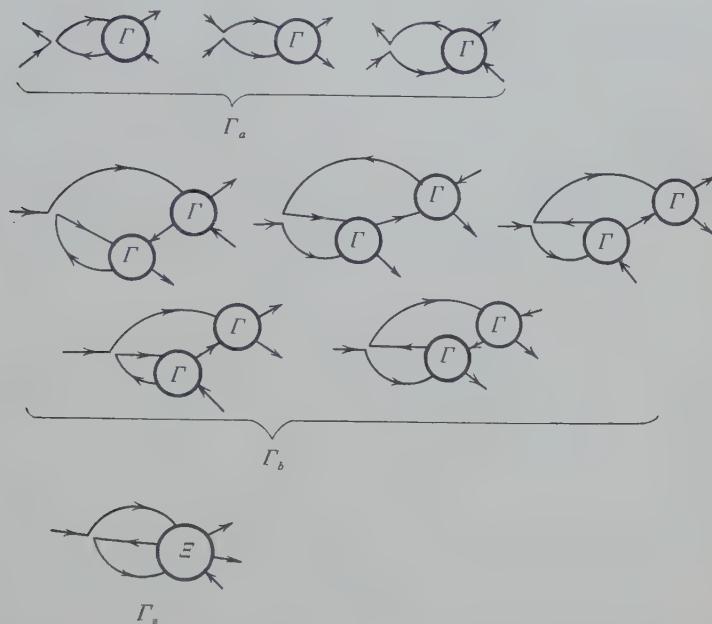


Fig. 1. Definitions of Γ_a , Γ_b and Γ_c in Eq. (9)

The equation for Γ is illustrated graphically. \mathfrak{M} is a divergent constant^t which can^e be removed by mass renormalization, and an analogous term does^r not appear in the cases of interactions among different fields.

§ 3. Asymptotic behaviours of the vertex part

In order that G can be represented in the spectral form of Lehmann

$$G(p) = \int d\xi \left\{ \frac{i\gamma p \rho_1(\xi) + \rho_2(\xi)}{p^2 - \xi - i\varepsilon} \right\}, \quad (10)$$

$\Sigma^*(p)$ may not increase more rapidly than $i\gamma p [\lg(|p^2/m^2|)]^n$ ($n < \infty$) for $|p^2| \rightarrow \infty$.

Now we consider what boundary condition should be adopted.

At first, we limit ourselves to considerations of cases of non-oscillating asymptotic forms.

In order that G has spectral representation and no divergent constant appears in Eq. (9), at first sight it is reasonable to choose boundary condition Γ which has asymptotic behaviour

$$\Gamma(p_1, p_2; p_3, p_1 + p_2 - p_3) = O((\sum_{i,j} |p_i p_j|)^{-2}). \quad (11)$$

Now (11) should be understood to mean that the lower bound of α , for which

$$\Gamma(p_1, p_2; p_3, p_4) / (\sum_{i,j} |p_i p_j|)^{\alpha} < \infty \quad (p_4 = p_1 + p_2 - p_3) \quad (12)$$

takes place independently of ratios among $p_i p_j$'s, is -2 . If Γ involves logarithmic factor, a set of α satisfying (12) is an open set. In such a case we adopt the lower bound of the closure of the set.

It should be noticed that (11) is not a necessary condition. Other possibilities shall be considered afterwards.

Substituting (11) into (9), we obtain

$$\Gamma_a \sim O((\sum |p_i p_j|)^{-1}), \quad (13)$$

$$\Gamma_b \sim O((\sum |p_i p_j|)^{-2}). \quad (14)$$

Then for validity of (11), the lowest order term of (9), $\sum_i g_i O_i \cdot O_i$ must be cancelled by other terms of the right-hand side of the equation in the limit $|p_i p_j| \rightarrow \infty$, then we obtain following conditions:

$$\begin{aligned} & \sum_i g_i \int dq_1 dq_2 O_i G(q_1) \cdot G(q_1 + q_2 - p_1) O_i G(q_2) \\ & \times \Xi(q_1, q_2, p_2; p_3, p_4, q_1 + q_2 - p_1) \rightarrow - \sum_i g_i O_i \cdot O_i - \Gamma_a, \end{aligned} \quad (15)$$

$$\Xi(p_1, p_2, p_3; p_4, p_5, p_6) \sim O((\sum |p_i p_j|)^{-5/2}), \quad (16)$$

therefore the last term of the right-hand side of (9) may not be ignored. In the equation for Ξ (Fig. 2) the last term (Fig. 2, e) must cancel the second term (Fig. 2, b), so that it may not be neglected.

If we adopt boundary condition Γ such as to have an asymptotic form, for example,

$$\Gamma \sim O((\sum |p_i^2|)^{-\alpha}) - O((\sum |p_i p_j|)^{-\alpha}), \quad (17)$$

and require that the leading terms in Σ^* contributed from the first and the second terms of (17) cancel each other in the limit of $|p^2| \rightarrow \infty$, the spectral condition for G may be valid. Substituting (17) into (9) we obtain

$$\Gamma_a \sim O((\sum |p_i p_j|)^{1-\alpha}), \quad (18)$$

$$\Gamma_b \sim O((\sum |p_i p_j|)^{2-2\alpha}). \quad (19)$$

In the case of $1 < \alpha < 2$, the condition (15) is modified, but (16) remains valid, then the circumstances are not changed essentially. Substituting (16) and (17) into the equation for Ξ , we see that the contributions of terms corresponding to Fig. 2, a, b, c and d are $O((\sum |p_i p_j|)^{(1/2)-2\alpha})$, $O((\sum |p_i p_j|)^{-3/2})$, $O((\sum |p_i p_j|)^{(3/2)-3\alpha})$ and $O((\sum |p_i p_j|)^{-(1/2)-\alpha})$, respectively; then these contributions must be cancelled by the last term.

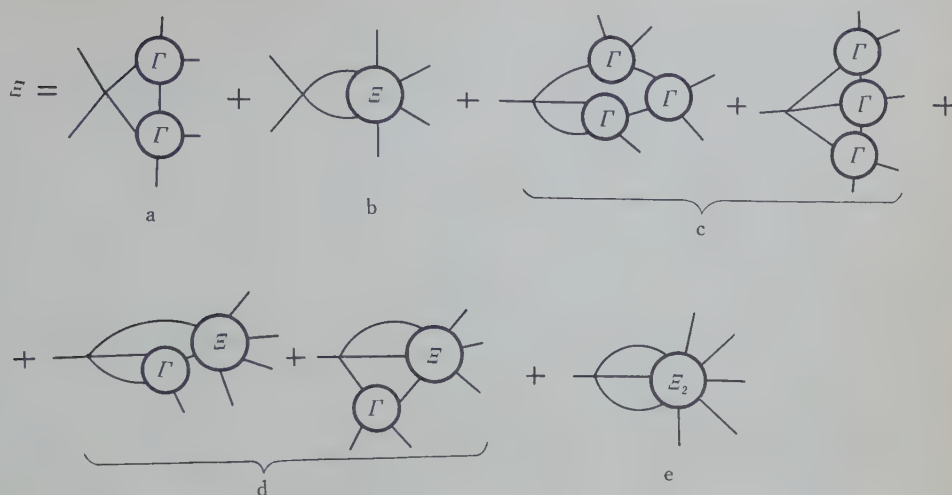


Fig. 2. Graphical illustration of the equation for Ξ
Directions of the lines are not distinguished for brevity.

In the case of $\alpha=1$, Γ_a and Γ_b become $O(1)$, accordingly $\sum g_i O_i \cdot O_i$ may be cancelled by $\Gamma_a + \Gamma_b$. But the circumstances about the equation for Ξ are similar to the case of $1 < \alpha < 2$, and also in the case of $\alpha < 1$.

Continuing such considerations, we conclude that it is not reasonable to consider a finite number of equations, namely to assume $\Xi_n(p_1, \dots, p_{n+2}; p'_1, \dots, p'_{n+2})$ with $n > n_0$ being negligible, where

$$\Xi_1 = \Xi, \quad (20)$$

$$\Xi_n = G^{-1} \frac{\partial^2 \Xi_{n-1}}{\partial \bar{\eta} \partial \eta} G^{-1}. \quad (21)$$

In the above considerations we have assumed the asymptotic form of Γ to be of form (11) or (17). Although the asymptotic form of Γ were such as $\exp\{-A \prod_{i,j} \lg(|p_i p_j|/M)\}$, (16) must remain valid; accordingly the circumstances are similar to the cases considered above.

§ 4. Comparison with the theory of Bogoljubov, Logunov and Shirkov

Comparison of our argument and the results of Bogoljubov, Logunov and Shirkov⁷⁾ is an interesting problem. They assumed that $\Gamma(p, p; p, p)$ has a spectral representation of the following form:

$$\Gamma(p, p; p, p) = 1 + p^2 \int_{m_0^2}^{\infty} \frac{\rho_r(\zeta) d\zeta}{\zeta(\zeta - p^2)} \quad (22)$$

and obtained an asymptotic expression for $\Gamma(p, p; p, p)$, summing the perturbation series for ρ_r :

$$\begin{aligned} \Gamma(p, p; p, p) &= 1 + g p^2 \int_{m_0^2}^{\infty} \frac{d\zeta}{(\zeta - p^2) [(1 + g \zeta \lg(\zeta/m^2))^2 + \pi^2 g^2 \zeta^2]} \\ &= \frac{1}{1 + g p^2 \lg(p^2/m^2)} - \frac{p^2}{g p_0^2 (p^2 - p_0^2) [1 + \lg(-p_0^2/m^2)]}, \end{aligned} \quad (23)$$

where p_0^2 is the root of the equation

$$1 + g p_0^2 \lg(-p_0^2/m^2) = 0. \quad (24)$$

Then, Bogoljubov, Logunov and Shirkov's result may be interpreted as a special case of (17) with $\alpha=0$, but $\Gamma(p_1, p_2; p_3, p_4)$ with different p_i 's must have asymptotic form not similar to that of $\Gamma(p, p; p, p)$.

Though the result of Bogoljubov, Logunov and Shirkov is not incompatible with the spectral condition for G , the cancellation of $gO \cdot O$ in the equation for Γ , (6), by other terms is not required in the procedure of obtaining Bogoljubov, Logunov and Shirkov's result, so we cannot obtain asymptotic expression for $\Gamma(p_1, p_2; p_3, p_4)$ with different p_i 's from such treatment and have no reason to choose $\Gamma(p_1, p_2; p_3, p_4)$ which has form (23) when $p_1=p_2=p_3=p_4 \rightarrow \infty$. Now it should be noticed that the result of their treatment is not compatible with the assumption that the asymptotic behaviours of Σ^* and Γ are determined by a closed system of equations for Σ^* and Γ .

§ 5. Conclusion and further outlook

In this note it is concluded that the spectral condition for G is not compatible with the assumption that the asymptotic forms of Σ^* and Γ are determined by a closed system of equations for them. In order that G has spectral representation and no divergent constant except for mass renormalization appears in the equations for Σ^* and Γ , cancellations must occur in the equation for Γ .

On the other hand, Σ^* or Γ should be given as boundary condition for the functional differential equation. Then our problem to be solved is whether it is possible to give a boundary condition compatible with the spectral condition for G and absence of divergent constant except for the mass renormalization.

The circumstances are similar to the case of pseudoscalar meson theory with pseudovector coupling.

The above considerations are related only to the cases of absence of a Konopinski-Uhlenbeck type interaction. If a Konopinski-Uhlenbeck type interaction exists, the condition for validity of the spectral representation for G is violated, hence it is not required that the asymptotic behaviour of $\Sigma^*(p)$ is $i\gamma p (\lg|p^2/m^2|)^n$. We shall consider such a case on some other chance.

The circumstances are not changed, even if we assume generalized spectral representation⁸⁾ for G

$$G(p) = (p^2 - m^2)^{n+1} \int d\zeta \frac{i\gamma p J_1(\zeta) + J_2(\zeta)}{(\zeta - m^2)^{n+1} (p^2 - \zeta)} + \sum_{l=0}^n (p^2 - m^2)^l (i\gamma p f_1^{(l)} + f_2^{(l)}) \quad (25)$$

which corresponds to the special case

$$\Sigma^*(p) \rightarrow -i\gamma p - m_0 + O((p^2)^{-n}), \quad |p^2| \rightarrow \infty. \quad (26)$$

In the above considerations we have assumed that the asymptotic forms of the propagators and the vertex parts are monotone. However, the question about the possibility of oscillating asymptotic forms is very interesting in connection with Heisenberg's argument that the propagator of non-linear field oscillates in neighbourhood of the light cone.

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Asymptotic Theory of Interacting Fields without Hamiltonian

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Asymptotic properties of the propagators and the vertex parts of quantized fields with interactions for which the Hamiltonian formalism is impossible are considered with use of Schwinger's theory of the Green functions. Possibilities of divergence-free theories are suggested.

§ 1. Introduction

It is expected that the introduction of interactions which do not satisfy the conditions for the validity of Lehmann's¹⁾ theorem is a possible method of avoiding the difficulties in the present quantum field theory, whose possibility was suggested by Umezawa,²⁾ who named such interactions the *c*-type interactions and estimated that such interactions may become predominant in the region of length $R_2 \sim 10^{-30}$ cm.

If such interactions exist, the singularities of the propagators on the light cone may be weaker than that of the so-called free propagators. In presence of the interaction which makes the field equations quasi-linear, the canonical formalism is impossible, therefore Heisenberg operators at different times are not connected by canonical transformations; accordingly the expansion with respect to the complete set of eigenstates of the total Hamiltonian is meaningless and Lehmann's theorem does not hold.

Of course, the perturbation theory is not applicable to such interactions. Then we treat the problem using Schwinger's³⁾ theory of the Green functions derived from the Lagrangean formalism.

In this note we consider the cases of the Konopinski-Uhlenbeck interaction and the case of boson-fermion interaction recently treated by Watanabe.⁴⁾

§ 2. Konopinski-Uhlenbeck type interaction

In this section we consider the Konopinski-Uhlenbeck type interaction as an example of the interactions for which the canonical formalism is impossible. In Fierz's⁵⁾ paper it is written that the canonical formalism is possible with suitable choice of the canonical variables. But validity of his theory is restricted in the second order with respect to the coupling constant, and rigorous Hamiltonian cannot be found. Therefore Lehmann's theorem does not take place, and it might be expected that

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the singularities of the propagators on the light cone are modified.

Now we define the propagators and obtain equations for them using Schwinger's method of functional differentiations.

In order to avoid unnecessary complications we consider a fermion field which has the following Lagrangean :

$$\begin{aligned}
 L(x) = & -\frac{1}{4} \left[\bar{\psi}(x), \left(\gamma_\mu \frac{\partial}{\partial x_\mu} + m_0 \right) \psi(x) \right] - \frac{1}{4} \left[\left(-\frac{\partial \bar{\psi}(x)}{\partial x_\mu} \gamma_\mu + m_0 \bar{\psi}(x) \right), \psi(x) \right] \\
 & + \frac{1}{4} \sum_{i=S, V, T, A, P} g_i \left\{ [\bar{\psi}(x), O_i \psi(x)] \left[\bar{\psi}(x), \gamma_\mu O_i \frac{\partial \psi(x)}{\partial x_\mu} \right] \right. \\
 & \left. - \left[\frac{\partial \bar{\psi}(x)}{\partial x_\mu}, O_i \gamma_\mu \psi(x) \right] [\bar{\psi}(x), O_i \psi(x)] \right\} \\
 & + \frac{1}{2} [\bar{\psi}(x), \eta(x)] + \frac{1}{2} [\bar{\eta}(x), \psi(x)], \quad (1)
 \end{aligned}$$

where η and $\bar{\eta}$ are fictitious fermion sources introduced by Schwinger. g_i 's are constants with dimension $[L^3]$.

Usually the Konopinski-Uhlenbeck type interaction is understood to be of V and A type, but S , T , and P type interactions do not vanish except for the lowest order of the perturbation expansion. Then we consider the generalized Konopinski-Uhlenbeck interaction.

From the Lagrangean (1) we obtain the following equation and a equation conjugate to it :

$$\begin{aligned}
 & \left(\gamma_\mu \frac{\partial}{\partial x_\mu} + m_0 \right) \psi(x) + \frac{1}{2} \sum_{i=S, \dots, P} g_i \left\{ [\bar{\psi}(x), O_i \psi(x)] \cdot \gamma_\mu O_i \frac{\partial \psi(x)}{\partial x_\mu} \right. \\
 & \left. + O_i \psi(x) \left[\bar{\psi}(x), \gamma_\mu O_i \frac{\partial \psi(x)}{\partial x_\mu} \right] - \left[\frac{\partial \bar{\psi}(x)}{\partial x_\mu}, O_i \gamma_\mu \psi(x) \right] O_i \psi(x) \right. \\
 & \left. + \frac{\partial}{\partial x_\mu} \left[O_i \gamma_\mu \psi(x) [\bar{\psi}(x), O_i \psi(x)] \right] \right\} = 0. \quad (2)
 \end{aligned}$$

We define the generating functional as follows :

$$\mathfrak{S}(\eta, \bar{\eta}) = P^* \exp \left[i \int_{-\infty}^{\infty} d^4 x L'(x; \eta, \bar{\eta}) \right] \quad (3)$$

$$L'(x; \eta, \bar{\eta}) = \frac{1}{2} [\bar{\psi}(x), \eta(x)] + \frac{1}{2} [\bar{\eta}(x), \psi(x)]. \quad (4)$$

Then the propagator can be defined as follows :

$$G(x, x'; \eta, \bar{\eta}) = \left\langle 0 \left| \frac{\delta^2 \mathfrak{S}(\eta, \bar{\eta})}{\delta \bar{\eta}(x) \delta \eta(x')} \right| 0 \right\rangle \langle 0 | \mathfrak{S}(\eta, \bar{\eta}) | 0 \rangle^{-1} \quad (5)$$

with suitable definition of "vacuum expectation values."

Then we obtain functional differential equation for this propagator :

$$\begin{aligned}
 & \left[\gamma_\mu \frac{\partial}{\partial x_\mu} + m_0 \right. \\
 & + \frac{1}{2} \sum_{i=S, \dots, P} g_i \left\{ \left[\left(\frac{\partial}{\partial \eta(x)} - \langle 0 | \bar{\psi}(x) | 0 \rangle \right), O_i \left(\frac{\partial}{\partial \bar{\eta}(x)} - \langle 0 | \psi(x) | 0 \rangle \right) \right] \gamma_\mu O_i \frac{\partial}{\partial x_\mu} \right. \\
 & + \left[\left(\frac{\partial}{\partial \eta(x)} - \langle 0 | \bar{\psi}(x) | 0 \rangle \right), \gamma_\mu O_i \frac{\partial}{\partial x_\mu} \left(\frac{\partial}{\partial \bar{\eta}(x)} - \langle 0 | \psi(x) | 0 \rangle \right) \right] O_i \\
 & - \left[\left(\frac{\partial}{\partial \eta(x)} - \langle 0 | \bar{\psi}(x) | 0 \rangle \right), \gamma_\mu O_i \left(\frac{\partial}{\partial \bar{\eta}(x)} - \langle 0 | \psi(x) | 0 \rangle \right) \right] \frac{\partial}{\partial x_\mu} O_i \\
 & - \left[\frac{\partial}{\partial x_\mu} \left(\frac{\partial}{\partial \eta(x)} - \langle 0 | \bar{\psi}(x) | 0 \rangle \right), O_i \gamma_\mu \left(\frac{\partial}{\partial \bar{\eta}(x)} - \langle 0 | \psi(x) | 0 \rangle \right) \right] O_i \\
 & + \left[\left(\frac{\partial}{\partial \eta(x)} - \langle 0 | \bar{\psi}(x) | 0 \rangle \right) \frac{\partial}{\partial x_\mu}, O_i \gamma_\mu \left(\frac{\partial}{\partial \bar{\eta}(x)} - \langle 0 | \psi(x) | 0 \rangle \right) \right] O_i \\
 & \left. + \frac{\partial}{\partial x_\mu} \left[\left(\frac{\partial}{\partial \eta(x)} - \langle 0 | \bar{\psi}(x) | 0 \rangle \right), O_i \left(\frac{\partial}{\partial \bar{\eta}(x)} - \langle 0 | \psi(x) | 0 \rangle \right) \right] O_i \gamma_\mu \right\} \\
 & \times G(x, x'; \eta, \bar{\eta}) = c \delta(x - x'). \quad (6)
 \end{aligned}$$

In this case the anticommutation relation between ψ and $\bar{\psi}$ at equal time is unknown, because of impossibility of the canonical formalism. Now we have adopted the condition that $\langle 0 | [\psi(\mathbf{x}, x_0), \bar{\psi}(\mathbf{x}', x_0)]_+ | 0 \rangle$ is equal to $c \delta^3(\mathbf{x} - \mathbf{x}')$ as the condition of the quantization, where c is a constant. The case of $c=0$ is excluded. If c is equal to zero, the equation for the propagator is homogeneous, then the following considerations in "momentum representation" is meaningless. We define the one-body propagator and the vertex part :

$$G(x - x') = G(x, x'; 0, 0) \quad (7)$$

$$\begin{aligned}
 \Gamma(x_1, x_2; x_3, x_4) &= \int dx_1' dx_2' dx_3' dx_4' G^{-1}(x_1 - x_1') G^{-1}(x_2 - x_2') \\
 &\times \frac{\delta^2 G(x_1', x_3'; \eta, \bar{\eta})}{\delta \bar{\eta}(x_2') \delta \eta(x_4')} \Big|_{\eta=\bar{\eta}=0} G^{-1}(x_3' - x_3) G^{-1}(x_4' - x_4). \quad (8)
 \end{aligned}$$

By usual procedure we obtain an integro-differential equation for G ,

$$\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + m_0 + \mathfrak{M} \left(\frac{\partial}{\partial x} \right) \right) G(x - x') + \int dx'' M(x, x'') G(x'' - x') = c \delta(x - x'), \quad (9)$$

$$\begin{aligned}
 \mathfrak{M} \left(\frac{\partial}{\partial x} \right) &= 2 \sum_{i=S, \dots, P} g_i \left\{ O_i G(0) \gamma_\mu O_i \frac{\partial}{\partial x_\mu} + \gamma_\mu O_i \frac{\partial G(y)}{\partial y_\mu} \Big|_{y=0} O_i \right. \\
 &\quad \left. - S p [O_i G(0)] \gamma_\mu O_i \frac{\partial}{\partial x_\mu} - S p \left[\gamma_\mu O_i \frac{\partial G(y)}{\partial y_\mu} \Big|_{y=0} \right] O_i \right\}, \quad (10)
 \end{aligned}$$

$$\begin{aligned}
 M(x, x') = & \sum_{i=8, \dots, P} g_i \int dx_1 dx_2 dx_3 \left\{ \gamma_\mu O_i \frac{\partial G(x-x_1)}{\partial x_\mu} \cdot G(x_2-x) O_i G(x-x_3) \right. \\
 & + O_i G(x-x_1) \cdot G(x_2-x) \gamma_\mu O_i \frac{\partial G(x-x_3)}{\partial x_\mu} \\
 & - O_i G(x-x_1) \cdot \frac{\partial G(x_2-x)}{\partial x_\mu} O_i \gamma_\mu G(x-x_3) \\
 & \left. + \frac{\partial}{\partial x_\mu} [O_i \gamma_\mu G(x-x_1) \cdot G(x_2-x) O_i G(x-x_3)] \right\} \Gamma(x_1, x_3; x_2, x'). \quad (11)
 \end{aligned}$$

We can make \mathfrak{M} be zero by suitable choice of g_i 's, and we do not consider it hereafter.

Similarly, we obtain the equation for Γ . As it is very complicated, we illustrate it graphically.

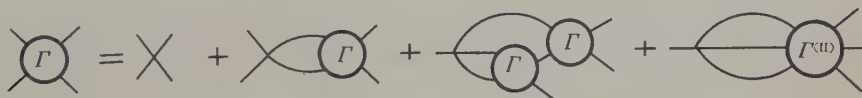


Fig. 1. Graphical illustration of the equation for Γ
Directions of the lines are not distinguished for brevity.

The system of equations for G , (9) ~ (11), is a system of non-linear integro-differential equations which involves another unknown function Γ . The equation for Γ involves $G^{-1}(\partial^2 \Gamma / \partial \bar{\eta} \partial \eta) |_{\eta=\bar{\eta}=0} G^{-1}$, and so on. On the other hand, G is boundary value of the solution of the functional differential equation (6) at $\eta=\bar{\eta}=0$, so it cannot be determined by solving the equation, but may be given arbitrarily as far as other requirements of the theory are not violated.

Then the problem to be solved is not to solve the equations for G and the equation for Γ as a system of non-linear integro-differential equations, but the problem whether it is possible that there exists $M(x, x')$ which makes the singularity of the propagator on the light cone weaker than that of the so-called free propagator, without violating the causality requirement and can be represented in form (11) with some function Γ . Now it should be noticed that Γ cannot be completely determined even if G were known.

We estimate the asymptotic behaviours of Fourier transforms (momentum representations) of G , Γ , and $\Gamma^{(11)}$, assuming as follows:

$$G(p) \sim O(|p^2|^{-\alpha/2}), \quad (12)$$

$$\Gamma(p_1, p_2; p_3, p_4) \sim O((\sum_{i,j} |p_i p_j|)^{\beta/2}) \quad (i, j=1, 2, 3; p_4=p_1+p_2-p_3), \quad (13)$$

$$\begin{aligned}
 \Gamma^{(11)}(p_1, p_2, p_3; p_4, p_5, p_6) = G^{-1} \frac{\partial^2 \Gamma}{\partial \bar{\eta} \partial \eta} \Big|_{\eta=\bar{\eta}=0} G^{-1} \sim O((\sum_{i,j} |p_i p_j|)^{\gamma/2}) \\
 (i, j=1, \dots, 5; p_6=p_1+p_2+p_3-p_4-p_5), \quad (14)
 \end{aligned}$$

where β and γ should be understood to be the lower bound of the sets of β' and γ' for which the following relations take place when all $|p_i p_j|$'s tend to ∞ :

$$\Gamma(p_1, p_2; p_3, p_4) / (\sum_{i,j} |p_i p_j|)^{\beta/2} < \infty, \quad (15)$$

$$\Gamma^{(11)}(p_1, p_2, p_3; p_4, p_5, p_6) / (\sum_{i,j} |p_i p_j|)^{\gamma/2} < \infty, \quad (16)$$

respectively. Here logarithmic factors are discarded because such factors do not change the order of powers. If G, Γ, \dots involve oscillating factors such as $\sin f(p_i)$, the following considerations may be modified, but we do not consider such a possibility here.

Assuming the absence of accidental cancellations, we obtain the following relations among α, β and γ :

$$\alpha = \text{Max}(9 + \beta - 3\alpha, 1), \quad (17)$$

$$\beta = \text{Max}(9 + 2\beta - 4\alpha, 5 + \beta - 2\alpha, 9 + \gamma - 3\alpha, 1). \quad (18)$$

Therefore we obtain

$$\alpha \geq \frac{5}{2}, \quad \beta = 4\alpha - 9, \quad \gamma = 3\alpha + \beta - 9. \quad (19)$$

Then the propagation character is modified. (In derivation of the last equality of (19), the equation for $\Gamma^{(11)}$ has been used.)

Now it is easily seen that if an accidental cancellation occurs in the equation for M , an accidental cancellation must occur also in the equation for Γ , and so on.

The relations (19) are not sufficient conditions for the absence of divergent constants. But, if $\alpha > 10/3$ and if Γ has, for example, asymptotic property such as

$$\Gamma \sim \prod_i |p_i|^2 / (\sum_{i,j} |p_i p_j|)^{-\beta/2 + 4\delta}, \quad (2\delta > \alpha), \quad (20)$$

no divergent constant appears in the equations for $M, \Gamma, \Gamma^{(11)}, \dots$, even if the so-called bare term of the vertex part is not cancelled.

Because it is possible to give Γ as boundary condition for the functional differential equation, instead of giving G or M , our problem is to investigate more concretely the conditions which must be satisfied by Γ in order that $M, \Gamma^{(11)}, \dots$ exist, not only in the asymptotic region.

It should be noticed that because of non-linearity of Eqs. (9)~(11) by which M should be determined when Γ was given, the uniqueness of the solution M is not guaranteed.

§ 3. Boson-fermion interaction

Now we consider the interaction between a fermion and a boson fields recently considered by Watanabe:⁸⁾

$$L_w(x) = g_w \bar{\psi}(x) \gamma' \psi(x) \square \phi(x) \quad (\gamma' = 1 \text{ or } i\gamma_5) \quad (21)$$

where g_w is a constant of dimension $[L^2]$. In order to avoid unnecessary complications, we assume ϕ be a neutral field.

Defining the fermion and boson propagators G, \mathfrak{G} by the usual procedure and adopting the quantization condition proposed in § 2, we obtain equations for them :

$$\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + m_0 + g_w \gamma' \left\langle 0 \left| \frac{\partial^2 \phi(x)}{\partial x_\mu \partial x_\mu} \right| 0 \right\rangle + g_w \gamma' \left[\frac{\partial^2}{\partial x_\mu \partial x_\mu}, \frac{\delta}{\delta J(x)} \right] \right) \times G(x, x'; \eta, \bar{\eta}, J) = c_1 \delta(x - x'), \quad (22)$$

$$\left(\frac{\partial^2}{\partial \xi_\mu \partial \xi_\mu} + \kappa_0^2 \right) \mathfrak{G}(\xi, \xi'; \eta, \bar{\eta}, J) + g_w S_p \left[\gamma' \frac{\partial^2}{\partial \xi_\mu \partial \xi_\mu} - \frac{\delta G(\xi, \xi'; \eta, \bar{\eta}, J)}{\delta J(\xi)} \right] = c_2 \delta(\xi - \xi'). \quad (23)$$

We define the one-body propagators and the vertex part :

$$G(x - x') = G(x, x'; 0, 0, 0), \quad (24)$$

$$\mathfrak{G}(\xi - \xi') = \mathfrak{G}(\xi, \xi'; 0, 0, 0), \quad (25)$$

$$g_w \Gamma(x; x'; \xi) = \int dx_1 dx_1' d\xi' \mathfrak{G}^{-1}(\xi - \xi') G^{-1}(x - x_1) \times \frac{\delta G(x; x'; \eta, \bar{\eta}, J)}{\delta J(\xi')} \Big|_{\eta=\bar{\eta}=J=0} G^{-1}(x_1' - x'). \quad (26)$$

In "momentum representation," we can write the equations for G, \mathfrak{G} and Γ as follows :

$$\left\{ i\gamma p + m_0 + g_w^2 \int dq \gamma' G(p - q) \Gamma(p - q, p; q) \mathfrak{G}(q) \right\} G(p) = 1, \quad (27)$$

$$\left\{ k^2 - \kappa_0^2 + g_w^2 k^2 \int dq S_p[\gamma' G(q) \Gamma(q, q - k; k) G(q - k)] \right\} \mathfrak{G}(k) = 1, \quad (28)$$

$$\begin{aligned} \Gamma(p, q; p - q) &= \gamma'(p - q)^2 + g_w^2 \int dk \gamma' G(p - k) \Gamma(p - k, q - k; p - q) \\ &\times G(q - k) \Gamma(q - k, q; -k) k^2 \mathfrak{G}(k) \\ &+ g_w \int dk \gamma' G(p - k) \Gamma^1(p - k, q; -k, p - q) k^2 \mathfrak{G}(k), \end{aligned} \quad (29)$$

where Γ^1 is Fourier transform of $\mathfrak{G}^{-1}(\partial \Gamma / \partial J) \Big|_{\eta=\bar{\eta}=J=0}$, and coefficients such as $(2\pi)^{-4}$ are discarded.

We estimate the asymptotic behaviours of G, \mathfrak{G} and Γ , assuming as follows :

$$G(p) \sim O(|p^2|^{-\alpha/2}), \quad (30)$$

$$\mathfrak{G}(k) \sim O(|k^2|^{-\beta/2}), \quad (31)$$

$$\Gamma(p, q; p - q) \sim O((|p^2| + |q^2| + |pq|)^{\gamma/2}). \quad (32)$$

Thus we obtain from (27) a relation among α, β and γ :

$$6 - 2\alpha - \beta + \gamma = 0. \quad (33)$$

The same relation is obtained from (28), hence two of α, β and γ are independent. Considering the equation for Γ we obtain the following condition :

$$2\alpha + \beta \geq 8. \quad (34)$$

The conditions obtained above are not in agreement with the results obtained by Watanabe in perturbation theory with cut-off.

If conditions

$$\alpha > 4, \quad \beta > 8 - \alpha \quad (35)$$

are satisfied and if Γ and Γ^I have asymptotic properties, such as

$$\Gamma(p, q; p-q) \sim O\left(\frac{|p^2|^\varepsilon |q^2|^\varepsilon |(p-q)^2|^{\varepsilon'}}{(|p^2| + |q^2| + |pq|)^{2\varepsilon + \varepsilon' - \gamma/2}}\right) \quad (36)$$

$$\Gamma^I(p_1, p_2; p_3, p_4) \sim O\left(\frac{|p_1^2|^\zeta |p_2^2|^\zeta |p_3^2|^{\zeta'} |p_4^2|^{\zeta'}}{(\sum |p_i p_j|)^{2\zeta + 2\zeta' - \delta/2}}\right) \quad (37)$$

$$2\varepsilon > \alpha, \quad \varepsilon' > \beta - 2, \quad \zeta + \zeta' > 6, \quad \delta = \alpha + \beta + \gamma - 6, \quad (38)$$

no divergent constant appears in the system of Eqs. (27) ~ (29).

Then a possibility of divergence-free theory is suggested.

§ 4. Conclusion and further outlook

In this note we have investigated asymptotic properties of the propagators in presence of the interactions which do not permit the canonical formalism. We have obtained some relations among the asymptotic properties of the propagators and the vertex parts, but could not determine them separately in any case. However, possibilities of divergence-free theories are suggested. For mathematically rigorous treatment of the problem, we must investigate the analytical properties of the propagators and the vertex parts. Though we have not made such a rigorous treatment in this note, we may think that the introduction of interactions such as have been treated above is a possible way of avoiding the divergence difficulties in the present quantum field theory.

The amplitudes of real processes depend only on $p_i p_j (i \neq j)$ because all p_i^2 's are equal to squares of corresponding masses, then we can conclude nothing about the high energy limit of the real processes from the above considerations.

It should be noticed also that there is no proportionality relation among coupling constants and the vertex part.

A new principle which determines the solution of the functional differential equations independently of the perturbation theory should be obtained but it will be a very difficult problem.

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S-Wave Pion- Σ -Hyperon Scattering

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Qualitative character of the S -wave pion- Σ -hyperon scattering is investigated, comparing with the corresponding S -wave pion-nucleon scattering. Information on S -wave pion- Σ -hyperon scattering is obtained from K^- capture experiments which have suggested large magnitudes of the phase shifts as well as large isotopic spin dependence. It is pointed out that these characteristics seem to be explained only when doublet approximation proposed by A. Pais is abandoned.

Effect of the K -coupling to the S -wave pion- Σ -hyperon and the S -wave K^- -nucleon scattering are also discussed and the result is that absorption process reflects rather significant effect on both scatterings, especially K^- -nucleon scattering, even when K -meson-baryon couplings are weaker than pion-baryon couplings by an order of magnitude.

§ 1. Introduction

To investigate S -wave pion- Σ -hyperon scattering seems to give us important information for understanding the structure of baryons. Especially its comparison with corresponding pion-nucleon scattering is important for testing a possible symmetry of pion-baryon interactions.¹⁾ Information on pion- Σ -hyperon scattering, however, is not directly obtained due to difficulties of the experimental technique. In this paper, therefore, we extract some of them from the data on K^- capture from hydrogen and try to investigate characteristic features of the pion- Σ -hyperon scattering, comparing with the corresponding pion-nucleon scattering.

Experimental data of K^- capture from hydrogen are now available in some detail from the work of the bubble chamber experiments.²⁾ For K^- capture at rest from hydrogen, experimental accuracy has considerably improved but still seems to be compatible with the old branching ratios, $4:2:2:\frac{1}{2}$, for $\Sigma^-:\Sigma^+:\Sigma^0:\Lambda^0$ reaction ratios.

For K^- capture in flight, the ratio Σ^-/Σ^+ has changed to become a value about unity at low energies.²⁾ We assume in the following analysis that K -meson is pseudoscalar, relative parity of nucleon and hyperon is even and the capture occurs through the S -state, the last of which is supported by a recent work carried out by T. B. Day et al.³⁾ and is also consistent with the isotropy of angular distributions at low energies. Under these assumptions, final state pion- Σ -hyperon

interaction becomes S -wave interaction and we can deduce (§ 2) from the data on K^- capture some information on the S -wave pion- Σ -hyperon scattering and compare with corresponding S -wave pion-nucleon scattering.

S -wave pion-nucleon scattering is characterized by small magnitudes of the phase shifts and their relatively large isotopic spin dependence, for which no satisfactory explanation has yet been given from the pion field theory⁴⁾. Recent investigation⁵⁾ on these points, however, suggests that at least qualitative features of S -wave pion-nucleon scattering seem to be explained under the following assumptions :

- I) Dilute nucleon-antinucleon pair extends to a range above $r_0 \approx (2\mu)^{-1}$; μ being a pion mass.
- II) Pion-pion interaction with attractive force ($\lambda\phi^4$; $\lambda < 0$) contributes to the S -wave pion-nucleon scattering.

In § 3, we shall investigate whether or not the characteristics of the S -wave pion- Σ -hyperon are explained with the same standpoint. In the case of pion- Σ -hyperon scattering, however, K -coupling effect would be expected to be larger than that of pion-nucleon scattering.*

In § 4, effects of the K -coupling to the S -wave pion Σ -hyperon scattering are, therefore, qualitatively investigated and finally we discuss in § 5 the results thus obtained.

§ 2. Phenomenological analysis

We shall first direct our attention to the branching ratio for the $K^- + p \rightarrow \Sigma^{\pm 0} + \pi^{\mp 0}$ reactions at rest and in flight. The matrix elements of these reactions are expressed in terms of eigenamplitudes for $T=0$ and $T=1$ if we assume charge independence :

$$\begin{aligned} M(K^- + p \rightarrow \Sigma^- + \pi^+) &= \sqrt{\frac{1}{6}} M_0 - \frac{1}{2} M_1 \\ M(K^- + p \rightarrow \Sigma^+ + \pi^-) &= \sqrt{\frac{1}{6}} M_0 + \frac{1}{2} M_1 \\ M(K^- + p \rightarrow \Sigma^0 + \pi^0) &= \sqrt{\frac{1}{6}} M_0. \end{aligned} \quad (2.1)$$

Eqs. (2.1), when combined with observed branching ratios reported at 1959 Annual International Conference on High Energy Physics in Kiev,²⁾ lead us to the following results.

* This fact has been emphasized by R. H. Dalitz et al.⁶⁾

Table I *

at rest	in flight $E=150$ Mev
$r = 0.4 \pm 0.1$	$r = 0.57$
$ \theta = 62^\circ \pm 4^\circ$	$ \theta = 90^\circ$
$\Sigma^- : \Sigma^+ : \Sigma^0 : \Lambda^0$	$\Sigma^- : \Sigma^+ : \Sigma^0 : \Lambda^0$
$= 45 : 21 : 27 : 7$	$\approx 9 : 9 : 6 : 7^{**}$

It may be noticed that r is rather insensitive to the incident K^- meson energy, while phase difference between M_1 and M_0 is large and energy sensitive.

We, next, prove that the above large phase difference at rest is entirely due to the final S -wave pion- Σ -hyperon scattering phase shift difference, i.e. $\delta_{T=1} - \delta_{T=0}$. For this purpose, we make use of the well-known fact that phase of a matrix element can in general be restricted under the assumption of Wigner's time reversal invariance, which was first applied to the analysis of photopion production⁷⁾ and which generalization was widely discussed by M. L. Goldberger et al.⁸⁾ and by S. Fubini et al.⁹⁾ The most general expression of S -matrix of a two-channel reaction, under the assumption of time reversal invariance, can be expressed in terms of the three real parameters.

$$S = \begin{pmatrix} De^{2i\delta} & iAe^{i(\delta+\eta)} \\ iAe^{i(\delta+\eta)} & De^{2i\eta} \end{pmatrix} \quad (2.2)$$

$$A^2 + D^2 = 1$$

where D , A , δ and η are real numbers.

If we identify δ and η to be real part S -wave phase shifts for K^-N and $\Sigma\pi$ scattering, expression (2.2) tells us that for $T=0$, in which case the $\Lambda^0\pi$ channel is not involved, the phase of the matrix element for $K^- + p \rightarrow \Sigma^{\pm 0} + \pi^{\mp 0}$ reaction is given by

$$\text{phase of the reaction amplitude} = \delta_{K^-N} + \delta_{\Sigma\pi} + \frac{\pi}{2} \quad (2.3)$$

where δ_{K^-N} and $\delta_{\Sigma\pi}$ mean the real parts of S -wave phase shifts for K^-N and $\Sigma\pi$ scattering, respectively. For $T=1$, where three channels K^-N , $\Lambda\pi$ and $\Sigma\pi$ are involved, we cannot generally obtain such a simple relation as (2.3). However, as we shall see later, the reaction $\Sigma + \pi \rightarrow \Lambda + \pi$ is not induced by S -wave pion in the lowest expansion of μ/M (μ and M are masses of a pion and a baryon, respectively). So, we can approximately neglect the contribution from the reaction $\Sigma + \pi \rightarrow \Lambda + \pi$. Then, the same relation (2.3) also holds for $T=1$ state.

* We have put $M_1/M_0 = re^{i\theta}$.

** Fluctuation of the branching ratio at 100~150 Mev seems to be fairly large, so these values need not be taken too seriously.

From the above consideration, the following conclusion may be derived that the large phase difference between M_1 and M_0 at rest is due to the phase shift difference in the final S -wave pion- Σ -hyperon scattering: i.e., $\delta_{\Sigma\pi}(T=1) - \delta_{\Sigma\pi}(T=0) \approx \pm 62^\circ$ at 90 Mev, since δ_{K^-N} is expected to vanish at zero energy.

§ 3. Phase shift of S -wave pion- Σ -hyperon scattering

We now investigate qualitative behaviour of the S -wave pion- Σ -hyperon scattering based on the analysis in § 2. For this purpose, we first neglect effects of K -meson-baryon coupling, which are considered later in § 4, and develop our arguments along that of the case for S -wave pion-nucleon scattering.⁵⁾ Then, effective Hamiltonian for the S -wave pion- Σ -hyperon scattering is given by

$$H = G_1(\bar{\Psi}_\Sigma \cdot \Psi_\Sigma)(\varphi_\pi \cdot \varphi_\pi) + G_2(\bar{\Psi}_\Sigma \cdot \varphi_\pi)(\Psi_\Sigma \cdot \varphi_\pi) \quad (3.1)$$

with $G_1 = \frac{1}{2M}g_{\Sigma\Sigma}^2$, $G_2 = \frac{1}{2M}(g_{\Sigma A}^2 - g_{\Sigma\Sigma}^2)$ M : baryon mass.

Expression (3.1) is obtained upon application of the Foldy transformation to the relativistic pseudoscalar pion-baryon interaction. We have neglected the mass difference between Σ and Λ^0 , which does not cause any serious effect on the present approximation. Corresponding Hamiltonian for the S -wave pion-nucleon scattering is given as ($\sim O(1/M)$)

$$H = \frac{1}{2M}g_{NN}^2(\bar{\psi}_N \cdot \psi_N)(\varphi_\pi \cdot \varphi_\pi). \quad (3.2)$$

Expression (3.2), as is well known, does not give any isotopic spin dependence for the S -wave phase shift. In contrast to the expression (3.2), Hamiltonian (3.1) gives rise to the splitting between $\delta_{\Sigma\pi}(T=1)$ and $\delta_{\Sigma\pi}(T=0)$ already in the lowest expansion of μ/M if $g_{\Sigma A}^2 \neq g_{\Sigma\Sigma}^2$, on account of which larger splitting than the S -wave pion-nucleon phase shift splitting $\delta_1 - \delta_3$ may be expected for appropriate pion-baryon coupling constants. It is also to be noticed that to the lowest order in μ/M , the reaction $\Sigma + \pi \rightarrow \Lambda^0 + \pi$ does not occur because we cannot construct isotopic spin scalar with the combination of Ψ_Σ , Ψ_Λ and φ_π . This reaction therefore occurs through the second order of the expansion in μ/M .

The Chew-Low theoretic approach for S -wave pion-nucleon scattering developed by S. D. Drell et al.¹¹⁾ can be applied easily to this case. The result is as follows:

$$a_\alpha(\omega) = C_\alpha^0 + C_\alpha^1\omega - \frac{\omega^2}{\pi} \int \frac{k}{\omega'^2} d\omega' \left[\frac{|a_\alpha(\omega')|^2}{\omega' - \omega - i\varepsilon} + C_{\alpha\beta} \frac{|a_\beta(\omega')|^2}{\omega' + \omega} \right] v^2(k) \quad (3.3)$$

where $a_\alpha(\omega)$ ($\alpha=0, 1$ and 2) represents the scattering amplitudes for $T=0, 1$ and 2 , respectively. $C_{\alpha\beta}$ is the crossing matrix and C_α^0 , C_α^1 are constants defined by

$$\begin{aligned}
 C^0 &= \begin{cases} \frac{1}{2M}(2g_{\Sigma A}^2 - g_{\Sigma \Sigma}^2) & \text{for } \alpha=0 \\ \frac{1}{4M}(-g_{\Sigma A}^2 + 3g_{\Sigma \Sigma}^2) & \text{for } \alpha=1 \\ \frac{1}{4M}(g_{\Sigma A}^2 + g_{\Sigma \Sigma}^2) & \text{for } \alpha=2 \end{cases} \quad (3.4) \\
 C_\alpha^1 &= -\frac{1}{\pi} \int \frac{k}{\omega^2} d\omega (\delta_{\alpha\beta} - C_{\alpha\beta}) \left\{ \left| \frac{1}{2} h_\beta^-(\omega) \right|^2 \right. \\
 &\quad \left. + \frac{1}{4} (h_\beta^{*-}(\omega) h_\beta^+(\omega) + h_\beta^-(\omega) h_\beta^{+*}(\omega)) \right\} v^2(k)
 \end{aligned}$$

where $h_\alpha^\pm(\omega)$ is defined analogous as in Drell's paper.

Main features of Eq. (3.3) are identical to the S -wave pion-nucleon scattering except that phase shift depends on the isotopic spin state at zero energy which does not contradict the crossing symmetry.¹²⁾

The contribution of pion-pion interaction to the S -wave pion- Σ -hyperon scattering is in general different from that to the S -wave pion-nucleon scattering since mesonic structure of the Σ -hyperon may be different from that of the nucleon. But if we make the assumption that mesonic structure of the Σ -hyperon is the same as the nucleon except magnitudes of the coupling constants and isotopic spin, we can estimate the contribution in the following way.

Analysis made in paper I⁵⁾ has shown that contribution of pion-pion interaction to the S -wave pion-nucleon scattering almost cancels out the isotopic spin independent repulsive term (3.2) in order to fit experiments. Thus we put as follows:

$$-\frac{1}{2M} g_{NN}^2 \hat{V}(0) \quad (3.5)$$

where $\hat{V}(q)$ mean the "potential" induced by pion-pion interaction and is calculated in paper I. Then, we can easily see that expression (3.4) turns out to be

$$\begin{aligned}
 C^0 &\approx \begin{cases} \frac{1}{2M}(2g_{\Sigma A}^2 - g_{\Sigma \Sigma}^2) - \frac{1}{2M} g_{\Sigma A}^2 \\ \frac{1}{2M}(-g_{\Sigma A}^2 + 3g_{\Sigma \Sigma}^2) - \frac{1}{2M} g_{\Sigma \Sigma}^2 \\ \frac{1}{4M}(g_{\Sigma A}^2 + g_{\Sigma \Sigma}^2) - \frac{1}{10M}(2g_{\Sigma A}^2 + 3g_{\Sigma \Sigma}^2) \end{cases} \quad (3.6)
 \end{aligned}$$

where second terms of the expression (3.6) represent contributions from pion-pion interaction.

Now, we investigate qualitative behaviour of phase shifts and roughly estimate magnitudes of the coupling constants in order to fit the phase splitting ($\delta_1 - \delta_0 \approx \pm 62^\circ$ at 90 Mev), using the Born approximation, since we are interested in qualitative differences between pion-nucleon and pion- Σ -hyperon scattering as al-

ready mentioned before. Magnitudes of the coupling constants are not determined separately because there is only one experiment which suggests $\delta_1 - \delta_0 \approx \pm 62^\circ$ at 90 Mev. So we examine two extreme cases, i. e. $g^2_{\Sigma\Sigma}/4\pi \approx 0$ (Case A) and $g^2_{\Sigma A}/4\pi \approx 0$ (Case B). The results are summarized in Table II.

Table II.

	Case A	Case B
phase shift $\tan \delta_\alpha = -(k/2\pi)C_\alpha^0$	$\delta_0, \delta_2 < 0$ repulsive $\delta_1 > 0$ attractive	$\delta_1, \delta_2 < 0$ repulsive $\delta_0 > 0$ attractive
magnitude of the coupling constant	$g^2_{\Sigma A}/4\pi \approx 7$ $\delta_1 - \delta_0 \approx 62^\circ$	$g^2_{\Sigma\Sigma}/4\pi \approx 7$ $\delta_0 - \delta_1 \approx 62^\circ$

We shall give some comments on the contents contained in Table II. For the case (A), which has been suggested by H. Miyazawa et al.,¹³⁾ only the $T=1$ state is attractive and a possible resonance for $T=1$ can be speculated. For the case (B), the $T=0$ state is attractive and in this case, resonance for the $T=0$ state is favorable for the enhancement of the Σ -production alone, suppressing A -production. In any case, experimental determination of the sign of the phase shifts for $T=0$ and $T=1$ state seems to be important to obtain relative magnitudes of the coupling constants $g^2_{\Sigma A}/4\pi$ and $g^2_{\Sigma\Sigma}/4\pi$.

To deduce definite conclusion from the above analysis seems dangerous since we have neglected the effect of K -meson-baryon couplings. In the next section we shall examine on this point.

§ 4. Effects of the K -coupling to the S -wave pion- Σ -hyperon scattering

We begin with our consideration on the K -coupling effect to the S -wave pion- Σ -hyperon scattering from K^- -nucleon scattering. Experiments on K^- -nucleon scattering have shown²⁾ that cross-section of K^- -proton elastic scattering at low energies ($\sigma_{K^-p}^{\text{el}} \approx 50$ mb) is much larger than that of K^+ -proton scattering ($\sigma_{K^+p} \approx 15$ mb) and of K^- -proton absorption reactions ($\sigma^{\text{ab}} \approx 15$ mb), the facts of which suggest that K^- absorption process might play an important role for the K^- -nucleon elastic scattering. Similar question is naturally raised also for the case of S -wave pion- Σ -hyperon scattering. From angular distributions for these scattering and reactions it is supposed that S -wave is dominant at least at low energies. This is supported by a recent phenomenological analysis on K^- -nucleon scattering developed by T. D. Jackson, D. G. Ravenhall and H. W. Wyld¹⁴⁾ who have succeeded in explaining these phenomena with two complex scattering lengths. These scattering lengths are calculated to be¹⁵⁾

$$\begin{aligned} \text{Solution A} \quad a_0 &= \pm 0.20 + 0.78 i & (\text{unit } 10^{-13} \text{cm}) \\ a_1 &= \pm 1.62 + 0.39 i \end{aligned}$$

$$\begin{aligned} \text{Solution B} \quad a_0 &= \pm 1.88 + 0.82 i \\ a_1 &= \pm 0.42 + 0.41 i. \end{aligned} \quad (4.1)$$

Sign of real parts of the scattering lengths are still undetermined.

Quantitative treatment based on the K -meson and pion field theory for these processes, however, seems very difficult. Nevertheless we try to estimate the effect of K -coupling qualitatively, using the Tamm-Dancoff method. Effective S -wave Hamiltonian is obtained by Foldy transformation analogous to the expression (3.1). Details of the calculation are shown in the Appendix. The results are as follows:

(A) S -wave K^-N scattering

$$\begin{aligned} \tan \delta_{KN}^0 &= -\frac{k}{2\pi} \frac{\alpha_1 \{1 - (\beta_1 + 2\beta_2) I_1(k)\} + \frac{3}{2} \delta_1^2 I_1(k)}{(1 - \alpha_1 I'(k)) \{1 - (\beta_1 + 2\beta_2) I_1(k)\} - \frac{3}{2} \delta_1^2 I_1(k) I'(k)}, \\ \tan \delta_{KN}^1 &= -\frac{k}{2\pi} \\ &\times \frac{\alpha_2 + \frac{\gamma^2}{2} I_1(k) \{1 - (\beta_1 + \beta_2) I_2(k)\}^{-1} + \delta_2^2 I_1(k) \left\{1 - \frac{2\beta_1 - \beta_2}{2} I_1(k)\right\}^{-1}}{1 - \alpha_2 I'(k) - \frac{\gamma^2}{2} I_1(k) I'(k) \{1 - (\beta_1 + \beta_2) I_2(k)\}^{-1} - \delta_2^2 I_1(k) I'(k) \left\{1 - \frac{2\beta_1 - \beta_2}{2} I_1(k)\right\}^{-1}}, \end{aligned}$$

(B) S -wave $\pi\Sigma$ scattering (above threshold)

$$\begin{aligned} \tan \delta_{\pi\Sigma}^0 &= -\frac{k}{2\pi} \frac{(\beta_1 + 2\beta_2) (1 - \alpha_1 I_1'(k)) + \frac{3}{2} \delta_1^2 I_1'(k)}{\{1 - (\beta_1 + 2\beta_2) I(k)\} \{1 - \alpha_1 I_1'(k)\} - \frac{3}{2} \delta_1^2 I(k) I_1'(k)}, \\ \tan \delta_{\pi\Sigma}^1 &= -\frac{k}{8\pi} \end{aligned} \quad (4.2)$$

$$\times \frac{(2\beta_1 - \beta_2) \left\{1 - \alpha_2 I_1'(k) - \frac{\gamma^2}{2} I_1(k) I_1'(k) (1 - (\beta_1 - \beta_2) I_2(k))\right\} - 4\delta_2^2 I_1'(k)}{\left\{1 - \frac{2\beta_1 - \beta_2}{2} I(k)\right\} \left\{1 - \alpha_2 I_1'(k) - \frac{\gamma^2}{2} I_1(k) I_1'(k) (1 - (\beta_1 - \beta_2) I_2(k))\right\} - \delta_2^2 I(k) I_1'(k)},$$

(C) $K^- + P \rightarrow Y + \pi$ absorption

$$M^0(K^- N \rightarrow \pi \Sigma) = \frac{k}{4\pi^2} \frac{\sqrt{6} \delta_1}{(1 - \alpha_1 I'(k)) \{1 - (\beta_1 + 2\beta_2) I_1(k)\} - \frac{3}{2} \delta_1^2 I_1(k) I'(k)},$$

$$\begin{aligned} M^1(K^- N \rightarrow \pi \Sigma) &= \frac{k}{4\pi^2} \frac{\sqrt{2} \gamma}{\{1 - (\beta_1 + \beta_2) I_2(k)\} [1 - \alpha_2 I'(k) - (\gamma^2/2) I'(k) I_1(k)]} \\ &\times \frac{1}{\{1 - (\beta_1 + \beta_2) I_2(k)\}^{-1} - \delta_2^2 I'(k) I_1(k) \{1 - (2\beta_1 - \beta_2) I_1(k)\}^{-1}}, \end{aligned}$$

$$\begin{aligned} M^1(K^- N \rightarrow \Lambda^0 \pi) &= \frac{k}{2\pi^2} \frac{\delta_2}{\{1 - (2\beta_1 - \beta_2) I_1(k)\} [1 - \alpha_2 I'(k) - (\gamma^2/2) I'(k) I_1(k)]} \\ &\times \frac{1}{\{1 - (\beta_1 + \beta_2) I_2(k)\}^{-1} - \delta_2^2 I'(k) I_1(k) \{1 - (2\beta_1 - \beta_2) I_1(k)\}^{-1}}. \end{aligned}$$

Now, we summarize the main characteristics of the expressions (4.2).

(i) Zero effective range approximation to the S -wave K^-N scattering holds good since energy dependence of $I_1(k)$ and $I'(k)$ is insensitive to the low energy incident K^- -meson energy. (See Figs. 1 and 2).

(ii) Zero effective range approximation to the S -wave pion- Σ -hyperon above threshold does not hold good since energy dependence of $I_1'(k)$ is sensitive to the incident pion energy. This difference is due to that of exothermic and endothermic reactions. Thus energy dependence of the phase shift of S -wave pion- Σ -hyperon scattering becomes more complicated than that of K^- -nucleon scattering.

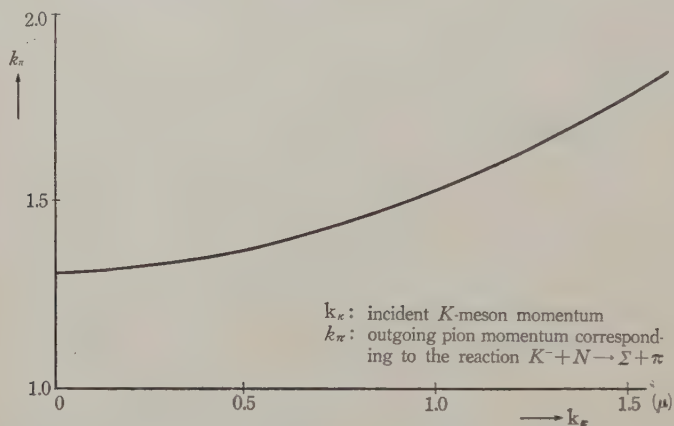


Fig. 1. Relation between incident K -meson momentum and outgoing pion momentum

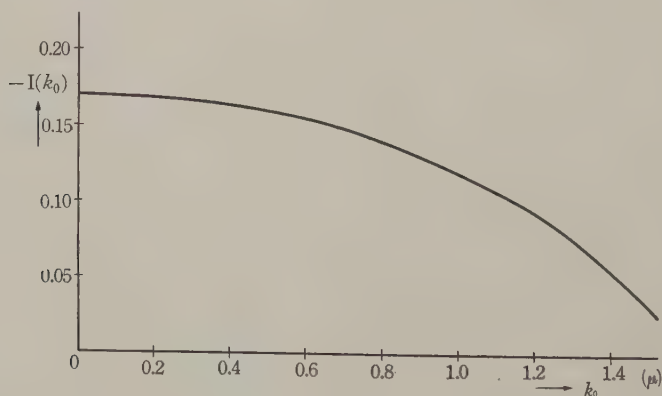


Fig. 2. Momentum dependence of $I(k)$

In order to estimate the effect of K -coupling on the S -wave pion- Σ -hyperon scattering, it is necessary to determine the magnitudes of coupling constants contained in the expressions (4.2). Since we are interested only in the qualitative behaviour of the phase shift, we assume in the following $f_{AN}^2/4\pi \approx 0$, $g_{\Sigma\Sigma}^2/4\pi \approx 0$,*

* These coupling constants mean, of course, bare coupling constants.

the assumption of which seems to be consistent with all the experiments up to now and has been proposed by H. Miyazawa et al.¹³⁾ from another point of view; and it determines the coupling constants $f_{\Sigma N}^2/4\pi$, $g_{\Sigma A}^2/4\pi$ so as to match the scattering length $a_0 + ib_0$ (4.1). Thereby we replace $I(k)$, $I'(k)$ with their average value, making use of the fact (i), i.e. $I(k)$ and $I'(k)$ are insensitive to the incident K -meson energy. Cut-off momentum is taken to be $\Lambda_s \approx 2\mu$ as assumed in paper I. It turned out, as a result, that the only solution A_+ ($0.2 + 0.78i$) had a following solution

$$\frac{g_{\Sigma A}^2}{4\pi} \approx 15, \quad \frac{f_{\Sigma N}^2}{4\pi} \approx 2 \sim 3.$$

We next examine the consistency of these coupling constants. Inserting these coupling constants into the expression (4.2), we found the following results.

(a) Scattering length for $T=1$ K^-N scattering

$$a_1 = -0.302 \quad b_1 = 0.232 \quad (10^{-13}\text{cm})$$

Agreement with solutions A_+ is not good.

(b) Branching ratio $\sigma(\Lambda^0)/\sigma(\Sigma^0)^*$

$$\begin{aligned} \sigma(\Lambda^0)/\sigma(\Sigma^0) &\approx 0.04 \quad \text{at rest} \\ &\approx 0.06 \quad \text{in flight} \end{aligned}$$

Experiments have shown that $\sigma(\Lambda^0)/\sigma(\Sigma^0) \approx 0.25$ at rest²⁾ and seems to increase with incident K -meson energy.

(c) Phase difference $\delta_{\Sigma\pi}(T=1) - \delta_{\Sigma\pi}(T=0)$

$$\tan \delta_1 = 0.26, \quad \tan \delta_0 = -0.47 \quad \delta_1 - \delta_0 = 40^\circ$$

These should be compared with the case of neglecting K -coupling effect.

$$\tan \delta_1 = 0.48, \quad \tan \delta_0 = -1.96 \quad \delta_1 - \delta_0 = 88^\circ$$

Experiments have shown $\delta_1 - \delta_0 \approx 62^\circ$. The qualitative agreement is to be noticed.

We do not attempt to discuss the behaviour of the phase shift based on the expression (4.2) in detail, because experimental data are still poor. It should be stressed, however, that the effect of K -coupling is rather large even if K -coupling is weaker by an order of magnitude than that of pion-nucleon coupling. This is due to the large absorption processes which significantly reflect their effects on the K^- -nucleon scattering and pion- Σ -hyperon scattering.

* To calculate the above ratio, we make use of the following relation and insert the experimental value for $r^2 = |M(1)/M(0)|^2 \approx 0.16$

$$\frac{\sigma(\Lambda^0)}{\sigma(\Sigma^0)} = 3 \left| \frac{M_A(1)}{M_\Sigma(0)} \right|^2 = 3 \left| \frac{M_\Sigma(1)}{M_\Sigma(0)} \right|^2 \left| \frac{M_A(1)}{M_\Sigma(1)} \right|^2 = 3 \times 0.16 \times \left| \frac{M_A(1)}{M_\Sigma(1)} \right|^2$$

§ 5. Concluding Remarks

In § 2 and § 3, we have investigated the scattering of pion with Σ -hyperon to find characteristic features of pion- Σ -hyperon interaction compared with the pion-nucleon scattering. To obtain information on pion- Σ -hyperon scattering is, however, very difficult. We, therefore, deduced some of them from the K^- -capture experiments which have shown that large S -wave phase shift splitting between δ_1 and δ_0 is required for obtaining the observed branching ratios of $K^- + p \rightarrow \Sigma + \pi$ reactions at rest. That this large phase shift splitting is hardly explained by a universal pion-baryon coupling proposed by M. Gell-Mann¹⁾ is easily seen when we compare with corresponding S -wave pion-nucleon scattering phase splitting¹⁵⁾ ($\delta_1 - \delta_3 \approx 20^\circ$ at 90 Mev). This fact remains true when $g_{\Sigma A}^2 = g_{\Sigma N}^2$, that is, doublet approximation¹⁶⁾ either is not favorable for the explanation of the branching ratio.

Quantitative treatment of S -wave pion-hyperon scattering, however, seems difficult than that of S -wave pion-nucleon scattering because two coupling constants of pion-baryon interaction and moreover K -meson-baryon couplings are involved. In § 3, we, therefore, first neglected the K -coupling and tried to find the characteristics of pion- Σ -hyperon scattering and we found that large splitting between δ_1 and δ_0 seems to be explained if we abandon doublet approximation.

Experimental determination of the sign of phase shifts for $T=0$, $T=1$ and $T=2$ states is very much desired to obtain more information about relative magnitude of the coupling constants $g_{\Sigma A}^2/4\pi$ and $g_{\Sigma N}^2/4\pi$. It should also be noticed that the sign of the phase shift for the $T=2$ state is always expected to be negative from the present treatment. Such a possibility has been discussed by R. H. Dalitz et al.⁶⁾ from another point of view.

Finally we examined in § 4 the effect of K -coupling to the S -wave pion- Σ -hyperon scattering, using the Tamm-Dancoff approximation. The result seems to suggest that the effect of K -coupling through absorption processes is significant for the quantitative discussion, especially for that of K^- -nucleon scattering. Similar argument is given by S. Minami¹⁷⁾ through a phenomenological analysis for K^- -nucleon scattering.

To conclude our discussion, we should like to stress the fact that in low energy phenomena involving pion, K -meson and baryons the S -wave effect is in general large and dominant over the P -wave effect, while smallness of the S -wave pion-nucleon scattering seems to belong to a special case.

The authors would like to express their deep gratitude to Professor H. Miyazawa for his guidance and kind encouragement.

Appendix

We start with the interaction Hamiltonian :

$$\begin{aligned}
H_{\text{int}} = & ig_{NN\pi} \bar{\Psi}_N \gamma_5 \tau_i \Psi_N \varphi_i \\
& + ig_{\Sigma A\pi} \bar{\Psi}_A \gamma_5 \Psi_\Sigma \varphi_i + h.c. \\
& - g_{\Sigma\Sigma\pi} (\bar{\Psi}_\Sigma \gamma_5 \times \Psi_\Sigma)_i \varphi_i \\
& + if_{ANK} \bar{\Psi}_A \gamma_5 \phi_K \Psi_N + h.c. \\
& + if_{\Sigma NK} \bar{\Psi}_\Sigma \gamma_5 \phi_K \tau_i \Psi_N + h.c.
\end{aligned} \tag{A.1}$$

where φ_i and ϕ_K mean π -meson field and K -meson field, respectively.

Applying the Foldy transformation—unitary transformation such that only the large elements of Dirac matrices retain—and neglecting the terms higher than $O(1/M)$, we get the following effective Hamiltonian:

$$\begin{aligned}
H_{\text{eff}} = & \frac{1}{2M} [g_{NN}^2 \bar{\Psi}_N \not{\epsilon}^2 \Psi_N + g_{\Sigma A}^2 \bar{\Psi}_A \not{\epsilon}^2 \Psi_A \not{\epsilon}^2 \\
& + g_{\Sigma\Sigma}^2 (\bar{\Psi}_\Sigma \cdot \Psi_\Sigma) (\not{\epsilon} \cdot \not{\epsilon}) + (g_{\Sigma A}^2 - g_{\Sigma\Sigma}^2) (\bar{\Psi}_\Sigma \cdot \not{\epsilon}) (\not{\epsilon} \cdot \Psi_\Sigma) \\
& + \{(-i)g_{\Sigma\Sigma} g_{\Sigma A} (\bar{\Psi}_\Sigma \times \not{\epsilon})_i \Psi_A \varphi_i + h.c.\}] \\
& + \frac{1}{2M} [f_{AN}^2 \bar{\Psi}_N \not{\epsilon} \phi_K^* \phi_K \Psi_N + f_{\Sigma A}^2 \bar{\Psi}_N \not{\epsilon} \phi_K^* \phi_K \tau_i \Psi_N \\
& + f_{AN}^2 \bar{\Psi}_A \not{\epsilon} \phi_K \phi_K^* \\
& + f_{\Sigma N}^2 (\Psi_\Sigma \cdot \Psi_\Sigma) (\phi_K \phi_K^*) + if_{\Sigma N}^2 \bar{\Psi}_\Sigma (\Psi_\Sigma \times \phi_K \tau \phi_K^*)_i \\
& + \{f_{\Sigma N} f_{AN} \bar{\Psi}_\Sigma \not{\epsilon} \phi_K \phi_K^* + h.c.\}] \\
& + \frac{1}{2M} [(g_{NN} f_{AN} + g_{\Sigma A} f_{\Sigma N}) \bar{\Psi}_A \not{\epsilon} \phi_K \tau_i \Psi_N \varphi_i + h.c.] \\
& + \{(g_{NN} f_{\Sigma N} + g_{\Sigma N} f_{AN}) \bar{\Psi}_\Sigma \not{\epsilon} \phi_K \Psi_N \varphi_i + h.c.\} \\
& + \{i(g_{\Sigma\Sigma} f_{\Sigma N} - g_{NN} f_{\Sigma N}) \bar{\Psi}_\Sigma (\phi_K \tau \Psi_N \times \not{\epsilon})_i + h.c.\}].
\end{aligned} \tag{A.2}$$

Here we have neglected the terms which change the orbital angular momentum, and for the sake of simplicity we neglect the mass differences of baryons.

We can now write down the Tamm-Dancoff equations for the K -meson-nucleon reaction. If we neglect the intermediate states which contain more than one meson, only the one meson states enter into our equations —i.e. zero-meson states cannot enter into the equations—, because of the bilinear form of the effective Hamiltonian with respect to meson operators. Also we are concerned only with the states of strangeness (-1) .

In the center of mass system:

$$\begin{aligned}
(E - M_N - E_l) M_{NK}^0(l) = & \frac{\alpha_1}{(2\pi)^3} \frac{1}{\sqrt{E_l}} \int d^3k \frac{1_{\Sigma\pi}}{\sqrt{E_\pi}} M_{NK}^0(k) \\
& + \sqrt{\frac{3}{2}} \frac{\delta_1}{(2\pi)^3} \frac{1}{\sqrt{E_l}} \int d^3k \frac{1}{\sqrt{E_k}} M_{\Sigma\pi}^0(k),
\end{aligned}$$

$$(E-M_{\pi}-\epsilon_l)M_{\pi\pi}^0(l)=\sqrt{\frac{3}{2}}\frac{\partial_1}{(2\pi)^3}\frac{1}{\sqrt{\epsilon_l}}\int d^3k\frac{1}{\sqrt{E_k}}M_{NK}^0(k) \\ +(\beta_1+2\beta_2)\frac{1}{(2\pi)^3}\frac{1}{\sqrt{\epsilon_l}}\int d^3k\frac{1}{\sqrt{\epsilon_k}}M_{\pi\pi}^0(k), \quad (\text{A}\cdot 3)$$

$$(E-M_N-E_l)M_{NK}^1(l)=-\frac{\alpha_2}{(2\pi)^3}\frac{1}{\sqrt{E_l}}\int d^3k\frac{1}{\sqrt{E_k}}M_{NK}^1(k) \\ +\sqrt{\frac{1}{2}}\frac{\gamma}{(2\pi)^3}\frac{1}{\sqrt{E_l}}\int d^3k\frac{1}{\sqrt{\epsilon_k}}M_{A\pi}^1(k) \\ +\frac{\delta_2}{(2\pi)^3}\frac{1}{\sqrt{E_l}}\int d^3k\frac{1}{\sqrt{\epsilon_k}}M_{\pi\pi}^1(k), \\ (E-M_A-\epsilon_l)M_{A\pi}^1(l)=\sqrt{\frac{1}{2}}\frac{\gamma}{(2\pi)^3}\frac{1}{\sqrt{\epsilon_l}}\int d^3k\frac{1}{\sqrt{E_k}}M_{NK}^1(k) \\ +\frac{(\beta_1+\beta_2)}{(2\pi)^3}\frac{1}{\sqrt{\epsilon_l}}\int d^3k\frac{1}{\sqrt{\epsilon_k}}M_{A\pi}^1(k), \quad (\text{A}\cdot 4)$$

$$(E-M_{\pi}-\epsilon_l)M_{\pi\pi}^1(l)=-\frac{\delta_2}{(2\pi)^3}\frac{1}{\sqrt{\epsilon}}\int d^3k\frac{1}{\sqrt{E_k}}M_{NK}^1(k) \\ +\left(\beta_1-\frac{\beta_2}{2}\right)\frac{1}{(2\pi)^3}\frac{1}{\sqrt{\epsilon_l}}\int d^3k\frac{1}{\sqrt{\epsilon_k}}M_{\pi\pi}^1(k),$$

where, for example, $M_{NK}^0(l)$ means the amplitude of the 0-isospin state that contains one nucleon and one K -meson with momentum l . E_l and ϵ_l mean the total energy of K -meson and π -meson with momentum l , respectively. $\alpha_1, \dots, \delta_2$ are defined as follows:

$$\alpha_1=f_{AN}^2/2M, \\ \alpha_2=f_{\pi N}^2/2M, \\ (\beta_1+2\beta_2)=1/2M(2g_{\pi A}^2-g_{\pi\pi}^2), \\ (\beta_1-\beta_2/2)=1/4M(3g_{\pi\pi}^2-g_{\pi A}^2), \quad (\text{A}\cdot 5) \\ (\beta_1+\beta_2)=1/2M(g_{\pi A}^2), \\ \gamma=1/2M(g_{NN}f_{AN}+g_{\pi A}f_{\pi N}), \\ \delta_1=1/2M(g_{NN}f_{\pi N}+g_{\pi A}f_{AN}), \\ \delta_2=1/2M(g_{\pi\pi}f_{\pi N}-g_{NN}f_{\pi N}).$$

Our next task is to solve Eqs. (A·3), and (A·4) under special boundary conditions.

S-wave K^-N scattering

We put the following form into Eqs. (A·3):

$$\begin{aligned}
M_{NK}^0(l) &= \delta(\omega_0' - E_l) + \frac{a_{NK}^0}{(\omega_0' - E_l)\sqrt{E_l}}, \\
M_{\Sigma\pi}^0(l) &= + \frac{a_{\Sigma\pi}^0}{(\omega_0 - \varepsilon_l - i\gamma_l)\sqrt{\varepsilon_l}}, \\
M_{NK}^1(l) &= \delta(\omega_0' - E_l) + \frac{a_{NK}^1}{(\omega_0' - E_l)\sqrt{E_l}}, \\
M_{\Lambda\pi}^1(l) &= + \frac{a_{\Lambda\pi}^1}{(\omega_0'' - \varepsilon_l - i\gamma_l)\sqrt{\varepsilon_l}}, \\
M_{\Sigma\pi}^1(l) &= + \frac{a_{\Sigma\pi}^1}{(\omega_0 - \varepsilon_l - i\gamma_l)\sqrt{\varepsilon_l}},
\end{aligned} \tag{A.6}$$

where

$$\begin{aligned}
\omega_0 &= E - M_\Sigma, \\
\omega_0' &= E - M_N, \\
\omega_0'' &= E - M_\Lambda
\end{aligned} \tag{A.7}$$

and γ_l is a small positive number. From (A.3) and (A.6) we get equations for a^0 and a^1 .

For example,

$$\begin{aligned}
a_{NK}^0 &= \frac{\alpha_1}{2\pi^2} k_0' \sqrt{\omega_0'} + \alpha_1 I' a_{NK}^0 + \sqrt{\frac{3}{2}} \delta_1 I_1 a_{\Sigma\pi}^0, \\
a_{\Sigma\pi}^0 &= \sqrt{\frac{3}{2}} \frac{\delta_1}{2\pi^2} k_0' \sqrt{\omega_0'} + \sqrt{\frac{3}{2}} \delta_1 I' a_{NK}^0 + (\beta_1 + 2\beta_2) I_1 a_{\Sigma\pi}^0,
\end{aligned} \tag{A.8}$$

where

$$\begin{aligned}
I(\omega_0') &= \frac{1}{2\pi^2} P \int_0^{2\mu} \frac{k^2 dk}{(\omega_0' - E_k) E_k}, \\
I_1(\omega_0) &= \frac{1}{2\pi^2} P \int_0^{2\mu} \frac{k^2 dk}{(\omega_0 - \varepsilon_k) \varepsilon_k} + i \frac{k_0}{2\pi} \quad (\text{above threshold}).
\end{aligned} \tag{A.9}$$

We can immediately obtain relation (4.2) by making use of the relation

$$\frac{1}{\pi} \tan \delta^0 = - \frac{a^0}{\sqrt{E_l}}. \tag{A.10}$$

The other relations of (4.2) can be easily obtained in the same manner.

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Photodisintegration of the Deuteron in the High Energy Range

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The differential and total cross sections for the photodisintegration of the deuteron are calculated for the incident photon energies in the range from 80 Mev to 300 Mev. In the calculation, a full expression of the electric interaction between the deuteron and radiation is used without expansion in terms of κr of each multipole or multipole transition. For the initial state, we adopt the pion-theoretical deuteron wave function which has almost 7% of D -state mixture. The plane wave is used for the final state. The conclusions are 1) κr expansion is not justified in the energy range $E_\gamma \geq 80$ Mev and gives an underestimate for the cross sections, 2) the multipoles higher than $E2$ do not affect for $E_\gamma \leq 80$ Mev, 3) the large D -state mixture is important, 4) the retardation effects are important at such high energy as the meson effects have to be considered.

It is shown that the excitation function agrees with the observed data in the energy range, if we add the meson effects to our result.

§ 1. Introduction

The photodisintegration of the deuteron serves as a source of knowledge for nuclear forces and interaction of the electromagnetic radiation with the deuteron. Especially, this reaction in the medium energy region ($20\text{Mev} \leq E_\gamma \leq 100\text{ Mev}$) of the incident photon has been investigated by many authors,¹⁻⁵⁾ because the reaction in this energy region does not seem to be affected by complicated effects such as virtual meson production but is to be determined mainly by the two nucleon interactions. And it has become rather convincing that one can explain the main features of the photodisintegration of deuteron in the medium energy region in terms of the electric transitions without renouncing Siegert's theorem, if one assumes a large D -state mixing in the deuteron and a strong positive tensor potential in the triplet odd final states. Both of them are natural consequences of the pion theory of nuclear forces.⁶⁾

In the high energy region, the importance will increase as to the multipole transitions. Concerning the electric multipole effects we can consider two kinds of effects. One is effects due to higher power terms of κr (κ is the photon momentum)* in each multipole and the other is effects of the multipole transition. Most of the authors, however, confined themselves to calculations of $E1$ (and $E2$) transitions and, furthermore, took up only the first term in the power series expansion

* Hsieh firstly investigated this effect for $E1$ transition.

in terms of κr of each multipole interaction. The present author and his collaborator have performed exact calculations in which the higher power terms (for $E1$ and $E2$ transitions) are all included and they found that the expansion of the power series is not justified even at an energy as low as 80 Mev.⁵⁾ Therefore, it seems worthwhile to investigate effects of the higher power terms of κr in a more general way and to see how far they affect the results of the simple calculations.

On the other hand, effects of the higher electric multipole transitions than the $E2$ transition have not yet been investigated. Of course, small contributions due to the higher multipole transitions are expected for the cross sections, because the higher the multipole transition is, the smaller the coefficient of the electric multipole interaction is. However, in the high energy region, the higher multipole transitions and, hence, the higher waves in the final states contribute to the reaction. Therefore, it is also interesting to see how much these contributions are effective to the results.

Concerning the photodisintegration of the deuteron at the higher energies, it is clear that one cannot reproduce the hump in the observed excitation function around 280 Mev in terms of the electric transition, because the latter predicts a smooth energy dependence of the excitation function. Some authors^{3, 7, 8)} have investigated that how far we can explain the excitation function using only nuclear forces. Of course, the hump is due to the resonance production⁹⁾ of virtual mesons by an incident photon. The recent investigations, however, suggested that the virtual meson effects give an appreciable contribution only to the spin flip $M1$ transition within the narrow range of energy around the observed hump. Then it is rather interesting to see whether one can reproduce all the observed data by just adding this contribution to the excitation function due to the electric transition.

Our present purpose is to show the result of our calculations in the energy range from 80 Mev to 300 Mev of the incident photon in which a complete expression including all multipoles for the electric interaction is used. This full expression of the electric interaction has the factor $\exp(i\kappa r)$ and it is not legitimate to expand with κr and retain the first few terms. The calculation at 80 Mev of the incident photon is very useful for seeing the contributions of the multipole transitions compared with the exact calculations for the $E1$ and $E2$ transitions.⁵⁾ For the final states, we assume the plane wave for the sake of simplicity of calculations. The approximation means to neglect the nuclear forces between nucleons in the final states, but can include all higher waves of the final states. However, this approximation is justified at the high energies as we shall see later. For the initial deuteron state, we adopt the pion-theoretical wave function,⁶⁾ which has about 7% of D -state mixture and reproduces the quadrupole moment of $2.7 \times 10^{-27} \text{ cm}^2$. As it is well known, the large D -wave mixture in the deuteron gives rise to a large isotropic part in the angular distribution of the $E1$ transition and can give nice agreement with experiment in the medium energy range. Therefore, it is most desirable to use the exact form of the pion-theoretical wave function at the present

case too. However, for simplicity of calculation we approximate this pion-theoretical wave function of the deuteron by an analytic form which has 7% of D -state mixture and $2.6 \times 10^{-27} \text{ cm}^2$ of the quadrupole moment.

We have emphasized the importance for the electric multipole transitions and the higher power effects of each multipole and have not touched the magnetic transition. However, it seems enough to see the order of the spin-flip magnetic transition which is largest in the magnetic effects.

In § 2, we explain the method for the present calculations of the transition matrix elements. In § 3, the exact and the approximate cross sections are given for several cases. The conventional $E1$ approximation is derived and the correspondence with the partial wave method is clarified. The results of the present calculations are discussed in § 4 and other effects such as magnetic transitions or virtual meson production are discussed in § 5. Conclusions are given in § 6. The detailed formula for the calculations are given in the Appendix.

§ 2. Electric transition matrix elements

To represent the interaction of a photon with the deuteron, we shall use the form of the interaction operator

$$H' = \int \mathbf{J} \cdot \mathbf{A} dV \quad (1)$$

where \mathbf{J} is the current operator and \mathbf{A} is the electromagnetic potential. This \mathbf{A} is given by a plane wave with photon momentum κ and polarization ϵ . Following the method given by Foldy^{11,12}, \mathbf{A} is separated into two parts, namely electric and magnetic radiations

$$\begin{aligned} \mathbf{A} \propto \epsilon \exp(i\kappa \mathbf{r}) = & \int_0^1 ds \{ \text{grad}[\epsilon \cdot \mathbf{r} \exp(is\kappa \mathbf{r})] \\ & - is\mathbf{r} \times [\kappa \times \epsilon] \exp(is\kappa \mathbf{r}) \}, \end{aligned} \quad (2)$$

where \mathbf{r} is the relative coordinate $\mathbf{r} = \mathbf{r}_p - \mathbf{r}_n$. The first term represents the complete electric radiation and the second term represents the magnetic radiation. Here the integration over s takes account of the so-called retardation effects or, more specifically, the higher multipoles and the higher power of $\kappa \mathbf{r}$ in each multipole. For the moment, we consider only the electric radiation. After the integration by part and then, utilizing the equation for the conservation of the local current, we can write the electric transition matrix element in the following form:

$$\langle \Psi_f | \int_0^1 ds \frac{1}{2} \epsilon \cdot \mathbf{r} \left[\exp\left(i \frac{s}{2} \kappa \mathbf{r}\right) - \exp\left(-i \frac{s}{2} \kappa \mathbf{r}\right) \right] \rho(\mathbf{r}) | \Psi_i \rangle,$$

where $\rho(\mathbf{r})$ represents the charge density of the total system. Then we assume that we can neglect the charge density due to the virtual pions and obtain the following expression :

$$\langle \Psi_f | \int_0^1 ds \frac{1}{2} \boldsymbol{\varepsilon} \cdot \mathbf{x} \left[\tau_1^p \exp\left(i \frac{s}{2} \boldsymbol{\kappa} \mathbf{x}\right) - \tau_2^p \exp\left(-i \frac{s}{2} \boldsymbol{\kappa} \mathbf{x}\right) \right] | \Psi_i \rangle, \quad (3)$$

where $\mathbf{x} = (\mu c / \hbar) \mathbf{r}$ and τ_i means the charge projection operator of the i -th proton $\tau_i^p = (1 + \tau_z^i)/2$. If one simply puts $s=0$ in (3), then one gets the conventional expression $(1/2) \boldsymbol{\varepsilon} \cdot \mathbf{r}$ for the electric dipole interaction. We calculate the matrix element (3) without expansion of the retardation factor, in order to include all the multipoles and then higher order terms in each multipole.

For each multipole transition, the final states are specified due to selection rules ${}^3S_1 + {}^3D_1 \rightarrow {}^3P + {}^3F$ for $E1$ transition, ${}^3S_1 + {}^3D_1 \rightarrow {}^3S_{\pm} + {}^3D + {}^3G$ for $E2$ transition, and ${}^3S_1 + {}^3D_1 \rightarrow {}^3P + {}^3F + {}^3H$ for $E3$ transition, and so on. In order to take into account all multipole transitions we must take all final states and the plane wave approximation is the most conventional form for the purpose. Thus we assume

$$\Psi_f = \frac{1}{\sqrt{2}} \{ \hat{\xi}_1 \eta_2 \exp[i \mathbf{k} \mathbf{x}] - \hat{\xi}_2 \eta_1 \exp[-i \mathbf{k} \mathbf{x}] \} \chi_m', \quad (4)$$

where \mathbf{k} is the relative propagation vector of the nucleons and $\hat{\xi}_i, \eta_i$, are the isotopic spin wave functions with $\hat{\xi}_i$ and η_i corresponding to the i -th particle being in a proton and a neutron states, respectively. χ_m' is the triplet spin function.

For the deuteron state, we adopt the pion-theoretical wave function given in Ref. 6), which has about 7 percent of D -state mixing and reproduces the quadrupole moment of $2.7 \times 10^{-27} \text{ cm}^2$. Thus,

$$\Psi_i = \frac{1}{\sqrt{4\pi}} \left[\frac{u(x)}{x} - \frac{1}{\sqrt{8}} S_{12} \frac{w(x)}{x} \right] \frac{1}{\sqrt{2}} (\hat{\xi}_1 \eta_2 - \eta_1 \hat{\xi}_2) \chi_m', \quad (5)$$

where S_{12} is the tensor potential operator, $u(x)$ and $w(x)$ are normalized as $\int [u(x)^2 + w(x)^2] dx = 1$. For simplicity of calculation, we use the following analytical form which approximates the wave function very well in the outer region :

$$\begin{aligned} u(x) &= A_s [\exp(-\alpha x) - \exp(-\beta x)], \\ w(x) &= D_1 \exp(-\alpha_1 x) + D_2 \exp(-\alpha_2 x) + D_3 \exp(-\alpha_3 x). \end{aligned} \quad (6)$$

The parameters are chosen as follows :

$$\begin{aligned} A_s &= 1.039 & D_1 &= 0.111 & \alpha_1 &= 0.4 \\ \alpha &= 0.328 & D_2 &= 0.656 & \alpha_2 &= 1.0 \\ \beta &= 1.972 & D_3 &= -0.767 & \alpha_3 &= 2.0. \end{aligned} \quad (7)$$

The wave function (6) and the parameters (7) reproduce the deuteron parameters $P_D = 7.0^\circ$, $Q = 2.6 \times 10^{-27} \text{ cm}^2$. In our present deuteron wave function (6), the hard

core is not taken into account. Therefore, the D -state mixture differs from the pion-theoretical one and rather becomes a little larger. This difference depends on the hard core radius. However, the inner part of $u(x)$ and $w(x)$ does not contribute to the transition. The transition amplitudes in the $E1$ transition ${}^3D_1 \rightarrow {}^3P$, ${}^3D \rightarrow {}^3F$, are mostly determined by the wave function in the outer region. And the higher the angular momentum of the final state is the more outer region affects the transition amplitude. Therefore, the approximation is justified in the present case. Thus the electric transition matrix can be calculated analytically by using (3), (4), (5), (6) without further approximations.

§3. Cross sections

When we take into account only the electric transition for the photodisintegration of deuteron, the differential cross section is written in the following form, taking the summation over the final spin states and averaging over the initial states and polarizations for the transition matrix element (3):

$$d\sigma = a + b \sin^2 \theta + c \sin^4 \theta + d \sin^6 \theta, \quad (8)$$

where θ is the angle between the incident photon direction κ and the outgoing photon direction \mathbf{k} . The coefficients a , b , c , and d are not constants but complicated functions of θ and one cannot express the differential cross section as a simplified form involving the retardation term. The detailed expressions of these coefficients will be given in the Appendix (i).

The differential cross section (8) involves all effects of the multipole transitions. Usually, we cannot separate these effects but only the dipole transition is taken out in the limit $s=0$ in (3). Then (3) gives rise to the conventional electric dipole operator $(1/2)(\boldsymbol{\varepsilon} \cdot \mathbf{r})$. Hereafter, we shall call this limit $s=0$ the "conventional $E1$ approximation". The integral over s is trivial and the coefficients c , and d , in (8) vanish. The coefficients a and b are expressed respectively as a_0 and b_0 in this case:

$$a_0 = B(k) \frac{9}{25} |\langle PD \rangle + \langle FD \rangle|^2, \quad (9)$$

$$b_0 = B(k) \left\{ \frac{3}{2} \langle PS \rangle^2 + \frac{3}{50} \langle PD \rangle^2 + \frac{9}{25} \langle FD \rangle^2 - \frac{27}{25} \langle PD \rangle \langle FD \rangle \right\}, \quad (10)$$

$$B(k) = \frac{1}{12} \frac{e^2}{\hbar c} \frac{ME_\gamma}{\hbar^2} \frac{1}{k} \left(\frac{\hbar}{\mu c} \right)^3, \quad (11)$$

where $\langle PS \rangle$, $\langle PD \rangle$, and $\langle FD \rangle$ are the transition amplitudes of the electric dipole transition. They mean transitions from the right side state to the left side state in the angular bracket. Detailed forms of these transition amplitudes are given in the Appendix (iii). The total cross section is reduced to the form

$$\sigma_T(s=0) = 4\pi B(k) \left[\langle PS \rangle^2 + \frac{2}{5} \langle PD \rangle^2 + \frac{3}{5} \langle FD \rangle^2 \right]. \quad (12)$$

Further, if we neglect the D -state mixture in the deuteron wave function ($s=0$, $w(r)=0$), then, only the transition amplitude $\langle PS \rangle$ does not vanish. The differential and total cross sections are given by the following form:

$$d\sigma(s=0, w=0) = B(k) \frac{3}{2} \langle PS \rangle^2 \sin^2 \theta, \quad (13)$$

$$\sigma_T(s=0, w=0) = 4\pi B(k) \langle PS \rangle^2.$$

This approximation corresponds to the approximation used by Schiff, Marshall and Guth (S.M.G. approximation).

§ 4. Results and discussions

In our calculation of the electric transition matrix element (3), we have kept the full expression of the interaction amplitudes without using the multipole expansion. In Fig. 1, we have plotted the differential cross sections given by our calculations of (8) and the conventional $E1$ approximation ($s=0$) of (9) and (10) at 80 Mev of the photon energy. For the sake of comparison, we have also plotted the differential cross section given by our previous calculations⁶⁾ in which we took into account the final state interactions, making use of the full expressions for the $E1$ and $E2$ interactions.

It is readily seen from Fig. 1 that the $s=0$ approximation gives rather poor results. This is not surprising since $\kappa R > 1$ (R is the deuteron radius) for the energy above 90 Mev and it is not legitimate to expand the interaction in terms of κr .

Hereafter, we shall denote our calculated results where the integration over s is performed by the symbol $\int ds$. The comparison

of our curve $\int ds$ with the $E1+E2$ curve previously calculated⁶⁾ shows that the effects of the multipoles higher than $E2$ are not very large. This is also conjectured from the following estimate. The effect of the higher power terms of the $E1$ transition operator is to increase the cross section about ten percent³⁾ at our energy region and, on the other hand, the cross section due to the $E2$ transition contributes about 1.3~1.5 percent of the $E1$ cross section.^{6),7)} This estimate for the cross section at our energy region is to be justified by our results listed in Table I. The interference terms between the $E1$ and $E2$ transi-

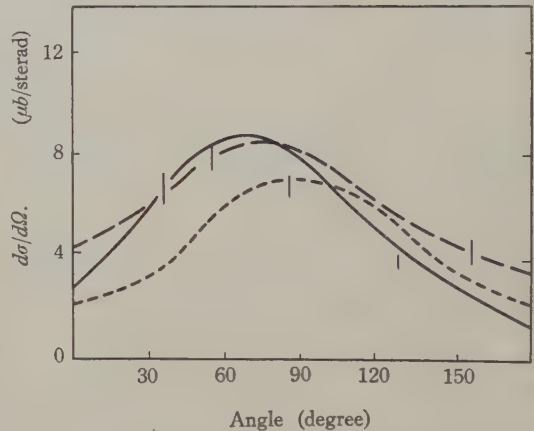


Fig. 1. Angular distribution of the $D(\gamma p)n$ reaction at $E_\gamma=80$ Mev

The solid curve represents our calculation. The dotted curve represents $s=0$. The dashed curve represents the $E1+E2$ transitions which involve the final state interaction.

tions do not affect on the total cross section. The effect of the $E2$ transition mainly shifts the peak of the angular distribution due to the $E1$ transition to the small angle through the interference terms. The angular distribution of our calculation $\int ds$, especially the shift of the peak, agrees well with the result of the $E1+E2$. If the transitions higher than $E2$ are effective, the peak of the angular distribution may shift to the smaller angle. If one assumes that the total cross sections of the higher multipoles decrease with the ratio $\sigma_T(E2)/\sigma_T(E1)$, we are led to the conclusion that the difference between the total cross sections of the conventional $E1$ approximation and our calculation $\int ds$ mainly depends on the higher power of $E1$ and $E2$. If one includes the final state interactions, the isotropic part of the angular distribution is expected to increase. The effects of the multipoles higher than $E2$ do not affect our previous results for $E_\gamma \leq 80$ Mev.

In Fig. 2, the angular distributions of the cases $s=0$, $\int ds$, $s=1$, are drawn. The effect of the higher multipoles in the angular distribution is to shift the

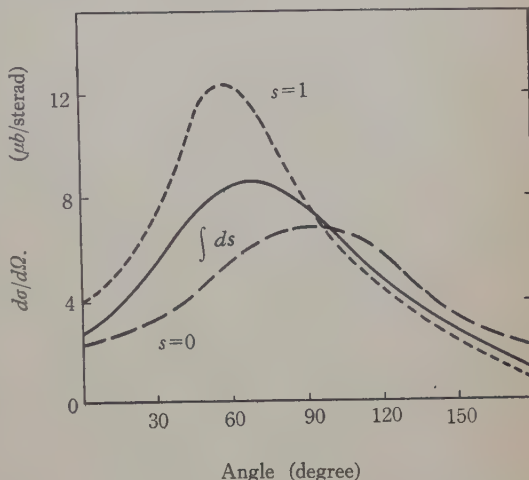


Fig. 2. Retardation effects on the angular distribution at $E_\gamma=80$ Mev

$\int ds$ means our calculation.

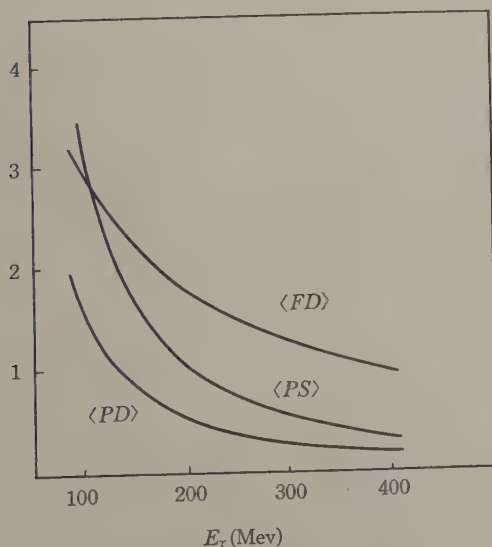


Fig. 3. The energy dependence of the transition amplitudes

peak to the small angle. The position of the peak does not so much change with the photon energy. But the height of the peak and the ratio of the maximum value to the minimum one of the differential cross section decrease with increasing energy. The total cross sections due to our calculation become larger compared with those by the conventional $E1$ approximation with increasing energy. Therefore we can understand the large contributions due to the retardation though we cannot estimate the contributions due to the multipole transition and the higher powers of each multipole separately. Furthermore, to discuss the angular distribution at 300 Mev is rather meaningless concerning the present calculation,

because there are many unknown effects which contribute to the angular distribution.

Next, let us consider the energy dependence of the angular distribution for the conventional $E1$ approximation. In the conventional $E1$ approximation, the transition amplitude $\langle PS \rangle$ of the ${}^3S_1 \rightarrow {}^3P$ transition is most important up to 100 Mev but decreases very rapidly with increasing energy. On the contrary, the transition amplitude $\langle FD \rangle$ of the ${}^3D_1 \rightarrow {}^3F$ transition does not so much decrease. The energy dependence of these transition amplitudes is shown in Fig. 3. The unisotropic parameter b_0 in the angular distribution formula (10) becomes smaller with increasing energy due to the negative interference term $\langle PD \rangle \langle FD \rangle$. On the other hand, the isotropic parameter a_0 does not contain any negative terms. The photo-disintegration parameters for the conventional $E1$ approximation a_0 , b_0 , σ_T ($s=0$) and our σ_T are given in Table I for the photon energies 80 Mev, 200 Mev, 300 Mev.

Concerning the D -state mixing in the deuteron state, the differential cross section and the total cross section are strongly dependent on the D -state wave function especially at high energies. If one neglects the D -state mixing in the deuteron ($w(r)=0$), the cross section formula is much simplified and reduces to the form calculated by Schiff,¹³⁾ Marshall and Guth.¹⁴⁾ They used Hulthen's function for the deuteron wave function (S.M.G. approximation). This S.M.G. approximation, of course, could not reproduce the experimental results but is only useful for seeing the dependence on the D -state mixture of the cross section. The S.M.G. approximation $P_D=0$ is compared with the conventional $E1$ approximation in Fig. 4. The total cross sections are given in Table I. σ_T (S.M.G.) decreases rapidly with the increasing photon energy and even at 80 Mev the total cross section is nothing but only 70 % of the conventional $E1$ approximation; moreover, the isotropic part is zero in the angular distributions as far as one also neglects the final state interaction. Therefore, one should naturally emphasize the importance of the large D -state probability.

Table I. Total cross sections in μb

E_γ (Mev)	80	200	300
σ_T (exact)	79.7		18.5
σ_T ($s=0$)	69.6	14.1	7.0
σ_T ($s=0, w(r)=0$)	48.0	4.6	1.4
a_0 ($\mu b/\text{sterad}$)	2.1	0.7	0.4
b_0 ($\mu b/\text{sterad}$)	5.0	0.6	0.2

In Fig. 5, we have plotted the excitation function given by our calculations, $\int ds$, the conventional $E1$ approximation ($s=0$), and the S.M.G. approximation ($s=0, w(r)=0$). For our calculations, $\int ds$, we have evaluated the cross section only at 80 Mev and 300 Mev of the photon energy and interpolated it between

these energies to get the curve in Fig 5. Since the excitation function of the conventional $E1$ approximation shows a smooth energy dependence, this interpolation will be justified. The prediction by our calculations $\int ds$ agrees with the observed results^{15,16)} fairly well except for the hump region. Thus, it is quite plausible that one can reproduce all the observed results on the photodisintegration of deuteron, if one also properly takes into account the virtual meson effects around the hump energy.

As we have already seen, the

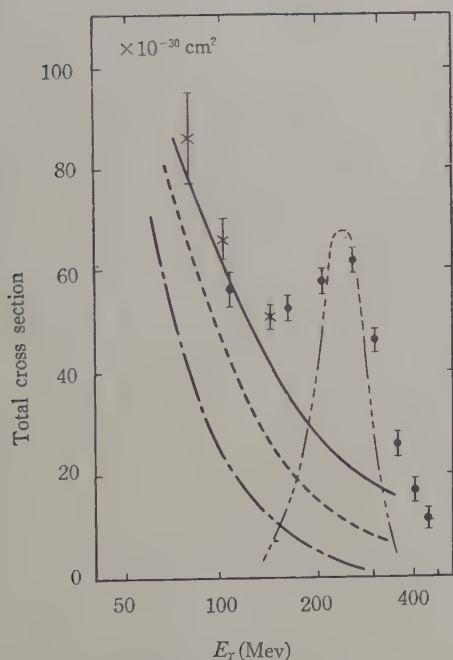


Fig. 5 The excitation functions
 — exact calculation,
 - - - $s=0$ approximation,
 - · - $s=0, w(r)=0$ approximation,
 ····· The virtual meson effect,¹⁰⁾
 × Illinois data¹⁶⁾
 ● California Tech. data,¹⁵⁾

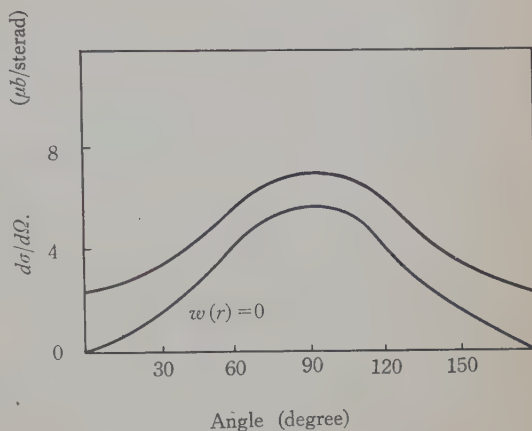


Fig. 4. The P_D dependence of the angular distribution

The upper curve corresponds to $s=0$.

The lower curve corresponds to $s=0, w(r)=0$.

multipole effects higher than $E2$ transition on the total cross section do not affect the results for $80 \text{ Mev} \geq E_\gamma$. But they increase with energy as one can see by the comparison of the curve $s=0$ with the curve $\int ds$ in Fig. 5. And the effects of higher order terms in κr become larger than the cross section with conventional $E1$ approximation ($s=0$) at 300 Mev of the photon energy. However, it would be meaningless to discuss the detailed angular distribution or the total cross section at these energy, because there are unknown virtual meson effects at this "resonance" energy.

§ 5. Other effects

Another important effect next to the electric transition is the magnetic transition. After the interaction operator (1) is separated into the electric part and the magnetic part, the magnetic part is further separated into the magnetic convection interaction, the magnetic spin interaction and the magnetic exchange interaction. The magnetic spin interaction plays an

important role in the very low energy region but its effect decreases with increasing energy. The last interactions have many ambiguities and are beyond the scope of the present paper. Here we shall content ourselves with estimating the order of the effect of the spin flip magnetic interaction.

The spin flip magnetic transition matrix element is written as

$$\begin{aligned} \left\langle \mathcal{P}_f \right| \frac{\hbar c \kappa}{\epsilon_0 - \epsilon_f} \frac{\mu}{M} \left\{ \frac{e}{2} \frac{e^{i(\kappa x/2)}}{2} [\mu_p(\tau_1^p + \tau_2^n) + \mu_n(\tau_1^n + \tau_2^p)] \right. \\ \left. - \frac{e}{2} \frac{e^{-i(\kappa x/2)}}{2} [\mu_p(\tau_1^n + \tau_2^p) + \mu_n(\tau_1^p + \tau_2^n)] \right\} \mathcal{E}' \cdot \frac{(\sigma^{(1)} - \sigma^{(2)})}{2} \left| \mathcal{P}_i \right\rangle. \end{aligned} \quad (14)$$

Comparing this interaction with the interaction in (3), the difference of factors are

$$\frac{\hbar c \kappa}{\epsilon_0 - \epsilon_f} \cdot \frac{\mu}{2M} (\mu_p - \mu_n). \quad (15)$$

$(\hbar c \kappa / \epsilon_0 - \epsilon_f) (1/2) (\mu_p - \mu_n)$ is the order of unity and $\mu/M \sim 1/6$. The overlap integrals are very similar to those of the electric transition. Therefore, the spin magnetic interaction may contribute about 3% to the cross section of the electric transitions. Although it gives a different angular distribution, the angular distribution of the total transition does not appreciably change because there is no interference between the electric transition and the magnetic spin flip interaction.

L. D. Perlstein and A. Klein¹⁰⁾ have investigated the exchange magnetic interactions in the hump energy region, assuming the pseudo-scalar nature of the pion and obtained reasonably good agreement with observation. If we add the excitation function calculated by them to the results of our exact calculations, then the total cross section around the hump region becomes too large to fit the observed values (Fig 5). But this discrepancy may be removed if one adopts a suitable singlet final state wave function for the resonance transition.

§ 6. Conclusions

In the high energy region, the electric transition gives the main contribution to the photodisintegration of deuteron. The multipole effects for the electric transition are clarified at 80 Mev of the photon energy by the discussions in § 4 and § 5. We could not treat the effect of the final state interaction in our special calculation. However, one can find the contributions of the deuteron wave function very well. From the comparison of our calculations with each approximation, we conclude the following points.

- (1) The κr expansion is not justified in the energy region $E_\gamma \geq 80$ Mev and gives an underestimate for cross sections.*

* These conclusions have been reached also in reference 3) on the basis of less extensive calculation.

- (2) The effects of the multipoles higher than $E2$ do not affect the results for $E_\gamma \leq 80$ Mev.
- (3) The large D -state mixing of the deuteron wave function is decisively important for the reaction, especially in reproducing the angular distribution and large cross section at high energies.
- (4) The effects of the final state interactions are not so decisive at high energies as at lower energies.
- (5) It is expected that one can reproduce all the observed results on the photodisintegration of deuteron, if one properly takes into account the multipole transitions and the virtual meson effects around the hump energy.

Appendix

(i) The exact form of the differential cross section

The exact calculation of the matrix element (3) is performed straightforwardly and the differential cross section can be written in the form (8). The coefficients a , b , c , and d in (8) are given as follows:

$$\begin{aligned}
 a &= B(k) k^2 16 Y'^2 \\
 b &= B(k) k^2 (12 X''^2 + 27 Y^2 + 3 Z^2 - 4 X'' Y + 32 X' Y' + 4 X'' Z + 10 Y Z) \\
 c &= B(k) k^2 (32 X'^2 + 28 X Y + 4 X Z - 8 X X'') \\
 d &= B(k) k^2 12 X^2.
 \end{aligned} \tag{A1}$$

Here X , X' , X'' , Y , Y' , Z , are the overlap integrals:

$$\begin{aligned}
 X &= 8^{-1/2} \int_0^1 ds (k/K)^3 \int j_3(Kx) (3/\sqrt{8}) w(x) x^2 dx, \\
 X' &= 8^{-1/2} \int_0^1 ds (k^2/K^3) \left(\frac{s}{2} \kappa - k \cos \theta \right) \int j_3(Kx) (3/\sqrt{8}) w(x) x^2 dx, \\
 X'' &= 8^{-1/2} \int_0^1 ds (k/K^3) \left(\frac{s}{2} \kappa - k \cos \theta \right)^2 \int j_3(Kx) (3/\sqrt{8}) w(x) x^2 dx, \\
 Y &= -8^{-1/2} \int_0^1 ds (k/K) \frac{2}{5} \int \{j_1(Kx) + j_3(Kx)\} (3/\sqrt{8}) w(x) x^2 dx, \\
 Y' &= -8^{-1/2} \int_0^1 ds (1/K) \left(\frac{s}{2} \kappa - k \cos \theta \right) \frac{2}{5} \int \{j_1(Kx) + j_3(Kx)\} \\
 &\quad \times (3/\sqrt{8}) w(x) x^2 dx, \\
 Z &= -2^{-1/2} \int_0^1 ds (k/K) \int j_1(Kx) \left[u(x) - \frac{1}{\sqrt{8}} w(x) \right] x^2 dx,
 \end{aligned} \tag{A2}$$

where

$$\begin{aligned} \mathbf{K} &= (s/2)\boldsymbol{\kappa} - \mathbf{k}, \\ |\mathbf{K}| &= \left[\left(\frac{s}{2}\boldsymbol{\kappa} - \mathbf{k} \cos \theta \right)^2 + k^2 \sin^2 \theta \right]^{1/2}. \end{aligned} \quad (\text{A3})$$

$j_l(Kx)$ is the spherical Bessel function of order l .

(ii) *The conventional E1 approximation ($s=0$)*

If we put $s=0$ in (A2), (A3), \mathbf{K} tends to \mathbf{k} and we get the X_0 , Y_0 , Z_0 .

$$\begin{aligned} X \rightarrow X_0 &= 8^{-1/2} \int j_3(kx) (3/\sqrt{8}) w(x) x^2 dx, \\ Y \rightarrow Y_0 &= -8^{-1/2} \int \frac{2}{5} [j_1(kx) + j_3(kx)] (3/\sqrt{8}) w(x) x^2 dx, \\ Z \rightarrow Z_0 &= -2^{-1/2} \int j_1(kx) \left[u(x) - \frac{1}{\sqrt{8}} w(x) \right] x^2 dx, \\ X' \rightarrow -X_0 \cos \theta, \quad X'' \rightarrow X_0 \cos^2 \theta, \quad Y' \rightarrow -Y_0 \cos \theta. \end{aligned} \quad (\text{A4})$$

(iii) *Correspondence with the partial wave method*

If we adopt the final l -state wave function as follows,

$$v_l(kx) = kx j_l(kx), \quad (\text{A5})$$

the transition amplitudes for E1 transition are written as

$$\begin{aligned} \langle PS \rangle &= \int kx j_1(kx) u(x) x dx, \\ \langle PD \rangle &= \int kx j_1(kx) w(x) x dx, \\ \langle FD \rangle &= \int kx j_3(kx) w(x) x dx. \end{aligned} \quad (\text{A6})$$

The final p -states of the conventional E1 approximation are the same for three values of J , 3P_0 state = 3P_1 state = 3P_2 state. Then, X_0 , Y_0 , Z_0 , in (A4) are expressed by (A6),

$$\begin{aligned} X_0 &= \frac{3}{8} \frac{1}{k} \langle FD \rangle, \\ Y_0 &= -\frac{3}{20} \frac{1}{k} \{ \langle PD \rangle + \langle FD \rangle \}, \\ Z_0 &= -\frac{1}{\sqrt{2}} \frac{1}{k} \left\{ \langle PS \rangle - \frac{1}{\sqrt{8}} \langle PD \rangle \right\}. \end{aligned} \quad (\text{A7})$$

The cross sections in the literature are obtained by these relations.

If one makes the transition amplitude $L(l, J)$ specified by l and J ,

$$\begin{aligned} L(10) &= \langle PS \rangle - \sqrt{2} \langle PD \rangle, & L(12) &= \langle PS \rangle - \frac{\sqrt{2}}{10} \langle PD \rangle, \\ L(11) &= \langle PS \rangle + \frac{\sqrt{2}}{2} \langle PD \rangle, & L(32) &= \frac{3}{5} \sqrt{3} \langle FD \rangle. \end{aligned} \quad (\text{A8})$$

The total cross section (12) becomes

$$\sigma_T(s=0) = 4\pi B(k) \frac{1}{9} [L^2(10) + 3L^2(11) + 5L^2(12) + 5L^2(32)].$$

This is just the total cross section formula for the $E1$ transition which is given by the partial wave method. Similarly, the differential cross section is written also. Thus, our conventional $E1$ approximation agrees with the partial wave method in which the final state interaction is neglected.

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Spinning Charged Test-Particles in General Relativity

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Equations of motion for charged test-particles in electromagnetic or vector-meson fields are derived from Fock-Papapetrou's method. Equations for spinning charged test-particles are obtained in a general covariant way.

§ 1. Introduction

The derivation of equations of motion from field equations alone is one of the important consequences of general relativity. Two distinct approaches to the same problem exist: one due to Einstein¹⁾ and his co-workers, and the other due to Fock,²⁾ and Papapetrou.³⁾ Motion charged test-particles have also been considered by Einstein,⁴⁾ Infeld and Wallace,⁵⁾ and Chase.⁶⁾ In the present paper we shall derive the motion equations of charged test-particles using the approach of Fock-Papapetrou. Papapetrou's⁷⁾ equations of spinning test-particles are also generalised to include the presence of electromagnetic fields. Our considerations are equally valid for the case of mesic charges in vector meson fields. The resulting equations of motion reduce, in the flat space approximation, to the equations obtained by Corben and Bhabha,⁸⁾ and Bhabha,⁹⁾ except for the explicit radiative terms.

§ 2. Equations of motion in the third order approximation

In our notation, latin indices will stand for spatial components and Greek indices will include time-component also. The velocity of light is taken to be unity. Moreover, the partial, covariant and material derivatives of a tensor A_{\dots} are denoted respectively by $\partial_{\mu} A_{\dots}$, $\nabla_{\mu} A_{\dots}$, $U^{\mu} \nabla_{\mu} A_{\dots} \equiv DA_{\dots}$.

The picture of a test-particle is a narrow world tube in space-time, inside which a line L will represent the motion of a particle. The coordinates of a point on L are denoted by $X^i(t)$, $X^4=t$. The main assumptions about the test-particle in this work are the following:

(i) The material energy-momentum tensor and charge-current vector $T^{\mu\nu} \neq 0$, $j^{\mu} \neq 0$ inside the world-tube, and they vanish immediately outside. On the boundary of separation junction conditions are $T^{\mu\alpha} n_{\alpha} = 0$, $j^{\alpha} n_{\alpha} = 0$, i.e., there is no normal energy-momentum or current flux across the boundary.

(ii) Any time-constant hyperplane slicing the material world-tube must be a

convex domain, i.e., no coordinate line can cross the boundary of the region more than twice.

(iii) $|x^i - X^i| = |\partial x^i| \leq R \rightarrow 0$; also $|T^{\mu\nu}|, |j^\mu|$ are bounded inside the tube. The last two conditions imply that background fields are predominant, so that gravitational or electromagnetic field intensities $\Gamma_{\mu\nu}^\alpha, F_\nu^\alpha$ may be uniform (homogeneous) within the tube. Our coordinate system is admissible in the sense that $g_{\mu\nu}$, and $\Gamma_{\nu\lambda}^\mu$ are continuous throughout. Field quantities will be expressed as power expansions in small parameters δx^i with this understanding.

The dynamical equations which furnish our starting point are consequences of combined Maxwell-Einstein field equations. These are

$$\nabla_\beta T^{\alpha\beta} = -\nabla_\beta E^{\alpha\beta} = -F_\nu^\alpha j^\nu, \quad (2.1)$$

where $T^{\alpha\beta}, E^{\alpha\beta}$ are the material and electromagnetic stress tensors, j^ν, F_ν^α are current vector and electromagnetic field quantities respectively. It can be shown that (2.1) represents the vector-meson case as well (provided the Lorentz-gauge condition $\nabla_\nu \phi^\nu = 0$ is assumed), so that all our subsequent considerations apply equally to both fields.

Now the covariant Gauss's theorem states that $\int_\Omega \nabla_\beta (T^{\alpha\beta} \lambda_\alpha) d\Omega = \int_{\sigma(\Omega)} \varepsilon(n) T^{\alpha\beta} n_\beta \lambda_\alpha d\sigma$; where $d\Omega, d\sigma$ are the invariant volume and hypersurface elements, indicator $\varepsilon(n) = \pm 1$, $n^\beta n_\beta = 1$, and λ_α is any arbitrary vector field. If we apply this theorem over the narrow four-dimensional volume element of the world-tube between two adjacent hyperplanes given by $t + \Delta t = \text{constant}$, and $t = \text{constant}$, with the boundary condition $T^{\alpha\beta} n_\beta = 0$ and (2.1), we get

$$\int_\Omega [-\lambda_\alpha F_\beta^\alpha j^\beta + T^{\mu\nu} (\partial_\mu \lambda_\nu - \Gamma_{\mu\nu}^\alpha \lambda_\alpha)] \sqrt{g} d^4 v = \Delta \int_\Omega T^{\alpha 4} \lambda_\alpha n_4 \sqrt{g'} d^3 v.$$

Now if we choose the arbitrary vector field such that $\lambda_1 = 1, \lambda_2 = \lambda_3 = \lambda_4 = 0$, and take $\lim \Delta t \rightarrow 0$, (remembering $n_4 \sqrt{g'} = \sqrt{g}$) we get

$$\frac{d}{dt} \int_{V_3(t)} \tau^{14} d^3 v + \int_{V_3(t)} \Gamma_{\mu\nu}^{14} \tau^{\mu\nu} d^3 v = - \int_{V_3(t)} F_\beta^1 j^\beta d^3 v.$$

Similarly we can derive

$$\frac{d}{dt} \int_{V_3(t)} \tau^{\alpha 4} d^3 v = - \int_{V_3(t)} \Gamma_{\mu\nu}^{\alpha 4} \tau^{\mu\nu} d^3 v - \int_{V_3(t)} F_\mu^\alpha j^\mu d^3 v, \quad (2.2a)$$

$$\frac{d}{dt} \int_{V_3(t)} x^\alpha \tau^{\beta 4} d^3 v = \int_{V_3(t)} \tau^{\alpha\beta} d^3 v - \int_{V_3(t)} x^\alpha \Gamma_{\mu\nu}^{\beta 4} \tau^{\mu\nu} d^3 v - \int_{V_3(t)} x^\alpha F_\mu^\beta j^\mu d^3 v, \quad (2.2b)$$

$$\begin{aligned} \frac{d}{dt} \int_{V_3(t)} x^\alpha x^\beta \tau^{\gamma 4} d^3 v &= \int_{V_3(t)} x^\alpha x^\beta \tau^{\gamma\beta} d^3 v + \int_{V_3(t)} x^\beta \tau^{\gamma\alpha} d^3 v \\ &\quad - \int_{V_3(t)} x^\alpha x^\beta \Gamma_{\mu\nu}^{\gamma 4} \tau^{\mu\nu} d^3 v - \int_{V_3(t)} x^\alpha x^\beta F_\mu^\gamma j^\mu d^3 v. \end{aligned} \quad (2.2c)$$

The relative coordinates in a time-constant hyper-plane with respect to the instantaneous $X^i(t)$ of the line L can be written as

$$\partial x^i = x^i - X^i, \quad \partial x^4 = 0. \quad (2.3)$$

Inside the world-tube the affinities and field intensities can be expressed according to the first mean value theorem (provided the assumption (ii) is valid), as

$$\begin{aligned} \Gamma_{\mu\nu}^{\alpha}(x^i, t) &= \Gamma_{(0)}^{\mu\nu\alpha}(X^i, t) + \partial x^{\sigma} \partial_{\sigma} \Gamma_{(\theta_1)}^{\mu\nu\alpha}(X^i + \theta_1 \partial x^i, t), \\ F_{\mu}^{\alpha}(x^i, t) &= F_{(0)}^{\mu\alpha}(X^i, t) + \partial x^{\sigma} \partial_{\sigma} F_{(\theta_1')}^{\mu\alpha}(X^i + \theta_1' \partial x^i, t). \end{aligned} \quad (2.4)$$

Inserting (2.4) into (2.2a) and (2.2b) with (2.3) we get

$$\frac{d}{dt} \int_{V_3} \tau^{\alpha 4} d^3 v = - \int_{V_3} \Gamma_{(0)}^{\mu\nu\alpha} d^3 v - \int_{V_3} F_{(0)}^{\beta\alpha} j^{\beta} d^3 v - \int_{V_3} \partial x^{\sigma} \partial_{\sigma} \Gamma_{(\theta_1')}^{\mu\nu\alpha} \tau^{\mu\nu} d^3 v - \int_{V_3} \partial x^{\sigma} \partial_{\sigma} F_{(\theta_1')}^{\mu\alpha} j^{\mu} d^3 v \quad (2.5)$$

$$\frac{dX^{\alpha}}{dt} \int_{V_3} \tau^{\beta 4} d^3 v = \int_{V_3} \tau^{\alpha\beta} d^3 v - \frac{d}{dt} \int_{V_3} \partial x^{\alpha} \tau^{\beta 4} d^3 v - \int_{V_3} \partial x^{\alpha} \Gamma_{\mu\nu}^{\beta} \tau^{\mu\nu} d^3 v - \int_{V_3} \partial x^{\alpha} F_{\mu}^{\beta} j^{\mu} d^3 v.$$

In asserting the order of smallness of the terms in the last equations we have made appeal to the assumption (iii). Retaining only up to the third order terms, one obtains from (2.5)

$$\begin{aligned} \frac{d}{ds} \left[\frac{M^{(\alpha 4)}}{U^4} \right] + \Gamma_{(0)}^{\mu\nu\alpha} M^{(\mu\nu)} &= - F_{(0)}^{\alpha} J^{(\mu)}, \\ \frac{U^{\alpha} M^{(\beta 4)}}{U^4} &= M^{(\alpha\beta)}, \end{aligned} \quad (2.6)$$

where, $U^{\alpha} = dX^{\alpha}/ds$, $ds = (g_{\alpha\beta} dX^{\alpha} dX^{\beta})^{1/2}$, and $M^{(\alpha\beta)} = U^4 \int_{V_3} \tau^{\alpha\beta} d^3 v$, $J^{(\mu)} = U^4 \int_{V_3} j^{\mu} d^3 v$. Moreover, writing $M = M^{(44)} (U^4)^{-2}$, from (2.6) the Lorentz equation of motion follows as

$$D(MU^{\alpha}) = - F_{(0)}^{\alpha} J^{(\mu)}. \quad (2.7)$$

This equation of motion for charged particle was also obtained by Chase⁹⁾ from the approach of Infeld and Schild.¹⁰⁾ In the third approximation Eq. (2.7) is valid for any point in the time-constant plane because the point $X^i(t)$ was chosen arbitrarily. (This arbitrariness can be removed by setting $\int_{V_3} \partial x^i \tau^{44} d^3 v = 0$).

§ 3. Equations of motion in the fourth order approximation

The second mean value theorem for $\Gamma_{\mu\nu}^{\alpha}$ and F_{μ}^{α} in the convex domain states that

$$\Gamma_{\mu\nu}^{\alpha}(x^i, t) = \Gamma_{(0)\mu\nu}^{\alpha}(X^i, t) + \partial x^{\sigma} \partial_{\sigma} \Gamma_{(0)\mu\nu}^{\alpha}(X^i, t) + \partial x^{\sigma} \partial x^{\rho} \partial_{\sigma} \partial_{\rho} \Gamma_{(0)\mu\nu}^{\alpha}(X^i + \theta_2 \partial x^i, t), \quad (3.1)$$

$$F_{\mu}^{\alpha}(x^i, t) = F_{(0)\mu}^{\alpha}(X^i, t) + \partial x^{\sigma} \partial_{\sigma} F_{(0)\mu}^{\alpha}(X^i, t) + \partial x^{\sigma} \partial x^{\rho} \partial_{\sigma} \partial_{\rho} F_{(0)\mu}^{\alpha}(X^i + \theta_2' \partial x^i, t).$$

Putting (2.3), (2.4), (3.1) into the set (2.2), one obtains

$$\begin{aligned} \frac{d}{dt} \int_{V_3} \tau^{\alpha 4} d^3 v &= - \int_{V_3} \Gamma_{(0)\mu\nu}^{\alpha} \tau^{\mu\nu} d^3 v - \partial_{\sigma} \Gamma_{(0)\mu\nu}^{\alpha} \int_{V_3} \partial x^{\sigma} \tau^{\mu\nu} d^3 v - \int_{V_3} \partial x^{\sigma} \partial x^{\rho} \partial_{\sigma} \partial_{\rho} \Gamma_{(0)\mu\nu}^{\alpha} \tau^{\mu\nu} d^3 v \\ &\quad 0[3] \quad 0[3] \quad 0[4] \quad 0[5] \\ &\quad - \int_{V_3} F_{(0)\mu}^{\alpha} j^{\mu} d^3 v - \partial_{\sigma} F_{(0)\mu}^{\alpha} \int_{V_3} \partial x^{\sigma} j^{\mu} d^3 v - \int_{V_3} \partial x^{\sigma} \partial x^{\rho} \partial_{\sigma} \partial_{\rho} F_{(0)\mu}^{\alpha} j^{\mu} d^3 v, \quad (3.2a) \\ &\quad 0[3] \quad 0[4] \quad 0[5] \end{aligned}$$

$$\begin{aligned} \frac{dX^{\alpha}}{dt} \int_{V_3} \tau^{\beta 4} d^3 v &= \int_{V_3} \tau^{\alpha\beta} d^3 v - \frac{d}{dt} \int_{V_3} \partial x^{\alpha} \tau^{\beta 4} d^3 v - \int_{V_3} \Gamma_{(0)\mu\nu}^{\alpha} \partial x^{\alpha} \tau^{\mu\nu} d^3 v \\ &\quad 0[3] \quad 0[3] \quad 0[4] \quad 0[4] \\ &\quad - \int_{V_3} \partial x^{\alpha} \partial x^{\sigma} \partial_{\sigma} \Gamma_{(0)\mu\nu}^{\beta} \tau^{\mu\nu} d^3 v - \int_{V_3} F_{(0)\mu}^{\beta} \partial x^{\alpha} j^{\mu} d^3 v - \int_{V_3} \partial x^{\alpha} \partial x^{\sigma} \partial_{\sigma} F_{(0)\mu}^{\beta} j^{\mu} d^3 v, \quad (3.2b) \\ &\quad 0[5] \quad 0[4] \quad 0[5] \end{aligned}$$

$$\begin{aligned} \frac{dX^{\alpha}}{dt} \int_{V_3} \partial x^{\beta} \tau^{\gamma 4} d^3 v + \frac{dX^{\beta}}{dt} \int_{V_3} \partial x^{\alpha} \tau^{\gamma 4} d^3 v &= \int_{V_3} \partial x^{\alpha} \tau^{\gamma\beta} d^3 v + \int_{V_3} \partial x^{\beta} \tau^{\gamma\alpha} d^3 v \\ &\quad 0[4] \quad 0[4] \quad 0[4] \quad 0[4] \\ &\quad - \frac{d}{dt} \int_{V_3} \partial x^{\alpha} \partial x^{\beta} \tau^{\gamma 4} d^3 v - \int_{V_3} \partial x^{\alpha} \partial x^{\beta} \Gamma_{(0)\mu\nu}^{\gamma} \tau^{\mu\nu} d^3 v - \int_{V_3} \partial x^{\alpha} \partial x^{\beta} F_{(0)\mu}^{\gamma} j^{\mu} d^3 v. \quad (3.2c) \\ &\quad 0[5] \quad 0[5] \quad 0[5] \end{aligned}$$

Now, keeping terms up to the fourth order, the last set, after some calculations, yield the equations of motion and spin, respectively, as

$$\begin{aligned} \frac{d}{ds} [M^{(\alpha 4)} / U^4] + \Gamma_{(0)\mu\nu}^{\alpha} M^{(\mu\nu)} - \partial_{\sigma} \Gamma_{(0)\mu\nu}^{\alpha} M^{(\sigma\mu\nu)} &= - F_{(0)\mu}^{\alpha} J^{(\mu)} + \partial_{\sigma} F_{(0)\mu}^{\alpha} \Sigma^{(\sigma\mu)}, \\ \frac{dS^{(\alpha\beta)}}{ds} + U^{\alpha} (U^4)^{-1} \frac{dS^{(\beta 4)}}{ds} - U^{\beta} (U^4)^{-1} \frac{dS^{(\alpha 4)}}{ds} \\ &\quad + [\Gamma_{(0)\mu\nu}^{\alpha} - U^{\alpha} (U^4)^{-1} \Gamma_{(0)\mu\nu}^4] M^{(\beta\mu\nu)} - [\Gamma_{(0)\mu\nu}^{\beta} - U^{\beta} (U^4)^{-1}] M^{(\alpha\mu\nu)} \\ &= [F_{(0)\mu}^{\beta} - U^{\beta} (U^4)^{-1}] \Sigma^{(\alpha\mu)} - [F_{(0)\mu}^{\alpha} - U^{\alpha} (U^4)^{-1} F_{(0)\mu}^4] \Sigma^{(\beta\mu)}, \quad (3.3) \end{aligned}$$

where we have introduced the relations

$$M^{(\alpha\mu\nu)} = M^{(\alpha\mu\nu)} = - U^4 \int_{V_3} \partial x^{\alpha} \tau^{\mu\nu} d^3 v,$$

$$\Sigma^{(\alpha\mu)} = -U^4 \int_{V_3} \partial x^\alpha j^\mu d^3v, \quad (3.4)$$

$$S^{(\alpha\beta)} = -S^{(\beta\alpha)} = \int_{V_3} \partial x^\alpha \tau^{\beta 4} d^3v - \int_{V_3} \partial x^\beta \tau^{\alpha 4} d^3v.$$

§ 4. Covariant forms of motion equations

An arbitrary space-time transformation may be effected by the superposition of two transformations viz., (i) $x^i = x^i(x'^\mu)$, $x^4 = x'^4$, and (ii) $x^i = x'^i$, $x^4 = x^4(x'^\mu)$. Furthermore, restricting ourselves, for the time being, to infinitesimal transformations only, we obtain the following transformation properties for the integrals, correct up to the fourth order approximation,

$$\begin{aligned} M^{(\lambda\alpha\beta)} &= \frac{\partial X^\alpha}{\partial X'^\mu} \frac{\partial X^\beta}{\partial X'^\nu} \left\{ \frac{\partial X^\lambda}{\partial X'^\rho} - U^\lambda (U^4)^{-1} \frac{\partial X^4}{\partial X'^\rho} \right\} M'^{(\rho\mu\nu)}, \\ \Sigma^{(\lambda\alpha)} &= \frac{\partial X^\alpha}{\partial X'^\mu} \left\{ \frac{\partial X^\lambda}{\partial X'^\rho} - U^\lambda (U^4)^{-1} \frac{\partial X^4}{\partial X'^\rho} \right\} \Sigma'^{(\rho\mu)}, \\ M^{(\alpha\beta)} &= \frac{\partial X^\alpha}{\partial X'^\mu} \frac{\partial X^\beta}{\partial X'^\nu} M'^{(\mu\nu)} \\ &\quad - \left\{ \frac{\partial^2 X^\alpha}{\partial X'^\mu \partial X'^\rho} \frac{\partial X^\beta}{\partial X'^\nu} + \frac{\partial X^\alpha}{\partial X'^\mu} \frac{\partial^2 X^\beta}{\partial X'^\nu \partial X'^\rho} \right\} M'^{(\rho\mu\nu)} \\ &\quad + \frac{d}{ds} \left\{ \frac{\partial X^\alpha}{\partial X'^\mu} \frac{\partial X^\beta}{\partial X'^\nu} \frac{\partial X^4}{\partial X'^\rho} (U^4)^{-1} M'^{(\rho\mu\nu)} \right\}, \\ J^{(\alpha)} &= \frac{\partial X^\alpha}{\partial X'^\mu} J'^{(\mu)} - \frac{\partial^2 X^\alpha}{\partial X'^\mu \partial X'^\rho} \Sigma'^{(\rho\mu)} + \frac{d}{ds} \left\{ \frac{\partial X^\alpha}{\partial X'^\mu} \frac{\partial X^4}{\partial X'^\rho} (U^4)^{-1} \Sigma'^{(\rho\mu)} \right\}. \quad (4.1) \end{aligned}$$

One can verify by direct calculations that (4.1) has the group property, so that transformation properties in (4.1) will remain intact for a finite transformation built up from a series of infinitesimal transformations characterised by (4.1). Furthermore, from (4.1) the following integrals, which obey formally the transformation properties of tensor, vector, or scalar, can be obtained:

$$\begin{aligned} S^{\alpha\beta} &= S^{(\alpha\beta)}, \quad J^\mu = J^{(\mu)} + \frac{d}{ds} \Sigma^{(\nu\mu)} U_\nu, \\ \Sigma^{\alpha\beta\gamma} &= -\Sigma^{\beta\alpha\gamma} = U^\alpha \Sigma^{(\beta\gamma)} - U^\beta \Sigma^{(\alpha\gamma)}, \\ A^{\alpha\beta} &= F_{(0)}^{\beta\mu} \Sigma^{(\alpha\mu)} - F_{(0)}^{\alpha\mu} \Sigma^{(\beta\mu)}, \\ M &= U_\alpha (U^4)^{-1} \{ M^{(\alpha 4)} + \Gamma_{(0)}^{\alpha\mu\nu} S^{\mu 4} U_\nu - F_{(0)}^{\alpha\mu} \Sigma^{\mu\nu} U_\nu \}. \quad (4.2) \end{aligned}$$

Introducing these quantities in (3.3), we finally obtain the covariant equations of motion and spin respectively as

$$D(MU^\alpha + U_\beta DS^{\alpha\beta}) + \frac{1}{2} S^{\mu\nu} U^\sigma R_{\nu\sigma\mu}{}^\alpha = -F_{(0)}^\alpha J^\mu - U_\beta \Sigma^{\sigma\beta\nu} \nabla_\sigma F_{(0)}^\alpha + D(U_\beta A^{\alpha\beta}), \quad (4.3)$$

$$DS^{\alpha\beta} + U^\alpha U_\sigma DS^{\beta\sigma} - U^\beta U_\sigma DS^{\alpha\sigma} = U_\sigma \{ \Sigma^{\alpha\sigma\mu} F_{(0)}^\beta - \Sigma^{\beta\sigma\mu} F_{(0)}^\alpha + \Sigma^{\alpha\beta\mu} F_{(0)}^\sigma \}.$$

Eqs. (4.3) reduce to equations of Papapetrou⁷⁾ in the absence of electromagnetic or vector meson fields, which in turn reduce to the results of Mathisson,¹¹⁾ and Lubanski,¹²⁾ in the limit of special relativity. Again, (4.3) can be regarded as the generalization of the works of Corben and Bhabha,⁸⁾ and Bhabha⁹⁾ into general relativity, except for the explicit radiative corrections.

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Some Remarks on the Born-Green-Rodriguez Theory of Condensation*

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We attempt to clarify the origin of the discrepancy in the singular point between the condensation theory of Born-Green-Rodriguez (BGR) using the integral equation method and that of Mayer using the series expansion method. We interpret the BGR theory as the approximate theory in which Mayer's "frame" is replaced by the "ring" (§ 2). In the framework of Mayer's theory (§ 3), it is shown that the maximum point of the BGR isotherm gives the condensation point, and that the characteristic temperature T_m for the BGR gas is absolute zero, and that the singular point of the BGR isotherm is analytically explainable but has no physical meaning (§§ 4 & 5). By applying the results of the author's previous paper to this case, the two-phase separation (the appearance of a "huge" cluster) and the horizontal line (starting from the maximum point of the isotherm) are deduced (§ 6). In connection with the present problem, some problems on the condensation theory are discussed: [1] A note on the integral equation method; 2) the different interpretations of an approximation; the rule of equal areas; van der Waals' equation; 3) the analytical properties of the condensation point; 4) the analytical behaviours of condensing systems; 5) the ideal Bose-Einstein gas] (§ 7).

§ 1. Introduction

From their integral equation Born, Green, and Rodriguez²⁾ (referred to as BGR hereafter) have derived the equation of state** $p = p(v)$ for a system of a very large number of molecules (particles). The isothermal curve*** obtained for a sufficiently low temperature is of the shape given in Fig. 1. They have interpreted the singularity Q on this isotherm as separating the superheated liquid from the supersaturated vapour, and have considered that the horizontal line representing the condensation range should be drawn by the rule of equal areas, just as in van der Waals' equation of state.

On the other hand, according to Mayer's theory^{3), 4), 5)} of condensation, the singularity on the isotherm must appear at the density of the saturated vapour, i. e., at the point (P in Fig. 2) at which the condensation begins. Thus we see a remarkable discrepancy (in position and interpretation of singularity) between

* The contents of this paper were originally read at the meeting of the Physical Society of Japan held at Nagoya University on April 2, 1955 [cf. the abstract book of the meeting, 7, p. 21], and were published in 1957 [reference 1] in Japanese.

** p is the pressure; $v [= V/N]$ is the volume per molecule.

*** Fig. 8 of ref. 2c).

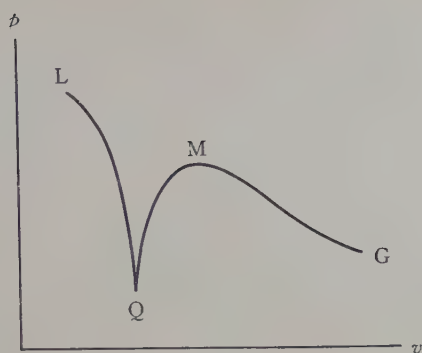


Fig. 1

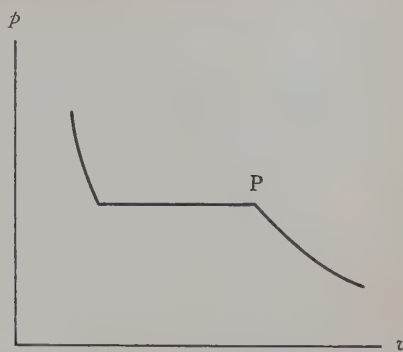


Fig. 2

the theories of Mayer and of BGR.

According to Van Hove's theorem,⁶⁾ the isothermal p - v curves obtained from the rigorous expression of the partition function for the real system should be non-increasing and so can never contain the unstable or even the metastable parts. Hence one may generally consider that the above-mentioned discrepancy arises from the approximations ("superposition" and "linearization") used in the BGR theory.

In this paper, however, with a view to clarifying more essential circumstances underlying this discrepancy, we shall give a detailed analysis of the origin of this discrepancy and the character of the BGR approximation, and especially show how the BGR theory can be interpreted in terms of Mayer's theory. In this connection we shall also discuss some analytical problems on the phase change.

Here we note that, as to Mayer's theory referred to in this paper, we leave the problem of the volume dependence of the cluster integrals out of account; this problem is discussed in another place.

§ 2. The BGR theory and the BGR gas

In the BGR theory,²⁾ an integral equation for the distribution function $n_2(r)$ of two molecules is obtained. Here this integral equation contains Kirkwood's superposition approximation [$n_3(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = n_2(\mathbf{r}_1, \mathbf{r}_2)n_2(\mathbf{r}_2, \mathbf{r}_3)n_2(\mathbf{r}_3, \mathbf{r}_1)/n_1^3$] about the molecular distribution functions and the approximation of linearizing the integral equation. Then one obtains the solution

$$n_2(r) = v^{-2} \exp\{-\phi(r)/\kappa T\} \cdot \left[1 + \frac{1}{2\pi^2} \int_0^\infty \frac{\epsilon^2 t^2 \{g(t)\}^2}{v - \epsilon g(t)} \cdot \frac{\sin(tr)}{tr} dt \right], \quad (1)$$

where $\phi(r)$ represents the pair potential (having a short range) and $g(t)$ is defined by Eq. (6),* and ϵ is a certain suitable constant, introduced in the approx-

* $f(r)$ and $g(t)$ of this paper represent $\alpha(r)$ and $(2\pi)^{3/2}\beta(t)$ of BGR, respectively. κ represents Boltzmann's constant.

imation procedure, and is considered to be of the order of magnitude 1. More rigorously, ϵ is considered as a function of v and T , but this refinement will not essentially affect the qualitative, analytical arguments stated in the following. Eq. (1) is substituted into the formula for the pressure

$$p = \kappa T v^{-1} - \frac{1}{6} \int_0^{\infty} n_2(r) \phi'(r) r \cdot 4\pi r^2 dr, \quad (2)$$

and thus the equation of state is calculated; hence the curve in Fig. 1.

Though the BGR theory covers both the gaseous and the liquid state, we shall here restrict ourselves to the gaseous state. For the gaseous state, i. e., for sufficiently large v , it is shown that (2), in which (1) is substituted for $n_2(r)$, is expressed in the form

$$p = \kappa T v^{-1} \left(1 - \sum_{k=1}^{\infty} \frac{k}{k+1} \beta_k v^{-k} \right), \quad (3)$$

which is nothing but the virial expansion. Here the coefficients β_k are given as follows.

$$\beta_1 = 4\pi \int_0^{\infty} r^2 f(r) dr, \quad (4a)$$

$$\beta_k = (\epsilon^k / 4\pi^2) \int_0^{\infty} t^2 \{g(t)\}^{k+1} dt \quad (k \geq 2), \quad (4b)^*$$

where

$$f(r) = \exp\{-\phi(r)/\kappa T\} - 1, \quad (5)$$

$$g(t) = 4\pi \int_0^{\infty} r^2 f(r) \cdot \frac{\sin(tr)}{tr} dr. \quad (6)$$

On the other hand, in Mayer's theory,^{3),4)} the equation of state for the gaseous state is given in the form of (3), but here the coefficients β_k are the so-called "irreducible cluster integrals"; and the set of $k+1$ molecules composing β_k is called an "irreducible cluster" or a "frame". Mayer's rigorous expression for β_k is

$$\beta_k = \frac{1}{k! V} \int_V \dots \int_V \sum_{k+1 \geq i > j \geq 1} \prod f_{ij} d\tau_1 d\tau_2 \dots d\tau_{k+1} \quad (7)$$

(sum over all products with all molecules more than singly connected)

where $f_{ij} = f(r_{ij}) = \exp\{-\phi(r_{ij})/\kappa T\} - 1$. It is very difficult to make explicit calculations of β_k for all k by using (7).

But the expression (4a, b) is easy to treat. It is an approximation to the

* Eq. (7.6) of ref. 2b).

rigorous expression (7). This approximation is a natural consequence of the approximations contained in the BGR integral equation. It should be remarked that each β_k of (4a, b) [with (6)], apart from the additional factors, represents an integral over a "ring" of $k+1$ molecules [connected by the functions $f(r)$]* (Fig. 3b), except for β_1 (Fig. 3a). [From this it is seen that β_1 and β_2 ($\epsilon=1$) of BGR are rigorous.] Thus we may say that the BGR approximation is such that "frames" are replaced by "rings".**

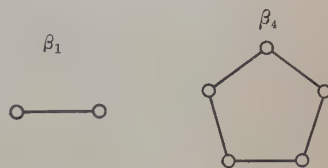


Fig. 3a

Fig. 3b

The hypothetical gas for which β_k is given by (4a, b) and which therefore consists of ring-shaped frames, may be called the "BGR gas." [cf. Fig. 4.]

Incidentally, the "ring-shaped frame" approximation has been used by Montroll and Mayer,⁷⁾ who have confined their discussion in the framework of Mayer's theory^{3),4)} from the beginning, and have directly given the ring integral expression to β_k without employing such an alternative method as the integral equation method. In this way Montroll and Mayer give to β_k the same expression as (4a, b) where $\epsilon=1$.

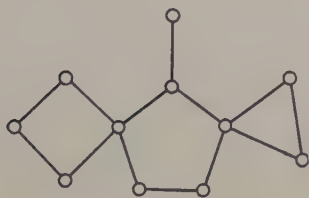


Fig. 4. [A cluster in the BGR gas]

Thus we may consider that, though the method of BGR is different from that of Mayer, the work of BGR is substantially equivalent*** to Montroll-Mayer's work (which is confined in the framework of Mayer's theory), at least if we restrict ourselves to the gaseous state.

Since the analytical structure of Mayer's theory is generally valid irrespective of the explicit expression for β_k , the BGR theory considered in the above way (i. e., considered within the framework of Mayer's theory) cannot conflict with Mayer's theory (e. g., in the properties of the singularity representing the condensation point). However, BGR have obtained the queer result shown in Fig. 1, which apparently conflicts with Mayer's theory. This paradox will be resolved in this paper.

§ 3. On Mayer's theory about the connection between singularity and condensation

Consider the analytic functions G_0 and G_1 defined by (analytic continuation of) the following series:

$$* \beta_k = \lim_{[V \rightarrow \infty]} (\epsilon k / 2V) \int \int_V \cdots \int f(r_{12}) f(r_{23}) \cdots f(r_{k, k+1}) f(r_{k+1, 1}) d\tau_1 d\tau_2 \cdots d\tau_{k+1}.$$

** The dependence of the ring integrals on the volume (V) of the gas container is neglected.

*** Whether $\epsilon=1$ or not, causes no essential difference in the analytical arguments.

$$G_0(v) = \sum_{k=1}^{\infty} \beta_k v^{-k}, \quad (8)$$

$$G_1(v) = \sum_{k=1}^{\infty} k \beta_k v^{-k}. \quad (9)$$

Then, according to Mayer's theory,^{(3),(4)} the starting point of condensation v_s is given by

$$v_s = \max\{v_1, v_0\}, \quad (10)$$

where v_1 is the largest positive real v (if any) for which

$$G_1(v) = 1, \quad (11)$$

and v_0 is the largest positive real v (if any) for which G_0 (and thus G_1) is singular.* It should be noted that v_0 , which is also the (largest positive real) singularity of the analytic function $p(v)$ defined by the series (3) [virial expansion], does not necessarily give the condensation point v_s . For low temperatures ($T \leq T_m$), v_s is certainly given by v_0 ; but, for high temperatures ($T_c > T > T_m$),** we have $v_1 > v_0$, hence v_s is given by v_1 and the singularity v_0 has no physical meaning but has only the mathematical meaning that it is the first singularity of the analytic continuation of the isotherm beyond the condensation point. If $T_c > T > T_m$, the slope of the isotherm is continuous at the point (v_s) of transition between the gaseous curve and the horizontal condensation line.

§ 4. Condensation of the BGR gas from the viewpoint of Mayer's theory

We shall now discuss the BGR theory (stated in § 2) within the analytical framework of Mayer's theory (stated in § 3).

For the BGR gas, we consider the analytic functions $G_0(v)$ and $G_1(v)$ [Eqs. (8) and (9)] where β_k ($k=1, 2, \dots$) is given by (4a, b). Then, from (4b), β_k is known⁽⁷⁾ to become

$$\beta_k \simeq C(k+1)^{-3/2} \beta_1^k \cdot \{1 + O(1/k)\} \quad (12)$$

asymptotically for large k . Here C is a constant independent of k , but may depend on the temperature and the intermolecular forces.

Here it should be noted that in the BGR theory, for all sufficiently low temperatures (such that the condensation occurs), β_1 is positive [owing to the dominance of the positive part of the function $f(r)$] and β_k for $k \geq 2$ is also positive.

From (12) we have

$$\lim_{k \rightarrow \infty} \{\beta_k / (k^{-3/2} \beta_1^k)\} = C(>0). \quad (13)$$

* If R is the radius of convergence of the series (8), (9), or (3), then $R \leq v_0^{-1}$. Especially, if all β_k are positive, then $R = v_0^{-1}$.

** Mayer et al.^{(3c),(d),(e)} suppose the existence of such a finite range $[T_m, T_c]$ of temperatures for the real gas.

Hence, for a given δ ($0 < \delta < C$), there exists a finite positive integer K such that

$$C - \delta \leq \beta_k / (k^{-3/2} \beta_1^k) \leq C + \delta \quad \text{for every } k \geq K. \quad (14)$$

Therefore, putting $c_1 = C - \delta$, $c_2 = C + \delta$, ($0 < c_1 < c_2$), we have

$$\sum_{k=1}^{K-1} \beta_k v^{-k} + c_1 \sum_{k=K}^{\infty} k^{-3/2} \beta_1^k v^{-k} \leq \sum_{k=1}^{\infty} \beta_k v^{-k} \leq \sum_{k=1}^{K-1} \beta_k v^{-k} + c_2 \sum_{k=K}^{\infty} k^{-3/2} \beta_1^k v^{-k}. \quad (15)$$

Hence* the series $\sum_{k=1}^{\infty} \beta_k v^{-k}$ is convergent for $v \geq \beta_1$ and is divergent for $v < \beta_1$; thus the radius of convergence of $\sum_{k=1}^{\infty} \beta_k v^{-k}$ is given by β_1^{-1} . Consequently, the function $G_0(v)$ has a singularity** at $v = \beta_1$ ($\equiv v_0$, say) and this singularity is a branch point and the value of the function at this point is finite. [Fig. 5a.] Similarly, we have from (14)

$$\sum_{k=1}^{K-1} k \beta_k v^{-k} + c_1 \sum_{k=K}^{\infty} k^{-1/2} \beta_1^k v^{-k} \leq \sum_{k=1}^{\infty} k \beta_k v^{-k} \leq \sum_{k=1}^{K-1} k \beta_k v^{-k} + c_2 \sum_{k=K}^{\infty} k^{-1/2} \beta_1^k v^{-k}. \quad (16)$$

Hence the series $\sum_{k=1}^{\infty} k \beta_k v^{-k}$ is convergent for $v > \beta_1$ and is divergent for $v \leq \beta_1$; thus the function $G_1(v)$ has a polar singularity at $v = \beta_1$ ($\equiv v_0$); namely, we have $G_1(v) \rightarrow +\infty$ as $v \rightarrow v_0 + 0$. [Fig. 5b.]

The above statements are true for all sufficiently low temperatures (such that the condensation occurs). Hence, for all such temperatures, it follows that $v_1 > v_0$ [where v_1 is the v for which $G_1(v) = 1$] [see Fig. 5b],*** and that consequently, according to § 3, the starting point of condensation is v_1 , which is different from the singularity (v_0) of G_0 , G_1 . Consequently, for the BGR gas we have $T_c > T > T_m$ for all temperatures below the critical temperature T_c . In other words, for the BGR gas, we have $T_m \leq 0$, that is, in fact, $T_m = 0$.

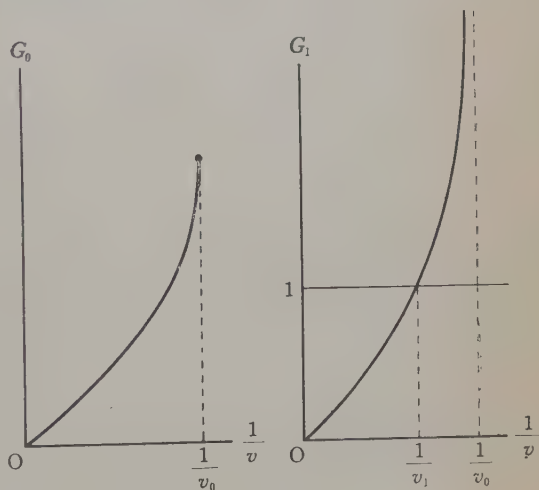


Fig. 5a

Fig. 5b

§ 5. Interpretation of the BGR isotherm

In the preceding section we have obtained the condensation point of the BGR

* Notice that the sum $\sum_{k=1}^{K-1}$ is always finite. Notice that $\sum_{k=1}^{\infty} k^{-3/2}$ converges (while $\sum_{k=1}^{\infty} k^{-1/2}$ diverges).

** Notice that $\beta_k > 0$ for every k . See the first footnote (*) on the preceding page.

*** In order that $v_1 > v_0$, it is sufficient (but not necessary) that $G_1(v_0 + 0) = +\infty$.

gas within the framework of Mayer's theory. We shall now show how the curious shape and the singular point of the BGR isotherm (Fig. 1) can be explained.

According to the analytical behaviour of the function $G_1(v)$, stated in § 4, and using (3) or

$$dp/dv = -\kappa T v^{-2} \{1 - G_1(v)\}, \quad (17)$$

we can obtain the gaseous isotherm and its analytic continuation beyond the condensation point. Thus we obtain a figure which is quite similar to Fig. 1 (obtained by BGR by numerical calculations^{2(c)}), excepting the part QL. The ranges $(0, v_1^{-1})$ and (v_1^{-1}, v_0^{-1}) in Fig. 5b correspond to the parts GM (gaseous state) and MQ of the curve in Fig. 1, respectively. The points v_1^{-1} and v_0^{-1} in Fig. 5b correspond to the maximum point M and the singular point Q of the curve in Fig. 1, respectively. Thus we may put

$$v_1 = v_M, \quad v_0 = v_Q. \quad (18)$$

The fact that $G_1(v_0 + 0) = +\infty$ in Fig. 5b, corresponds to the fact that the slope on the right side at the singularity Q is $+\infty$ in Fig. 1. [cf. (17).]

We will not go into the detailed discussion of the part QL of the BGR isotherm, which may be considered to be an analytic continuation of the curve MQ beyond the singularity Q into the liquid range.

Thus in terms of Mayer's theory we have explained the BGR isotherm; namely, it is regarded as consisting of the gaseous isotherm and its analytic continuation. In such consideration there is no theoretical inconsistency concerning the singular point Q of the BGR isotherm and the condensation point M deduced by Mayer's theory; the point Q is the first singularity in the analytic continuation of the gaseous isotherm beyond M, hence it is explainable in Mayer's theory, but has no physical meaning.

Here it may be noted that the virial expansion (3) is convergent for $v_M > v \geq v_Q$ as well as for $v \geq v_M$ (gaseous state), and is divergent only for $v < v_Q$.

§ 6. Appearance of a "huge" cluster in the BGR gas

Now, for the BGR gas, we can obtain the horizontal part of the isotherm and prove the appearance of the liquid phase, by employing the rigorous proof of condensation given by the author.⁸⁾ Here it may be noted that we have defined the BGR gas as the gas for which the β_k 's in the following rigorous expression of the configurational partition function (\mathcal{Q}_N) are replaced by the ring integrals (4a, b).*

$$\mathcal{Q}_N = \sum_{m_l \geq 0} \left(\sum_{l=1}^N l m_l = N \right) \prod_{l=1}^N \frac{(V b_l)^{m_l}}{m_l!}, \quad (19)$$

where the b_l 's are the so-called "cluster integrals", which in turn are expressed as

* Here we neglect the volume dependence of the b_l 's and the β_k 's.

$$b_l = \sum_{n_k \geq 0} \left(\sum_{k=1}^{l-1} k n_k = l-1 \right) \prod_{k=1}^{l-1} \frac{(V \beta_k)^{n_k}}{n_k!}. \quad (20)$$

The theory of condensation is valid irrespective of the explicit expression for β_k . Thus for $v < v_s$ ($=v_1=v_M$) we have* [by reference 8] (cf. p. 678 of ref. 8b))

$$\ln \Omega \simeq \ln \bar{\varphi} + \ln \sum \bar{\varphi}, \quad (21)$$

where

$$\ln \bar{\varphi} \simeq V \sum_l b_l z_s^l - \bar{N}_1 \ln z_s \text{ (saturated vapour), } [z_s \equiv b_0^{-1}], \quad (22)$$

$$\ln \sum \bar{\varphi} \simeq \ln b_{\bar{N}_2} \simeq \bar{N}_2 \ln b_0 \text{ (liquid drop),} \quad (23)$$

and

$$N = \bar{N}_1 + \bar{N}_2 = \sum_l l V b_l z_s^l + \bar{N}_2. \quad (24)$$

From these formulas, we see (i) that a "huge" cluster ($\ln b_{\bar{N}_2}$) [which represents the liquid phase] appears and coexists with the saturated set of "small" clusters [which represents the saturated vapour] if $v < v_M$, and (ii) that the isotherm for $v < v_M$ is horizontal, whence we deduce the horizontal line (representing the condensation range) drawn from the maximum point M of the BGR isotherm. [See Fig. 6.] Hence there is no discontinuity in slope at the starting point (M) of condensation for every $T(<T_c)$; this is natural since $T_m=0$.

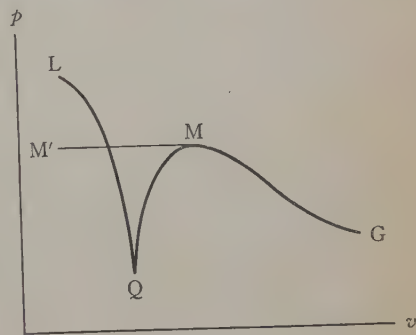


Fig. 6

According to the discussions in § A5 of Appendix A of reference 8b), we have the following expression for the "huge" cluster integral for all sufficiently low temperatures (since $T_m=0$).

$$\ln b_{\bar{N}_2} \simeq \bar{N}_2 \sum_k \beta_k y_0^k - \bar{N}_2 \ln y_0, \quad (25)$$

where y_0 is determined by

$$\sum_{k=1}^{\infty} k \beta_k y_0^k = 1 \quad (26)$$

and the β_k are the ring integrals (4a, b). This means that the "huge" cluster, i. e., the liquid phase (in thermodynamic equilibrium) contains no "huge" frame (ring) but consists of a large number of "small" rings only.

If we considered the volume dependence of the cluster integrals, we should obtain the isotherm for the pure liquid phase. The liquid isotherm thus obtained would not necessarily be identical with the liquid part of the BGR isotherm, ob-

* In these formulas the concise notations and expressions are used. For the corresponding rigorous formulas and their proofs, see reference 8b).

tained as a solution of the integral equation, because the approximation in which the β_k in Eqs. (19) and (20) are replaced by the ring integrals and the approximation of superposition and linearization in the BGR integral equation are certainly equivalent for the gaseous state, but may not necessarily be equivalent for the liquid state.

§ 7. Concluding remarks

In connection with the BGR isotherm discussed above, we shall now give some remarks on the analytical theory of condensation.

1. If one derives rigorously the pressure p as a function of v , one must obtain a stable isotherm of usual shape having a horizontal condensation line. This may be true irrespective of whether one starts from the rigorous expression of the partition function or one uses a rigorous method (if any) of integral equations. In this case, in the integral equations, the possibility of states of two-phase coexistence—that is, states of macroscopically inhomogeneous density—should not be excluded. This may mean that the one-molecule distribution function (or the number density) n_1 is not necessarily constant in space, and the two-molecule distribution function n_2 is not necessarily a function of $r \equiv |\mathbf{r}_2 - \mathbf{r}_1|$ only (but also, of \mathbf{r}_1 or \mathbf{r}_2),* and so on. [The situations may differ according as the gravitation is assumed or not, that is, whether or not the liquid phase is located at the bottom of the vessel (with a horizontal definite boundary between vapour and liquid).]

1a. The BGR integral equation does not satisfy the above requirements, and moreover, contains the approximations of superposition and linearization; hence it is not surprising that an isotherm of curious shape has been obtained as its solution.

2. Now we may consider that, in interpreting the BGR approximation, there are two views:

(I) One interprets the BGR isotherm as an approximation to the isotherm** which would be obtained if one forcibly assumed the existence of only one homogeneous phase for every v . From this viewpoint every part of the isotherm—even the part MQ—has a physical meaning at any rate, though some parts represent metastable and unstable states. Thus we may apply the rule of equal areas to obtain the stable isotherm and hence the horizontal condensation line. [The line AA' in Fig. 7.]

(II) One interprets the BGR approximation as an approximation to the rigorous

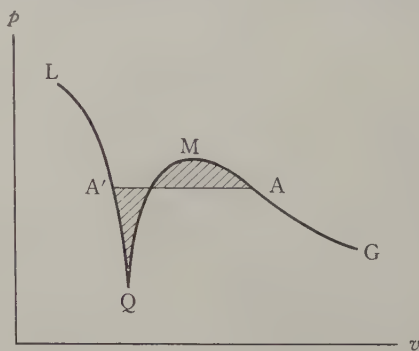


Fig. 7

* \mathbf{r}_1 and \mathbf{r}_2 represent the positions of the two molecules.

** Whether or not the rigorous isotherm thus obtained would really have such a singularity as Q seems to be open to question.

expression for β_k . From this viewpoint the part MQ (and the singularity Q) cannot have any physical meaning, and so the rule of equal areas (which presupposes every part of the isotherm—even the unstable part—to have a physical meaning on assumption of single-phase since for its proof the pressure in every part is used for integration in evaluating the Gibbs free energy) is not applicable. In §§ 4, 5, and 6 of the present paper, we have taken the view (II) and obtained the horizontal condensation line MM'. [Fig. 6.]

2a. Thus we see that by the different methods [(I) and (II)] of interpretation of the approximation, one obtains the different ranges (points) of condensation from the BGR isotherm. [Here we will not go into the discussion of the question which method of interpretation is more suitable from the physical and mathematical viewpoint.] Of course, in the rigorous theory, such a situation does not occur.

2b. For other approximate theories, we can also use the above two methods of interpretation. For example, consider van der Waals' equation

$$(p + a/v^2)(v - b) = \kappa T. \quad (27)$$

If we take the view (I), the usual method applying the rule of equal areas is valid [see the line AA' in Fig. 9]. If we take the view (II), Eq. (27) is written in the form of (3), where

$$\beta_1 = 2\{(a/\kappa T) - b\}, \quad (28a)$$

$$\beta_k = -b^k \cdot (k+1)/k \quad (k \geq 2). \quad (28b)$$

And, for all temperatures below the critical temperature, the condensation point is given by the largest v (say v_1) such that $G_1(v) = \sum_{k=1}^{\infty} k \beta_k v^{-k} = 1$. Here we may

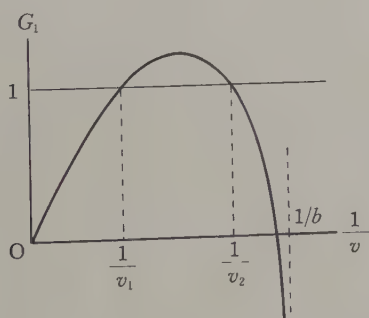


Fig. 8

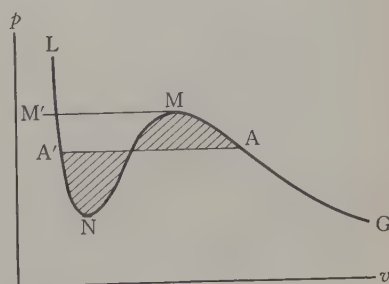


Fig. 9

note that also in this case, $T_m = 0$. [See Fig. 8.] v_1 corresponds to the maximum point M in Fig. 9; (thus $v_1 = v_M$, $v_2 = v_N$); and we obtain the horizontal condensation line MM'.

3. The analytical structure of the theories of BGR and of van der Waals considered from the viewpoint (II) will be realized (in a more general form [cf.

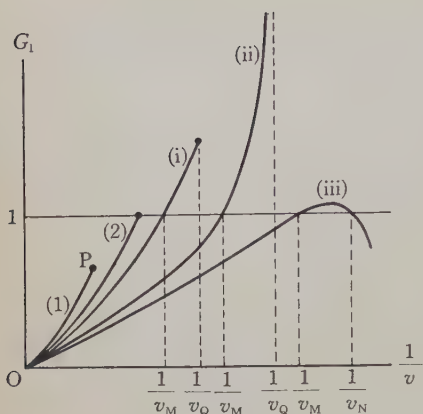


Fig. 10. (See footnote*)

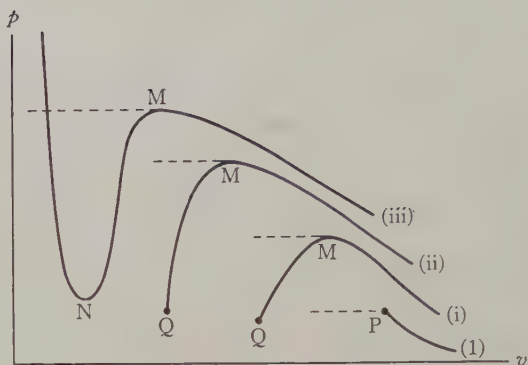


Fig. 11

Figs. 10 and 11, (i)] for $T > T_m$ in the theory of the real gas (cf. § 3). For $T > T_m$, if there is a singularity (v_Q say) of the virial expansion, then v_Q is smaller than the starting point of condensation v_M (which is the maximum point of the curve consisting of the gaseous isotherm and its analytic continuation); thus, for $T > T_m$ the divergence of the virial expansion is *not* connected with condensation. In fact, the condensation point v_M corresponds to the first singularity** (on the positive real axis of z) of the analytic functions $p(z)$ and $v(z)$ defined by the following series:

$$p = \kappa T \sum_{l=1}^{\infty} b_l z^l, \quad (29)$$

$$1/v = \sum_{l=1}^{\infty} l b_l z^l, \quad (30)$$

that is,

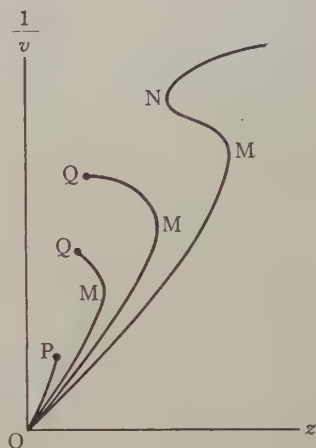


Fig. 12

* In Figs. 10, 11, 12, and 13, the analytical behaviour of G_1 vs. v^{-1} , of p vs. v , of v^{-1} vs. z , and of p vs. z are shown for all (or some) of the following cases:

- | | |
|--|-------------|
| (1): $T < T_m$ (P denoting the first singularity), | } real gas, |
| (2): $T = T_m$, | |
| (i): $T_m < T < T_c$ (general case), | |
| (ii): BGR gas ($T < T_c$) [viewpoint (II)], | |
| (iii): van der Waals ($T < T_c$) [viewpoint (II)]. | |

To economize space, all cases are included in one figure, except for Fig. 13.

** This singularity is "analytical" (in the phraseology defined in reference 10)), since we neglect the volume dependence of the b_l 's in the present paper. For discussions of the question whether the singularity is "analytical" or "non-analytical" in the case of volume-dependent cluster integrals, see references 10) and 11).

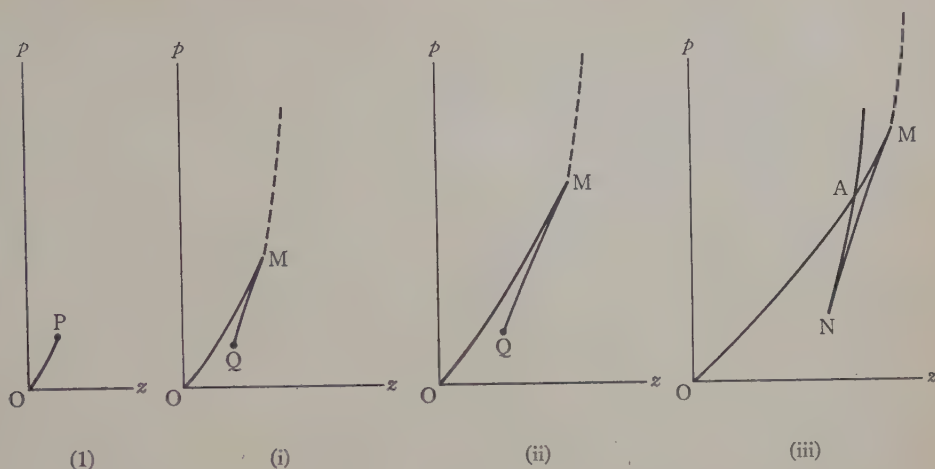


Fig. 13

$$z = v^{-1} \exp \{ -G_0(v) \}, \quad (30')$$

z representing the activity (or fugacity) of the system. But the analytic function $p(v)$ (with its analytic continuation) is regular* at v_M . [Thus the variable v may be said to be a “(locally) uniformizing parameter”; by the transformation of the variable from z to v , the ramified element of the analytic function p changes into a rational element.] A maximum point of a function is of course regular but may be likened to a sort of *singularity* since the point corresponds to an algebraic branch point of the inverse function. v_M is such a point of $p(v)$, and thus corresponds to the condensation point, in the case that $T > T_m$.

4. In the theory of the real gas for $T > T_m$, and in the theories of BGR and of van der Waals considered from the viewpoint (II), the functions $1/v$ and p of z [=activity] (and their analytic continuation) behave in the manner shown in Figs. 12 and (13) [(i), (ii), (iii)] (the solid curves).** Here the point M is an *algebraic* branch point; at M, dp/dz must be equal in the two branches. In this case, if the liquid function [the broken curve in Fig. 13, (i), (ii), (iii)] were obtained, it would, in general, be an analytic function other than the gaseous function, since, at M, dp/dz must be different in the gaseous curve and in the liquid curve. This means that the type (c) in the classification of analytical behaviours, which has been given by the author [ref. 9); p. 378 of ref. 11)], must be realized in this case.***

5. At this point we may refer to the ideal Bose-Einstein gas. In this gas the cluster integrals b_l are expressed by the ring integrals in which the function corresponding to the intermolecular connection (due to the apparent attraction by BE statistics) is given by $f(r) = \exp \{ -(\pi/\lambda^2) r^2 \}$; thus $b_l = \lambda^{3(l-1)/l^{5/2}}$, where $\lambda = h/\sqrt{2\pi m \kappa T}$. This should be compared with the BGR gas in which the β_k are

* Note that also $z(v)$ [Eq. (30')] is regular at v_M .

** Cf. Eq. (30') and $d\{\ln z\}/d\{\ln(v^{-1})\} = 1 - G_1(v)$ and $dp/dz = \kappa T (zv) = \kappa T \exp G_0(v)$.

*** For $T < T_m$, we have not given this conclusion. About the question whether the singularity

P (in Figs. 10-13) is *algebraic* or not, we shall argue in another place.

ring integrals. [For a more detailed comparison between the ideal BE gas and the BGR gas, see § 9 of reference 12).] It is interesting to note that, in the isothermal p - v curve of the ideal BE gas, there is no discontinuity in slope at the point (v_0) of BE condensation for every $T > 0$ [see § 5 of reference 5) for example]; thus we have $T_m = 0$, just as in the case of the BGR gas. This situation is considered to be due to the rather *weak* connections (attractions) between the molecules:—the *ring-shaped* frames (BGR) or the *ring-shaped* clusters (BE). In the theory of the real gas, the temperatures $T > T_m$ are considered to be high enough to make the attraction effects so weak that a case analogous to the BGR case [viewpoint (II)] occurs. From this it is also understandable that the ring approximation is rather good for relatively high temperatures. As to the van der Waals case [viewpoint (II)], it may be noted that, since β_k for $k \geq 2$ is negative, the attraction effects are rather weak for all temperatures, so that $T_m = 0$.

6. In conclusion it should be remarked that, starting from the analysis of the curious shape and the singular point of the BGR isotherm, we have been able to make several suggestive arguments on the condensation phenomena from the analytical viewpoint.

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One-Particle Motions in Many-Particle Systems and the Optical Model in Nuclear Reactions

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A possible scheme of the systematic one-particle motion in a many-particle system is presented from the first principle as a time-dependent formalism. The theory is formulated and interpreted exclusively for the optical model in nuclear reactions, although the present formalism can be utilized to study various problems in solid state physics. First the one-particle amplitude is so defined as to describe the processes of elastic scattering. Then it is shown that the systematic part of the amplitude, corresponding to the coarse-grained motion of the system, obeys the one-particle Schrödinger equation with the optical potential, and that the fluctuating part of the amplitude is governed by the Langevin-like equation with the same optical potential and by the fluctuation-dissipation theorem. This is just the scheme assumed in the previous paper from the semi-phenomenological point of view. The optical potential can be calculated from its definition given as the Fourier transform of the so-called "self-energy" part appearing in the equation of the one-particle Green function in the medium of the target nucleus. From the definition it is easily seen that the optical potential is, in general, non-local and slightly energy-dependent. The optical potential is decomposed into two parts, one being the static (or energy-independent) part to be observed in the target nucleus in the fixed ground state and the other representing reactions of nuclear excitations. It is inferred that the former would not be so different from the corresponding term of the one-particle potential as expected in the ordinary shell model, and that the latter is small due to the average effect originating in the energy spread of the incident beam. The former is purely real, while the latter has an imaginary part which is responsible for the probability dissipation of elastic scattering. The face of the optical potential may be of the same type irrespective of the question whether the incident beam is a simple short wave-packet or a mixed beam, so far as the coarse-grained motions are pursued. Finally it is proved that the fluctuation-dissipation theorem holds for the correlation function of the fluctuating source or amplitude if the system is excited in quasi-equilibrium.

§ 1. Introduction and summary

The one-particle approximation is extensively used to treat various many-particle problems in theoretical physics. As is easily understood, the one-particle approximation should be justified only when for each particle the surrounding particles behave like a continuous medium to its motion. Such a situation is often realized in familiar cases. For example, the Brownian motion is well explained by an illustrative use of the one-particle approximation. In the previous paper,¹⁾ the optical model in nuclear reactions at low energies was discussed in a way analogous to the theory of Brownian motion. There a neutron was regarded as a Brownian

particle moving in a medium formed by surrounding nucleons, under the assumption that the neutron amplitude obeys a Langevin-like equation with the optical potential and the fluctuating source. The purpose of the present paper is to formulate such a scheme of one-particle motion in a many-particle system through derivation of the Langevin-like equation from the first principle. The procedure contains construction of the optical potential for one particle, but detailed calculations will here be left for a forthcoming paper.

The basic idea consists of separation of the averaged (or coarse-grained) one-particle motion from the complicated motion in the many-particle system. The same idea was already formulated to investigate the hydrodynamical model for the meson cloud produced in nucleon-nucleon collisions at super-high energies.²⁾ It may also be of interest to apply the present formalism to the problems in matter physics, for example, solid state physics.³⁾ Throughout this paper, however, we shall exclusively investigate nuclear reactions caused by collisions of a low energy nucleon with a large nucleus. It may be easy to extend the theory so as to deal with problems of some systematic interactions among two or more particles in a medium.

A particle moving in a many-particle system experiences successive collisions with surrounding particles. If the motion of the particle is recorded on a microscopic time scale, we can observe all the possible types of fine interactions, such as virtual or provisional pair-excitations of the medium. In some practical cases, however, we have only to discuss the observation on a more rough time scale, on which it is impossible to record truly the fine variations of motion in a time-interval shorter than a characteristic time. The characteristic time may be considered to be at most of the order of the lifetime of provisional states produced by pair-excitations. On such a time scale we may observe a coarse-grained one-particle motion. This one-particle motion is regarded as the average of microscopic motions over each time-interval whose length is of the order of the characteristic time. Deviations of microscopic motions from the average may be recognized as random fluctuations, so far as each time-interval of the order of the characteristic time can asymptotically be regarded as an instant in observations. Of course, the randomness comes from the large degrees of freedom of the system. However, one must pay close attention to the randomness. If we use a wave packet with a time-length long enough to discriminate the average spacing of energy levels of organized oscillations in the whole system, we could observe a new type of systematic motion of the system (for example, the compound nucleus). Hence we should restrict ourselves to observations in a time-interval

$$\tau_0 \lesssim t \ll T_0$$

for usage of a single wave packet. Here $\Gamma \equiv (\hbar/\tau_0)$ is the width of a "giant resonance" and $\bar{D} \equiv (\hbar/T_0)$ the average spacing of fine-structure levels. Such observations would be realized by making use of a short wave packet with the time-length $t_0 (\tau_0 \ll t_0 \ll T_0)$ or with the energy spread $\Delta E_p \simeq (\hbar/t_0)$. Then one can record

the random fluctuations or some kind of irreversible process, as is interpreted by Hayakawa et al.⁴⁾ and the previous paper. However, it may hardly be acceptable to consider the above short wave packet as the one produced in actual instruments. One may rather prefer considering the actual beam like a mixed beam to regarding it as a single and short wave packet. The mixed beam consists of a random mixture of a number of long wave packets, in which each wave packet has a sharply determined energy but its center of energy distributes over the range ΔE_{μ} . The randomness in the mixed beam comes from the random mixture of wave packets, while the short wave packet shows an apparent random interaction with large systems. In both cases we can expect the same aspect of irreversible processes from discussions given by van Hove and Toda.⁵⁾ At any rate, it becomes necessary for derivation of the coarse-grained one-particle motion to discard or to average the fine interactions in each time-interval ($t \lesssim \tau_0$).^{*} To discard the interval ($t \lesssim \tau_0$) is equivalent to ignoring the high frequency parts.

The coarse-grained motion may come to be of a two-particle motion in a medium. For a while, we shall deal with the case in which the coarse-grained motion can be of a one-particle motion. In such a case, it is most convenient to define the one-particle amplitude $\chi(\mathbf{x}, t)$ so as to describe the coarse-grained one-particle motion. The amplitude $\chi(\mathbf{x}, t)$ would obey the one-particle Schrödinger equation with the so-called optical potential.

In § 2 we formulate the relationships between the one-particle amplitude and the state vector of the scattering problem, by means of the Green function. There it will be shown that the conventional one-particle amplitude (one-particle Feynman amplitude) may be separated into two parts, the one representing the average or coarse-grained one-particle motion of the system and the other the deviations or fluctuations of the amplitude from the average. If the former does not contain the high frequency oscillations and the latter is subject to the fluctuation-dissipation theorem, it will be inferred that the system is specified by the systematic or coarse-grained one-particle motion. There we shall take the averaging procedure into account, although its full use will be given in § 5. Moreover, we shall have to pay attention to the distribution of the target nucleus or the center-of-mass motion of the target nucleus. Its center-of-mass motions are to be connected with the uniformity in space and time of the Green function and the optical potential. In § 3 the equation of the one-particle Green function is constructed by making use of the functional derivatives with respect to the scalar external field φ artificially introduced. One may see that, by introducing φ , formal calculations of the Green function or its self-energy part become much simpler than the customary method using the spinor external field. The method has already been used by some authors^{6), 7)} in quantum

* The choice $\tau_0 = (\hbar/\Gamma)$ is the over-estimated value for the characteristic time. Strictly speaking, the characteristic time must be of the order of the relaxation time τ_1 associated with the quantity η defined in § 5. One may expect that $\tau_0 \gg \tau_1$.

field theory, and it is especially useful in the case of the standard two-particle interaction. The self-energy part will be decomposed into two parts to prepare the static and fluctuation parts of the optical potential. Some formulas of the self-energy part will be obtained in perturbation theory. The various quantities can be represented graphically in a way similar to the method given by Klein and Prange⁸⁾ and others in the case of the uniform media. § 4 is devoted to definition and interpretation of the optical potential. The optical potential is defined by the Fourier transform of the self-energy part.* Its static part is real, and is non-local due to the exchange effect. The fluctuation part has the imaginary part and the non-locality representing the virtual nuclear excitations caused by the incident nucleon. The energy spread of the incident beam has an effect on the observed values of the optical potential through a sort of averaging procedure. § 5 is connected with discussions on the fluctuation part of the amplitude and the Langevin-like equation as assumed in the previous paper. There we should examine the fluctuation-dissipation theorem to be satisfied by the fluctuation part of the amplitude. If the fluctuation part of the amplitude were not subject to the fluctuation-dissipation theorem, the fluctuation part would contain another type of systematic motions different from the one-particle motions and the systematic motions of the system would not be of the one-particle motion. The fluctuation-dissipation theorem may hold if the excited states caused by impact of nucleon can be regarded as in equilibrium in a somewhat long period, in other words, if the compound nucleus is formed and the oscillations with various energies are superposed during a somewhat long interval. Such situations are consistent with the discussions given by Hayakawa et al.⁴⁾ In conclusion we shall briefly illustrate the method by treating the systematic two-particle motion in a medium.

§ 2. One-particle amplitudes and Green functions

In the present paper we shall consider the elastic scattering of a nucleon by a large nucleus with mass number A . Denote simply the Heisenberg state vector of the target nucleus in the ground state by a ket-vector $|A\rangle$ being defined by

$$|A\rangle \equiv \frac{1}{\sqrt{A!}} \iint \cdots \int d^3x_1 d^3x_2 \cdots d^3x_A \psi^*(x_1, t) \psi^*(x_2, t) \cdots \psi^*(x_A, t) |\text{vac}\rangle \\ \times \mathcal{P}_A(x_1, x_2, \cdots x_A, t), \quad (2.1)$$

where $\psi(x, t)$ is the Heisenberg operator of the nucleon field and $\mathcal{P}_A(x_1, x_2, \cdots x_A; t)$ the amplitude of the target nucleus in the ground state. Here the reader must read the letter A in the symbol $|A\rangle$ as indicating the mass number and the ground state of the target nucleus. Now we must pay attention to the center-of-mass

* After preparation of the manuscript, the author found the work by J. S. Bell and E. J. Squires, Phys. Rev. Letter **3** (1959), 96. They have formulated the optical potential by the procedures quite similar to our method. But their letter does not contain discussions about the fluctuation part of the amplitude.

motion of the target nucleus. If the center-of-mass motion were in a state with a definite momentum, in other words, if it were described by a plane wave, then the target nucleus would uniformly distribute over the whole space and, consequently, the incident nucleon would behave as if it were moving in a uniform potential. For a target nucleus bound in a region with atomic size the incident nucleon would move in an apparent potential with some extension of the order of atomic size. At any rate, the center-of-mass motion of the target nucleus is characterized by very low frequencies and, due to the large mass, is never modified by impact of low energy nucleons. Consequently, the center-of-mass of the target nucleus may be treated as if fixed in a very small region of order of the nuclear radius, with the understanding that the spatial average follows. For the fixed center-of-mass of the target nucleus, we certainly observe a non-uniform potential with a size of the order of the nuclear radius. Thus, for our purpose, we must consider that the state $|A\rangle$ is never an eigenstate of the total momentum operator \mathbf{P} , but it can be assumed that the expectation value of \mathbf{P} vanishes in the state $|A\rangle$, that is, $\langle A|\mathbf{P}|A\rangle = 0$. This means that the center of the target wave packet is always at rest. Now we suppose that the center-of-mass motion of the target nucleus is described by a wave packet with the spatial dimensions $\Delta X \sim R_0$, R_0 being the nuclear radius. Since the momentum fluctuation is of the order of $\Delta P \sim (\hbar/R_0)$, the spatial extension becomes of the order of $R_1 \equiv (\hbar/MR_0)t_0$ in a time $t_0 \sim (\hbar/\Delta E_p)$, ΔE_p being the energy spread of the incident beam. Thus we may accept the assumption of the fixed center if $(R_1/R_0) \lesssim 1$. Using the formulas

$$\lambda \equiv (\hbar/p) = 4.5 E_p^{-1/2} \cdot 10^{-13} \text{ cm} \quad (E_p \text{ in Mev}),$$

$$R_0 \equiv 1.47 A^{1/3} \cdot 10^{-13} \text{ cm},$$

one gets

$$(R_1/R_0) \simeq 9 A^{-5/8} \Delta E_p^{-1}.$$

The condition $(R_1/R_0) \lesssim 1$ can be realized in actual cases, so we may always have a wave packet with size $\Delta X \sim R_0$. In contrast with the total momentum, the total energy operator (total Hamiltonian) has the state vector $|A\rangle$ as a member of its approximate eigenvectors. Since the condition $(R_1/R_0) \lesssim 1$ implies the inequality $(\Delta P)^2/(2M) \lesssim \Delta E_p$, the energy fluctuation is smaller than the energy spread of the incident beam. Thus the state $|A\rangle$ can be considered to be an eigenstate of the total Hamiltonian H , if the quantities of the order of ΔE_p are discarded. Thus we obtain the eigenvalue equation

$$H|A\rangle = E_A|A\rangle + O(\Delta E_p).$$

In what follows, we shall choose the zero point of energy to be such that $E_A = 0$. Hence the above equation is rewritten as

$$H|A\rangle = O(\Delta E_p). \quad (2 \cdot 2)$$

Now we consider the scattering states. Denote the scattering state corresponding to a nearly monochromatic beam by the ket-vector $|E_m^\pm; A+1\rangle$, where the symbol \pm stands for the outgoing and ingoing wave condition and $A+1$ means the collision of a single nucleon with a large nucleus having mass number A . As mentioned in § 1, there may be two kinds of incident beams: (i) the short wave packet and (ii) the mixed beam. The system is described in case (i) by the state vector

$$|E_p^+; \hbar; t_0\rangle \equiv \sum_m |E_m^+; A+1\rangle a_m \quad (2.3)$$

or in case (ii) by the density matrix

$$\Xi_p \equiv \sum_m |E_m^+; A+1\rangle w_m \langle E_m^+; A+1|. \quad (2.4)$$

Here both a_m and w_m are smooth functions having non-zero values only in the energy range ΔE_p around the center E_p . Particularly, w_m is the probability of finding the m -th state in the mixed state. For the transition $p \rightarrow q$, we have the transition probability

$$W_{p \rightarrow q} = \langle E_q^- | \Xi_p | E_q^- \rangle = \sum_m |\langle E_q^- | E_m^+; A+1 \rangle|^2 w_m \quad (2.5)$$

in the case (ii). The time evolution of the system is described by

$$|E_p^+; \hbar; t_0\rangle_t = \exp\{(-i/\hbar)(Ht)\} |E_p^+; \hbar; t_0\rangle = \sum_m |E_m^+; A+1\rangle_t a_m$$

or

$$\begin{aligned} \Xi_p(t; t') &= \exp\{(-i/\hbar)(Ht)\} \Xi_p \exp\{(i/\hbar)(Ht')\} \\ &= \sum_m |E_m^+; A+1\rangle_t w_m \langle E_m^+; A+1|, \end{aligned} \quad (2.6)$$

where

$$|E_m^+; A+1\rangle_t \equiv \exp\{(-i/\hbar)(Ht)\} |E_m^+; A+1\rangle.$$

The vector $|E_m^+; A+1\rangle_t$ could be decomposed into two parts, one corresponding to the elastic scattering and the other to the inelastic scattering. We may write this as follows:

$$|E_m^+; A+1\rangle_t = |E_m^+; A+1\rangle_t^{el} + |E_m^+; A+1\rangle_t^{inel},$$

where the superscripts *el* and *inel* stand for the elastic and inelastic parts respectively. Both the vectors $|\dots\rangle_t^{el}$ and $|\dots\rangle_t^{inel}$ must be expressed by some linear combinations of vectors

$$\psi^*|A\rangle, \psi^*\psi\psi^*|A\rangle, \psi^*\psi\psi^*\psi\psi^*|A\rangle, \dots$$

Asymptotically in the remote future, however, the vector $|\dots\rangle_t^{el}$ will become a simple vector of the type $\psi^*|A\rangle$, while the vector $|\dots\rangle_t^{inel}$ will lose the component proportional to this type $\psi^*|A\rangle$ and will be expressed by vectors orthogonal to the vector $\psi^*|A\rangle$. Therefore it may be acceptable to pursue the motions of a nucleon

in elastic scattering by the $\phi^*|A\rangle$ -component of $|E_m^+; A+1\rangle_t$. We therefore define the amplitude

$$\chi_m(\mathbf{q}, t) \equiv \langle A|\phi(\mathbf{q})|E_m^+; A+1\rangle_t = \langle A|\phi(\mathbf{q}, t)|E_m^+; A+1\rangle. \quad (2.7)$$

Here we have considered that the nucleon after scattering will be found in a state of momentum \mathbf{q} and energy E_q equal to the initial energy E_m . The above decomposition is rewritten as

$$\chi_m(\mathbf{q}, t) = \chi_m^{el}(\mathbf{q}, t) + \chi_m^{inel}(\mathbf{q}, t) \quad (2.8)$$

for amplitude, χ_m^{el} and χ_m^{inel} being the $\phi^*|A\rangle$ -components of the vectors $|\dots\rangle_t^{el}$ and $|\dots\rangle_t^{inel}$ respectively. It is natural that the appearance of the vector $|\dots\rangle_t^{inel}$ originates in the pair-excitations caused by impact of the incident nucleon, and that the one-nucleon part of $|\dots\rangle_t^{inel}$ represents the probability amplitude of finding *virtual* pair-excitation. Consequently the amplitude $\chi_m^{inel}(\mathbf{q}, t)$ oscillates with frequencies higher than (E_m/\hbar) by the inverse lifetime of excited states and asymptotically vanishes as t tends to ∞ . In contrast with χ_m^{inel} the amplitude $\chi_m^{el}(\mathbf{q}, t)$ oscillates with frequencies near (E_m/\hbar) and never vanishes in the remote future. Thus one may infer that the time average of χ_m^{inel} can be negligible in comparison with the similar average of χ_m^{el} . Moreover the change in the phase of χ_m^{inel} relative to that of χ_m^{el} may be regarded as random for varying m , so that we obtain

$$\sum_m a_m \chi_m^{inel}(\mathbf{q}, t) = 0, \quad (2.9)$$

or

$$\sum_m a_m \chi_m(\mathbf{q}, t) = \sum_m a_m \chi_m^{el}(\mathbf{q}, t).$$

Now we define the amplitude

$$\chi_p(\mathbf{q}, t) \equiv \langle A|\phi(\mathbf{q}, t)|E_p^+, \hbar/t_0\rangle,$$

which is written in the form

$$\chi_p(\mathbf{q}, t) = \sum_m a_m \chi_m(\mathbf{q}, t) = \sum_m a_m \chi_m^{el}(\mathbf{q}, t).$$

To the approximation $\Delta E_p \simeq 0$ we may write

$$\chi_m^i = \chi_p + O(\Delta E_p).$$

Then the decomposition of χ_m becomes

$$\chi_m = \chi_p + \chi_m^{inel}, \quad (2.10)$$

which is nothing but the form expected in the previous paper¹⁾ in which the functions χ , χ_0 and $\hat{\epsilon}$ should be regarded respectively as χ_m , χ_p and χ_m^{inel} in the present paper. Similar situations can also be observed in the case of a mixed beam. Thus the motions of one nucleon under consideration can be decomposed into the systematic part and the fluctuation part. The systematic part or the coarse-grained motion can be described by the function, in the coordinate representation,

$$\left. \begin{aligned} \chi_p(\mathbf{x}, t) &\equiv \langle A | \phi(\mathbf{x}, t) | E_p^+; \hbar/t_0 \rangle \\ \text{or} \\ A_p(\mathbf{x}, t; \mathbf{x}', t') &\equiv \langle A | \phi(\mathbf{x}, t) \Xi_p \phi^*(\mathbf{x}', t') | A \rangle, \end{aligned} \right\} \quad (2.11)$$

provided that χ_p or A_p does not include rapid oscillations with frequencies higher than the inverse lifetime of virtual or provisional excitation. χ_p or A_p is written in terms of χ_m as follows:

$$\begin{aligned} \chi_p(\mathbf{x}, t) &= \sum_m a_m \chi_m(\mathbf{x}, t) \\ A_p(\mathbf{x}, t; \mathbf{x}', t') &= \sum_m \chi_m(\mathbf{x}, t) \omega_m \chi_m^*(\mathbf{x}', t'), \end{aligned} \quad (2.12)$$

where

$$\chi_m(\mathbf{x}, t) \equiv \langle A | \phi(\mathbf{x}, t) | E_m^+; A+1 \rangle.$$

As suggested by van Hove and Toda,⁵⁾ the functions χ_p and A_p (suppressing the \mathbf{x}' , t' -dependence) would obey the same sort of equation so far as the coarse-grained motions are concerned. For the sake of simplicity we shall formulate the theory by making use of the amplitude χ_p .

Next we shall study the relationships between the amplitude χ_p (or A_p) and the one-particle Green function. First, consider the vector

$$|E_p; \hbar/t_0\rangle = \int d^3\mathbf{x} \int dt \phi^*(\mathbf{x}, t) |A\rangle \varphi_p(\mathbf{x}) \exp\{(-i/\hbar)(E_p t)\} g(t-t_1; t_0), \quad (2.13)$$

where $\varphi_p(\mathbf{x})$ is the wave packet function of the incident nucleon. $g(t-t_1; t_0)$ is a function slowly varying in the region $|t-t_1| \lesssim t_0$ and vanishing in the region $|t-t_1| \gtrsim t_0$, and is characterized by

$$\int g(t-t_1; t_0) dt = 1, \quad \frac{dg}{dt} \simeq \frac{1}{t_0} g.$$

Now it is quite easy to prove that the vector $|E_p; \hbar/t_0\rangle$ represents an eigenstate of H belonging to the eigenvalue E_p with the accuracy $\Delta E_p \simeq (\hbar/t_0)$. That is to say, we get

$$(H - E_p) |E_p; \hbar/t_0\rangle = O(\Delta E_p). \quad (2.14)$$

Moreover, it can be easily shown that the vector $|E_p; \hbar/t_0\rangle$ does not depend on t_1 with the same accuracy, because

$$i\hbar \cdot d/dt_1 |E_p; \hbar/t_0\rangle = (E_p - H) |E_p; \hbar/t_0\rangle + O(\Delta E_p) = O(\Delta E_p). \quad (2.15)$$

To specify g corresponds to a choice of the boundary conditions. Here consider the functions $g^{(\pm)}$ defined by

$$\begin{aligned} g^{(+)}(t-t_1; t_0) &= \begin{cases} (1/t_0) \exp\{(1/t_0)(t-t_1)\} & t < t_1 \\ 0 & t > t_1, \end{cases} \\ g^{(-)}(t-t_1; t_0) &= \begin{cases} 0 & t < t_1 \\ (1/t_0) \exp\{(-1/t_0)(t-t_1)\} & t > t_1. \end{cases} \end{aligned}$$

For $g^{(+)}$, we obtain the vector $|E_p^+; \hbar/t_0\rangle$ subject to the outgoing-wave boundary condition. This will be seen in the following way: The expression (2.13) can be rewritten as

$$\begin{aligned} |E_p^+; \hbar/t_0\rangle &= (1/t_0) \int_{-\infty}^{t_1} dt \exp\{(i/\hbar)(H-E_p)t + (1/t_0)(t-t_1)\} |\varphi_p\rangle \\ &= \exp\{(i/\hbar)(H-E_p)t\} \frac{(\hbar/t_0)}{i(H-E_p) + (\hbar/t_0)} |\varphi_p\rangle, \end{aligned}$$

where we have used the equation

$$\phi(\mathbf{x}, t) = \exp\{(i/\hbar)(Ht)\} \phi(\mathbf{x}) \exp\{(-i/\hbar)(Ht)\} \quad (2.16)$$

and the abbreviation

$$|\varphi_p\rangle \equiv \int d^3\mathbf{x} \phi^*(\mathbf{x}) |A\rangle \varphi_p(\mathbf{x}). \quad (2.17)$$

$|\varphi_p\rangle$ is the initial state of the system. Applying the operator $\exp[-(i/\hbar)(H-E_p)t_1]$ to the above equation from the left and using (2.14), one gets the equation

$$\begin{aligned} |E_p^+; \hbar/t_0\rangle &= \frac{(\hbar/t_0)}{i(H-E_p) + (\hbar/t_0)} |\varphi_p\rangle + O(\Delta E_p) \\ &= |\varphi_p\rangle + \frac{1}{E_p - H_0 + i(\hbar/t_0)} H_I |E_p^+; \hbar/t_0\rangle + O(\Delta E_p), \end{aligned} \quad (2.18)$$

where $H = H_0 + H_I$. Eq. (2.18) is nothing but the Lippmann-Schwinger equation for the scattering state subject to the outgoing-wave condition and $(\hbar/t_0) \simeq \Delta E_p$ certainly means the energy spread in the wave packet state.⁹⁾ This is just what was expected when we wrote Eq. (2.13). On the other hand we easily obtain the vector $|E_p^-; \hbar/t_0\rangle$ subject to the incoming-wave condition from the choice of $g^{(-)}$. Thus we can write the amplitude χ_p as

$$\begin{aligned} \chi_p(\mathbf{x}, t) &= \int d^3\mathbf{x}' \int_{-\infty}^{t_1} dt' \langle A | \phi(\mathbf{x}, t) \phi^*(\mathbf{x}', t') | A \rangle \\ &\quad \times \varphi_p(\mathbf{x}') \exp\{(-i/\hbar)(E_p t')\} g^{(+)}(t' - t_1; t_0). \end{aligned}$$

This expression suggests to us the definition of the one-particle Green function in the medium A as follows:

$$G(x, x') \equiv (i\hbar)^{-1} \langle A | T(\phi(x) \phi^*(x')) | A \rangle, \quad (2.19)$$

where $T(\dots)$ stands for the well-known time ordering product and x or x' is a space-time point. In terms of G , the amplitude χ_p is rewritten as

$$\chi_p(\mathbf{x}, t) = i\hbar \int d^3\mathbf{x}' \int_{-\infty}^{t_1} dt' G(x, x') \varphi_p(\mathbf{x}') \exp\{(-i/\hbar)(E_p t')\} g^{(+)}(t' - t_1; t_0), \quad (2.20)$$

provided that the time t_1 is chosen in the remote past. Hence, if G obeys the equation of the type

$$\mathcal{D}_{x,t} G(\mathbf{x}, t; \mathbf{x}', t') = \delta(\mathbf{x} - \mathbf{x}') \delta(t - t'), \quad (2.21)$$

the equation of χ_p becomes

$$\mathcal{D}_{x,t} \chi_p(\mathbf{x}, t) = 0. \quad (2.22)$$

From the definition (2.8) we can see the time-dependence of χ_p in the following form:

$$\chi_p(\mathbf{x}, t) = \exp\{(-i/\hbar)(E_p t)\} \chi_p(\mathbf{x}, 0) + O(\Delta E_p). \quad (2.23)$$

In terms of χ_p or G , we can write the element of the S -matrix for the transition from p to q as follows:

$$\begin{aligned} S_{pq} &= \langle E_q^-; \hbar/t_0 | E_p^+; \hbar/t_0 \rangle = \int d^3\mathbf{x} \int_{t_1}^{\infty} dt \exp\{(i/\hbar)(E_q t)\} g^{(-)}(t - t_1; t_0) \varphi_q^*(\mathbf{x}) \chi_p(\mathbf{x}, t) \\ &= (i\hbar) \int d^3\mathbf{x} \int_{t_1}^{\infty} dt \int_{-\infty}^{t'} d^3\mathbf{x}' \int_{-\infty}^{t'} dt' \exp\{(i/\hbar)(E_q t)\} g^{(-)}(t - t_1; t_0) \varphi_q^*(\mathbf{x}) \\ &\quad \times G(\mathbf{x}, \mathbf{x}') \varphi_p(\mathbf{x}') \exp\{(-i/\hbar)(E_p t')\} g^{(+)}(t' - t_1'; t_0). \end{aligned}$$

As is well-known, the Green function with time $t < t'$ can be used for the description of hole-propagation in the medium A . Detailed discussions of the one-particle Green function will be given in the next section. Here we discuss a slight modification of the definition for the Green function in the medium whose state is not a pure state but a mixed state. Then it is plausible to use the definition

$$\begin{aligned} G(\mathbf{x}, \mathbf{x}') &\equiv (i\hbar)^{-1} \sum_A b_A \langle A | T(\phi(\mathbf{x}) \phi^*(\mathbf{x}')) | A \rangle / \sum_A b_A \langle A | A \rangle \\ &= (i\hbar)^{-1} \text{Tr} \{ \rho T(\phi(\mathbf{x}) \phi^*(\mathbf{x}')) \} / \text{Tr} \{ \rho \}, \end{aligned} \quad (2.24)$$

where ρ is the density matrix describing the state of the medium, that is,

$$\rho \equiv \sum_A |A\rangle b_A \langle A|, \quad (2.25)$$

b_A being the statistical weight assigned to the state A . The definition (2.24) has already been used by the present author²⁾ and others.¹⁰⁾ This is useful to investigate the systematic motions or microscopic motions in macroscopic bodies such as gases, liquids, solids and nuclear matter with infinite extension. In what follows, we shall often write the expectation value of an operator Q in a pure or mixed state by the unique symbol $\langle Q \rangle$, that is,

$$\begin{aligned} \langle Q \rangle &\equiv \langle A | Q | A \rangle / \langle A | A \rangle \\ &\equiv \text{Tr} \{ \rho Q \} / \text{Tr} \{ \rho \}. \end{aligned} \quad (2.26)$$

For example, we simply write (2.19) and (2.24) as

$$G(x, x') = (i\hbar)^{-1} \langle T(\phi(x) \phi^*(x')) \rangle. \quad (2.27)$$

Similarly the many-particle Green functions can be defined by

$$\left. \begin{aligned} G_{II}(x_1, x_2; x'_1, x'_2) &\equiv (i\hbar)^{-2} \langle T(\phi(x_1) \phi(x_2) \phi^*(x'_1) \phi^*(x'_2)) \rangle, \\ G_{III}(x_1, x_2, x_3; x'_1, x'_2, x'_3) &\equiv (i\hbar)^{-3} \\ &\times \langle T(\phi(x_1) \phi(x_2) \phi(x_3) \phi^*(x'_1) \phi^*(x'_2) \phi^*(x'_3)) \rangle, \\ &\dots\dots\dots \\ &\dots\dots\dots \end{aligned} \right\} \quad (2.28)$$

They may describe a kind of systematic motions among two or more particles occurring in a many-particle system, just as the one-particle Green function may be associated with a systematic (or coarse-grained) one-particle motion in a medium. In fact, the many-nucleon Green function may be utilized in problems containing the surface interactions or some direct interactions in nuclear reactions, provided that one identifies the closed shell of the target nucleus with a medium and regards the incident nucleon and extra nucleons in the target nucleus as particles in a systematic motion. In various problems of transport phenomena, it is well known that the two-particle Green function is most useful for the calculation of transport coefficients.

Finally, we shall explain a qualitative relation between the space-time variation of the Green function and the uniformity of the medium. Differentiating the Green function with respect to \mathbf{x} , one gets

$$\nabla G(\mathbf{x}, t; \mathbf{x}', t') = -\nabla' G(\mathbf{x}, t; \mathbf{x}', t') + \hbar^{-2} \langle [T(\phi(\mathbf{x}, t) \phi^*(\mathbf{x}', t')), \mathbf{P}] \rangle. \quad (2.29)$$

Here we used the equation

$$\nabla \phi(\mathbf{x}, t) = (i, \hbar) [\mathbf{P}, \phi(\mathbf{x}, t)], \quad (2.30)$$

\mathbf{P} being the total momentum operator

$$\mathbf{P} \equiv \int \phi^*(\mathbf{x}, t) (\hbar/i) \nabla \phi(\mathbf{x}, t) d^3\mathbf{x}. \quad (2.31)$$

If the state $|A\rangle$ were an eigenstate \mathbf{P} or the density matrix ρ commuting with \mathbf{P} , in other words, if the medium were uniform, the last term of (2.29) would vanish and the function G would satisfy the equation

$$(\nabla + \nabla') G(\mathbf{x}, t; \mathbf{x}', t') = 0. \quad (2.32)$$

Eq. (2.32) means that G is a function depending only on the difference $\mathbf{x} - \mathbf{x}'$ with respect to the spatial coordinates, that is to say, a uniform function irrelevant to the origin of the coordinate system. Returning to our problem for nuclear reactions in which the state $|A\rangle$ is not an eigenstate of \mathbf{P} , Eq. (2.32) never holds and consequently, the function G is not uniform. On the other hand, we can regard G as a uniform function with respect to the time coordinates, as long as we neglect quantities of the order of ΔE_n , for we obtain the relation

$$\begin{aligned} \frac{\partial}{\partial t} G(\mathbf{x}, t; \mathbf{x}', t') &= (i\hbar)^{-1} \delta^{(4)}(\mathbf{x} - \mathbf{x}') + (i\hbar)^{-1} \left\langle T \left\{ \frac{\partial \psi(\mathbf{x}, t)}{\partial t} \psi^*(\mathbf{x}', t') \right\} \right\rangle \\ &= - \frac{\partial}{\partial t'} G(\mathbf{x}, t; \mathbf{x}', t') + O(\Delta E_p), \end{aligned} \quad (2.33)$$

where Eq. (2.14) was used. (The delta function $\delta^{(4)}(\mathbf{x} - \mathbf{x}') \equiv \delta(\mathbf{x} - \mathbf{x}') \delta(t - t')$ in the first member of (2.33) comes from the jump of G at its discontinuous point $t = t'$.) Thus we can treat G as a function of $t - t'$ alone with respect to the time variables.

§ 3. One-particle Green function

We shall restrict ourselves to systems described by the Lagrangian density

$$\begin{aligned} L \equiv i\hbar \psi^*(x) \frac{\partial \psi(x)}{\partial t} - \frac{\hbar^2}{2m} \nabla^2 \psi^*(x) \cdot \nabla \psi(x) \\ - \frac{1}{2} \int \psi^*(x) \psi^*(x') U(x - x') \psi(x') \psi(x) d^4 x', \end{aligned} \quad (3.1)$$

where $d^4 x' \equiv d^3 \mathbf{x}' dt'$ and $U(x - x') \equiv V(\mathbf{x} - \mathbf{x}') \delta(t - t')$, $V(\mathbf{x} - \mathbf{x}')$ being the potential for the two-particle interaction. Moreover, we assume that $U(x - x')$ is a symmetric function of x and x' . From the Lagrangian density (3.1), we get the Hamiltonian

$$\begin{aligned} H = \int \psi^*(x) \left[- \frac{\hbar^2}{2m} \nabla^2 \right] \psi(x) d^3 \mathbf{x} \\ + \frac{1}{2} \iint \psi^*(x) \psi^*(x') U(x - x') \psi(x') \psi(x) d^4 x' d^3 \mathbf{x} \end{aligned} \quad (3.2)$$

and the field equation

$$i\hbar \frac{\partial}{\partial t} \psi(x) = \left[- \frac{\hbar^2}{2m} \nabla^2 + \int \psi^*(x') U(x - x') \psi(x') d^4 x' \right] \psi(x). \quad (3.3)$$

For the sake of mathematical convenience we may introduce a c -number external field $\varphi(x)$ by adding the term

$$- \psi^*(x) \psi(x) \varphi(x) \quad (3.4)$$

to the Lagrangian density. This results in the modification

$$- \frac{\hbar^2}{2m} \nabla^2 \rightarrow - \frac{\hbar^2}{2m} \nabla^2 + \varphi(x)$$

in (3.2) and (3.3). Thus $\psi(x)$ obeys the operator equation

$$i\hbar \frac{\partial \psi(x)}{\partial t} = \left[- \frac{\hbar^2}{2m} \nabla^2 + \varphi(x) + \int \psi^*(x') U(x - x') \psi(x') d^4 x' \right] \psi(x). \quad (3.5)$$

The external field $\varphi(x)$ must be switched on adiabatically at the beginning and

off at the end of the process. Although $\varphi(x)$ never modifies the specifications of the state of the medium, the Hamiltonian turns out to be a time-dependent operator due to the occurrence of φ . Hence we distinguish between the initial state $|A_I\rangle$ and the final state $|A_F\rangle$ of the medium, and then understand the fact that $\langle A_F|A_I\rangle \neq 1$. Now we know Schwinger's dynamical principle¹¹⁾

$$\delta \langle A_F|A_I \rangle = (i/\hbar) \langle A_F | \delta \int L d^4x | A_I \rangle, \quad (3.6)$$

from which we get the useful formulas

$$\left. \begin{aligned} \frac{\delta}{\delta \varphi(x)} \langle A_F|A_I \rangle &= (i\hbar)^{-1} \langle A_F | \psi^*(x) \psi(x) | A_I \rangle, \\ \frac{\delta}{\delta \varphi(x)} \langle A_F | Q | A_I \rangle &= (i\hbar)^{-1} \langle A_F | T \{ Q \psi^*(x) \psi(x) \} | A_I \rangle, \end{aligned} \right\} \quad (3.7)$$

Q being an operator. In the presence of φ , the definition (2.26) of $\langle Q \rangle$ should be modified as follows:

$$\begin{aligned} \langle Q \rangle &\equiv \langle A_F | Q | A_I \rangle / \langle A_F | A_I \rangle \\ &\equiv \text{Tr} \{ \rho Q \} / \text{Tr} \{ \rho \}, \end{aligned} \quad (3.8)$$

or

where

$$\rho \equiv \sum_A |A_I\rangle b_A \langle A_F|.$$

Thus one gets the formula

$$i\hbar \frac{\delta}{\delta \varphi(x)} \langle Q \rangle = \langle T \{ Q \psi^*(x) \psi(x) \} \rangle - \langle Q \rangle \langle \psi^*(x) \psi(x) \rangle. \quad (3.9)$$

Now we proceed to derive the equation for the one-particle Green function. Substituting (3.3) into the first member of (2.33), we immediately obtain the equation

$$\begin{aligned} \left[i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 \right] G(x, x') &= \delta^{(4)}(x - x') \\ &+ (i\hbar)^{-1} \int d^4x'' U(x - x'') \langle T(\psi^*(x'') \psi(x'') \psi(x) \psi^*(x')) \rangle \end{aligned}$$

for the one-particle Green function. The last term on the right-hand side can be written in terms of the two-particle Green function, that is

$$\begin{aligned} \left[i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 \right] G(x, x') &= \delta^{(4)}(x - x') \\ &+ i\hbar \int d^4x'' U(x - x'') G_{II}(x'', x; x'', x'). \end{aligned} \quad (3.10)$$

Thus the one-particle Green function is coupled with the two-particle Green function. Similarly the equation of G_{II} includes the one-particle Green function G and the three-particle Green function G_{III} . In general, the equation of the N -particle Green

function G_N is participated with the $(N+1)$ -particle Green function. Hence we have arrived at the set of simultaneous equations for an infinite number of Green functions, $G, G_{II}, G_{III}, \dots$. We may as well obtain, in a compact form, the equation to be satisfied by the one-particle Green function alone, by eliminating other Green functions. This procedure may be achieved by means of functional differentiation with respect to $\varphi(x)$ introduced at the beginning of the present section.

Denote the Green function in the external field φ by G_φ . G_φ obeys the equation

$$\left[i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 - \varphi(x) \right] G_\varphi(x, x') = \delta^{(4)}(x - x') \\ + (i\hbar)^{-1} \int d^4 x'' U(x - x'') \langle T(\psi^*(x'') \psi(x'') \psi(x) \psi^*(x')) \rangle.$$

Using (3.9), we can rewrite the last term on the right-hand side as follows:

$$(i\hbar)^{-1} \int d^4 x'' U(x - x'') \langle T(\psi^*(x'') \psi(x'') \psi(x) \psi^*(x')) \rangle \\ = \bar{V}_\varphi(x) G_\varphi(x, x') + i\hbar \int d^4 x'' U(x - x'') \frac{\delta}{\delta \varphi(x'')} G_\varphi(x, x'),$$

where we have used the abbreviation

$$\bar{V}_\varphi(x) \equiv \int d^4 x'' U(x - x'') \langle \psi^*(x'') \psi(x'') \rangle \\ = -i\hbar \int d^4 x'' U(x - x'') G_\varphi(x'', t''; x'', t'' + 0). \quad (3.11)$$

Hence the equation for G_φ becomes

$$\left[i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 - \varphi(x) - \bar{V}_\varphi(x) \right. \\ \left. - i\hbar \int d^4 x'' U(x - x'') \frac{\delta}{\delta \varphi(x'')} \right] G_\varphi(x, x') = \delta^{(4)}(x - x'). \quad (3.12)$$

This includes only the one-particle Green function. Here it must be remembered that the function $\varphi(x)$ is to be put equal to zero everywhere after all calculations have been completed. The function defined by

$$\bar{V}(x) \equiv \lim_{\varphi \rightarrow 0} \bar{V}_\varphi(x) = \int \langle A | \psi^*(\mathbf{x}'', t) \psi(\mathbf{x}'', t) | A \rangle V(\mathbf{x} - \mathbf{x}'') d^3 \mathbf{x}'' \\ = \sum_{i=1}^A \int \mathcal{P}_A^*(\mathbf{x}_1, \dots, \mathbf{x}_A) V(\mathbf{x} - \mathbf{x}_i) \mathcal{P}_A(\mathbf{x}_1, \dots, \mathbf{x}_A) d^3 \mathbf{x}_1 \dots d^3 \mathbf{x}_A, \quad (3.13)$$

is interpreted as the *static average potential** for a nucleon in the target nucleus whose state is always fixed. It is easy to prove that $\bar{V}(x)$ is real. $\bar{V}(x)$ would

* It is readily shown that $\bar{V}(x)$ is independent of t as long as we neglect quantities of the order of ΔE_p .

naturally be reduced to a constant independent of x if G were uniform, in other words, if the target nucleus were in an eigenstate of the total momentum operator. This is immediately proved from the definition. On the other hand, the term

$$\lim_{\varphi \rightarrow 0} i\hbar \int d^4 x'' U(x-x'') \frac{\partial}{\partial \varphi(x'')} G_{\varphi}(x, x') \quad (3.14)^*$$

represents the *exchange effects* between the incident nucleon and the core nucleon or among core nucleons and the *pair-excitation effects* of the nucleus caused by impact of the incident nucleon, because the operation $\partial/\partial \varphi(x)$ means the creation of a nucleon-hole pair at x . The term (3.14) is to be regarded as the result of elimination of many-particle Green functions in the set of equations, in which their participation means possible exchanges among nucleons and possible excitations of nucleus. Such a situation (or effects of $(\partial/\partial \varphi)$) is well visualized by introducing the graphical method⁽¹²⁾ developed in quantum field theory, as will be seen later.

In order to introduce the graphical method analogous to relativistic vacuum field theories, it is of essential importance to utilize extensively the fact that the annihilation operator ψ can be interpreted as a creation operator of a hole in the medium when applied to the state vector of the medium. This resembles the creation operator for anti-particles in relativistic vacuum field theory. However, there is an essential difference between relativistic vacuum field theory and the present theory. In the present theory ψ cannot always be interpreted as a creation operator of a hole, because the medium has only a finite number of degrees of freedom. Hence it must be understood that the complete parallelism between both theories holds if we discard the terms vanishing at the limit of infinite freedom of the medium.

For the purpose of formulating the graphical method, we first assume the existence of the inverse function $G_{\varphi}^{-1}(x, x')$ satisfying the relations

$$\left. \begin{aligned} \int G_{\varphi}^{-1}(x, x'') G_{\varphi}(x'', x') d^4 x'' &= \delta^{(4)}(x-x'), \\ \int G_{\varphi}(x, x'') G_{\varphi}^{-1}(x'', x') d^4 x'' &= \delta^{(4)}(x-x'). \end{aligned} \right\} \quad (3.15)$$

In other words, the inverse function G_{φ}^{-1} is nothing but the operator applied to G_{φ} , although it must be written as an integral operator. Now define the quantities $\langle \varphi(x) \rangle$ and $A_{\varphi}(x, x')$ by

$$\left. \begin{aligned} \langle \varphi(x) \rangle &\equiv \varphi(x) + \bar{V}_{\varphi}(x) - \bar{V}(x), \\ A_{\varphi}(x, x') &\equiv \int d^4 x'' U(x-x'') \frac{\partial \langle \varphi(x') \rangle}{\partial \varphi(x'')} \end{aligned} \right\} \quad (3.16)$$

* It may be worth while to remark that the Green function is free from divergences because the functional differentiation occurs only in the integral with the kernel $U(x-x')$. In contrast with this, the equation of the Green function contains the corresponding term $\lim_{\varphi \rightarrow 0} (\partial/\partial \varphi(x)) G_{\varphi}(x, x')$ in the cases of some divergent field theories.

Then Eq. (3.12) becomes

$$\left[i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 - \bar{V}(x) - \langle \varphi(x) \rangle \right. \\ \left. - i\hbar \int d^4x'' \mathcal{A}_\varphi(x, x'') \frac{\delta}{\delta \langle \varphi(x'') \rangle} \right] G_\varphi(x, x') = \delta^{(4)}(x - x'). \quad (3.17)$$

The functions $\langle \varphi(x) \rangle$ and $\mathcal{A}_\varphi(x, x')$ are regarded respectively as the external field and the two-particle potential modified by interaction with the medium. Differentiating one of (3.16) with respect to $\langle \varphi(x) \rangle$, we immediately obtain the formula

$$\frac{\delta G_\varphi(x, x')}{\delta \langle \varphi(x'') \rangle} = - \iint G_\varphi(x, \xi) \frac{\delta G_\varphi^{-1}(\xi, \zeta)}{\delta \langle \varphi(x'') \rangle} G_\varphi(\zeta, x') d^4\xi d^4\zeta. \quad (3.18)$$

Replacing the functional derivative in (3.17) with (3.18), one can find the explicit form of the integral operator G_φ^{-1} as follows:

$$G_\varphi^{-1}(x, x') = \left[i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 - \bar{V}(x) - \langle \varphi(x) \rangle \right] \delta^{(4)}(x - x') - \Pi_\varphi(x, x'), \quad (3.19)$$

where we have used the abbreviation

$$\Pi_\varphi(x, x') \equiv i\hbar \iint d^4x'' d^4\xi \mathcal{A}_\varphi(x, x'') G_\varphi(x, \xi) \Gamma_\varphi(\xi, x'; x''), \quad (3.20)$$

$\Gamma_\varphi(\xi, x'; x'')$ being defined by

$$\Gamma_\varphi(\xi, x'; x'') \equiv - \frac{\delta G_\varphi^{-1}(\xi, \zeta)}{\delta \langle \varphi(x'') \rangle}. \quad (3.21)$$

Here let us define the quantity Σ_φ by

$$\Sigma_\varphi(x, x') \equiv \bar{V}(x) \delta^{(4)}(x - x') + \Pi_\varphi(x, x'). \quad (3.22)$$

Then G_φ^{-1} is rewritten as

$$G_\varphi^{-1}(x, x') = \left[i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 - \langle \varphi(x) \rangle \right] \delta^{(4)}(x - x') - \Sigma_\varphi(x, x'), \quad (3.23)$$

from which we find the relation

$$\Gamma_\varphi(\xi, \zeta; x'') = \delta^{(4)}(\xi - \zeta) \delta^{(4)}(\xi - x'') + \frac{\delta \Sigma_\varphi(\xi, \zeta)}{\delta \langle \varphi(x'') \rangle}. \quad (3.24)$$

For \mathcal{A}_φ and $\langle \varphi \rangle$ we obtain the equations

$$\mathcal{A}_\varphi(x, x') = U(x - x') - i\hbar \iiint d^4x'' d^4\xi' d^4\xi'' d^4\zeta' \mathcal{A}_\varphi(x, \xi') \\ \times \Gamma_\varphi(\xi', \xi''; \zeta') G_\varphi(\xi'', x'') G_\varphi(x'', \xi') U(x'' - x') \quad (3.25)$$

and

$$\langle \varphi(x) \rangle = \varphi(x) - i\hbar \int d^4x'' U(x - x'') [G_\varphi(x'', t''; x'', t'' + 0) - G(x'', t''; x'', t'' + 0)] \quad (3.26)$$

from (3.11) and (3.16). Therefore we have formulated a set of five equations in five unknowns, G_φ , Δ_φ , Σ_φ , Γ_φ and $\langle\varphi\rangle$, which is rewritten in matrix form as follows:

$$\left[p_0 - \frac{1}{2m} \mathbf{p}^2 - \langle\varphi\rangle - \Sigma_\varphi\right] G_\varphi = 1, \quad (3.27a)$$

$$\Sigma_\varphi = -i\hbar U G_\varphi + i\hbar \Delta_\varphi G_\varphi \Gamma_\varphi, \quad (3.27b)$$

$$\Delta_\varphi = U - i\hbar \Delta_\varphi \Gamma_\varphi G_\varphi U, \quad (3.27c)$$

$$\Gamma_\varphi = 1 + \frac{\partial \Sigma_\varphi}{\partial \langle\varphi\rangle}, \quad (3.27d)$$

$$\langle\varphi\rangle = \varphi - i\hbar U (G_\varphi - G), \quad (3.27e)$$

where we used the matrix representations such as $\langle x|p_0|x'\rangle \equiv (i\hbar\partial/\partial t)\delta^{(4)}(x-x')$, $\langle x|\mathbf{p}|x'\rangle \equiv (\hbar/i)\nabla\delta^{(4)}(x-x')$, $\langle x|\Sigma_\varphi|x'\rangle \equiv \Sigma_\varphi(x, x')$, $\langle x|G_\varphi|x'\rangle \equiv G_\varphi(x, x')$ and so on.

All the quantities needed are obtained from the φ -dependent functions by the limit operation $\varphi \rightarrow 0$. Denote every function at the limit $\varphi=0$ by the same letter without subscript φ , for example,

$$\Gamma(\xi, \zeta; x'') \equiv \lim_{\varphi \rightarrow 0} \Gamma_\varphi(\xi, \zeta; x'').$$

We then obtain the equation

$$\left[i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \nabla^2\right] G(x, x') - \int d^4 x'' \Sigma(x, x'') G(x'', x') = \delta^{(4)}(x-x') \quad (3.28)$$

for the one-particle Green function, where

$$\left. \begin{aligned} \Sigma(x, x') &= \bar{V}(x) \delta^{(4)}(x-x') + \Pi(x; x'), \\ \Pi(x, x') &= i\hbar \int d^4 x'' d^4 \xi \Delta(x, x'') G(x, \xi) \Gamma(\xi, x'; x''). \end{aligned} \right\} \quad (3.29)$$

Evidently the function Π includes the term

$$\Pi^{ee}(x, x') \equiv i\hbar U(x-x') G(x, x') \quad (3.30)$$

which is interpreted as the exchange effect between two nucleons corresponding to the first term \bar{V} in Σ . This fact will be seen from the comparison of the two terms

$$\left. \begin{aligned} \bar{V}(x) G(x, x') &= -i\hbar \int d^4 \xi U(x-\xi) G(\xi, \xi) G(x, x'), \\ \int \Pi^{ee}(x, \xi) G(\xi, x') d^4 \xi &= i\hbar \int d^4 \xi U(x-\xi) G(x, \xi) G(\xi, x'). \end{aligned} \right\} \quad (3.31)$$

Then we divide Σ into two parts as follows:

$$\Sigma(x, x') \equiv \Sigma^S(x, x') + \Sigma^F(x, x'), \quad (3.32)$$

where

$$\left. \begin{aligned} \Sigma^s(x, x') &\equiv \bar{V}(x) \delta^{(4)}(x-x') + \Pi^{ex}(x, x'), \\ \Sigma^F(x, x') &\equiv \Sigma(x, x') - \Pi^{ex}(x, x'). \end{aligned} \right\} \quad (3.33)$$

The meaning or role of the division will later be clarified in the graphical representation and in construction of the optical potential.

Now we can explain the graphical representation. $G(x, x')$ is graphically represented by drawing a heavy solid line, connecting points x and x' , with an arrow toward x from x' . According as $t > t'$ or $t < t'$, the line corresponds to a nucleon line or a hole line in a medium.

$\Delta(x, x')$ is represented by a heavy broken line without an arrow connecting x and x' . $\Gamma(\hat{\xi}, \hat{\xi}; x)$ is the vertex part with an outgoing nucleon point $\hat{\xi}$, an incoming nucleon point $\hat{\xi}$ and a force point x . The unperturbed functions G_0 and $\Delta_0 = U$ are, respectively, represented by a fine solid line and a fine broken line. Now we can draw a graph

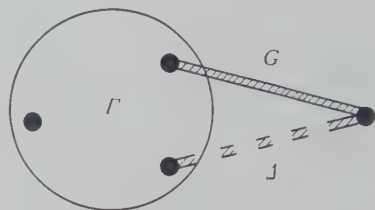


Fig. 1. The graph of Π

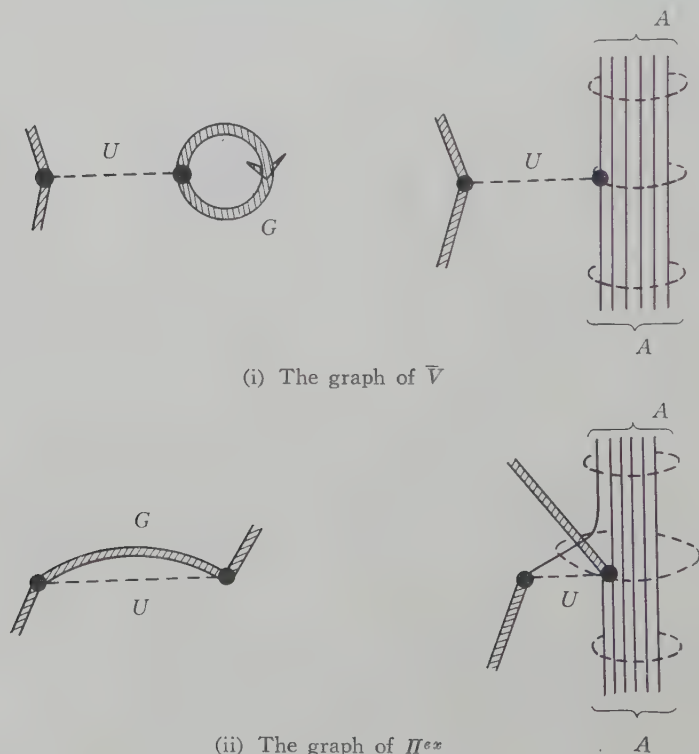


Fig. 2. The graphs of Σ^s

The right figures in (i) and (ii) show explicitly all lines of the medium particles.

corresponding to Π in a way suggested by its definition (3.29). Fig. 1 shows the graph of Π . This has the structure similar to the proper self-energy graph of an electron in quantum electrodynamics. The terms of Σ^s are shown in Fig. 2, in which the above-mentioned exchange is well visualized. There the graphs showing the medium particles are also drawn.

If we had complete knowledge about the wave function belonging to the ground state of the target nucleus, we could calculate Σ^s and might use it as the starting point of the perturbation theory. However, it is difficult to prepare the correct wave function of the target nucleus, so we should proceed to the perturbation calculations of G together with solving the ground state of the target nucleus. First we assume that the wave functions of the system have been found for the approximate Hamiltonian H_0 defined by

$$H_0 \equiv \int \psi^*(x) \left[-\frac{\hbar^2}{2m} \nabla^2 \psi(x) \right] d^3x + \iint \psi^*(x) \Xi(x, x') \psi(x') d^4x' d^3x, \quad (3.34)$$

where $\Xi(x, x')$ is the non-local potential appropriately chosen. In this case the one-particle Green function G_0 obeys the equation

$$\left[i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 \right] G_0(x, x') - \int d^4x'' \Xi(x, x'') G_0(x'', x') = \delta^{(4)}(x - x'). \quad (3.35)$$

If the function G_0 is used as the starting point of perturbation theory, it is convenient to rewrite the equation of G in the following integral equation

$$G(x, x') = G_0(x, x') + \iint d^4x'' d^4x''' G_0(x, x'') [\Sigma'(x'', x''') - \Xi(x'', x''')] G(x''', x') \quad (3.36)$$

or symbolically,

$$G = G_0 + G_0 [\Sigma - \Pi] G. \quad (3.37)$$

The convergence or accuracy of the perturbation theory essentially relies on the choice of Ξ . If in passing of the incident beam the target nucleus is almost always in its ground state, we may consider the static approximation to be good and choose Ξ as

$$\Xi = \Sigma^s[G_0]. \quad (3.38)$$

This may be regarded as simple application of the Hartree-Fock method to our problem. For this choice we should solve the equation

$$\left[p_0 - \frac{1}{2m} \mathbf{p}^2 + \Sigma^s[G_0] \right] G_0 = 1, \quad (3.39)$$

for example, by the successive iteration. The effects of the fixed center-of-mass of the target nucleus are, in practice, taken into account through the explicit form of $\Sigma^s[G_0]$ prepared at the first stage of this iteration. Besides (3.38), we have other

possibilities of choosing Ξ , for example, ones suggested by Brueckner's method. Detailed discussions will be postponed for a forthcoming paper. At any rate we can write the Green function G_0 in the form

$$G_0(\mathbf{x}, t; \mathbf{x}', t') = \left. \begin{aligned} &= \frac{1}{i\hbar} \sum_n \chi_n^{(0)}(\mathbf{x}) \exp\{-(i/\hbar)E_n^{(0)}(t-t')\} \chi_n^{(0)*}(\mathbf{x}') \\ &= -\frac{1}{i\hbar} \sum_{\bar{n}} \chi_{\bar{n}}^{(0)*}(\mathbf{x}) \exp\{(i/\hbar)E_{\bar{n}}^{(0)}(t-t')\} \chi_{\bar{n}}^{(0)}(\mathbf{x}') \end{aligned} \right\} \begin{array}{l} t > t', \\ t < t', \end{array} \quad (3.40)$$

where the functions $\chi_n^{(0)}$ and $\chi_{\bar{n}}^{(0)}$ are, respectively, the amplitudes of a nucleon in the n -th state and of a hole in the \bar{n} -th state determined by H_0 . They can be defined by

$$\left. \begin{aligned} \chi_n^{(0)}(\mathbf{x}) &\equiv \langle \Phi_A | \psi(\mathbf{x}) | \Phi_{n,A+1} \rangle, \\ \chi_{\bar{n}}^{(0)}(\mathbf{x}) &\equiv \langle \Phi_A | \psi^*(\mathbf{x}) | \Phi_{\bar{n},A+1} \rangle, \end{aligned} \right\} \quad (3.41)$$

where Φ_A is the vector representing the ground state of the A -particle system and $\Phi_{n,A+1}$ the n -th state of the $(A+1)$ -particle system governed by H_0 .

The perturbation series of \mathcal{A} is immediately obtained as follows:

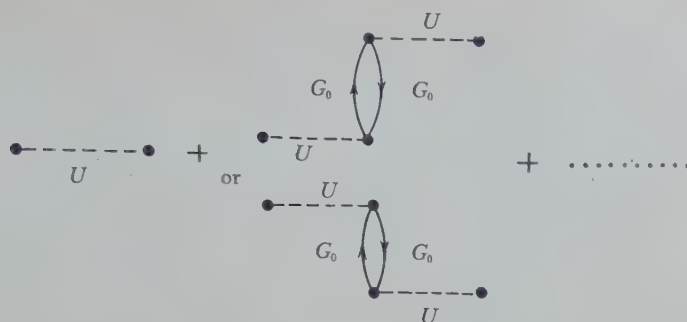
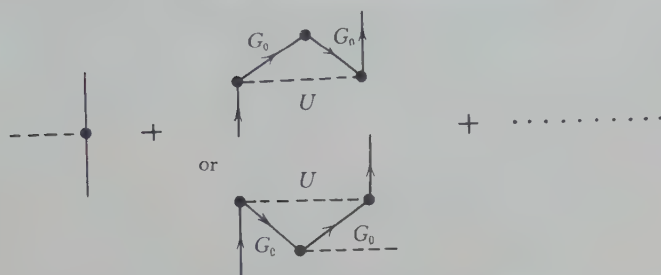
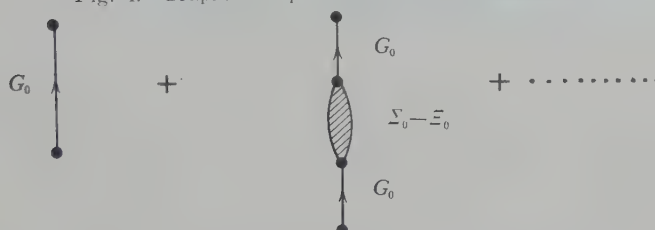
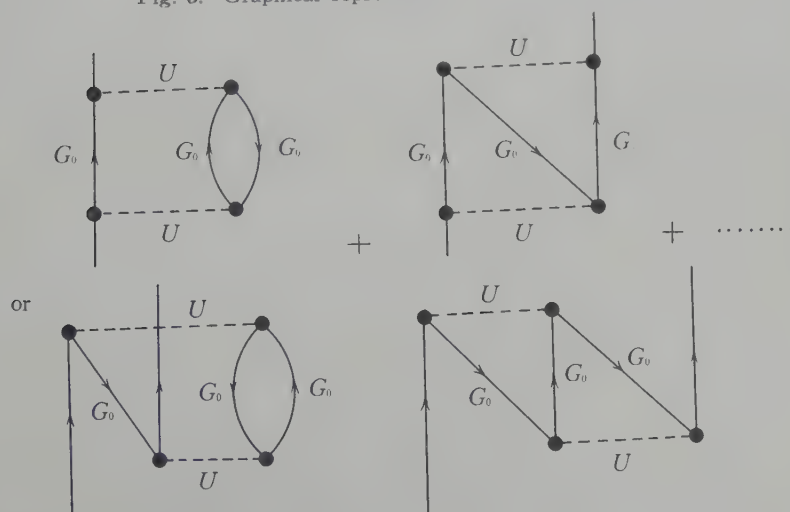
$$\begin{aligned} \mathcal{A}(x, x') &= U(x-x') - (i\hbar) \iint d^4x'' d^4\zeta' U(x'-x'') G_0(x'', \zeta') G_0(\zeta', x'') U(x-\zeta') \\ &+ \dots \end{aligned} \quad (3.42)$$

Similarly we get

$$\begin{aligned} \Gamma(\xi, \zeta; x'') &= \delta^{(4)}(\xi-\zeta) \delta^{(4)}(\xi-x'') + i\hbar U(\xi-\zeta) G_0(\xi, x'') G_0(x'', \zeta) \\ &+ \dots \end{aligned} \quad (3.43)$$

The graphs corresponding to the perturbation series of \mathcal{A} are shown in Fig. 3 and the graphs of Γ in Fig. 4. The series produced by iteration of (3.36) or (3.37) is graphically represented as in Fig. 5, but the concrete form of $(\Sigma - \Xi)$ depends on the choice of Ξ . The graphs of Σ^S can simply be drawn by substituting the graphs of G into the heavy solid line in Fig. 2. Finally we write the perturbation series for Σ^F

$$\begin{aligned} \Sigma^F(x, x') &= -(i\hbar)^2 \iint d^4x'' d^4\zeta' U(x'-x'') G_0(x'', \zeta') G_0(\zeta', x'') U(x-\zeta') G_0(x, x') \\ &+ (i\hbar)^2 \iint d^4x'' d^4\zeta U(x-x'') G_0(x, \xi) U(\xi-x') G_0(\xi, x'') G_0(x'', x') \\ &+ \dots \end{aligned} \quad (3.44)$$


 Fig. 3. Graphical representation of the series of A

 Fig. 4. Graphical representation of the series of F

 Fig. 5. Graphical representation of the series of G

 Fig. 6. Graphical representation of the series of Σ^F

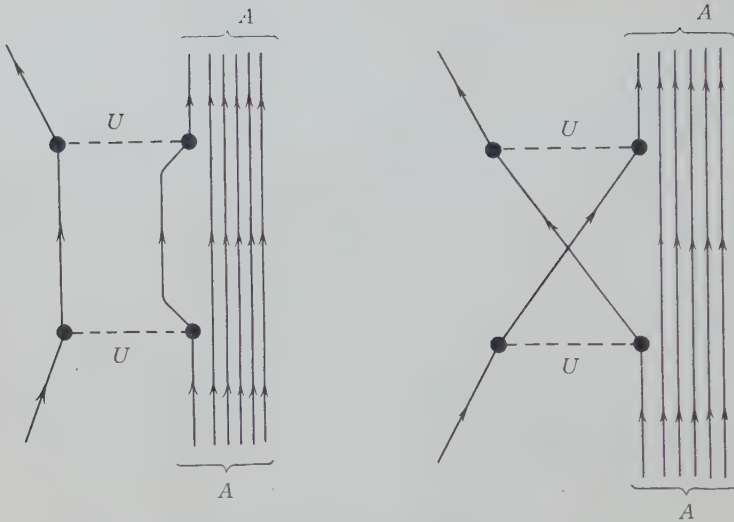


Fig. 7. Interpretation of the first two terms of (3.44)

which is graphically represented in Fig. 6. In particular, the first two terms are interpreted as the exchange-pair between two excited nucleons or holes. See Fig. 7. As understood from Fig. 6 or Fig. 7, the function Σ^F describes all the reactions to the one-nucleon (or hole) motions from virtual or provisional excitations caused by impact of the nucleon.

Before concluding this section, it may be of interest to remark the resemblance of the equation

$$\left[i\hbar \int d^4x'' \Delta_\varphi(x, x'') \frac{\delta}{\delta \langle \varphi(x'') \rangle} \right] G_\varphi(x, x') = \int d^4\zeta \Pi_\varphi(x, \zeta) G_\varphi(\zeta, x') \quad (3.45)$$

to the ordinary eigenvalue problem of a differential operator. The ordinary eigenvalue problem is characterized by the equation

$$Ly = \lambda y$$

and a boundary condition imposed on y , where L is a differential operator, λ its eigenvalue and y its eigenfunction belonging to λ . There λ and y depend sensitively on the boundary condition. Corresponding to different boundary conditions, one must find different sets of eigenvalues and eigenfunctions of the same operator L . Here equate the operator $i\hbar \int d^4x'' \Delta_\varphi(x, x'') (\delta/\delta \langle \varphi(x'') \rangle)$ with L , Π_φ with λ , and G_φ with y , respectively. Then Π_φ may be regarded as an eigenvalue of a functional differential operator. In fact, Eq. (3.17) always has the same form whatever may be the medium in which the particle moves and whatever boundary conditions may be imposed on G_φ . However, the equation, after replacement (3.45), is sensitive to the boundary conditions or the state of the medium. Therefore the replacement (3.45) may be regarded as a sort of eigenvalue equation accompanied with an appropriate boundary condition.

§ 4. Optical potential

In the present section we shall formulate the optical potential to be observed in nuclear reactions. The basic idea has already been suggested in the previous paper¹⁾ and explained in the preceding sections of the present paper.

In last section we have derived the equation

$$\left[i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 \right] G(x, x') - \int d^4 x'' \Sigma(x, x'') G(x'', x') = \delta^{(4)}(x - x') \quad (4.1)$$

for the one-nucleon Green function. From the discussions given in § 2, it is easily shown that the amplitude $\chi_p(\mathbf{x}, t)$, describing the coarse-grained motion of one nucleon in the elastic scattering under consideration, obeys the equation

$$\left[i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 \right] \chi_p(\mathbf{x}, t) - \int d^4 x'' \Sigma(x, x'') \chi_p(\mathbf{x}'', t') = 0. \quad (4.2)$$

Here we recall that the function $\Sigma(x, x')$ is a function of the difference $(t - t')$ with respect to the time variables and the amplitude $\chi_p(\mathbf{x}, t)$ oscillates harmonically in time so far as all quantities of the order of $\Delta E_p \simeq (\hbar/t_0)$ can be discarded. Consequently, the last term in the left-hand side of (4.2) can be reduced to the form

$$\int d^4 x'' \Sigma(x, x'') \chi_p(\mathbf{x}'', t'') = \int d^3 \mathbf{x}'' \langle \mathbf{x} | \mathcal{V}_p | \mathbf{x}'' \rangle \chi_p(\mathbf{x}'', t), \quad (4.3)$$

where the function $\langle \mathbf{x} | \mathcal{V}_p | \mathbf{x}' \rangle$ is defined by

$$\langle \mathbf{x} | \mathcal{V}_p | \mathbf{x}' \rangle \equiv \int_{-\infty}^{\infty} dt' \exp \{ (i/\hbar) E_p(t - t') \} \Sigma(x, x'). \quad (4.4)$$

Thus Eq. (4.2) can be rewritten as

$$\left[i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 \right] \chi_p(\mathbf{x}, t) - \int d^3 \mathbf{x}' \langle \mathbf{x} | \mathcal{V}_p | \mathbf{x}' \rangle \chi_p(\mathbf{x}', t) = 0. \quad (4.5)$$

This form of the equation permits us to interpret the function $\langle \mathbf{x} | \mathcal{V}_p | \mathbf{x}' \rangle$ as the optical potential for one nucleon. Therefore the optical potential is, in general, non-local and also dependent on the energy of the incident particle, and is time-independent with the accuracy ΔE_p . Only when the wave-length of the incident nucleon is much longer than the characteristic length for non-locality (in other words, the correlation length), we can treat \mathcal{V}_p as the local potential

$$\mathcal{V}_p(\mathbf{x}) \simeq \int d^3 \mathbf{x}_0 \langle \mathbf{x} - \frac{1}{2} \mathbf{x}_0 | \mathcal{V}_p | \mathbf{x} + \frac{1}{2} \mathbf{x}_0 \rangle. \quad (4.6)$$

As seen in the last section, the non-locality of the optical potential comes from exchange effects among nucleons and virtual or provisional excitations of the target. Neglecting the exchange effects, we may treat \mathcal{V}_p as if it were local, when the static Hartree field is a good approximation and is a slowly varying function over the region of the order of the nucleon de Broglie wave-length. In uniform media the optical potential becomes uniform, that is, a function of $\mathbf{x} - \mathbf{x}'$ alone.

First we discuss the properties of \mathcal{V}_p^N , corresponding to Σ^N defined by (3.33), which is regarded as the contribution of the target in the fixed state. From (3.33) and (4.4) we immediately get the formula

$$\langle \mathbf{x} | \mathcal{V}_p^s | \mathbf{x}' \rangle = \bar{V}(\mathbf{x}) \delta(\mathbf{x} - \mathbf{x}') + i\hbar G(\mathbf{x}, 0; \mathbf{x}', 0) V(\mathbf{x} - \mathbf{x}'). \quad (4.7)$$

Therefore the static part \mathcal{V}_p^s never depends on the energy of the incident nucleon. Furthermore, it is easy to prove the hermitian property of \mathcal{V}_p^s , that is

$$\langle \mathbf{x} | \mathcal{V}_p^s | \mathbf{x}' \rangle = \{ \langle \mathbf{x}' | \mathcal{V}_p^s | \mathbf{x} \rangle \}^*. \quad (4.8)$$

The hermitian property of \mathcal{V}_p^s tells us the facts that \mathcal{V}_p^s is never responsible for probability dissipation of the amplitude of the elastic scattering, in other words, for the inelastic scattering or formation of any compound nucleus. Such properties of \mathcal{V}_p^s show that \mathcal{V}_p^s is worthy to be called the *static* part of the optical potential.

The non-hermitian property of the optical potential originates in the fluctuation part Σ^F defined by (3.33). Denote the contribution of Σ^F to the optical potential by \mathcal{V}_p^F . The *fluctuation* part \mathcal{V}_p^F is, in general, decomposed into the hermitian part \mathcal{V}_p^{Fr} and the anti-hermitian part \mathcal{V}_p^{Fi} , that is,

$$\mathcal{V}_p^F \equiv \mathcal{V}_p^{Fr} - i\mathcal{V}_p^{Fi}. \quad (4.9)$$

Both the matrices \mathcal{V}_p^{Fr} and \mathcal{V}_p^{Fi} are hermitian. In order to illustrate the concrete form of \mathcal{V}_p^F , we shall calculate the function $\langle \mathbf{x} | \mathcal{V}_p^F | \mathbf{x}' \rangle$ to the lowest order in the perturbation theory formulated in the last section. Substituting (3.33) into the definition of \mathcal{V}_p^F and using the explicit forms (3.40) and (3.44), we obtain, after some calculation, the contribution of the first two terms of (3.44) to the optical potential as follows:

$$\begin{aligned} \langle \mathbf{x} | \mathcal{V}_p^F | \mathbf{x}' \rangle &= (i\hbar)^{-1} \sum_{lm\bar{n}} \int_{-\infty}^t dt' \exp \{ (i/\hbar) (E_p - E_l^{(0)} - E_{\bar{m}}^{(0)} - E_n^{(0)}) (t - t') \} F_{lm\bar{n}}(\mathbf{x}) F_{lm\bar{n}}^*(\mathbf{x}') \\ &- (i\hbar)^{-1} \sum_{lm\bar{n}} \int_t^{\infty} dt' \exp \{ (i/\hbar) (E_p + E_l^{(0)} + E_{\bar{m}}^{(0)} + E_n^{(0)}) (t - t') \} F_{lm\bar{n}}^*(\mathbf{x}) F_{lm\bar{n}}(\mathbf{x}'), \end{aligned} \quad (4.10)$$

where we have used the abbreviations

$$\left. \begin{aligned} F_{lm\bar{n}}(\mathbf{x}) &\equiv \int d^3\mathbf{x}'' V(\mathbf{x} - \mathbf{x}'') \chi_m^{(0)}(\mathbf{x}'') \chi_{ln}^{(0)}(\mathbf{x}, \mathbf{x}''), \\ F_{lm\bar{n}}(\mathbf{x}) &\equiv \int d^3\mathbf{x}'' V(\mathbf{x} - \mathbf{x}'') \chi_m^{(0)}(\mathbf{x}'') \chi_{ln}^{(0)}(\mathbf{x}, \mathbf{x}''). \end{aligned} \right\} \quad (4.11)$$

Here $\chi_{ln}^{(0)}$ and $\chi_{l\bar{n}}^{(0)}$ are, respectively, defined by

$$\left. \begin{aligned} \chi_{ln}^{(0)}(\mathbf{x}, \mathbf{x}'') &\equiv \frac{1}{\sqrt{2}} \{ \chi_l^{(0)}(\mathbf{x}) \chi_n^{(0)}(\mathbf{x}'') - \chi_n^{(0)}(\mathbf{x}) \chi_l^{(0)}(\mathbf{x}'') \}, \\ \chi_{l\bar{n}}^{(0)}(\mathbf{x}, \mathbf{x}'') &\equiv \frac{1}{\sqrt{2}} \{ \chi_l^{(0)}(\mathbf{x}) \chi_{\bar{n}}^{(0)}(\mathbf{x}'') - \chi_{\bar{n}}^{(0)}(\mathbf{x}) \chi_l^{(0)}(\mathbf{x}'') \}. \end{aligned} \right\} \quad (4.12)$$

The functions $F_{lm\bar{n}}$ and $F_{lm\bar{n}}$ have the properties

$$F_{l\bar{m}\bar{n}} = -F_{\bar{n}\bar{m}l}, \quad F_{\bar{l}m\bar{n}} = -F_{\bar{n}m\bar{l}} \quad (4.13)$$

due to the anti-symmetry of $\chi_{in}^{(0)}$ and $\chi_{\bar{i}\bar{n}}^{(0)}$. To evaluate the time integrals in (4.10), we can use the formulas

$$\begin{aligned} (i\hbar)^{-1} \int_{-\infty}^t dt' \exp\left[\frac{i}{\hbar} a(t-t')\right] &= -i\pi\delta(a) + P\frac{1}{a}, \\ - (i\hbar)^{-1} \int_t^{\infty} dt' \exp\left[\frac{i}{\hbar} a(t-t')\right] &= i\pi\delta(a) + P\frac{1}{a}. \end{aligned} \quad (4.14)$$

Strictly speaking, however, the delta-function and the principal part should be treated as the functions

$$\delta(a) = \frac{1}{\pi} \cdot \frac{(\hbar/t_0)}{a^2 + (\hbar/t_0)^2}, \quad P\frac{1}{a} = \frac{a}{a^2 + (\hbar/t_0)^2} \quad (4.15)$$

in the calculations under consideration. Only when (\hbar/t_0) can be regarded as zero on a rough energy scale, they can be put equal to the delta-function and the correct principal part, respectively. The explicit forms such as (4.15) may be obtained by introducing the factor $\exp[-|t-t'|/t_0]$ into the integrand. This is evidently justified in the case of the short wave packet as mentioned in § 2. Hence we have the integral

$$\int_{-\infty}^{\infty} d(t-t') \exp\left[\frac{i}{\hbar} E_p(t-t') - \frac{1}{t_0} |t-t'| \right] \Sigma^F(\mathbf{x}, \mathbf{x}'; t-t'),$$

which gives us the Fourier transform $\tilde{\Sigma}^F$ of Σ^F at a complex energy point with the imaginary part of the order of \hbar/t_0 . As is well-known (see Appendix or reference 13)), the Fourier transform $\tilde{\Sigma}^F$ at such a point is nearly equal to the average of $\tilde{\Sigma}^F$ on the real axis of energy over the range of the order of \hbar/t_0 . The latter, that is, the average, is nothing but the quantity wanted in the case of the mixed beam as mentioned in § 2. Therefore we have obtained the same form for the optical potential in the two cases, the short wave packet and the mixed beam. In spite of these discussions, we still use the notation $\delta(a)$ and $P(1/a)$ for the sake of simplicity. Readers must understand them to be given by the definitions (4.15). By making use of the formulas (4.15), we can immediately write down $\mathcal{U}_p^{(0)Fp}$ and $\mathcal{U}_p^{(0)Fi}$ as follows:

$$\langle \mathbf{x} | \mathcal{U}_p^{(0)Fp} | \mathbf{x}' \rangle = P \sum_{l\bar{m}\bar{n}} \frac{F_{l\bar{m}\bar{n}}(\mathbf{x}) F_{l\bar{m}\bar{n}}^*(\mathbf{x}')}{E_p - E_l^{(0)} - E_m^{(0)} - E_n^{(0)}} + P \sum_{l\bar{m}\bar{n}} \frac{F_{l\bar{m}\bar{n}}^*(\mathbf{x}) F_{l\bar{m}\bar{n}}(\mathbf{x}')}{E_p + E_{\bar{l}}^{(0)} + E_m^{(0)} + E_n^{(0)}}, \quad (4.16)$$

$$\begin{aligned} \langle \mathbf{x} | \mathcal{U}_p^{(0)Fi} | \mathbf{x}' \rangle &= \pi \sum_{l\bar{m}\bar{n}} \delta(E_p - E_l^{(0)} - E_m^{(0)} - E_n^{(0)}) F_{l\bar{m}\bar{n}}(\mathbf{x}) F_{l\bar{m}\bar{n}}^*(\mathbf{x}') \\ &\quad - \pi \sum_{l\bar{m}\bar{n}} \delta(E_p + E_{\bar{l}}^{(0)} + E_m^{(0)} + E_n^{(0)}) F_{l\bar{m}\bar{n}}^*(\mathbf{x}) F_{l\bar{m}\bar{n}}(\mathbf{x}'). \end{aligned} \quad (4.17)$$

It is easy to prove the hermiticity of $\mathcal{V}_p^{0, Fp}$ and $\mathcal{V}_p^{0, Fi}$. The last term in the right-hand side of (4.17) vanishes in the cases of nucleon propagation, because it is impossible to realize the energy relation, $E_p + E_l^{(0)} + E_m^{(0)} + E_n^{(0)} = 0$, corresponding to spontaneous excitations of the target nucleus in the ground state. Inversely, the first term vanishes for hole propagation, since the energy relation $E_p - E_l^{(0)} - E_m^{(0)} - E_n^{(0)} = 0$ is never realized. Thus we have only to keep the first term in the case of elastic scattering of a nucleon, that is,

$$\langle \mathbf{x} | \mathcal{V}_p^{0, Fi} | \mathbf{x}' \rangle = \pi \sum_{l\bar{m}n} \delta(E_p - E_l^{(0)} - E_m^{(0)} - E_n^{(0)}) F_{l\bar{m}n}(\mathbf{x}) F_{l\bar{m}n}^*(\mathbf{x}'). \quad (4.18)$$

It is evident that the E_p -dependence of the optical potential just obtained is not so sensitive as expected in the case of monochromatic beam, owing to the smoothness of the functions (4.15).

From (4.16) and (4.18) we can calculate the expectation value of $\mathcal{V}_p^{0, Fp}$ for the amplitude χ_p as follows:

$$\begin{aligned} (\chi_p, \mathcal{V}_p^{0, Fp} \chi_p) &\equiv \iint \chi_p^*(\mathbf{x}) \langle \mathbf{x} | \mathcal{V}_p^{0, Fp} | \mathbf{x}' \rangle \chi_p(\mathbf{x}') d^3\mathbf{x} d^3\mathbf{x}' \\ &= P \sum_{l\bar{m}n} \frac{|\langle p | V | l\bar{m}n \rangle|^2}{E_p - E_l^{(0)} - E_m^{(0)} - E_n^{(0)}} + P \sum_{l\bar{m}n} \frac{|\langle A | V | p\bar{l}m\bar{n} \rangle|^2}{E_p + E_l^{(0)} + E_m^{(0)} + E_n^{(0)}}, \end{aligned} \quad (4.19)$$

$$(\chi_p, \mathcal{V}_p^{0, Fi} \chi_p) \equiv \iint \chi_p^*(\mathbf{x}) \langle \mathbf{x} | \mathcal{V}_p^{0, Fi} | \mathbf{x}' \rangle \chi_p(\mathbf{x}') d^3\mathbf{x} d^3\mathbf{x}' = \frac{\hbar}{2} w_p, \quad (4.20)$$

where

$$w_p \equiv \frac{2\pi}{\hbar} \sum_{l\bar{m}n} \delta(E_p - E_l^{(0)} - E_m^{(0)} - E_n^{(0)}) |\langle p | V | l\bar{m}n \rangle|^2. \quad (4.21)$$

Here we used the abbreviations

$$\left. \begin{aligned} \langle p | V | l\bar{m}n \rangle &\equiv \int \chi_p^*(\mathbf{x}) F_{l\bar{m}n}(\mathbf{x}) d^3\mathbf{x}, \\ \langle A | V | p\bar{l}m\bar{n} \rangle &\equiv \int \chi_p(\mathbf{x}) F_{l\bar{m}n}(\mathbf{x}) d^3\mathbf{x}. \end{aligned} \right\} \quad (4.22)$$

From (4.20) and (4.21) we know that $\mathcal{V}_p^{0, Fi}$ has a positive expectation value, and that w_p is nothing but the time rate* of the transition probability for exciting the target nucleus or for decay of its excited state. This is the situation just expected, because the antihermitian part $\mathcal{V}_p^{0, Fi}$, responsible for the probability dissipation of the one-nucleon amplitude, is one of the contributions of the fluctuation part Σ^F represented by Fig. 6 to the optical potential. It is noted that the probability dissipation of the one-nucleon amplitude is caused by the pair-excitations of the target nucleus. (For hole propagation we have the negative expectation value of

* The characteristic time τ_0 in discussions in § 1 is at most of the order of the inverse of w_p . Hence it follows that $\tau_0 \simeq w_p^{-1}$ or $\simeq (\hbar/\Gamma)$, Γ being the width of a giant resonance.

$\hat{\mathcal{U}}_p^{Fz}$. It is reasonable, however, because hole propagation can be regarded as the time-reversed process of particle propagation.) The expression (4.19) for the expectation value of $\hat{\mathcal{U}}_p^{Fz}$ has a strong resemblance to the self-energy of a particle or to the level-shift in quantum field theory. In fact, the hermitian part $\hat{\mathcal{U}}_p^{Fz}$ is responsible for the shifts of the resonance level. Therefore the fluctuation part $\hat{\mathcal{U}}_p^{Fz}$ represents reactions of virtual or provisional excitations of the target nucleus to the one-nucleon motion in the elastic scattering.

Finally we may state the criterion for the possibility of observing the optical potential for the elastic scattering of one nucleon in nuclear reactions in the following way: The elastic scattering under consideration can be described by the so-called "optical potential" if our perturbation series, which starts from the unperturbed potential appropriately chosen, rapidly converges and $\hat{\mathcal{U}}_p^{Fz}$ is relatively insensitive to the energy of the incident nucleon. Another criterion will be formulated as the fluctuation-dissipation theorem for the fluctuation part of the amplitude in the next section.

§ 5. Langevin-like equation and fluctuation-dissipation theorem

In § 2 we concluded that the one-nucleon amplitude χ_m may be expressed as the sum

$$\chi_m(\mathbf{x}, t) = \chi_p(\mathbf{x}, t) + \chi_m^{inel}(\mathbf{x}, t) \quad (5.1)$$

with the accuracy ΔE_p . χ_p describes the systematic or coarse-grained motion, while χ_m^{inel} represents the fluctuation part oscillating rapidly. Hence the decomposition given in (5.1) is to be compared with Eq. (13) in the previous paper,¹⁾ which defines the fluctuation part $\hat{\xi}$ of the amplitude χ by the deviation from the average χ_0 . Therefore we are inclined to put

$$\chi = \chi_m, \quad \chi_0 = \chi_p, \quad \hat{\xi} = \chi_m^{inel}. \quad (5.2)$$

In the present section we shall justify the identification (5.2) by obtaining the equation to be satisfied by the fluctuation part $\hat{\xi}$ and by formulating its statistical properties.

Here we rewrite the Heisenberg equation (3.3) of the nucleon field $\phi(\mathbf{x}, t)$ as

$$\left[i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 \right] \phi(\mathbf{x}, t) - \int d^3\mathbf{x}' \langle \mathbf{x} | \hat{\mathcal{U}}_p | \mathbf{x}' \rangle \phi(\mathbf{x}', t) = \eta(\mathbf{x}, t), \quad (5.3)$$

where

$$\eta(\mathbf{x}, t) \equiv \int d^3\mathbf{x}' \phi^*(\mathbf{x}', t) V(\mathbf{x} - \mathbf{x}') \phi(\mathbf{x}', t) \phi(\mathbf{x}, t) - \int d^3\mathbf{x}' \langle \mathbf{x} | \hat{\mathcal{U}}_p | \mathbf{x}' \rangle \phi(\mathbf{x}', t). \quad (5.4)$$

From (5.3) and (2.12) one immediately obtains the equation

$$\left[i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 \right] \chi_m(\mathbf{x}, t) - \int d^3\mathbf{x}' \langle \mathbf{x} | \hat{\mathcal{U}}_p | \mathbf{x}' \rangle \chi_m(\mathbf{x}', t) = f_m(\mathbf{x}, t), \quad (5.5)$$

where

$$f_m(\mathbf{x}, t) \equiv \langle A | \gamma(\mathbf{x}, t) | E_m^+; A+1 \rangle. \quad (5.6)$$

Taking into account the relation

$$\chi_p \equiv \sum_m a_m \chi_m$$

and Eq. (4.5) for χ_p , we can readily prove the property

$$\sum_m a_m f_m = 0.$$

Let us now define a sort of average \bar{Q} of a function Q_m by

$$\bar{Q} \equiv \sum_m a_m Q_m / \sum_m a_m^{inel} \langle E_m^+; A+1 | E_m^+, A+1 \rangle^{inel}. \quad (5.7)$$

Then it is quite easy to see the relations

$$\chi_0 = \bar{\chi} \propto \chi_p, \quad \bar{f} = 0, \quad \bar{\xi} = 0, \quad (5.8)$$

which are just the properties assumed in the previous paper. Moreover, it is evident that $\hat{\xi}$ or χ_m^{inel} obeys the equation

$$\left[i\hbar \frac{\partial}{\partial t} - \mathcal{H} \right] \hat{\xi} = f, \quad (5.9)$$

where \mathcal{H} is symbolically written as

$$\mathcal{H} \equiv -\frac{\hbar^2}{2m} \nabla^2 + \mathcal{V}_p. \quad (5.10)$$

Eqs. (5.5) and (5.9) together with (5.8) should be identified with the Langevin-like equation assumed in the previous paper, if the fluctuation part $\hat{\xi}$ (i. e. χ_m^{inel}) or the fluctuating source function f is subject to the fluctuation-dissipation theorem.

To prove the fluctuation-dissipation theorem, we shall calculate the correlation function of $\hat{\xi}$ or f namely,

$$\overline{\hat{\xi}(\mathbf{x}, t) \hat{\xi}^*(\mathbf{x}', t')} \quad \text{or} \quad \overline{f(\mathbf{x}, t) f^*(\mathbf{x}', t')}.$$

If nuclear excitations caused by successive pair-excitations continue during far longer periods than the lifetime of a single decay, we may regard oscillations of $\hat{\xi}$ or f as a sort of stationary random fluctuations. Thus the correlation functions become functions of $(t-t')$ and, in particular, the correlation function of $\hat{\xi}$ satisfies the equation for χ_0 . Hence we easily obtain the formula

$$\begin{aligned} \overline{\hat{\xi}(\mathbf{x}, t) \hat{\xi}^*(\mathbf{x}', t')} &= \exp \{ -(i/\hbar) \mathcal{H}_x(t-t') \} \overline{\hat{\xi}(\mathbf{x}, t') \hat{\xi}^*(\mathbf{x}', t')} \quad t > t', \\ &= \exp \{ (i/\hbar) \mathcal{H}_{x'}^*(t'-t) \} \overline{\hat{\xi}(\mathbf{x}, t) \hat{\xi}^*(\mathbf{x}', t)} \quad t < t'. \end{aligned} \quad (5.11)$$

Using the definition of $\hat{\xi}$ or χ_m^{inel} , one gets

$$\overline{\hat{\xi}(\mathbf{x}, t) \hat{\xi}^*(\mathbf{x}', t)} = \langle A | \psi(\mathbf{x}) \Xi_t^{inel} \psi^*(\mathbf{x}') | A \rangle, \quad (5.12)$$

where

$$\Xi_t^{inel} \equiv \sum_m |E_m^+; A+1\rangle_t^{inel} a_m^{inel} \langle E_m^+; A+1| / \sum_m a_m^{inel} \langle E_m^+; A+1| E_m^+; A+1 \rangle_t^{inel}. \quad (5.13)$$

Owing to the randomness of $\chi_m^{inel}(\mathbf{x}, t)$, it may be expected that the excited nucleus is in thermal equilibrium with temperature T and the operator Ξ_t^{inel} can be replaced by the operator* proportional to $\exp[-H/kT]$. Therefore (5.12) is reduced to

$$\begin{aligned} \overline{\xi(\mathbf{x}, t) \xi^*(\mathbf{x}', t)} &= \sum_\nu n(\nu, T) \langle \mathbf{x} | \nu \rangle \langle \tilde{\nu} | \mathbf{x}' \rangle, \\ &= \sum_\nu n(\nu, T) \langle \mathbf{x} | \tilde{\nu} \rangle \langle \nu | \mathbf{x}' \rangle, \end{aligned} \quad (5.14)$$

where $|\nu\rangle$ are $|\tilde{\nu}\rangle$ are respectively the eigenstate and its time-reversed state of \mathcal{H} belonging to an eigenvalue E_ν , and $n(\nu, T)$ is the state density of the ν -th state. In the derivation of (5.14) we used the assumption of random phase for the matrix elements of Ξ_t^{inel} , that is,

$$\langle \nu; A+1 | \Xi_t^{inel} | \nu'; A+1 \rangle = n(\nu, T) \delta_{\nu\nu'}, \quad (5.15)$$

where

$$\begin{aligned} |\nu; A+1\rangle &= \alpha_\nu^* |A\rangle \quad \text{or} \quad \alpha_{\nu'}^* |A\rangle, \\ \phi(\mathbf{x}) &= \sum_\nu \langle \mathbf{x} | \nu \rangle \alpha_\nu = \sum_\nu \langle \mathbf{x} | \tilde{\nu} \rangle \alpha_{\nu'}. \end{aligned}$$

Substituting (5.14) in (5.11), we find the correlation function

$$\begin{aligned} \overline{\xi(\mathbf{x}, t) \xi^*(\mathbf{x}', t')} &= \sum_\nu n(\nu, T) \exp\{-(i/\hbar)E_\nu(t-t')\} \langle \mathbf{x} | \nu \rangle \langle \tilde{\nu} | \mathbf{x}' \rangle \quad t > t', \\ &= \sum_\nu n(\nu, T) \exp\{(i/\hbar)E_\nu^*(t'-t)\} \langle \mathbf{x} | \tilde{\nu} \rangle \langle \nu | \mathbf{x}' \rangle \quad t < t'. \end{aligned}$$

Now this expression can be replaced by the integral

$$\overline{\xi(\mathbf{x}, t) \xi^*(\mathbf{x}', t')} = \frac{i\hbar}{2\pi} \int_{-\infty}^{\infty} d\omega \exp\{-i\omega(t-t')\} n(\omega, T) \langle \mathbf{x} | \frac{1}{\hbar\omega - \mathcal{H}} - \frac{1}{\hbar\omega - \mathcal{H}^*} | \mathbf{x}' \rangle, \quad (5.16)$$

because, in the complex ω -plane, the function $\langle \mathbf{x} | (\hbar\omega - \mathcal{H})^{-1} | \mathbf{x}' \rangle$ has all its poles in the lower half, while all poles of the function $\langle \mathbf{x} | (\hbar\omega - \mathcal{H}^*)^{-1} | \mathbf{x}' \rangle$ are found in the upper half. Here $n(\omega, T)$ is so defined as to become equal to $n(\nu, T)$ at $\omega = (E_\nu/\hbar)$, and is related to the mean energy $E(\omega, T)$ near frequency ω as follows:

$$n(\omega, T) = E(\omega, T)/\hbar\omega,$$

$E(\omega, T)$ being reduced to kT for $\hbar\omega \ll kT$. From (5.16) we immediately obtain the Fourier transform of the correlation function, that is,

* Note that Ξ_t^{inel} becomes this type irrespective of the initial distribution a_m in observations on a rough time scale on which we can consider to be τ_0 (relaxation time) $\simeq 0$.

$$\begin{aligned}\overline{\hat{\xi}_\omega(\mathbf{x})\hat{\xi}_\omega^*(\mathbf{x}')}&=\frac{i}{\omega}E(\omega, T)\langle\mathbf{x}|\frac{1}{\hbar\omega-\mathcal{H}}-\frac{1}{\hbar\omega-\mathcal{H}^*}|\mathbf{x}'\rangle\delta(\omega-\omega')\\&=\frac{1}{\omega}E(\omega, T)\langle\mathbf{x}|\frac{1}{\hbar\omega-\mathcal{H}}[-2\operatorname{Im}\mathcal{V}_p]\frac{1}{\hbar\omega-\mathcal{H}^*}|\mathbf{x}'\rangle\delta(\omega-\omega'),\end{aligned}\quad (5\cdot17)$$

where $\hat{\xi}_\omega(\mathbf{x})$ is the Fourier transform of $\hat{\xi}(\mathbf{x}, t)$ and $\operatorname{Im}\mathcal{V}_p$ stands for the anti-hermitian part of \mathcal{V}_p . By making use of the equation

$$(\hbar\omega-\mathcal{H})\hat{\xi}_\omega(\mathbf{x})=f_\omega(\mathbf{x}),$$

the correlation function of $\hat{\xi}$ gives us that of f , that is,

$$\overline{f_\omega(\mathbf{x})f_\omega^*(\mathbf{x}')}= \frac{1}{\omega}E(\omega, T)\langle\mathbf{x}|-2\operatorname{Im}\mathcal{V}_p|\mathbf{x}'\rangle\delta(\omega-\omega').\quad (5\cdot18)$$

This is nothing but the fluctuation-dissipation theorem or the Nyquist theorem assumed in the previous paper. By derivation of (5·18) we have completed the interpretation that the imaginary part of the optical potential is responsible for the probability dissipation of elastic scattering coming from nuclear excitations.

In the present section we have derived the fluctuation-dissipation theorem (5·17) or (5·18) on the basis of (5·14) or (5·15). The formula (5·14) or (5·15) plays one of the most essential roles in our discussions. It is not so easy to prove the formula (5·14) or (5·15) exactly. Nevertheless, we think that its use would be justified in observations on a rough time-scale as mentioned in § 1. Strictly speaking, the use of (37) or (59) in the previous paper must be justified by proving (5·14) or (5·15).

There formula (5·15) may be proved if the excited nucleus can be considered to be in equilibrium, in other words, if the compound nucleus is formed and its oscillations with various frequencies are superposed in a somewhat long interval. Thus, under the same condition, we could expect the coarse-grained one-particle motion and the optical model for the many-particle system. The condition for appearance of equilibrium may be formulated in the following way. If we have the much smaller value of the relaxation time τ_1 associated with f (or η) than the time $\tau_0=(\hbar/\omega_p)$, namely

$$\tau_0\gg\tau_1,$$

we may expect that the system is in equilibrium. The time τ_1 may be obtained as the damping time of the function given by

$$\begin{aligned}B(t-t')&=\iint\mathcal{L}_p^*(\mathbf{x}, 0)\langle A|\frac{1}{2}\{\eta(\mathbf{x}, t)\eta^*(\mathbf{x}', t')+\eta^*(\mathbf{x}', t')\eta(\mathbf{x}, t)\}|A\rangle\\&\quad\times\mathcal{L}_p(\mathbf{x}', 0)d^3\mathbf{x}d^3\mathbf{x}'.\end{aligned}\quad (5\cdot19)$$

The characteristic time used in discussions of § 1 should be identified with τ_1 as mentioned in the footnote of § 1. When the condition $\tau_0\gg\tau_1$ breaks down, it becomes necessary to find another type of systematic motion (different from the one-

particle motion), for example, the two-particle motion as suggested in the next section.

§ 6. Conclusions

In the present paper we have formulated, from first principles, a possible scheme for the one-particle approximation in many-particle systems, in particular, in nuclear reactions. The systematic part of the one-particle amplitude describes the coarse-grained motion of the system and obeys the one-particle Schrödinger equation with the so-called optical potential. The fluctuation part is governed by a Langevin-like equation with the same optical potential and with the fluctuating source function, and is subject to the fluctuation-dissipation theorem. The optical potential is decomposed into the static part and the fluctuation part. The former is to be observed for the target nucleus in the fixed ground state, while the latter represents reactions of nuclear excitations. The static part would not be so different from the one-particle potential as expected in the nuclear shell model. The fluctuation part is considerably smaller due to the function (4.15), as is seen from its explicit form. Such an effect comes from the energy spread of the incident beam and has already been pointed out by Bloch¹³⁾ and Hayakawa et al.⁴⁾ The fluctuation part of the optical potential has an imaginary part as a result of the probability dissipation for elastic scattering. As is seen from the fluctuation-dissipation theorem, the imaginary part is proportional to the correlation function of the fluctuating source function, which may become larger near the nuclear surface rather than inside the nucleus. Therefore it is very interesting to investigate the imaginary part of the optical potential near the nuclear surface from the formula (4.17). The problem may become easy to handle by assuming the one-sided extension of the infinite nuclear matter.¹⁴⁾

We have so far discussed the one-particle motions in a many-particle system. There it has been expected that the coarse-grained motion of the system becomes a sort of one-particle motion. Such an expectation, however, is not necessarily realized in every case. In some cases we should be concerned with systematic motions among two or more particles as the coarse-grained motions of the system. For example, let us consider the elastic scattering of a nucleon with a relatively high energy by a large nucleus which is composed of a large closed shell and an extra nucleon loosely bound by the shell. In this problem, we can no longer expect to describe the coarse-grained motion by the one-particle amplitude χ_p . (Moreover, the remaining amplitude χ_n^{inel} would not be subject to the fluctuation-dissipation theorem.) Thus we should utilize the two-particle amplitude and the two-particle Green function by identifying the closed shell with the medium. Now we could write the equation of the two-particle Green function G_{II} as follows:

$$G_{II}(1, 2; 1', 2') = G_{II}^0(1, 2; 1', 2') \\ + \iint G(1, 3)G(2, 4)I(3, 4; 3', 4')G_{II}(3', 4'; 1', 2')d^4(3)d^4(4)d^4(3')d^4(4'),$$

where G is the one-particle Green function, I the interaction part between two particles and $G_{II}^0(1, 2; 1', 2') \equiv G(1, 1')G(2, 2') - G(1, 2')G(2, 1')$. Therefore the two-particle amplitude $\chi_{II}(1, 2)$ to describe the coarse-grained motion obeys the equation

$$\chi_{II}(1, 2) = \chi_{II}^0(1, 2) + \iint G(1, 3)G(2, 4)I(3, 4; 3', 4')\chi_{II}(3', 4')d^4(3)d^4(4)d^4(3')d^4(4'),$$

where $\chi_{II}^0(1, 2) \equiv \chi_b(1)\chi_s(2) - \chi_b(2)\chi_s(1)$. Here χ_s and χ_b are, respectively, the one-particle amplitudes in the medium of the closed shell corresponding to the scattered nucleon and the bound nucleon. This is an example of the surface interaction or direct interaction. The deuteron stripping reaction may be treated by this formalism with some modifications. But we cannot directly apply the present formalism to the problems of inelastic scattering. Nevertheless, it might be useful to describe some inelastic scatterings by the optical model as suggested by Bloch.¹³⁾

Appendix

Here we shall briefly explain the situation that the Fourier transform $\tilde{\Sigma}^F$ of Σ^F at the complex point $E = E_p \pm i(\hbar/t_0)$ is nearly equal to the average of $\tilde{\Sigma}^F$ on the real axis over the range centred at E_p with spread of the order $\Delta E_p = (\hbar/t_0)$. First we decompose $\Sigma^F(\tau)$ into two parts as follows:

$$\Sigma^F(\tau) = \Sigma_1^F(\tau) + \Sigma_2^F(\tau),$$

where $\Sigma_1^F(\tau) = 0$ for $\tau < 0$ and $\Sigma_2^F(\tau) = 0$ for $\tau > 0$. The Fourier transformation is defined by

$$\tilde{\Sigma}^F(E) \equiv \int_{-\infty}^{\infty} \Sigma^F(\tau) \exp\{(i/\hbar)E\tau\} d\tau$$

on the real axis. The wanted average is written as

$$\begin{aligned} \overline{\tilde{\Sigma}^F(E)} &\equiv \int_{-\infty}^{\infty} \tilde{\Sigma}^F(E) A(E - E_p; \Delta E_p) dE \\ &= \int_{-\infty}^{\infty} d\tau \exp\{(i/\hbar)E_p\tau\} \Sigma^F(\tau) \left[\int_{-\infty}^{\infty} d\tilde{\epsilon} \exp\{(i/\hbar)\tilde{\epsilon}\tau\} A(\tilde{\epsilon}, \Delta E_p) \right], \end{aligned}$$

where $A(\tilde{\epsilon}; \Delta E_p)$ is a smooth function which has non-zero values only near and in a region centred at the origin with the spread $\Delta E_p = (\hbar/t_0)$. Since the function A is well approximated by $(\Delta E_p/\pi)[\tilde{\epsilon}^2 + (\Delta E_p)^2]^{-1}$, the above integral with respect to $\tilde{\epsilon}$ becomes

$$\int_{-\infty}^{\infty} d\tilde{\epsilon} \exp\{(i/\hbar)\tilde{\epsilon}\tau\} A(\tilde{\epsilon}; \Delta E_p) = \exp\{-|\tau|/t_0\}.$$

Thus it follows that

$$\overline{\tilde{Z}^F(E)} = \int_{-\infty}^{\infty} d\tau \exp \{ (i/\hbar) E_p \tau - |\tau|/t_0 \} Z^F(\tau) = \tilde{Z}_1^F \left(E_p + i \frac{\hbar}{t_0} \right) + \tilde{Z}_2^F \left(E_p - i \frac{\hbar}{t_0} \right).$$

This relation is just what we wanted to prove.

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Note added in proof: (i) The optical potential obtained in the present paper is of the Lorentzian type. The problem whether the processes are Lorentzian or Gaussian was discussed by Hayakawa et al. (ii) The state vector $|E_m^+; A+|_t\rangle$ used in § 2 would rather be replaced with $\exp[-(i/\hbar)H(t-t_0)]|\varphi_m\rangle$, where $|\varphi_m\rangle$ is the initial wave packet with a narrow spread in energy. This state vector asymptotically approaches to that used in § 2. More detailed discussions should be required.

Mass Levels of Baryons and Mesons

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In this paper we investigate the relation between the observed mass levels of baryons and mesons as well as the resonance levels in pion-nucleon and kaon-nucleon reactions and the various configurations of particle states derived from Ikeda-Ogawa-Ohnuki's symmetry theory which is based on Sakata's composite particle model. It is found that there is a close correspondence between the theoretical levels and the experimental evidence.

§ 1. Introduction

In order to understand the success of the Nakano—Nishijima—Gell-Mann scheme for the strongly interacting particles on the realistic ground, Sakata¹⁾ has proposed the idea of the composite particle model for elementary particles. According to his composite theory, only proton, neutron, Λ -particle and their antiparticles are assumed to be the basic particles and other mesons and heavy unstable hyperons are compound states composed of these basic particles.

Due to the lack of knowledge about the dynamics of composite particles which probably needs an essentially new law yet unknown, very few analyses have been made so far to develop the composite theory concretely. Now, in his original article, Sakata compared the introduction of Λ -particle as a fundamental particle to the introduction of neutron in the early stage of the history of nuclear theory. Along this line of thought Matumoto²⁾ considered that the mass levels of mesons and baryons other than proton, neutron and Λ -particle, as well as the resonance levels of pion-nucleon scattering, have a similar significance to what the nuclei and the resonance levels of nuclei have in low-energy nuclear physics, and he proposed the semi-empirical formula for these mass levels. The formula given by Matumoto seems to be naive and phenomenological, but we feel that the investigation in his way will be rather effective at the present stage of the composite particle theory and that a certain clue to the future development must be obtained in this way.

Recently Ikeda, Ogawa and Ohnuki³⁾ have made a forward step in the development of Sakata's idea. Ogawa supposed the existence of a certain kind of symmetry among proton, neutron and Λ -particle in Sakata's composite model. He first assumed that in the limiting case of the equal mass the physical laws are invariant under the exchange between any pair of proton (p), neutron (n) and Λ -particle (Λ), and constructed a theory of three-dimensional unitary transformation in col-

laboration with Ikeda and Ohnuki. Then they took account of the mass difference between Λ -particle and nucleon. This new symmetry is very attractive, since it seems to have abstracted some important aspect of Sakata's composite model. In this theory new quantum numbers are automatically introduced by which various configurations of particle states are systematically classified. Thus in order to see the contents of the new theory it is very necessary to clarify how the newly obtained levels of particle state correlate with those found in nature.

In this paper we shall investigate what relations may exist between the theoretical levels derived from the I-O-O theory and the experimental evidences, and attempt to find out simple laws hidden in nature. We discuss the theory up to five-body configurations. This will cover the energy region where the reliable experiments have been or will be performed in the near future.

As we see in the following, rather remarkable correspondence seems to exist between the I-O-O theory and the experimental facts. This suggests us that Sakata's composite particle model has obtained another strong evidence and the I-O-O theory based on the composite particle model may be a very prospective one in understanding the nature of strongly interacting particles and their interactions.

In section 2 we discuss the mass formula which we use as the means for the comparison of the theory with the experimental data. In sections 3 and 4 we give the discussions on the fermion states and boson states respectively. The final section is devoted to the summary and discussion of the results. The explicit expressions of the configuration up to five-body system which are necessary for the analyses in sections 3 and 4 are given in Appendix I. In Appendix II we summarize the theoretical and experimental values of the masses of the states.

§ 2. Mass formula for the composite particles

In discussing the relations of the theoretically expected states with the experimental data, we shall mainly examine their energy level, since it is the most easily obtainable information.

In the I-O-O theory the basis vector of an irreducible constituent of the $(m+n)$ -body system consisting of m particles and n antiparticles will be expressed as

$$\sum_{i,j=1,2,3} C(i_1, i_2, \dots, i_m; j_1, j_2, \dots, j_n) \chi_{i_1} \chi_{i_2} \dots \chi_{i_m} \chi_{j_1} \chi_{j_2} \dots \chi_{j_n}, \quad (1)$$

where χ_1 , χ_2 and χ_3 are proton, neutron and Λ -particle, and $\bar{\chi}_1$, $\bar{\chi}_2$ and $\bar{\chi}_3$ are their antiparticles respectively. Information about $C(i_1, i_2, \dots, i_m; j_1, j_2, \dots, j_n)$ will be obtained from Appendix I. Now we assume that the mass of the particle corresponding to this basis-vector is given by

$$m = \sum_{i,j=1,2,3} C(i_1, i_2, \dots, i_m; j_1, j_2, \dots, j_n)^2 m(\chi_{i_1} \chi_{i_2} \dots \chi_{i_m} \bar{\chi}_{j_1} \bar{\chi}_{j_2} \dots \bar{\chi}_{j_n}), \quad (2)$$

where $m(\chi_{i_1 i_2} \cdots \chi_{i_m}^{j_1 j_2} \chi_{i_m}^{j_n})$ is the mass of the "sub-particle" $\chi_{i_1 i_2}^{j_1 j_2} \cdots \chi_{i_m}^{j_n}$.

In applying this formula we further assume that the mass of sub-particle is independent of M , M' and I , where M and M' are the quantum numbers specifying each irreducible constituent, and I is the isotopic spin. We also assume in the following analysis that there is no spin dependence in the mass value of sub-particle.

In the following discussion we neglect the mass difference between proton and neutron (we express proton and neutron by common symbol N), which results in giving the same value to all the particles belonging to a charge multiplet.

Now the problem is to find out the masses of "sub-particles". This will not be supplied directly from experiment except in few cases. For its rough evaluation we propose the following mass formula which is analogous to Matumoto's but contains a little simplification,

$$m(\chi_{i_1 i_2} \cdots \chi_{i_m}^{j_1 j_2} \chi_{i_m}^{j_n}) \sim (n_N + n_{\bar{N}})m_N + (n_A + n_{\bar{A}})m_A + (n_{N\bar{N}} - n_{N\bar{N}} - n_{\bar{N}\bar{N}})V(N\bar{N}) \\ + (n_{N\bar{A}} + n_{A\bar{N}} - n_{NA} - n_{\bar{N}\bar{A}})V(N\bar{A}) + (n_{A\bar{A}} - n_{AA} - n_{\bar{A}\bar{A}})V(A\bar{A}), \\ V(N\bar{A}) = V(N\bar{N}) + \Delta V, \quad V(A\bar{A}) = V(N\bar{N}) + 2\Delta V. \quad (3)$$

In the above Formula (3) $m_N(m_A)$ is mass of nucleon (A -particle), and $n_N(n_A)$ and $n_{\bar{N}}(n_{\bar{A}})$ are the number of nucleon (A -particle) and antinucleon (anti A -particle) respectively. $n_{N\bar{N}}$, etc., are the numbers of nucleon-antinucleon pairs in "sub-particle", etc. We call this formula the "two-body approximation" together with (2) for the mass of composite particle. In Formula (3) $V(N\bar{N})$ and ΔV are determined so as to give the experimental masses of pion and kaon correctly. This mass formula differs from Matumoto's in that this is a two-parameter formula ($V(N\bar{N})$ and ΔV) instead of Matumoto's three-parameter formula ($V(N\bar{N})$, $V(N\bar{A})$ and $V(A\bar{A})$). In Matumoto's formula the "binding energy" $V(A\bar{A})$ of A - A bond is determined so as to fit the mass of Ξ -particle, but here in the new formula it is the linear extrapolation of $V(N\bar{A})$ value. The reason for taking this is that $V(A\bar{A}) (= -V(A\bar{A}))$ will be approximately given by

$$V(A\bar{A}) = V(m_N + \Delta m, m_N + \Delta m) \cong V(m_N, m_N + \Delta m) + \Delta m V' \\ \cong V(m_N, m_N) + 2\Delta m V', \quad (4)$$

where $V(m_N, m_N + \Delta m) = V(N\bar{A})$ and $V(m_N, m_N) = V(N\bar{N})$, and Δm is A -particle—nucleon mass difference. We have no reason to believe that the value of $V(A\bar{A})$ determined from the mass of Ξ -particle gives a better estimate to the real value than the above linear extrapolation, since it is hardly probable that the mass of Ξ -particle is correctly given by the two-body approximation, as will be seen in the case of ($I=3/2, J=3/2$) resonance of π - N scattering at 190 Mev laboratory pion kinetic energy. The most reliable value will be supplied by the mass of $\pi^{0'}$ -meson if we can know it by experiment. Our intention to take (3) is not to reduce the

number of parameters, but to extract simple relations among the masses of particles and emphasize that in the limiting case of equal mass of Λ -particle and nucleon there must be no mass split in Formula (3).

In the following discussion when we apply this formula, it will always be necessary to take into account the theoretical error of the order of about 200 Mev.

Notations for states

We use the following notations to represent fermion and boson states. For a fermion we write like $F_j^i(S, I)$ where subscript j expresses the number of particles plus antiparticles of which the fermion is composed, and we denote by superscript i the class to which this particle belongs (each class corresponds to each irreducible constituent, see Appendix I). S and I in the bracket are its strangeness and isotopic-spin. These notations will supply enough information for finding the explicit configuration which is given in Appendix I. For boson we use similar notation $B_j^i(S, I)$, the meaning of the suffices being the same as before.

§ 3. Fermions

In this section we discuss the fermion states of baryon number 1. Many experimental data will be available for these states and we can expect interesting results.

3-1. The system of one baryon

p , n and Λ belong to this. These are the basic particle from which all the particles are constructed. We have nothing to say about it at present except that the physical nucleon and Λ -particle are, of course, the superposition of one-body state, three-body states, etc., having the same $M(=3/2)$, $M'(=5/2)$, spin and parity.

3-2. The system of two baryons and one antibaryon

There will appear $3^3=27$ states in this three-body system. However, essentially new states which have different M and M' from those of one-body system are 21. These states belong to classes 1 and 3.* They are the characteristic states of three-body system. The remaining 6 states which representations are equivalent to that of one-body system may be new states if they have different spin and parity from those of the corresponding one-body states. But as we shall see below, such a case does not seem to have occurred in reality although it will still be premature to draw a definite conclusion. In the following we shall assume that the states having the same M , M' , I and S have the same spin and parity and discuss in terms of the states of the lowest configuration.

The characteristic states of three-body system seem to have fairly good correspondence with the actually observed hyperons and the resonance states in the

* The classes 1, 2, 3 and 4 of the three-body system in this paper correspond to the classes IV, III, II and I of I-O-O's article³⁾ respectively.

meson-nucleon scattering. Let us see it briefly in the following.

(i) Strange particles

$F_s^1(-2, 1/2)$ may be assigned to Ξ -particles

and

$F_s^1(-1, 1)$ and $F_s^3(-1, 1)$ may be assigned to Σ -particles.

The masses of $F_s^1(-1, 1)$ and $F_s^3(-1, 1)$ are the same as is easily seen from their configurations and mass formula. So the observed Σ -particle could be $F_s^1(-1, 1)$ or $F_s^3(-1, 1)$ particles or possibly their mixture. Because they belong to different classes, they may or may not have different spin-parity. The present experimental data seem to support spin $1/2$ for Σ -particles. They might have different parity, if they both have spin $1/2$. In future we may be able to find such a complexity of Σ -particles.

(ii) Resonance states in π - N scattering

In the experimental data of π - N scattering below 1 BeV it is observed that there exist three resonance states, the well-known $(3, 2, 3, 2)$ resonance at 190 Mev laboratory pion kinetic energy, $(1/2, 3/2)$ state at about 650 Mev and $(1, 2, >5, 2)$ state at about 950 Mev.⁴⁾ We assign $F_s^1(0, 3/2)$, $F_s^3(0, 1/2)$ and $F_s^1(0, 1/2)$ to these resonance states respectively. The consistency of this assignment will be supported by the following fact. The masses of these states are given by the following expressions from Formula (2),

$$\begin{aligned} m(F_s^1(0, 3/2)) &= m(NN\bar{N}) \\ m(F_s^3(0, 1/2)) &= \{m(NN\bar{N}) + m(N\bar{A}\bar{A})\}/2 \\ m(F_s^1(0, 1/2)) &= \{m(NN\bar{N}) + 3m(N\bar{A}\bar{A})\}/4, \end{aligned} \quad (5)$$

where $m(NN\bar{N})$ and $m(N\bar{A}\bar{A})$ mean the masses of $NN\bar{N}$ and $N\bar{A}\bar{A}$ sub-particles respectively. If we determine the $m(NN\bar{N})$ value by 190 Mev resonance and $m(N\bar{A}\bar{A})$ by 950 Mev resonance, we immediately obtain about 670 Mev for the resonance energy of $F_s^3(0, 1/2)$ state.* This fits remarkably with experiment and may support our assumption that the spin dependency of energy level is very small and may guarantee the approximation to neglect the correction to the configuration coefficients $C(i_1, i_2, \dots, i_m; j_1, j_2, \dots, j_n)$ due to actual A -particle-nucleon mass difference.**

* For the Saclay's data which give resonances at about 600 Mev and 900 Mev pion kinetic energy, we can see similar relations.

** In this paper we take the total energy in the center of mass system at resonance as the mass of the "particle" corresponding to the resonance level. Strictly speaking, this is not correct. We must take account of the energy shift between the actual resonance energy and the level energy. See, for example, J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics*, Chap. VIII, London. We thank Prof. S. Hayakawa for his calling our attention to this fact. However, we do not consider this problem here, since the discussion of this problem about all observed resonance states is practically impossible now, and the arguments of the present article will not suffer essential change from it.

(iii) Resonance states in K - N scattering

The K - N scattering experiments are still at preliminary stage, so the experimental information is rather poor and not definite, unlike the π - N case. We have two states which would be observed as resonance states in K - N scattering. They are $F_3^3(1, 0)$ and $F_3^1(1, 1)$. These two states having $I=0$ and $I=1$ are energetically degenerate.* We think it will be appropriate to assume that these states contribute to the resonance-like behavior around 200 Mev laboratory kaon kinetic energy in K^+-n scattering,^{4)***} if the resonance really takes place there, since this will be the lowest resonance level of K - N scattering and the strange particle having strangeness 1 and mass $< m_N + m_K$ (m_K is the mass of kaon) has not so far been found. The experimental clarification of degeneracy of these resonance states is very interesting, since we have not had such levels before.

 (iv) Resonance states in \bar{K} - N scattering

In the K^--p scattering it is observed that the elastic cross section at very low energy region is very large.⁴⁾ At 25 Mev of laboratory kaon kinetic energy it is even four times of K^+-p cross section and it gradually decreases as the energy increases. This feature has been discussed by many authors. Among others Matthews and Salam⁵⁾ have proposed the existence of a resonance state at 25 Mev. We also maintain this is the case and assume that this corresponds to the state of $I=0$ in class 1, $F_3^1(-1, 0)$. In order to confirm this, it is necessary to perform the experiment of K^--n elastic scattering and to see that no resonance occurs in this process.

(v) Empirical mass formula for the three-body system of two baryons and one antibaryon

We have not discussed much about the absolute value of mass of each level or the relation of the value predicted by the two-body approximation with the experimental value. The values calculated with use of a two-body approximation are given in Appendix II, together with the experimental values. It gives a fit to the experimental data within the error of 200 Mev. This may be satisfactory agreement if we remember the roughness of the argument. In order to look further into the problem of energy levels, it is required to have the knowledge of the details of the construction of these levels and the mechanism of the interactions.

Here we shall refer to a notable feature which is observed in the difference between the prediction of two-parameter formula and the experimental data. It is

$$V(NN\bar{N}) \simeq V(N\bar{N}\bar{N})$$

* These resonance levels and their energy degeneracy have been emphasized by Matumoto²⁾ as the important consequences of Sakata's composite model and Matumoto's mass formula.

** The present $K^++n \rightarrow K^0+p$ data do not show such a resonance feature. This is very interesting in connection with the degeneracy of $I=0$ and $I=1$ states. However, the experiment in this case seems to be rather difficult. Further investigation will be expected to clarify this point.

where $V(\chi\chi\chi)_{i,j}^k$ is the difference between the experimental value and the estimation of mass Formula (3):

$$V(\chi\chi\chi)_{i,j}^k = \text{experimental value} - \text{Formula (3)}.$$

Including this relation, the experimental data are expressed approximately by the next formula,

$$\begin{aligned} m(\chi\chi\chi)_{i,j}^k &= \text{Formula (3)} + V(\chi\chi\chi)_{i,j}^k, \\ V(\chi\chi\chi)_{i,j}^k &\cong (-1)^\sigma \{U - (n_\Lambda + n_{\bar{\Lambda}}) \Delta U\}, \\ \sigma &= (n_\Lambda + n_{\bar{\Lambda}})(n_\Lambda n_{\bar{\Lambda}}/2 + 1). \end{aligned} \quad (6)$$

If we choose U and ΔU appropriately, the fit of this formula to the present experimental data can be made within 50 Mev. Of course the above expression (6) is one example of representing the experimental data in correlation with Formula (3). It will be interesting here to note some possible contents which are involved in the term $V(\chi\chi\chi)_{i,j}^k$. It may contain contributions from various origins. For example:

(i) The correction may arise from the fact that we use the parameters adjusted to the data of the one-body system and two-body system. Even if the two-body approximation is a very accurate one, the contribution of the higher configuration states to the lowest configuration state will not be so simple as to be expressed by the "effective" two-body approximation.

(ii) The three-body forces might be present. This can be probable since the phenomena under discussion are at short distance.

(iii) There may be spin-dependence in the energy value of sub-particle. However, this does not seem to be large if we remember the fairly good agreement of mass Formula (2) with the resonance levels below 1 Bev in π - N scattering. We need precise values of the masses of these levels in order to obtain more definite conclusion of the problem (cf. footnote of page 666).

(iv) The correction to the configuration coefficients due to Λ -particle-nucleon mass difference, since the configuration coefficients used in the discussion of this section is obtained for the case when nucleon and Λ -particle have the same mass. This will be the second order correction to the mass difference.

(v) Although we have assumed $V(\chi\chi)_{i,j} = -V(\chi\chi)_{i,j}$ in Formula (3), the actual configuration will not be similar for particle and antiparticle. (For example, in the sub-particle $nn\bar{n}$, we may expect that neutron-neutron distance is longer than neutron-antineutron distance).

We probably could not explain $V(\chi\chi\chi)_{i,j}^k$ in terms of the particular one of these causes. This seems especially true for the case (ii). It is noted that the two-body

approximation gives lower values for ordinary particles, while it gives higher values for strange particles.

We have not so far discussed the logical consistency of the results obtained in this section with Ikeda-Ogawa-Ohnuki's theory.

If our assignments in this section are correct, we have the state with spin $1/2(F_3^1(-1, 1))$, the state with spin $3/2(F_3^1(0, 3/2))$ and the state with higher spin $(F_3^1(0, 1/2))$, in the same class (class 1). This is apparently contradictory with the fact that all of the states belonging to the same class must have the same spin and parity. We do not know now what implications such an inconsistency has. We might think that the interaction responsible for the Λ -particle—nucleon mass difference has an unusual character, and make a change in spin and parity of the state.

In connection with this we remark one interesting feature of the three-body binding energy $V(\chi\chi\chi)_k$. It does not give the same mass for all states of a class even if we make $\Delta m = 0$.*.** If this is really the case, it suggests a very interesting fact that we may not be able to switch on the interaction responsible for Λ - N mass difference *adiabatically*. This interaction induces a change in spin and parity of the system, still leaving the original configuration nearly unaffected. We have not met such a peculiar kind of interaction before. This paradox seems to have a profound meaning and needs further investigation.

Finally we shall emphasize that we have a necessary and sufficient number of states to explain the experiments and there do not appear surplus states which can not be related with the experimental evidences, so far as the present experimental data are concerned. This is one of the reasons why we have neglected the three-body configurations having the same M and M' as those of one-body.

3-3. The system of three baryons and two antibaryons

There are $3^5=243$ states in the five-body system. In these 243 states the essentially new states which have different M and M' from those of one-body and three-body system are 105. These 105 states are the states belonging to the classes 1, 4, 6, and 10. It is noted that particles of class 10 are physically indistinguishable from those of class 6 (See Appendix I).

In discussing these states the experimental data available for comparison are too scanty yet, except for a few cases, and also we can give only crude theoretical

* There is no trouble to assume that the states having the same mass in the limit of $\Delta m=0$ must have the same spin-parity so far as the present experimental data are concerned.

** It will be interesting to see that the mass of $(3/2, 3/2)$ level of π - N scattering is the same with that of Ξ -particle in the case $\Delta m=0$. This may still suggest that Ξ -particle is spin $3/2$ particle³⁾, although the strictness of the argument is somewhat weakened. Similarly, the mass of the resonance states in K - N scattering at about 200 Mev coincides with the mass of Σ -particle in the limit of $\Delta m=0$. This suggests that 200 Mev resonance states have $J=1/2$. This is consistent with the present experimental data.

estimation based on Formula (3) about the values of the energy levels of these states. We use this two-parameter formula when rough estimation is necessary to see the order of levels.

i) New particles

Several new particles may appear in this five-body configuration, but the number of new particles is not so large as one might expect. The possible states which might be observed as new particles are $F_5^4(-3, 0)$, $F_5^1(-3, 1)$, $F_5^1(-2, 3/2)$, $F_5^6(-2, 3/2)$, $F_5^{10}(-2, 3/2)$, $F_5^1(-1, 2)$, $F_5^6(-1, 2)$ and $F_5^{10}(-1, 2)$. $F_5^4(-3, 0)$ and $F_5^1(-3, 1)$ are energetically degenerated. $F_5^1(-2, 3/2)$, $F_5^6(-2, 3/2)$ and $F_5^{10}(-2, 3/2)$ are degenerated for their energy and isotopic spin, and the same circumstances are observed for $F_5^1(-1, 2)$, $F_5^6(-1, 2)$ and $F_5^{10}(-1, 2)$. Of these particles our feeling is that the states with strangeness -3 will be observed as particle, but it is questionable whether the states with strangeness -2 and -1 will be realized not as resonance states but as meta-stable particles. In this respect two-parameter Formula (3) is not so adequate, since it gives value too critical for such a discussion. If the experiments can detect new particles in future, these new particles will be abovementioned particles. The states with strangeness ≥ 1 will appear with large mass, so they will be observed as resonance states in kaon-nucleon scattering or kaon-hyperon scattering.

ii) Resonance states in π - N scattering

It has been observed that in the π^+-p scattering experiments the cross section shows somewhat resonance-like behavior around 1.3 BeV spread from 1.2 BeV to 1.4 BeV.⁴⁾

This feature is interesting when we consider the existence of four states having $I=3/2$ and corresponding energies. They are $F_5^1(0, 3/2)$, $F_5^4(0, 3/2)$, $F_5^6(0, 3/2)$ and $F_5^{10}(0, 3/2)$. Two of these four states ($F_5^6(0, 3/2)$ and $F_5^{10}(0, 3/2)$) are energetically degenerate. These states lie in space of about 100 MeV. The estimation of the energies of these states by the two-parameter formula seems to give a little lower value than that of the experimental observation if we associate these theoretical levels with the resonance-like behavior around 1.3 BeV. However, such a difference is to be expected, since even in the case of three-body system we have already met the situation that the experimental data of π - N scattering is higher than the theoretical value of Formula (3) by about 150 MeV.

For the mass of five-body state, it will be interesting to make the following estimation. The mass of a sub-particle, for example, $m(NNN\bar{N})$, is given in the case of two-body approximation as follows,

$$m(NNN\bar{N}) = m(NN\bar{N}) + m(N\bar{N}). \quad (7)$$

In this equation we make some correction to the two-body approximation. In the right-hand side of the above Equation (7) we replace the theoretical value given by two-parameter Formula (3), $m(NN\bar{N})$, by the experimental value, $m_{\text{exp}}(NN\bar{N})$, i.e., the total energy in center of mass system of $(3/2, 3/2)$ resonance in π - N scattering,

$$m(NNN\bar{N}\bar{N}) = m_{\text{exp}}(NN\bar{N}) + m(N\bar{N}). \quad (8)$$

Applying a similar method in the case of $m(NN\bar{N}\bar{N})$,* we obtain a new value which gives remarkable fit to the experimental value of the resonance around 1.3 Bev. These values are given in Appendix II.

Of course, we have no assurance that this procedure will always give good results. By applying the similar method to $I=1/2$ case, we find the existence of resonance states at about 1.7 Bev and 2.2 Bev laboratory pion kinetic energies. The experimental data are very poor at this energy region. We must wait for the future experiments.

These resonance states in pion-nucleon scattering is given in Fig. 1, together with the present experimental data.

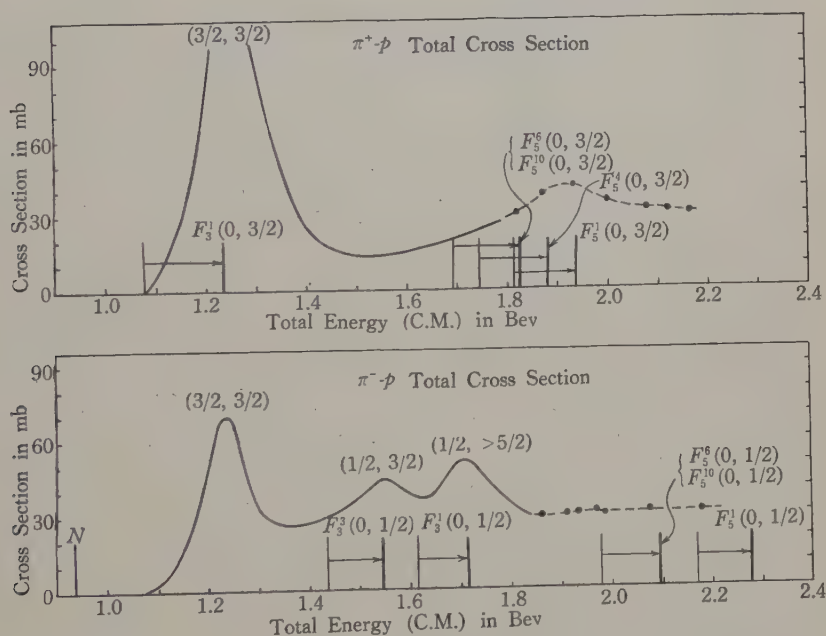


Fig. 1. The theoretical levels of resonance states and the experimental cross sections in π - N scattering

In Figs. 1...4 the levels indicated by thick lines are those of the experimental values for the three-body system and of Formula (8) for the five-body system, and the levels indicated by thin lines are those of Formula (3) for both cases.

* There are two ways of décomposition of $m(NN\bar{N}\bar{N})$, i. e., $m(N\bar{N}\bar{N}) + m(NN)$ and $m(NNN) + m(N\bar{N})$. However, they differ to each other only by about 70 Mev. This might support the above approximation.

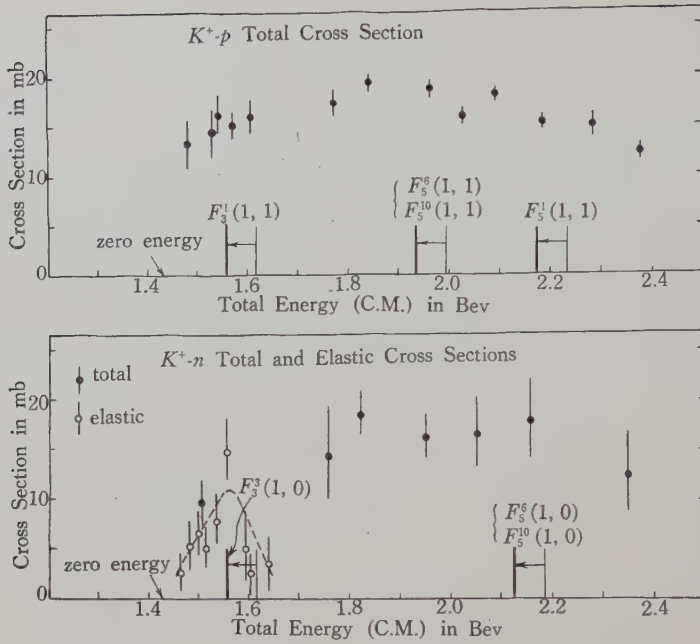


Fig. 2. The theoretical levels of resonance states and the experimental cross sections in K - N scattering

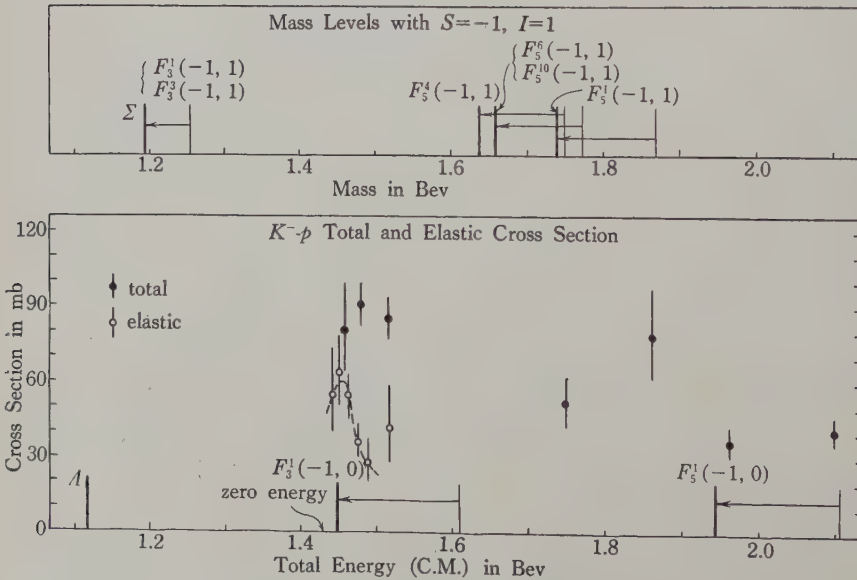


Fig. 3. The theoretical levels of resonance states and the experimental cross sections in \bar{K} - N scattering

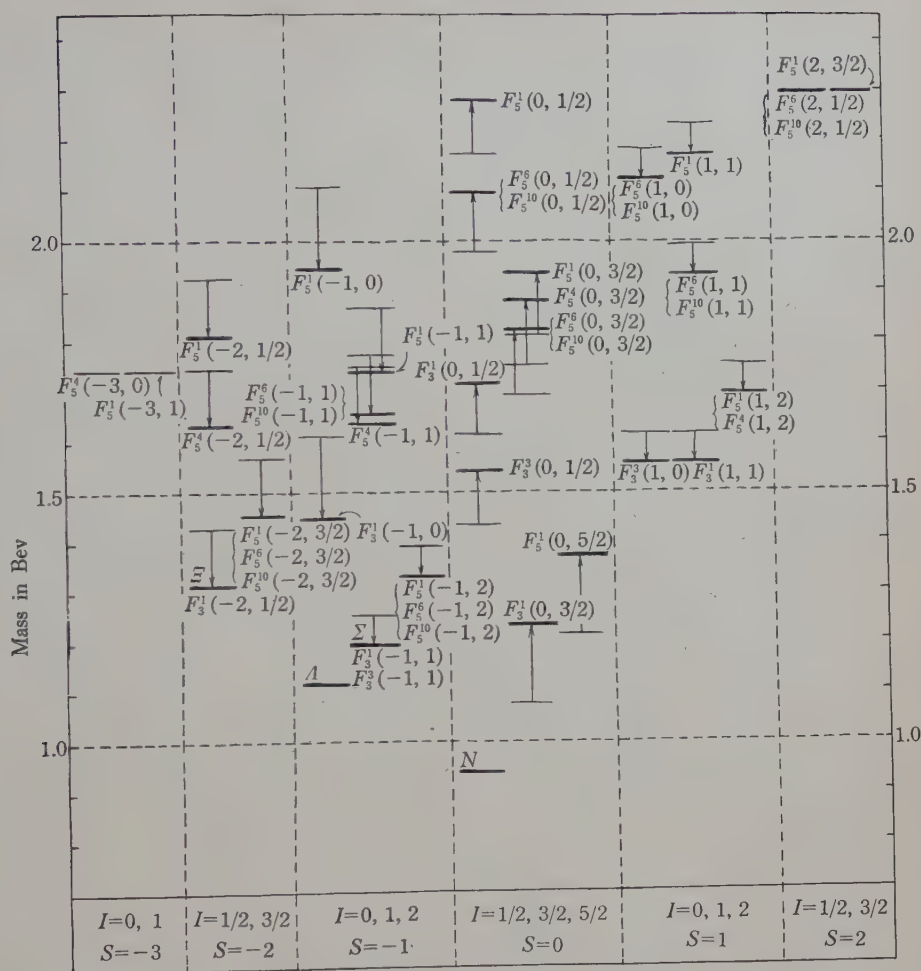


Fig. 4. The level scheme of fermions

iii) Resonance states in $K-N$ scattering and in $\bar{K}-N$ scattering

The experiments of K^+ and K^- scattering by nucleons are at the preliminary stage, and we have not reliable data with which the theoretical prediction is to be compared. At present we may say at most that the gross structure of the energy spectrum of the present experimental cross section may not be inconsistent with the theoretical predictions. The theoretical levels are given in Figs. 2 and 3 with the experimental $K-N$ and $\bar{K}-N$ scattering cross sections.

The whole scheme of fermions up to the five-body system is summarized in Fig. 4.

§ 4. Bosons

Contrary to the case of fermions, the main interest is in "particles", rather

than in "resonance states", since the experimental information about the latter is at present scanty.* Thus the estimation of the masses of these composite states becomes much more important for discussing which levels are stable against rapid decay. However, our knowledge about this point is confined to the two-parameter formula. The two-parameter formula always gives value too critical for such a discussion and we cannot very well base our discussions on it, except in a few cases in which the instability of states seems to be rather apparent.

4-1. The system of one baryon and one antibaryon

$3^2=9$ states are involved in this system. Seven states of them are identified with the observed particles and used to determine the parameters of Formula (3).

The remaining two states are strangeness 0, iso-scalar particles, $B_2^1(0, 0)$ and $B_2^2(0, 0)$. We denote them as $\pi^{0'}$ and $\pi^{0''}$ respectively.

The estimation of the masses for these particles based on Formula (3) gives 615 Mev for $\pi^{0'}$ and 377 Mev for $\pi^{0''}$. For such mass the $\pi^{0'}$ decay into two pions, three pions and four pions are energetically possible. This implies that $\pi^{0'}$ would

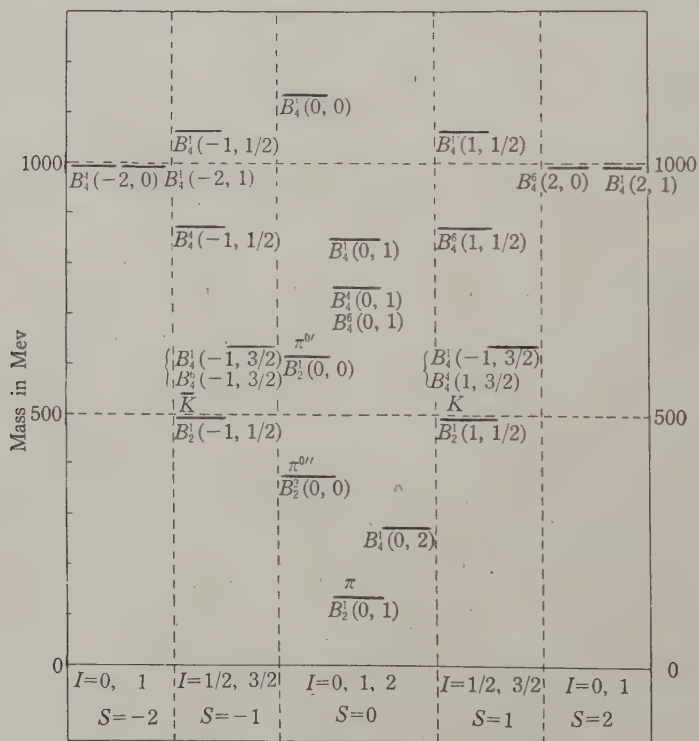


Fig. 5. The level scheme of bosons
The indicated levels are those of Formula (3).

* $B(0, I)$'s are very interesting in connection with the strong pion-pion interaction.

decay rapidly whatever its spin and parity may be, and will not be observed as a particle.

$\pi^{0''}$ is more interesting. Its theoretical value, 377 Mev, is surprisingly close to our previous estimation of the mass of "second neutral meson"* from K_{e3}^+ electron energy spectrum.⁶⁾ $\pi^{0''}$ must be 0^- or 1^+ in order to be observed as a particle.

Besides this reason, we have also another reason to be interested in $\pi^{0''}$. $\pi^{0''}$ is not a characteristic state to the system of one baryon and one anti-baryon. The corresponding lowest configuration state having $M=M'=0$ is the "vacuum". So the argument that $\pi^{0''}$ will not exist may be possible. We do not know whether $\pi^{0''}$ would exist or not. The investigation of this problem will serve for the clarification of the essence of the "vacuum".

4-2. The system of two baryons and two antibaryons

Among the $3^4=81$ states, the new states characteristic of the four-body system are 47. The interest in the four-body system is of course in the point how many states will appear as new particles. For the discussion of this problem, it is necessary to have correct information about the spin, parity and mass of these states. About these points, to our regret, we have no information except the rough estimation of mass by two-parameter formula. Some of them seem to be stable against the rapid decay caused by strong interactions. They are $B_4^4(-2, 0)$, $B_4^1(-2, 1)$, $B_4^1(-1, 3/2)$ and $B_4^0(-1, 3/2)$, and their antiparticles, $B_4^0(2, 0)$, $B_4^1(2, 1)$, $B_4^1(1, 3/2)$ and $B_4^4(1, 3/2)$. Recently some possible evidence for a negative heavy meson with strangeness -2 has been reported.⁷⁾ This particle, if it exists, will be $B_4^4(-2, 0)$ or $B_4^1(-2, 1)$, and we can expect the meson of nearly the same mass and double charge.

The whole scheme of bosons up to the four-body system is summarized in Fig. 5.

§ 5. Summary and Discussion

In this paper we have analysed the relations between the theoretical predictions of Ikeda-Ogawa-Ohnuki's theory and the experimental evidences about the levels of baryon and meson system. And we have found that a rather close relation may be established between them, as far as we are concerned with the reliable experimental data. We have more interesting predictions for which we do not yet have definite experimental data but more accurate experimental investigations is now expected to proceed. About these predictions we shall emphasize the importance

* This value may be a little larger, if we attempt the explanation of the behavior of K_{e3}^+ energy spectrum by the contribution of the mode $K_{e3}^+ \rightarrow \pi^{0''} + e^+ + \nu$ as was examined in our previous article. However, the present estimation about the mass of $A\bar{A}$ sub-particle may contain an error. Whether $\pi^{0''}$ be the main cause for the bimodal feature of the K_{e3}^+ spectrum or not, the leptonic three-body decay of K-meson seems to be the unique process which gives the information about $\pi^{0''}$ in the decay process, if it exists.

of the following points, of which experimental clarification is yet insufficient.

(a) $I=3/2$ resonance states in π^+-p scattering around 1.3 Bev laboratory pion kinetic energies

It is a very interesting problem whether we can clearly separate the present prolonged resonance into three peaks by experiments, as once occurred in $I=1/2$ resonance below 1 Bev. Since the spacing between levels is relatively small, it might not show clear separation of each resonance state.

(b) $I=0$ and $I=1$ resonance in $K-N$ scattering at relatively low energy (~ 200 Mev laboratory kaon kinetic energy)

Experimental information about these levels is relatively scanty. These levels are particularly interesting, since the levels having different isotopic spin are energetically degenerate.

(c) $I=0$ resonance of $\bar{K}-N$ scattering at extremely low energy

It will be very important to confirm our assumption that the level is of $I=0$ by $K^- - n$ scattering experiment.

(d) Among the new particles whose existence is expected, we are particularly interested in $\pi^{0''}$. Information about the mass of $\pi^{0''}$ will give us a more accurate value for the $A-\bar{A}$ binding energy. Its relatively small mass may rather facilitate our experimental detection.

It is possible to construct the states with higher configuration in the I-O-O theory than those we have discussed so far. In the case of fermion there will appear a large number of states which are to be related to resonances in very high energy nuclear reactions. However, the spacing between these higher levels will become smaller, and furthermore the intermixture of levels of the higher configuration states and the lower configuration states might occur. Then we shall recognize them not as individual resonance peaks but as very broad resonance with no peaks.

Among the states of many-body systems we may also have new hyperons with large negative strangeness and mesons with large strangeness, but it will be very hard to find them by experiment since their masses are expected to be very large.

In this paper we have not given the conventional field theoretical arguments for these levels, especially for resonance states. It might be a very difficult problem for the current theory to give the explanation of the independency (or very weak dependency) on M , M' , I and J (spin of particle) of the mass of sub-particle. These curious simplicities together with the spin-paradox pointed out in section 3 suggest us the need of drastical changes in our way of thinking and necessity of a new approach to the problem.

Even if the situation does not need any drastical change and usual mesodynamics has any meaning, we are very doubtful whether such a theory with Yukawa type interaction can succeed in explaining the many resonance levels observed in high energy reaction. If the discussion given in this paper is correct, the states observed as particles and resonance states are the characteristic states which have M and M' different from those of lower configurations, and it will

not be easy to construct such irreducible states from the theory in which the formation of particular mesons and baryons has already occurred. Or it will be very difficult to separate the essentially new states that are responsible for the resonances even if such theory touches the reality in some aspects.

We may say at least that the explanation of the second and third resonances in π^-p scattering in terms of pion-pion interaction will not be adequate,⁸⁾ since the analysis in the present article has shown that A -particle plays an important and essential role in these states.

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Appendix I

Here we give explicit expressions of the basis vectors of the irreducible representations of the system of two-body (one baryon and one antibaryon), three-body (two baryons and one antibaryon), four-body (two baryons and two antibaryons) and five-body (three baryons and two antibaryons).

The mathematical procedures to decompose each system into its irreducible constituents and to obtain the basis vectors of each irreducible representation are detailed in I-O-O's article.³⁾ So we give only information necessary for understanding the mutual correspondence between the classes and the irreducible constituents.

The transformation property of the system of m -baryons and n -antibaryons is equivalent to that of mixed tensor $T_{\mu_1 \mu_2 \dots \mu_m}^{\alpha_1 \alpha_2 \dots \alpha_m}$ of contravariant valence m and covariant valence n . The discussion of the system of two-body and three-body has been given in I-O-O's article. So the problem to be discussed here is the decomposition of $T_{\mu\nu}^{\alpha\beta}$ and $T_{\mu\nu}^{\alpha\beta\tau}$ into their irreducible constituents. In the following we give the decomposition into irreducible constituents and their characterizing equations. The notations keep correspondence with those of reference 3).

(1) The system of two baryons and two antibaryons, $T_{\mu\nu}^{\alpha\beta}$

$T_{\mu\nu}^{\alpha\beta}$ is first decomposed with respect to the upper and lower indices by Young's diagram

$$T_{\mu\nu}^{\alpha\beta} = T_{(\mu\nu)}^{(\alpha\beta)} + T_{[\mu\nu]}^{(\alpha\beta)} + T_{(\mu\nu)}^{[\alpha\beta]} + T_{[\mu\nu]}^{[\alpha\beta]}.$$

Next we decompose $T_{(\mu\nu)}^{(\alpha\beta)}$, $T_{[\mu\nu]}^{(\alpha\beta)}$, $T_{(\mu\nu)}^{[\alpha\beta]}$ and $T_{[\mu\nu]}^{[\alpha\beta]}$ by contraction operation:*

* It is to be noted that the decomposition by the contraction operation is in general not unique.

$$T_{(\mu\nu)}^{(\alpha\beta)} = T_1^{\alpha\beta} + T_2^{\alpha\beta} + T_3^{\alpha\beta},$$

$$T_{[\mu\nu]}^{(\alpha\beta)} = T_4^{\alpha\beta} + T_5^{\alpha\beta},$$

$$T_{(\mu\nu)}^{[\alpha\beta]} = T_6^{\alpha\beta} + T_7^{\alpha\beta},$$

$$T_{[\mu\nu]}^{[\alpha\beta]} = T_8^{\alpha\beta} + T_9^{\alpha\beta},$$

where

$$T_2^{\alpha\beta} = \partial_\mu^\alpha T_2^\beta + \partial_\nu^\alpha T_2^\beta + \partial_\mu^\beta T_2^\alpha + \partial_\nu^\beta T_2^\alpha,$$

$$T_3^{\alpha\beta} = (\partial_\mu^\alpha \partial_\nu^\beta + \partial_\nu^\alpha \partial_\mu^\beta) T,$$

$$T_5^{\alpha\beta} = \partial_\mu^\alpha T_5^\beta - \partial_\nu^\alpha T_5^\beta + \partial_\mu^\beta T_5^\alpha - \partial_\nu^\beta T_5^\alpha,$$

$$T_7^{\alpha\beta} = \partial_\mu^\alpha T_7^\beta + \partial_\nu^\alpha T_7^\beta - \partial_\mu^\beta T_7^\alpha - \partial_\nu^\beta T_7^\alpha,$$

$$T_8^{\alpha\beta} = \partial_\mu^\alpha T_8^\beta - \partial_\nu^\alpha T_8^\beta - \partial_\mu^\beta T_8^\alpha + \partial_\nu^\beta T_8^\alpha,$$

$$T_9^{\alpha\beta} = (\partial_\mu^\alpha \partial_\nu^\beta - \partial_\nu^\alpha \partial_\mu^\beta) T,$$

and besides the symmetry properties, $T_{\mu\nu}^\alpha$, $T_{\mu\nu}^\beta$, $T_{\mu\nu}^{\alpha\beta}$, $T_{\mu\nu}^\alpha$, $T_{\mu\nu}^{\alpha\beta}$, $T_{\mu\nu}^\alpha$ and $T_{\mu\nu}^\alpha$ are characterized by

$$T_{\lambda\nu}^{\lambda\beta} = T_{\lambda}^\lambda = T_{\lambda\nu}^{\lambda\beta} = T_{\lambda}^\lambda = T_{\lambda\nu}^{\lambda\beta} = T_{\lambda}^\lambda = T_{\lambda}^\lambda = 0.$$

$T_{\mu\nu}^{\alpha\beta}$, \dots , $T_{\mu\nu}^{\alpha\beta}$ correspond to class 1, \dots , class 9 of the system of four-body respectively.

(2) The system of three baryons and two antibaryons, $T_{\mu\nu}^{\alpha\beta\gamma}$

We first decompose $T_{\mu\nu}^{\alpha\beta\gamma}$ by Young's diagram with respect to upper and lower indices,

$$T_{\mu\nu}^{\alpha\beta\gamma} = T_{(\mu\nu)}^{(\alpha\beta\gamma)} + T_{[\mu\nu]}^{(\alpha\beta\gamma)} + T_{(\mu\nu)}^{[\alpha\beta\gamma]} + T_{(\mu\nu)}^{[\alpha\beta\gamma]} + T_{[\mu\nu]}^{(\alpha\beta\gamma)} + T_{[\mu\nu]}^{[\alpha\beta\gamma]} + T_{(\mu\nu)}^{[\alpha\beta\gamma]} + T_{[\mu\nu]}^{(\alpha\beta\gamma)},$$

where $T_{(\mu\nu)}^{[\alpha\beta\gamma]}$ and $T_{(\mu\nu)}^{[\alpha\beta\gamma]}$ satisfy

$$T_{(\mu\nu)}^{[\alpha\beta\gamma]} + T_{(\mu\nu)}^{[\beta\gamma\alpha]} + T_{(\mu\nu)}^{[\gamma\alpha\beta]} = 0,$$

and

$$T_{(\mu\nu)}^{[\alpha\beta\gamma]} + T_{(\mu\nu)}^{[\beta\gamma\alpha]} + T_{(\mu\nu)}^{[\gamma\alpha\beta]} = 0,$$

respectively, and $T_{[\mu\nu]}^{[\alpha\beta\gamma]}$ and $T_{[\mu\nu]}^{[\alpha\beta\gamma]}$ satisfy similar conditions.

$T_{(\mu\nu)}^{(\alpha\beta\gamma)}$, $T_{[\mu\nu]}^{(\alpha\beta\gamma)}$, $T_{(\mu\nu)}^{[\alpha\beta\gamma]}$, $T_{(\mu\nu)}^{[\alpha\beta\gamma]}$, $T_{[\mu\nu]}^{(\alpha\beta\gamma)}$ and $T_{[\mu\nu]}^{[\alpha\beta\gamma]}$ are decomposed further keeping their symmetricities by applying the contraction operation as follows:

$$T_{(\mu\nu)}^{(\alpha\beta\gamma)} = T_1^{\alpha\beta\gamma} + T_2^{\alpha\beta\gamma} + T_3^{\alpha\beta\gamma},$$

$$T_{[\mu\nu]}^{(\alpha\beta\gamma)} = T_4^{\alpha\beta\gamma} + T_5^{\alpha\beta\gamma},$$

$$T_{(\mu\nu)}^{[\alpha\beta\gamma]} = T_6^{\alpha\beta\gamma} + T_7^{\alpha\beta\gamma} + T_8^{\alpha\beta\gamma} + T_9^{\alpha\beta\gamma},$$

$$T_{(\mu\nu)}^{\alpha\beta\tau} = T_{10}^{\alpha\beta\tau} + T_{11}^{\alpha\beta\tau} + T_{12}^{\alpha\beta\tau} + T_{13}^{\alpha\beta\tau},$$

$$T_{[\mu\nu]}^{\alpha\beta\tau} = T_{14}^{\alpha\beta\tau} + T_{15}^{\alpha\beta\tau} + T_{16}^{\alpha\beta\tau},$$

$$T_{[\mu\nu]}^{\alpha\beta\tau} = T_{17}^{\alpha\beta\tau} + T_{18}^{\alpha\beta\tau} + T_{19}^{\alpha\beta\tau}$$

where

$$T_{2\mu\nu}^{\alpha\beta\tau} = \partial_{\mu}^{\alpha} T_{2\nu}^{\beta\tau} + \partial_{\nu}^{\beta} T_{2\mu}^{\alpha\tau} + \partial_{\mu}^{\beta} T_{2\nu}^{\tau\alpha} + \partial_{\nu}^{\alpha} T_{2\mu}^{\tau\beta} + \partial_{\mu}^{\tau} T_{2\nu}^{\alpha\beta} + \partial_{\nu}^{\tau} T_{2\mu}^{\alpha\beta},$$

$$T_{3\mu\nu}^{\alpha\beta\tau} = (\partial_{\mu}^{\alpha} \partial_{\nu}^{\beta} + \partial_{\nu}^{\alpha} \partial_{\mu}^{\beta}) T_{3\tau}^{\tau} + (\partial_{\mu}^{\beta} \partial_{\nu}^{\alpha} + \partial_{\nu}^{\beta} \partial_{\mu}^{\alpha}) T_{3\tau}^{\alpha} + (\partial_{\mu}^{\alpha} \partial_{\nu}^{\beta} + \partial_{\nu}^{\alpha} \partial_{\mu}^{\beta}) T_{3\tau}^{\beta},$$

$$T_{5\mu\nu}^{\alpha\beta\tau} = \partial_{\mu}^{\alpha} T_{5\nu}^{\beta\tau} - \partial_{\nu}^{\beta} T_{5\mu}^{\alpha\tau} + \partial_{\mu}^{\beta} T_{5\nu}^{\tau\alpha} - \partial_{\nu}^{\alpha} T_{5\mu}^{\tau\beta} + \partial_{\mu}^{\tau} T_{5\nu}^{\alpha\beta} - \partial_{\nu}^{\tau} T_{5\mu}^{\alpha\beta},$$

$$T_{7\mu\nu}^{\alpha\beta\tau} = \partial_{\mu}^{\beta} T_{7\nu}^{\alpha\tau} + \partial_{\nu}^{\alpha} T_{7\mu}^{\beta\tau} - \partial_{\mu}^{\tau} T_{7\nu}^{\alpha\beta} - \partial_{\nu}^{\tau} T_{7\mu}^{\alpha\beta},$$

$$T_{8\mu\nu}^{\alpha\beta\tau} = \partial_{\mu}^{\beta} T_{8\nu}^{\alpha\tau} + \partial_{\nu}^{\alpha} T_{8\mu}^{\beta\tau} - \partial_{\mu}^{\tau} T_{8\nu}^{\alpha\beta} - \partial_{\nu}^{\tau} T_{8\mu}^{\alpha\beta},$$

$$T_{9\mu\nu}^{\alpha\beta\tau} = (\partial_{\mu}^{\alpha} \partial_{\nu}^{\beta} + \partial_{\nu}^{\alpha} \partial_{\mu}^{\beta}) T_{9\tau}^{\tau} - (\partial_{\mu}^{\alpha} \partial_{\nu}^{\beta} + \partial_{\nu}^{\alpha} \partial_{\mu}^{\beta}) T_{9\tau}^{\beta},$$

$$T_{11\mu\nu}^{\alpha\beta\tau} = \partial_{\mu}^{\alpha} T_{11\nu}^{\beta\tau} + \partial_{\nu}^{\beta} T_{11\mu}^{\alpha\tau} - \partial_{\mu}^{\beta} T_{11\nu}^{\alpha\tau} - \partial_{\nu}^{\alpha} T_{11\mu}^{\beta\tau},$$

$$T_{12\mu\nu}^{\alpha\beta\tau} = \partial_{\mu}^{\alpha} T_{12\nu}^{\beta\tau} + \partial_{\nu}^{\beta} T_{12\mu}^{\alpha\tau} - \partial_{\mu}^{\beta} T_{12\nu}^{\alpha\tau} - \partial_{\nu}^{\alpha} T_{12\mu}^{\beta\tau},$$

$$T_{13\mu\nu}^{\alpha\beta\tau} = (\partial_{\mu}^{\alpha} \partial_{\nu}^{\beta} + \partial_{\nu}^{\alpha} \partial_{\mu}^{\beta}) T_{13\tau}^{\tau} - (\partial_{\mu}^{\alpha} \partial_{\nu}^{\beta} + \partial_{\nu}^{\alpha} \partial_{\mu}^{\beta}) T_{13\tau}^{\alpha},$$

$$T_{14\mu\nu}^{\alpha\beta\tau} = \partial_{\mu}^{\beta} T_{14\nu}^{\alpha\tau} - \partial_{\nu}^{\beta} T_{14\mu}^{\alpha\tau} - \partial_{\mu}^{\tau} T_{14\nu}^{\alpha\beta} + \partial_{\nu}^{\tau} T_{14\mu}^{\alpha\beta},$$

$$T_{15\mu\nu}^{\alpha\beta\tau} = \partial_{\mu}^{\beta} T_{15\nu}^{\alpha\tau} - \partial_{\nu}^{\beta} T_{15\mu}^{\alpha\tau} - \partial_{\mu}^{\tau} T_{15\nu}^{\alpha\beta} + \partial_{\nu}^{\tau} T_{15\mu}^{\alpha\beta},$$

$$T_{16\mu\nu}^{\alpha\beta\tau} = (\partial_{\mu}^{\alpha} \partial_{\nu}^{\beta} - \partial_{\nu}^{\alpha} \partial_{\mu}^{\beta}) T_{16\tau}^{\tau} - (\partial_{\mu}^{\alpha} \partial_{\nu}^{\beta} - \partial_{\nu}^{\alpha} \partial_{\mu}^{\beta}) T_{16\tau}^{\beta},$$

$$T_{17\mu\nu}^{\alpha\beta\tau} = \partial_{\mu}^{\alpha} T_{17\nu}^{\beta\tau} - \partial_{\nu}^{\alpha} T_{17\mu}^{\beta\tau} - \partial_{\mu}^{\beta} T_{17\nu}^{\alpha\tau} + \partial_{\nu}^{\beta} T_{17\mu}^{\alpha\tau},$$

$$T_{18\mu\nu}^{\alpha\beta\tau} = \partial_{\mu}^{\alpha} T_{18\nu}^{\beta\tau} - \partial_{\nu}^{\alpha} T_{18\mu}^{\beta\tau} - \partial_{\mu}^{\beta} T_{18\nu}^{\alpha\tau} + \partial_{\nu}^{\beta} T_{18\mu}^{\alpha\tau},$$

$$T_{19\mu\nu}^{\alpha\beta\tau} = (\partial_{\mu}^{\alpha} \partial_{\nu}^{\beta} - \partial_{\nu}^{\alpha} \partial_{\mu}^{\beta}) T_{19\tau}^{\tau} - (\partial_{\mu}^{\alpha} \partial_{\nu}^{\beta} - \partial_{\nu}^{\alpha} \partial_{\mu}^{\beta}) T_{19\tau}^{\alpha},$$

and $T_{1\mu\nu}^{\alpha\beta\tau}$, $T_{2\mu\nu}^{\alpha\beta}$, $T_{4\mu\nu}^{\alpha\beta\tau}$, $T_{5\mu\nu}^{\alpha\beta}$, $T_{6\mu\nu}^{\alpha\beta\tau}$, $T_{7\mu\nu}^{\alpha\beta}$, $T_{8\mu\nu}^{\alpha\beta}$, $T_{10\mu\nu}^{\alpha\beta\tau}$, $T_{11\mu\nu}^{\alpha\beta}$, $T_{12\mu\nu}^{\alpha\beta}$, $T_{14\mu\nu}^{\alpha\beta}$, $T_{15\mu\nu}^{\alpha\beta}$, $T_{17\mu\nu}^{\alpha\beta}$ and $T_{18\mu\nu}^{\alpha\beta}$ are characterized by

$$\begin{aligned} T_{1\lambda\nu}^{\lambda\beta\tau} &= T_{2\lambda}^{\lambda\beta} = T_{4\lambda\nu}^{\lambda\beta\tau} = T_{5\lambda}^{\lambda\beta} = T_{6\lambda\nu}^{\lambda\beta\tau} = T_{7\lambda}^{\lambda\beta} = T_{8\lambda\nu}^{\lambda\beta} = T_{10\lambda\nu}^{\lambda\beta\tau} = T_{11\lambda}^{\lambda\beta} = T_{12\lambda}^{\lambda\beta} \\ &= T_{14\lambda}^{\lambda\beta} = T_{15\lambda}^{\lambda\beta} = T_{17\lambda}^{\lambda\beta} = T_{18\lambda}^{\lambda\beta} = 0. \end{aligned}$$

$T_{2\mu}^{\alpha\beta}$, $T_{5\mu}^{\alpha\beta}$, $T_{7\mu}^{\alpha\beta}$, $T_{11\mu}^{\alpha\beta}$, $T_{14\mu}^{\alpha\beta}$ and $T_{17\mu}^{\alpha\beta}$ are symmetric while $T_{8\mu}^{\alpha\beta}$, $T_{12\mu}^{\alpha\beta}$, $T_{15\mu}^{\alpha\beta}$ and $T_{18\mu}^{\alpha\beta}$ are antisymmetric with respect to the upper indices α and β . $T_{(\mu\nu)}^{\alpha\beta\tau}$ and $T_{[\mu\nu]}^{\alpha\beta\tau}$ are no more decomposed:

$$T_{(\mu\nu)}^{[\alpha\beta\gamma]} = T_{20}^{\alpha\beta\gamma} = \partial_{\mu}^{\alpha} T_{20}^{\beta\gamma} + \partial_{\nu}^{\alpha} T_{20}^{\beta\gamma} + \partial_{\mu}^{\beta} T_{20}^{\gamma\alpha} + \partial_{\nu}^{\beta} T_{20}^{\gamma\alpha} + \partial_{\mu}^{\gamma} T_{20}^{\alpha\beta} + \partial_{\nu}^{\gamma} T_{20}^{\alpha\beta},$$

$$T_{21}^{[\alpha\beta\gamma]} = T_{21}^{\alpha\beta\gamma} = (\partial_{\mu}^{\alpha} \partial_{\nu}^{\beta} - \partial_{\nu}^{\alpha} \partial_{\mu}^{\beta}) T_{21}^{\gamma} + (\partial_{\mu}^{\beta} \partial_{\nu}^{\gamma} - \partial_{\nu}^{\beta} \partial_{\mu}^{\gamma}) T_{21}^{\alpha} + (\partial_{\mu}^{\gamma} \partial_{\nu}^{\alpha} - \partial_{\nu}^{\gamma} \partial_{\mu}^{\alpha}) T_{21}^{\beta},$$

where $T_{\mu}^{\alpha\beta}$ has the symmetry property, $T_{20}^{\alpha\beta} = -T_{20}^{\beta\alpha}$.

$T_{1}^{\alpha\beta\gamma}$, ..., $T_{21}^{\alpha\beta\gamma}$ correspond to class 1, ..., class 21 of the system of five-body respectively.

Each class is characterized by quantum number M and M' which are given by*

$$M = \frac{1}{2} [q_0^2 + s_0^2 + (n_B - q_0 + s_0)^2 + 2(q_0 + s_0)],$$

$$M' = \frac{1}{2} [q_0^3 - s_0^3 + (n_B - q_0 + s_0)^3 + 3(q_0^2 - s_0^2) - (q_0 + s_0) + 4l_0]$$

where n_B is baryon number, s_0 is the maximum value of strangeness in the class, l_0 is the maximum eigen-value of I_3 of the states having strangeness s_0 , and q_0 is the charge of the state having s_0 and l_0 , i. e., $q_0 = l_0 + (n_B + s_0)/2$.

Notations used in the following tables

$$(AB) = A(x)B(y) + B(x)A(y),$$

$$[AB] = A(x)B(y) - B(x)A(y),$$

$$(ABC) = A(x)B(y)C(z) + A(x)C(y)B(z) + B(x)C(y)A(z) \\ + B(x)A(y)C(z) + C(x)A(y)B(z) + C(x)B(y)A(z),$$

$$(AAB) = A(x)A(y)B(z) + A(x)B(y)A(z) + B(x)A(y)A(z),$$

$$[ABC] = A(x)B(y)C(z) - A(x)C(y)B(z) + B(x)C(y)A(z), \\ - B(x)A(y)C(z) + C(x)A(y)B(z) - C(x)B(y)A(z),$$

where A , B and C are all different from each other.

In the case of five-body system the configurations of classes 10, 11, 12, 13; 17, 18, 19 are obtained from classes 6, 7, 8, 9; 14, 15, 16 by the exchange $A[BC] \rightarrow [BC]A$.

Table I. Configuration of the system of one baryon and one antibaryon

Class 1. $M=3$, $M'=0$

$B_2^{-1} (-1, 1/2)$	$-A\bar{n}$ $A\bar{p}$
$B_2^{-1} (0, 0)$	$\{-p\bar{p} - n\bar{n} + 2A\bar{A}\}/\sqrt{6}$

* M and M' in this paper are different from those of I-O-O's article³⁾ by factors 1/2 and 1/8 respectively. The expression of M' in I-O-O's is, unfortunately, not correct about the sign of the second term.

$B_2^1 (0, 1)$	$-p\bar{n}$ $\{p\bar{p}-n\bar{n}\}/\sqrt{2}$ $n\bar{p}$
$B_2^1 (1, 1/2)$	$p\bar{A}$ $n\bar{A}$

 Class 2. $M=0, M'=0$

$B_2^3 (0, 0)$	$\{p\bar{p}+n\bar{n}+A\bar{A}\}/\sqrt{3}$
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Table II. Configuration of the system of two baryons and one antibaryon

 Class 1. $M=11/2, M'=17/2$

$F_3^1 (-2, 1/2)$	$-AA\bar{n}$ $AA\bar{p}$
$F_3^1 (-1, 0)$	$\{-(pA)\bar{p}-(nA)\bar{n}+2A\bar{A}\}/2\sqrt{2}$
$F_3^1 (-1, 1)$	$-(pA)\bar{n}/\sqrt{2}$ $\{(pA)\bar{p}-(nA)\bar{n}\}/2$ $(nA)\bar{p}/\sqrt{2}$
$F_3^1 (0, 1/2)$	$\{-2p\bar{p}-(pn)\bar{n}+3(pA)\bar{A}\}/2\sqrt{6}$ $\{-(pn)\bar{p}-2nn\bar{n}+3(nA)\bar{A}\}/2\sqrt{6}$
$F_3^1 (0, 3/2)$	$-p\bar{p}\bar{n}$ $\{p\bar{p}\bar{p}-(pn)\bar{n}\}/\sqrt{3}$ $\{(pn)\bar{p}-nn\bar{n}\}/\sqrt{3}$ $nn\bar{p}$
$F_3^1 (1, 1)$	$p\bar{p}\bar{A}$ $(pn)\bar{A}/\sqrt{2}$ $nn\bar{A}$

 Class 2. $M=3/2, M'=5/2$

$F_3^2 (-1, 0)$	$\{(pA)\bar{p}+(nA)\bar{n}+2A\bar{A}\}/2\sqrt{2}$
$F_3^2 (0, 1/2)$	$\{2p\bar{p}\bar{p}+(pn)\bar{n}+(pA)\bar{A}\}/2\sqrt{2}$ $\{(pn)\bar{p}+2nn\bar{n}+(nA)\bar{A}\}/2\sqrt{2}$

 Class 3. $M=7/2, M'=-1/2$

$F_3^3 (-1, 1)$	$-[p\bar{A}]\bar{n}/\sqrt{2}$ $\{[pA]\bar{p}-[nA]\bar{n}\}/2$ $[nA]\bar{p}/\sqrt{2}$
$F_3^3 (0, 1/2)$	$\{-[pn]\bar{n}+[pA]\bar{A}\}/2$ $\{[pn]\bar{p}+[nA]\bar{A}\}/2$
$F_3^3 (1, 0)$	$[pn]\bar{A}/\sqrt{2}$

Class 4. $M=3/2$, $M'=5/2$

F_8^4 $(-1, 0)$	$\{[pA]\bar{p} + [nA]\bar{n}\}/2$
F_8^4 $(0, 1/2)$	$\{[pn]\bar{n} + [pA]\bar{A}\}/2$ $\{-[pn]\bar{p} + [nA]\bar{A}\}/2$

Table III. Configuration of the system of two baryons and two antibaryons

Class 1. $M=8$, $M'=0$

B_4^1 $(-2, 1)$	$\Lambda A \bar{n} \bar{n}$ $-\Lambda A (\bar{p} \bar{n})/\sqrt{2}$ $\Lambda A \bar{p} \bar{p}$
B_4^1 $(-1, 1/2)$	$\{(pA)(\bar{p} \bar{n}) + 2(nA)\bar{n} \bar{n} - 3\Lambda A (\bar{n} \bar{A})\}/\sqrt{30}$ $\{-2(pA)\bar{p} \bar{p} - (nA)(\bar{n} \bar{p}) + 3\Lambda A (\bar{p} \bar{A})\}/\sqrt{30}$
B_4^1 $(-1, 3/2)$	$(pA)\bar{n} \bar{n}/\sqrt{2}$ $\{-(pA)(\bar{p} \bar{n}) + (nA)\bar{n} \bar{n}\}/\sqrt{6}$ $\{(pA)\bar{p} \bar{p} - (nA)(\bar{p} \bar{n})\}/\sqrt{6}$ $(nA)\bar{p} \bar{p}/\sqrt{2}$
B_4^1 $(0, 0)$	$\{2p\bar{p}\bar{p} + (pn)(\bar{p} \bar{n}) + 2n\bar{n} \bar{n} - 3(pA)(\bar{p} \bar{A}) - 3(nA)(\bar{n} \bar{A}) + 6\Lambda A \bar{A} \bar{A}\}/2\sqrt{30}$
B_4^1 $(0, 1)$	$\{p\bar{p}(\bar{p} \bar{n}) + (pn)\bar{n} \bar{n} - 2(pA)(\bar{n} \bar{A})\}/2\sqrt{5}$ $\{-p\bar{p}\bar{p} + n\bar{n} \bar{n} + (pA)(\bar{p} \bar{A}) - (nA)(\bar{n} \bar{A})\}/\sqrt{10}$ $\{-(pn)\bar{p} \bar{p} - n\bar{n}(\bar{p} \bar{n}) + 2(nA)(\bar{p} \bar{A})\}/2\sqrt{5}$
B_4^1 $(0, 2)$	$p\bar{p}\bar{n} \bar{n}$ $\{-p\bar{p}(\bar{p} \bar{n}) + (pn)\bar{n} \bar{n}\}/2$ $\{p\bar{p}\bar{p} - (pn)(\bar{p} \bar{n}) + n\bar{n} \bar{n}\}/\sqrt{6}$ $\{(pn)\bar{p} \bar{p} - n\bar{n}(\bar{p} \bar{n})\}/2$ $n\bar{n}\bar{p} \bar{p}$
B_4^1 $(1, 1/2)$	$\{-2p\bar{p}(\bar{p} \bar{A}) - (pn)(\bar{n} \bar{A}) + 3(pA)\bar{A} \bar{A}\}/\sqrt{30}$ $\{-(pn)(\bar{p} \bar{A}) - 2n\bar{n}(\bar{n} \bar{A}) + 3(nA)\bar{A} \bar{A}\}/\sqrt{30}$
B_4^1 $(1, 3/2)$	$-p\bar{p}(\bar{n} \bar{A})/\sqrt{2}$ $\{p\bar{p}(\bar{p} \bar{A}) - (pn)(\bar{n} \bar{A})\}/\sqrt{6}$ $\{(pn)(\bar{p} \bar{A}) - n\bar{n}(\bar{n} \bar{A})\}/\sqrt{6}$ $n\bar{n}(\bar{p} \bar{A})/\sqrt{2}$
B_4^1 $(2, 1)$	$p\bar{p}\bar{A} \bar{A}$ $(pn)\bar{A} \bar{A}/\sqrt{2}$ $n\bar{n}\bar{A} \bar{A}$

Class 2. $M=3$, $M'=0$

B_4^2 $(-1, 1/2)$	$\{-(pA)(\bar{p} \bar{n}) - 2(nA)\bar{n} \bar{n} - 2\Lambda A (\bar{n} \bar{A})\}/2\sqrt{5}$ $\{2(pA)\bar{p} \bar{p} + (nA)(\bar{p} \bar{n}) + 2\Lambda A (\bar{p} \bar{A})\}/2\sqrt{5}$
B_4^2 $(0, 0)$	$\{-4p\bar{p}\bar{p} - 2(pn)(\bar{p} \bar{n}) - 4n\bar{n} \bar{n} + (pA)(\bar{p} \bar{A}) + (nA)(\bar{n} \bar{A}) + 8\Lambda A \bar{A} \bar{A}\}/2\sqrt{30}$

$B_4^2 (0, 1)$	$\{-2p\bar{p}(\bar{p}\bar{n}) - 2(pn)\bar{n}\bar{n} - (pA)(\bar{n}\bar{A})\}/2\sqrt{5}$ $\{4p\bar{p}\bar{p}\bar{p} - 4nn\bar{n}\bar{n} + (pA)(\bar{p}\bar{A}) - (nA)(\bar{n}\bar{A})\}/2\sqrt{10}$ $\{2(pn)\bar{p}\bar{p} + 2nn(\bar{p}\bar{n}) + (nA)(\bar{p}\bar{A})\}/2\sqrt{5}$
$B_4^2 (1, 1/2)$	$\{2p\bar{p}(\bar{p}\bar{A}) + (pn)(\bar{n}\bar{A}) + 2(pA)\bar{A}\bar{A}\}/2\sqrt{5}$ $\{(pn)(\bar{p}\bar{A}) + 2nn(\bar{n}\bar{A}) + 2(nA)\bar{A}\bar{A}\}/2\sqrt{5}$

 Class 3. $M=0, M'=0$

$B_4^3 (0, 0)$	$\{2p\bar{p}\bar{p}\bar{p} + (pn)(\bar{p}\bar{n}) + 2nn\bar{n}\bar{n} + (pA)(\bar{p}\bar{A}) + (nA)(\bar{n}\bar{A}) + 2AA\bar{A}\bar{A}\}/2\sqrt{6}$
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 Class 4. $M=6, M'=9$

$B_4^4 (-2, 0)$	$AA[\bar{p}\bar{n}]/\sqrt{2}$
$B_4^4 (-1, 1/2)$	$\{(pA)[\bar{p}\bar{n}] + AA[\bar{n}\bar{A}]\}/\sqrt{6}$ $\{(nA)[\bar{p}\bar{n}] - AA[\bar{p}\bar{A}]\}/\sqrt{6}$
$B_4^4 (0, 1)$	$\{p\bar{p}[\bar{p}\bar{n}] + (pA)[\bar{n}\bar{A}]\}/\sqrt{6}$ $\{(pn)[\bar{p}\bar{n}] - (pA)[\bar{p}\bar{A}] + (nA)[\bar{n}\bar{A}]\}/2\sqrt{3}$ $\{nn[\bar{p}\bar{n}] - (nA)[\bar{p}\bar{A}]\}/\sqrt{6}$
$B_4^4 (1, 3/2)$	$p\bar{p}[\bar{n}\bar{A}]/\sqrt{2}$ $\{-p\bar{p}[\bar{p}\bar{A}] + (pn)[\bar{n}\bar{A}]\}/\sqrt{6}$ $\{-(pn)[\bar{p}\bar{A}] + nn[\bar{n}\bar{A}]\}/\sqrt{6}$ $-nn[\bar{p}\bar{A}]/\sqrt{2}$

 Class 5. $M=3, M'=0$

$B_4^5 (-1, 1/2)$	$\{-(pA)[\bar{p}\bar{n}] + 2AA[\bar{n}\bar{A}]\}/2\sqrt{3}$ $\{-(nA)[\bar{p}\bar{n}] - 2AA[\bar{p}\bar{A}]\}/2\sqrt{3}$
$B_4^5 (0, 0)$	$\{(pA)[\bar{p}\bar{A}] + (nA)[\bar{n}\bar{A}]\}/2\sqrt{2}$
$B_4^5 (0, 1)$	$\{-2p\bar{p}\bar{p}[\bar{n}] + (pA)[\bar{n}\bar{A}]\}/2\sqrt{3}$ $\{-2(pn)[\bar{p}\bar{n}] - (pA)[\bar{p}\bar{A}] + (nA)[\bar{n}\bar{A}]\}/2\sqrt{6}$ $\{-2nn[\bar{p}\bar{n}] - (nA)[\bar{p}\bar{A}]\}/2\sqrt{3}$
$B_4^5 (1, 1/2)$	$\{2p\bar{p}[\bar{p}\bar{A}] + (pn)[\bar{n}\bar{A}]\}/2\sqrt{3}$ $\{(pn)[\bar{p}\bar{A}] + 2nn[\bar{n}\bar{A}]\}/2\sqrt{3}$

 Class 6. $M=6, M'=-9$

$B_4^6 (-1, 3/2)$	$[pA]\bar{n}\bar{n}/\sqrt{2}$ $\{-[pA](\bar{p}\bar{n}) + [nA]\bar{n}\bar{n}\}/\sqrt{6}$ $\{[pA]\bar{p}\bar{p} - [nA](\bar{p}\bar{n})\}/\sqrt{6}$ $[nA]\bar{p}\bar{p}/\sqrt{2}$
$B_4^6 (0, 1)$	$\{[pn]\bar{n}\bar{n} - [pA](\bar{n}\bar{A})\}/\sqrt{6}$ $\{-[pn](\bar{p}\bar{n}) + [pA](\bar{p}\bar{A}) - [nA](\bar{n}\bar{A})\}/2\sqrt{3}$ $\{[pn]\bar{p}\bar{p} + [nA](\bar{p}\bar{A})\}/\sqrt{6}$

$B_4^6 (1, 1/2)$	$\{-[pn](\bar{n}\bar{A}) + [pA]\bar{A}\bar{A}\}/\sqrt{6}$ $\{[pn](\bar{p}\bar{A}) + [nA]\bar{A}\bar{A}\}/\sqrt{6}$
$B_4^6 (2, 0)$	$[pn]\bar{A}\bar{A}/\sqrt{2}$

Class 7. $M=3, M'=0$

$B_4^7 (-1, 1/2)$	$\{-[pA](\bar{p}\bar{n}) - 2[nA]\bar{n}\bar{n}\}/2\sqrt{3}$ $\{2[pA]\bar{p}\bar{p} + [nA](\bar{p}\bar{n})\}/2\sqrt{3}$
$B_4^7 (0, 0)$	$\{[pA](\bar{p}\bar{A}) + [nA](\bar{n}\bar{A})\}/2\sqrt{2}$
$B_4^7 (0, 1)$	$\{2[pn]\bar{n}\bar{n} + [pA](\bar{n}\bar{A})\}/2\sqrt{3}$ $\{-2[pn](\bar{p}\bar{n}) - [pA](\bar{p}\bar{A}) + [nA](\bar{n}\bar{A})\}/2\sqrt{6}$ $\{2[pn]\bar{p}\bar{p} - [nA](\bar{p}\bar{A})\}/2\sqrt{3}$
$B_4^7 (1, 1/2)$	$\{-[pn](\bar{n}\bar{A}) - 2[pA]\bar{A}\bar{A}\}/2\sqrt{3}$ $\{[pn](\bar{p}\bar{A}) - 2[nA]\bar{A}\bar{A}\}/2\sqrt{3}$

Class 8. $M=3, M'=0$

$B_4^8 (-1, 1/2)$	$-[pA][\bar{p}\bar{n}]/2$ $-[nA][\bar{p}\bar{n}]/2$
$B_4^8 (0, 0)$	$\{-2[pn][\bar{p}\bar{n}] + [pA][\bar{p}\bar{A}] + [nA][\bar{n}\bar{A}]\}/2\sqrt{6}$
$B_4^8 (0, 1)$	$-[pA][\bar{n}\bar{A}]/2$ $\{[pA][\bar{p}\bar{A}] - [nA][\bar{n}\bar{A}]\}/2\sqrt{2}$ $[nA][\bar{p}\bar{A}]/2$
$B_4^8 (1, 1/2)$	$-[pn][\bar{n}\bar{A}]/2$ $[pn][\bar{p}\bar{A}]/2$

Class 9. $M=0, M'=0$

$B_4^9 (0, 0)$	$\{[pn][\bar{p}\bar{n}] + [pA][\bar{p}\bar{A}] + [nA][\bar{n}\bar{A}]\}/2\sqrt{3}$
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Table IV. Configuration of the system of three baryons and two antibaryons

Class 1. $M=23/2, M'=35/2$

$F_8^1 (-3, 1)$	$\Lambda\Lambda\Lambda\bar{n}\bar{n}$ $-\Lambda\Lambda\Lambda(\bar{p}\bar{n})/\sqrt{2}$ $\Lambda\Lambda\Lambda\bar{p}\bar{p}$
$F_8^1 (-2, 1/2)$	$\{(p\Lambda\Lambda)(\bar{p}\bar{n}) + 2(n\Lambda\Lambda)\bar{n}\bar{n} - 3\Lambda\Lambda\Lambda(\bar{n}\bar{A})\}/6$ $\{-2(p\Lambda\Lambda)\bar{p}\bar{p} - (n\Lambda\Lambda)(\bar{p}\bar{n}) + 3\Lambda\Lambda\Lambda(\bar{p}\bar{A})\}/6$
$F_8^1 (-2, 3/2)$	$(p\Lambda\Lambda)\bar{n}\bar{n}/\sqrt{3}$ $\{- (p\Lambda\Lambda)(\bar{p}\bar{n}) + (n\Lambda\Lambda)\bar{n}\bar{n}\}/3$ $\{(p\Lambda\Lambda)\bar{p}\bar{p} - (n\Lambda\Lambda)(\bar{p}\bar{n})\}/3$ $(n\Lambda\Lambda)\bar{p}\bar{p}/\sqrt{3}$

$F_8^1 (-1, 0)$	$\{2(p p \Lambda) \bar{p} \bar{p} + (p n \Lambda) (\bar{p} \bar{n}) + 2(n n \Lambda) \bar{n} \bar{n} - 3(p \Lambda \Lambda) (\bar{p} \bar{\Lambda}) - 3(n \Lambda \Lambda) (\bar{n} \bar{\Lambda}) + 6 \Lambda \Lambda \Lambda \bar{\Lambda} \bar{\Lambda}\} / 6 \sqrt{5}$
$F_8^1 (-1, 1)$	$\{(p p \Lambda) (\bar{p} \bar{n}) + (p n \Lambda) \bar{n} \bar{n} - 2(p \Lambda \Lambda) (\bar{n} \bar{\Lambda})\} / 6$ $\{- (p p \Lambda) \bar{p} \bar{p} + (n n \Lambda) \bar{n} \bar{n} + (p \Lambda \Lambda) (\bar{p} \bar{\Lambda}) - (n \Lambda \Lambda) (\bar{n} \bar{\Lambda})\} / 3 \sqrt{2}$ $\{- (p n \Lambda) \bar{p} \bar{p} - (n n \Lambda) (\bar{p} \bar{n}) + 2(n \Lambda \Lambda) (\bar{p} \bar{\Lambda})\} / 6$
$F_8^1 (-1, 2)$	$(p p \Lambda) \bar{n} \bar{n} / \sqrt{3}$ $\{- (p p \Lambda) (\bar{p} \bar{n}) + (p n \Lambda) \bar{n} \bar{n}\} / 2 \sqrt{3}$ $\{(p p \Lambda) \bar{p} \bar{p} - (p n \Lambda) (\bar{p} \bar{n}) + (n n \Lambda) \bar{n} \bar{n}\} / 3 \sqrt{2}$ $\{(p n \Lambda) \bar{p} \bar{p} - (n n \Lambda) (\bar{p} \bar{n})\} / 2 \sqrt{3}$ $(n n \Lambda) \bar{p} \bar{p} / \sqrt{3}$
$F_8^1 (0, 1/2)$	$\{3 p p p \bar{p} \bar{p} + (p p n) (\bar{p} \bar{n}) + (p n n) \bar{n} \bar{n} - 4(p p \Lambda) (\bar{p} \bar{\Lambda}) - 2(p n \Lambda) (\bar{n} \bar{\Lambda}) + 6(p \Lambda \Lambda) \bar{\Lambda} \bar{\Lambda}\} / 3 \sqrt{30}$ $\{(p p n) \bar{p} \bar{p} + (p n n) (\bar{p} \bar{n}) + 3 n n n \bar{n} \bar{n} - 2(p n \Lambda) (\bar{p} \bar{\Lambda}) - 4(n n \Lambda) (\bar{n} \bar{\Lambda}) + 6(n \Lambda \Lambda) \bar{\Lambda} \bar{\Lambda}\} / 3 \sqrt{30}$
$F_8^1 (0, 3/2)$	$\{3 p p p (\bar{p} \bar{n}) + 2(p p n) \bar{n} \bar{n} - 5(p p \Lambda) (\bar{n} \bar{\Lambda})\} / 6 \sqrt{5}$ $\{-6 p p p \bar{p} \bar{p} + (p p n) (\bar{p} \bar{n}) + 4(p n n) \bar{n} \bar{n} - 5(p n \Lambda) (\bar{n} \bar{\Lambda}) + 5(p p \Lambda) (\bar{p} \bar{\Lambda})\} / 6 \sqrt{15}$ $\{-4(p p n) \bar{p} \bar{p} - (p n n) (\bar{p} \bar{n}) + 6 n n n \bar{n} \bar{n} + 5(p n \Lambda) (\bar{p} \bar{\Lambda}) - 5(n n \Lambda) (\bar{n} \bar{\Lambda})\} / 6 \sqrt{15}$ $\{-2(p n n) \bar{p} \bar{p} - 3 n n n (\bar{p} \bar{n}) + 5(n n \Lambda) (\bar{p} \bar{\Lambda})\} / 6 \sqrt{5}$
$F_8^1 (0, 5/2)$	$p p p \bar{n} \bar{n}$ $\{-p p p (\bar{p} \bar{n}) + (p p n) \bar{n} \bar{n}\} / \sqrt{5}$ $\{p p p \bar{p} \bar{p} - (p p n) (\bar{p} \bar{n}) + (p n n) \bar{n} \bar{n}\} / \sqrt{10}$ $\{(p p n) \bar{p} \bar{p} - (p n n) (\bar{p} \bar{n}) + n n n \bar{n} \bar{n}\} / \sqrt{10}$ $\{(p n n) \bar{p} \bar{p} - n n n (\bar{p} \bar{n})\} / \sqrt{5}$ $n n n \bar{p} \bar{p}$
$F_8^1 (1, 1)$	$\{-3 p p p (\bar{p} \bar{\Lambda}) - (p p n) (\bar{n} \bar{\Lambda}) + 4(p p \Lambda) \bar{\Lambda} \bar{\Lambda}\} / 6 \sqrt{2}$ $\{- (p p n) (\bar{p} \bar{\Lambda}) - (p n n) (\bar{n} \bar{\Lambda}) + 2(p n \Lambda) \bar{\Lambda} \bar{\Lambda}\} / 6$ $\{- (p n n) (\bar{p} \bar{\Lambda}) - 3 n n n (\bar{n} \bar{\Lambda}) + 4(n n \Lambda) \bar{\Lambda} \bar{\Lambda}\} / 6 \sqrt{2}$
$F_8^1 (1, 2)$	$-p p p (\bar{n} \bar{\Lambda}) / \sqrt{2}$ $\{p p p (\bar{p} \bar{\Lambda}) - (p p n) (\bar{n} \bar{\Lambda})\} / 2 \sqrt{2}$ $\{(p p n) (\bar{p} \bar{\Lambda}) - (p n n) (\bar{n} \bar{\Lambda})\} / 2 \sqrt{3}$ $\{(p n n) (\bar{p} \bar{\Lambda}) - n n n (\bar{n} \bar{\Lambda})\} / 2 \sqrt{2}$ $n n n (\bar{p} \bar{\Lambda}) / \sqrt{2}$
$F_8^1 (2, 3/2)$	$p p p \bar{\Lambda} \bar{\Lambda}$ $(p p n) \bar{\Lambda} \bar{\Lambda} / \sqrt{3}$ $(p n n) \bar{\Lambda} \bar{\Lambda} / \sqrt{3}$ $n n n \bar{\Lambda} \bar{\Lambda}$

Class 2. $M=11/2$, $M'=17/2$

$F_8^2 (-2, 1/2)$	$\{- (p \Lambda \Lambda) (\bar{p} \bar{n}) - 2(n \Lambda \Lambda) \bar{n} \bar{n} - 3 \Lambda \Lambda \Lambda (\bar{n} \bar{\Lambda})\} / 6$ $\{2(p \Lambda \Lambda) \bar{p} \bar{p} + (n \Lambda \Lambda) (\bar{p} \bar{n}) + 3 \Lambda \Lambda \Lambda (\bar{p} \bar{\Lambda})\} / 6$
$F_8^2 (-1, 0)$	$\{-2(p p \Lambda) \bar{p} \bar{p} - (p n \Lambda) (\bar{p} \bar{n}) - 2(n n \Lambda) \bar{n} \bar{n} + 6 \Lambda \Lambda \Lambda \bar{\Lambda} \bar{\Lambda}\} / 6 \sqrt{2}$

$F_5^2 (-1, 1)$	$\begin{aligned} & \{-(p p A)(\bar{p} \bar{n}) - (p n A) \bar{n} \bar{n} - (p A A)(\bar{n} \bar{A})\}/3\sqrt{2} \\ & \{2(p p A) \bar{p} \bar{p} - 2(n n A) \bar{n} \bar{n} + (p A A)(\bar{p} \bar{A}) - (n A A)(\bar{n} \bar{A})\}/6 \\ & \{(p n A) \bar{p} \bar{p} + (n n A)(\bar{p} \bar{n}) + (n A A)(\bar{p} \bar{A})\}/3\sqrt{2} \end{aligned}$
$F_5^2 (0, 1/2)$	$\begin{aligned} & \{-6 p p p \bar{p} \bar{p} - 2(p p n)(\bar{p} \bar{n}) - 2(p n n) \bar{n} \bar{n} + 2(p p A)(\bar{p} \bar{A}) + (p n A)(\bar{n} \bar{A}) \\ & \quad + 6(p A A) \bar{A} \bar{A}\}/6\sqrt{6} \\ & \{-2(p p n) \bar{p} \bar{p} - 2(p n n)(\bar{p} \bar{n}) - 6 n n n \bar{n} \bar{n} + (p n A)(\bar{p} \bar{A}) + 2(n n A)(\bar{n} \bar{A}) \\ & \quad + 6(n A A) \bar{A} \bar{A}\}/6\sqrt{6} \end{aligned}$
$F_5^2 (0, 3/2)$	$\begin{aligned} & \{-3 p p p (\bar{p} \bar{n}) - 2(p p n) \bar{n} \bar{n} - (p p A)(\bar{n} \bar{A})\}/6 \\ & \{6 p p p \bar{p} \bar{p} - (p p n)(\bar{p} \bar{n}) - 4(p n n) \bar{n} \bar{n} + (p p A)(\bar{p} \bar{A}) - (p n A)(\bar{n} \bar{A})\}/6\sqrt{3} \\ & \{4(p p n) \bar{p} \bar{p} + (p n n)(\bar{p} \bar{n}) - 6 n n n \bar{n} \bar{n} + (p n A)(\bar{p} \bar{A}) - (n n A)(\bar{n} \bar{A})\}/6\sqrt{3} \\ & \{2(p n n) \bar{p} \bar{p} + 3 n n n (\bar{p} \bar{n}) + (n n A)(\bar{p} \bar{A})\}/6 \end{aligned}$
$F_5^2 (1, 1)$	$\begin{aligned} & \{3 p p p (\bar{p} \bar{A}) + (p p n)(\bar{n} \bar{A}) + 2(p p A) \bar{A} \bar{A}\}/6 \\ & \{(p p n)(\bar{p} \bar{A}) + (p n n)(\bar{n} \bar{A}) + (p n A) \bar{A} \bar{A}\}/3\sqrt{2} \\ & \{(p n n)(\bar{p} \bar{A}) + 3 n n n (\bar{n} \bar{A}) + 2(n n A) \bar{A} \bar{A}\}/6 \end{aligned}$

Class 3. $M=3/2$, $M'=5/2$

$F_5^3 (-1, 0)$	$\begin{aligned} & \{2(p p A) \bar{p} \bar{p} + (p n A)(\bar{p} \bar{n}) + 2(n n A) \bar{n} \bar{n} + 2(p A A)(\bar{p} \bar{A}) + 2(n A A)(\bar{n} \bar{A}) \\ & \quad + 6 A A A \bar{A} \bar{A}\}/2\sqrt{30} \end{aligned}$
$F_5^3 (0, 1/2)$	$\begin{aligned} & \{6 p p p \bar{p} \bar{p} + 2(p p n)(\bar{p} \bar{n}) + 2(p n n) \bar{n} \bar{n} + 2(p p A)(\bar{p} \bar{A}) + (p n A)(\bar{n} \bar{A}) \\ & \quad + 2(p A A) \bar{A} \bar{A}\}/2\sqrt{30} \\ & \{2(p p n) \bar{p} \bar{p} + 2(p n n)(\bar{p} \bar{n}) + 6 n n n \bar{n} \bar{n} + (p n A)(\bar{p} \bar{A}) + 2(n n A)(\bar{n} \bar{A}) \\ & \quad + 2(n A A) \bar{A} \bar{A}\}/2\sqrt{30} \end{aligned}$

Class 4. $M=19/2$, $M'=53/2$

$F_5^4 (-3, 0)$	$A A A [\bar{p} \bar{n}]/\sqrt{2}$
$F_5^4 (-2, 1/2)$	$\begin{aligned} & \{(p A A) [\bar{p} \bar{n}] + A A A [\bar{n} \bar{A}]\}/2\sqrt{2} \\ & \{(n A A) [\bar{p} \bar{n}] - A A A [\bar{p} \bar{A}]\}/2\sqrt{2} \end{aligned}$
$F_5^4 (-1, 1)$	$\begin{aligned} & \{(p p A) [\bar{p} \bar{n}] + (p A A) [\bar{n} \bar{A}]\}/2\sqrt{3} \\ & \{(p n A) [\bar{p} \bar{n}] - (p A A) [\bar{p} \bar{A}] + (n A A) [\bar{n} \bar{A}]\}/2\sqrt{6} \\ & \{(n n A) [\bar{p} \bar{n}] - (n A A) [\bar{p} \bar{A}]\}/2\sqrt{3} \end{aligned}$
$F_5^4 (0, 3/2)$	$\begin{aligned} & \{p p p [\bar{p} \bar{n}] + (p p A) [\bar{n} \bar{A}]\}/2\sqrt{2} \\ & \{(p p n) [\bar{p} \bar{n}] - (p p A) [\bar{p} \bar{A}] + (p n A) [\bar{n} \bar{A}]\}/2\sqrt{6} \\ & \{(p n n) [\bar{p} \bar{n}] + (n n A) [\bar{n} \bar{A}] - (p n A) [\bar{p} \bar{A}]\}/2\sqrt{6} \\ & \{n n n [\bar{p} \bar{n}] - (n n A) [\bar{p} \bar{A}]\}/2\sqrt{2} \end{aligned}$
$F_5^4 (1, 2)$	$\begin{aligned} & p p p [\bar{n} \bar{A}]/\sqrt{2} \\ & \{-p p p [\bar{p} \bar{A}] + (p p n) [\bar{n} \bar{A}]\}/2\sqrt{2} \\ & \{-(p p n) [\bar{p} \bar{A}] + (p n n) [\bar{n} \bar{A}]\}/2\sqrt{3} \\ & \{-(p n n) [\bar{p} \bar{A}] + n n n [\bar{n} \bar{A}]\}/2\sqrt{2} \\ & -n n n [\bar{p} \bar{A}]/\sqrt{2} \end{aligned}$

Class 5. $M=11/2$, $M'=17/2$

F_5^5 $(-2, 1/2)$	$\{-(p\bar{A}A) [\bar{p}\bar{n}] + 3\bar{A}AA [\bar{n}\bar{A}]\}/2\sqrt{6}$ $\{-(n\bar{A}A) [\bar{p}\bar{n}] - 3\bar{A}AA [\bar{p}\bar{A}]\}/2\sqrt{6}$
F_5^5 $(-1, 0)$	$\{(p\bar{A}A) [\bar{p}\bar{A}] + (n\bar{A}A) [\bar{n}\bar{A}]\}/2\sqrt{3}$
F_5^5 $(-1, 1)$	$\{-(p\bar{p}A) [\bar{p}\bar{n}] + (p\bar{A}A) [\bar{n}\bar{A}]\}/2\sqrt{3}$ $\{-(p\bar{n}A) [\bar{p}\bar{n}] - (p\bar{A}A) [\bar{p}\bar{A}] + (n\bar{A}A) [\bar{n}\bar{A}]\}/2\sqrt{6}$ $\{-(n\bar{n}A) [\bar{p}\bar{n}] - (n\bar{A}A) [\bar{p}\bar{A}]\}/2\sqrt{3}$
F_5^5 $(0, 1/2)$	$\{2(p\bar{p}A) [\bar{p}\bar{A}] + (p\bar{n}A) [\bar{n}\bar{A}]\}/6$ $\{(p\bar{n}A) [\bar{p}\bar{A}] + 2(n\bar{n}A) [\bar{n}\bar{A}]\}/6$
F_5^5 $(0, 3/2)$	$\{-3p\bar{p}p [\bar{p}\bar{n}] + (p\bar{p}A) [\bar{n}\bar{A}]\}/2\sqrt{6}$ $\{-3(p\bar{p}n) [\bar{p}\bar{n}] - (p\bar{p}A) [\bar{p}\bar{A}] + (p\bar{n}A) [\bar{n}\bar{A}]\}/6\sqrt{2}$ $\{-3(p\bar{p}n) [\bar{p}\bar{n}] - (p\bar{n}A) [\bar{p}\bar{A}] + (n\bar{n}A) [\bar{n}\bar{A}]\}/6\sqrt{2}$ $\{-3nnn [\bar{p}\bar{n}] - (n\bar{n}A) [\bar{p}\bar{A}]\}/2\sqrt{6}$
F_5^5 $(1, 1)$	$\{3p\bar{p}p [\bar{p}\bar{A}] + (p\bar{p}n) [\bar{n}\bar{A}]\}/2\sqrt{6}$ $\{(p\bar{p}n) [\bar{p}\bar{A}] + (p\bar{p}n) [\bar{n}\bar{A}]\}/2\sqrt{3}$ $\{(p\bar{p}n) [\bar{p}\bar{A}] + 3nnn [\bar{n}\bar{A}]\}/2\sqrt{6}$

 Class 6. $M=17/2$, $M'=-1/2$

F_5^6 $(-2, 3/2)$	$A[pA] \bar{n}\bar{n}/\sqrt{2}$ $\{-A[pA] (\bar{p}\bar{n}) + A[nA] \bar{n}\bar{n}\}/\sqrt{6}$ $\{A[pA] \bar{p}\bar{p} - A[nA] (\bar{p}\bar{n})\}/\sqrt{6}$ $A[nA] \bar{p}\bar{p}/\sqrt{2}$
F_5^6 $(-1, 1)$	$\{p[pA] (\bar{p}\bar{n}) - p[nA] \bar{n}\bar{n} + 3n[pA] \bar{n}\bar{n} + 4A[pn] \bar{n}\bar{n} - 4A[pA] (\bar{n}\bar{A})\}/2\sqrt{30}$ $\{-p[pA] \bar{p}\bar{p} + p[nA] (\bar{p}\bar{n}) - n[pA] (\bar{p}\bar{n}) + n[nA] \bar{n}\bar{n} - 2A[pn] (\bar{p}\bar{n})$ $\quad - 2A[nA] (\bar{n}\bar{A}) + 2A[pA] (\bar{p}\bar{A})\}/2\sqrt{15}$ $\{-3p[nA] \bar{p}\bar{p} + n[pA] \bar{p}\bar{p} + 4A[pn] \bar{p}\bar{p} - n[nA] (\bar{p}\bar{n}) + 4A[nA] (\bar{p}\bar{A})\}/2\sqrt{30}$
F_5^6 $(-1, 2)$	$p[pA] \bar{n}\bar{n}/\sqrt{2}$ $\{n[pA] \bar{n}\bar{n} - p[pA] (\bar{p}\bar{n}) + p[nA] \bar{n}\bar{n}\}/2\sqrt{2}$ $\{p[pA] \bar{p}\bar{p} - n[pA] (\bar{p}\bar{n}) - p[nA] (\bar{p}\bar{n}) + n[nA] \bar{n}\bar{n}\}/2\sqrt{3}$ $\{p[nA] \bar{p}\bar{p} - n[nA] (\bar{p}\bar{n}) + n[pA] \bar{p}\bar{p}\}/2\sqrt{2}$ $n[nA] \bar{p}\bar{p}/\sqrt{2}$
F_5^6 $(0, 1/2)$	$\{p[pn] (\bar{p}\bar{n}) + 2n[pn] \bar{n}\bar{n} - p[pA] (\bar{p}\bar{A}) + p[nA] (\bar{n}\bar{A}) - 2n[pA] (\bar{n}\bar{A})$ $\quad - 3A[pn] (\bar{n}\bar{A}) + 3A[pA] \bar{A}\bar{A}\}/3\sqrt{10}$ $\{-2p[pn] \bar{p}\bar{p} - n[pn] (\bar{p}\bar{n}) - 2p[nA] (\bar{p}\bar{A}) + n[pA] (\bar{p}\bar{A}) - n[nA] (\bar{n}\bar{A})$ $\quad + 3A[pn] (\bar{p}\bar{A}) + 3A[nA] \bar{A}\bar{A}\}/3\sqrt{10}$
F_5^6 $(0, 3/2)$	$\{p[pn] \bar{n}\bar{n} - p[pA] (\bar{n}\bar{A})\}/\sqrt{6}$ $\{n[pn] \bar{n}\bar{n} - p[pn] (\bar{p}\bar{n}) + p[pA] (\bar{p}\bar{A}) - p[nA] (\bar{n}\bar{A}) - n[pA] (\bar{n}\bar{A})\}/3\sqrt{2}$ $\{p[pn] \bar{p}\bar{p} - n[pn] (\bar{p}\bar{n}) + p[nA] (\bar{p}\bar{A}) + n[pA] (\bar{p}\bar{A}) - n[nA] (\bar{n}\bar{A})\}/3\sqrt{2}$ $\{n[pn] \bar{p}\bar{p} + n[nA] (\bar{p}\bar{A})\}/\sqrt{6}$
F_5^6 $(1, 0)$	$\{-p[pn] (\bar{p}\bar{A}) - n[pn] (\bar{n}\bar{A}) - p[nA] \bar{A}\bar{A} + 2A[pn] \bar{A}\bar{A} + n[pA] \bar{A}\bar{A}\}/2\sqrt{5}$

$F_5^6 (1, 1)$	$\{-p[pn](\bar{n}\bar{A}) + p[pA]\bar{A}\bar{A}\}/\sqrt{6}$ $\{p[pn](\bar{p}\bar{A}) - n[pn](\bar{n}\bar{A}) + p[nA]\bar{A}\bar{A} + n[pA]\bar{A}\bar{A}\}/2\sqrt{3}$ $\{n[pn](\bar{p}\bar{A}) + n[nA]\bar{A}\bar{A}\}/\sqrt{6}$
$F_5^6 (2, 1/2)$	$p[pn]\bar{A}\bar{A}/\sqrt{2}$ $n[pn]\bar{A}\bar{A}/\sqrt{2}$

Class 7. $M=11/2$, $M'=17/2$

$F_5^7 (-2, 1/2)$	$\{-2A[nA]\bar{n}\bar{n} - A[pA](\bar{p}\bar{n})\}/2\sqrt{3}$ $\{2A[pA]\bar{p}\bar{p} + A[nA](\bar{p}\bar{n})\}/2\sqrt{3}$
$F_5^7 (-1, 0)$	$\{-2p[pA]\bar{p}\bar{p} - p[nA](\bar{p}\bar{n}) - n[pA](\bar{p}\bar{n}) - 2n[nA]\bar{n}\bar{n} + 3A[pA](\bar{p}\bar{A}) + 3A[nA](\bar{n}\bar{A})\}/4\sqrt{6}$
$F_5^7 (-1, 1)$	$\{-p[pA](\bar{p}\bar{n}) - 2p[nA]\bar{n}\bar{n} + 2A[pn]\bar{n}\bar{n} + A[pA](\bar{n}\bar{A})\}/2\sqrt{6}$ $\{2p[pA]\bar{p}\bar{p} + p[nA](\bar{p}\bar{n}) - n[pA](\bar{p}\bar{n}) - 2n[nA]\bar{n}\bar{n} - 2A[pn](\bar{p}\bar{n}) - A[pA](\bar{p}\bar{A}) + A[nA](\bar{n}\bar{A})\}/4\sqrt{3}$ $\{2n[pA]\bar{p}\bar{p} + n[nA](\bar{p}\bar{n}) + 2A[pn]\bar{p}\bar{p} - A[nA](\bar{p}\bar{A})\}/2\sqrt{6}$
$F_5^7 (0, 1/2)$	$\{p[pn](\bar{p}\bar{n}) + 2n[pn]\bar{n}\bar{n} + 5p\bar{p}\bar{A} + 4p[nA](\bar{n}\bar{A}) + n[pA](\bar{n}\bar{A}) - 3A[pn](\bar{n}\bar{A}) - 6A[pA]\bar{A}\bar{A}\}/12\sqrt{2}$ $\{-2p[pn]\bar{p}\bar{p} - n[pn](\bar{p}\bar{n}) + 5n[nA](\bar{n}\bar{A}) + p[nA](\bar{p}\bar{A}) + 4n[pA](\bar{p}\bar{A}) + 3A[pn](\bar{p}\bar{A}) - 6A[nA]\bar{A}\bar{A}\}/12\sqrt{2}$
$F_5^7 (0, 3/2)$	$\{2p[pn]\bar{n}\bar{n} + p[pA](\bar{n}\bar{A})\}/2\sqrt{3}$ $\{-2p[pn](\bar{p}\bar{n}) + 2n[pn]\bar{n}\bar{n} - p[pA](\bar{p}\bar{A}) + p[nA](\bar{n}\bar{A}) + n[pA](\bar{n}\bar{A})\}/6$ $\{2p[pn]\bar{p}\bar{p} - 2n[pn](\bar{p}\bar{n}) - p[nA](\bar{p}\bar{A}) - n[pA](\bar{p}\bar{A}) + n[nA](\bar{n}\bar{A})\}/6$ $\{2n[pn]\bar{p}\bar{p} - n[nA](\bar{p}\bar{A})\}/2\sqrt{3}$
$F_5^7 (1, 1)$	$\{-p[pn](\bar{n}\bar{A}) - 2p[pA]\bar{A}\bar{A}\}/2\sqrt{3}$ $\{p[pn](\bar{p}\bar{A}) - n[pn](\bar{n}\bar{A}) - 2p[nA]\bar{A}\bar{A} - 2n[pA]\bar{A}\bar{A}\}/2\sqrt{6}$ $\{n[pn](\bar{p}\bar{A}) - 2n[nA]\bar{A}\bar{A}\}/2\sqrt{3}$

Class 8. $M=7/2$, $M'=-1/2$

$F_5^8 (-1, 1)$	$\{-p[pA](\bar{p}\bar{n}) - 2p[nA]\bar{n}\bar{n} - 2A[pn]\bar{n}\bar{n} - A[pA](\bar{n}\bar{A})\}/2\sqrt{6}$ $\{2p[pA]\bar{p}\bar{p} + p[nA](\bar{p}\bar{n}) - n[pA](\bar{p}\bar{n}) + 2A[pn](\bar{p}\bar{n}) - 2n[nA]\bar{n}\bar{n} + A[pA](\bar{p}\bar{A}) - A[nA](\bar{n}\bar{A})\}/4\sqrt{3}$ $\{2n[pA]\bar{p}\bar{p} - 2A[pn]\bar{p}\bar{p} + n[nA](\bar{p}\bar{n}) + A[nA](\bar{p}\bar{A})\}/2\sqrt{6}$
$F_5^8 (0, 1/2)$	$\{-p[pn](\bar{p}\bar{n}) - 2n[pn]\bar{n}\bar{n} + p[pA](\bar{p}\bar{A}) + A[pn](\bar{n}\bar{A}) - n[pA](\bar{n}\bar{A}) + 2p[nA](\bar{n}\bar{A}) + 2A[pA]\bar{A}\bar{A}\}/4\sqrt{3}$ $\{2p[pn]\bar{p}\bar{p} + n[pn](\bar{p}\bar{n}) - p[nA](\bar{p}\bar{A}) + 2n[pA](\bar{p}\bar{A}) - A[pn](\bar{p}\bar{A}) + n[nA](\bar{n}\bar{A}) + 2A[nA]\bar{A}\bar{A}\}/4\sqrt{3}$
$F_5^8 (1, 0)$	$\{p[pn](\bar{p}\bar{A}) + n[pn](\bar{n}\bar{A}) - 2p[nA]\bar{A}\bar{A} + 2n[pA]\bar{A}\bar{A}\}/2\sqrt{6}$

Class 9. $M=3/2$, $M'=5/2$

$F_5^9 (-1, 0)$	$\{p[nA](\bar{p}\bar{n}) + 2p[pA]\bar{p}\bar{p} + n[pA](\bar{p}\bar{n}) + 2n[nA]\bar{n}\bar{n} + A[pA](\bar{p}\bar{A}) + A[nA](\bar{n}\bar{A})\}/4\sqrt{2}$
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$F_8^9 (0, 1/2)$	$\{-p[pn](\bar{p}\bar{n}) - 2n[pn]\bar{n}\bar{n} - p[pA](\bar{p}\bar{A}) - n[pA](\bar{n}\bar{A}) - A[pA](\bar{n}\bar{A}) - 2A[pA]\bar{A}\bar{A}\}/4\sqrt{2}$ $\{2p[pn]\bar{p}\bar{p} + n[pn](\bar{p}\bar{n}) - p[nA](\bar{p}\bar{A}) - n[nA](\bar{n}\bar{A}) + A[pn](\bar{p}\bar{A}) - 2A[nA]\bar{A}\bar{A}\}/4\sqrt{2}$
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Class 14. $M=11/2, M'=17/2$

$F_8^{13} (-2, 1/2)$	$A[pA][\bar{p}\bar{n}]/2$ $A[nA][\bar{p}\bar{n}]/2$
$F_8^{14} (-1, 0)$	$\{-p[nA][\bar{p}\bar{n}] + n[pA][\bar{p}\bar{n}] + 2A[pn][\bar{p}\bar{n}] - A[pA][\bar{p}\bar{A}] - A[nA][\bar{n}\bar{A}]\}/4\sqrt{2}$
$F_8^{14} (-1, 1)$	$\{p[pA][\bar{p}\bar{n}] + A[pA][\bar{n}\bar{A}]\}/2\sqrt{2}$ $\{p[nA][\bar{p}\bar{n}] + n[pA][\bar{p}\bar{n}] - A[pA][\bar{p}\bar{A}] + A[nA][\bar{n}\bar{A}]\}/4$ $\{n[nA][\bar{p}\bar{n}] - A[nA][\bar{p}\bar{A}]\}/2\sqrt{2}$
$F_8^{14} (0, 1/2)$	$\{3p[pn][\bar{p}\bar{n}] - p[pA][\bar{p}\bar{A}] + n[pA][\bar{n}\bar{A}] - 2p[nA][\bar{n}\bar{A}] + 3A[pn][\bar{n}\bar{A}]\}/4\sqrt{6}$ $\{3n[pn][\bar{p}\bar{n}] - n[nA][\bar{n}\bar{A}] + p[nA][\bar{p}\bar{A}] - 2n[pA][\bar{p}\bar{A}] - 3A[pn][\bar{p}\bar{A}]\}/4\sqrt{6}$
$F_8^{14} (0, 3/2)$	$p[pA][\bar{n}\bar{A}]/2$ $\{-p[pA][\bar{p}\bar{A}] + p[\bar{n}A][\bar{n}\bar{A}] + n[pA][\bar{n}\bar{A}]\}/2\sqrt{3}$ $\{-p[nA][\bar{p}\bar{A}] + n[\bar{n}A][\bar{n}\bar{A}] - n[pA][\bar{p}\bar{A}]\}/2\sqrt{3}$ $-n[nA][\bar{p}\bar{A}]/2$
$F_8^{14} (1, 1)$	$p[pn][\bar{n}\bar{A}]/2$ $\{-p[pn][\bar{p}\bar{A}] + n[pn][\bar{n}\bar{A}]\}/2\sqrt{2}$ $-n[pn][\bar{p}\bar{A}]/2$

Class 15. $M=7/2, M'=-1/2$

$F_8^{15} (-1, 1)$	$\{-p[pA][\bar{p}\bar{n}] + A[pA][\bar{n}\bar{A}]\}/2\sqrt{2}$ $\{-p[nA][\bar{p}\bar{n}] - n[pA][\bar{p}\bar{n}] - A[pA][\bar{p}\bar{A}] + A[nA][\bar{n}\bar{A}]\}/4$ $\{-n[nA][\bar{p}\bar{n}] - A[nA][\bar{p}\bar{A}]\}/2\sqrt{2}$
$F_8^{15} (0, 1/2)$	$\{-p[pn][\bar{p}\bar{n}] + p[pA][\bar{p}\bar{A}] + n[pA][\bar{n}\bar{A}] + A[pn][\bar{n}\bar{A}]\}/4$ $\{-n[pn][\bar{p}\bar{n}] + p[nA][\bar{p}\bar{A}] + n[nA][\bar{n}\bar{A}] - A[pn][\bar{p}\bar{A}]\}/4$
$F_8^{15} (1, 0)$	$\{p[pn][\bar{p}\bar{A}] + n[pn][\bar{n}\bar{p}]\}/2\sqrt{2}$

Class 16. $M=3/2, M'=5/2$

$F_8^{16} (-1, 0)$	$\{-p[nA][\bar{p}\bar{n}] + n[pA][\bar{p}\bar{n}] + A[pA][\bar{p}\bar{A}] + A[nA][\bar{n}\bar{A}]\}/4$
$F_8^{16} (0, 1/2)$	$\{p[pn][\bar{p}\bar{n}] + p[pA][\bar{p}\bar{A}] + n[pA][\bar{n}\bar{A}] - A[pn][\bar{n}\bar{A}]\}/4$ $\{n[pn][\bar{p}\bar{n}] + n[nA][\bar{n}\bar{A}] + p[nA][\bar{p}\bar{A}] + A[pn][\bar{p}\bar{A}]\}/4$

Class 20. $M=7/2, M'=-1/2$

$F_8^{20} (-1, 1)$	$[pnA]/\bar{n}\bar{n}\sqrt{6}$ $- [pnA](\bar{p}\bar{n})/2\sqrt{3}$ $[pnA]\bar{p}\bar{p}/\sqrt{6}$
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$F_5^{20} (0, 1/2)$	$- [pnA] (\bar{n}\bar{A})/2\sqrt{3}$ $[pnA] (\bar{p}\bar{A})/2\sqrt{3}$
$F_5^{20} (1, 0)$	$[pnA] \bar{A}\bar{A}/\sqrt{6}$

Class 21. $M=3/2, M'=5/2$

$F_5^{21} (-1, 0)$	$[pnA] [\bar{p}\bar{n}]/2\sqrt{3}$
$F_5^{21} (1, 0/2)$	$[pnA] [\bar{n}\bar{A}]/2\sqrt{3}$ $- [pnA] [\bar{p}\bar{A}]/2\sqrt{3}$

Appendix II

In this Appendix II we give the theoretical and experimental masses of fermions and besons. The following numerical values are taken for the parameters in Formulas (3) and (6).

$m_N=939$ Mev
 $m_\Lambda=1115$ Mev
 $V(N\bar{N})=-1740$ Mev
 $\Delta V=182$ Mev= $1.03\times\Delta m$
 $U=170$ Mev
 $\Delta U=132$ Mev= $0.75\times\Delta m$

In Tables (V) and (VI) for the experimental value the arithmetical mean of the masses of particles belonging to a charge multiplet is given as the mass of the multiplet.

Table V. The theoretical and experimental values of masses of bosons

S	I	State	Formula (3) (two-body approx.)	Experimental value	Remarks
-1	1/2	$B_2^1 (-1, 1/2)$	496 (Mev)	496 (Mev)	\bar{K}
0	0	$B_2^1 (0, 0)$	615		π^0
		$B_2^3 (0, 0)$	377		$\pi^{0'}$
	1	$B_2^1 (0, 1)$	138	138	π
1	1/2	$B_2^1 (1, 1/2)$	496	496	K

-2	0	$B_4^4 (-2, 0)$	992		stable for $\rightarrow 2\bar{K}?$
	1	$B_4^1 (-2, 1)$	992		
-1	1/2	$B_4^1 (-1, 1/2)$	1064		stable for $\rightarrow \pi + \bar{K}?$
		$B_4^4 (-1, 1/2)$	873		
	3/2	$B_4^1 (-1, 3/2)$	634		
		$B_4^6 (-1, 3/2)$	634		
0	0	$B_4^1 (0, 0)$	1135		
	1	$B_4^1 (0, 1)$	849		
		$B_4^4 (0, 1)$	753		
		$B_4^6 (0, 1)$	753		
	2	$B_4^1 (0, 2)$	276		
1	1/2	$B_4^1 (1, 1/2)$	1064		stable for $\rightarrow \pi + K?$
		$B_4^6 (1, 1/2)$	873		
	3/2	$B_4^1 (1, 3/2)$	634		
		$B_4^4 (1, 3/2)$	634		
2	0	$B_4^6 (2, 0)$	992		stable for $\rightarrow 2K?$
	1	$B_4^1 (2, 1)$	992		

Table VI. The theoretical and experimental values of masses of fermions of two baryons and one antibaryon

S	I	State	Formula (3) (two-body approx.)	Formula (6)	Experimental value	Remarks
-2	1/2	$F_3^1 (-2, 1/2)$	1429 (Mev)	1335 (Mev)	1315 (Mev)	Ξ
-1	0	$F_3^1 (-1, 0)$	1611	1479	1448	resonance at 25 Mev in $K^- - p$
	1	$F_3^1 (-1, 1)$	1253	1215	1193	Σ
		$F_3^3 (-1, 1)$	1253	1215		
0	1/2	$F_3^1 (0, 1/2)$	1614	1727	1716	(1/2, >5/2) resonance at 950 Mev in $\pi^- - p$
		$F_3^3 (0, 1/2)$	1435	1567	1543	(1/2, 3/2) resonance at 650 Mev in $\pi^- - p$
	3/2	$F_3^1 (0, 3/2)$	1077	1247	1233	(3/2, 3/2) resonance at 190 Mev in $\pi^+ - p$
1	0	$F_3^3 (1, 0)$	1617	1579	1559	resonance at 200 Mev in $K^+ - n$
	1	$F_3^1 (1, 1)$	1617	1579	1559	

Table VII. The theoretical and experimental value of masses of fermions of three baryons and two antibaryons

S	I	State	Formula (3)	Formula (8)*,**	Remarks
-3	0	$F_5^4 (-3, 0)$	1743 (Mev)	— (Mev)	
	1	$F_5^1 (-3, 1)$	1743	—	
-2	1/2	$F_5^1 (-2, 1/2)$	1925	1811	
		$F_5^4 (-2, 1/2)$	1746	1632	
	3/2	$F_5^1 (-2, 3/2)$	1567	1453	
		$F_5^6 (-2, 3/2)$	1567	1453	
		$F_5^{10} (-2, 3/2)$	1567	1453	
-1	0	$F_5^1 (-1, 0)$	2107	1944	
	1	$F_5^1 (-1, 1)$	1868	1740	
		$F_5^4 (-1, 1)$	1749	1638	
		$F_5^6 (-1, 1)$	1773	1658	
		$F_5^{10} (-1, 1)$	1773	1658	
	2	$F_5^1 (-1, 2)$	1391	1331	
		$F_5^6 (-1, 2)$	1391	1331	
		$F_5^{10} (-1, 2)$	1391	1331	
	1/2	$F_5^1 (0, 1/2)$	2170	2278	
		$F_5^6 (0, 1/2)$	1979	2096	
		$F_5^{10} (0, 1/2)$	1979	2096	
	3/2	$F_5^1 (0, 3/2)$	1812	1938	prolonged resonance around 1.3 Bev in $\pi^+ - p$ scattering
		$F_5^4 (0, 3/2)$	1752	1881	
		$F_5^6 (0, 3/2)$	1692	1824	
		$F_5^{10} (0, 3/2)$	1692	1824	
	5/2	$F_5^1 (0, 5/2)$	1215	1371	
1	0	$F_5^6 (1, 0)$	2185	2127	
		$F_5^{10} (1, 0)$	2185	2127	
	1	$F_5^1 (1, 1)$	2232	2174	
		$F_5^6 (1, 1)$	1994	1936	
		$F_5^{10} (1, 1)$	1994	1936	
	2	$F_5^1 (1, 2)$	1755	1697	
		$F_5^4 (1, 2)$	1755	1697	
2	1/2	$F_5^6 (2, 1/2)$	2295	—	
		$F_5^{10} (2, 1/2)$	2295	—	
	3/2	$F_5^1 (2, 3/2)$	2295	—	

* When there are two ways of decomposition, we take their arithmetical mean.

** We use $m_{\text{exp}}(N\Lambda)$ adjusted to the resonance at 950 Mev in $\pi^- - p$ scattering.

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A Note on the Electromagnetic Response of Normal Metals

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The general formalism of the transverse conductivity, developed in the previous note, is applied to the simplest example, i.e. the free electron gas, and shown to result in the same extremely anomalous skin effect as the usual method of Reuter and Sondheimer.

It is also pointed out that the deficiency from the optical sum rule in this example indeed leads to the Landau diamagnetism as required by our general formula.

§ 1. Introduction

In the previous note,¹⁾ Kubo's formalism of conductivity²⁾ was generalized to include a space and time dependent electromagnetic disturbance. Though it is general, the formalism looks so much different from the conventional theory of conductivity that one might wonder how it works in actual applications and can lead to any result of practical use. The purpose of the present note is to answer these questions by applying the formalism to the simplest example, i.e. the extremely anomalous skin effect of normal metals.

In the usual theory of the anomalous skin effect,³⁾ an integral relation between current and electric field is obtained from the Boltzmann-Bloch transport equation and coupled with Maxwell equations under appropriate boundary conditions. In our formalism, on the other hand, the kernel of the integral relation is obtained from the general expression for the transverse conductivity derived in the previous note.¹⁾ Hence the problem is to calculate the transverse conductivity. The calculation is particularly simple in the extremely anomalous region, because here one may regard the system of conduction electrons as a free Fermi gas, ignoring scattering by impurities or phonons. The well-known result of Reuter and Sondheimer³⁾ is obtained in this way. Though the example is rather trivial, it may be helpful in understanding our general formalism and for future studies of more complicated cases.

In the previous note,¹⁾ it was pointed out that the absorption v.s. frequency curve in general does not satisfy the well-known optical sum rule²⁾ and that the deficiency is related with the diamagnetic moment of a metal. In the final part of the present note, this general argument will be applied to the free electron gas to show that the deficiency in this case is related indeed with the Landau diamagnetism. Although the effect is numerically negligible, the example is helpful in understanding the case of a superconductor, in which the deficiency from the sum rule

becomes quite appreciable because of a large London-Pippard diamagnetism as discussed by Tinkham and Ferrell.⁴⁾

§ 2. A general expression for the surface impedance

When a transverse electromagnetic field proportional to $\exp\{i(\mathbf{q} \cdot \mathbf{r} - \omega t)\}$ exists in a metal, the current response of conduction electrons is characterized by the transverse conductivity¹⁾

$$\sigma(q, \omega) = \lim_{\alpha \rightarrow +0} \int_0^{\infty} dt e^{-\alpha t + i\omega t} S_0(q, t). \quad (1)$$

Here

$$S_0(q, t) = \frac{1}{2} \sum_{\mu, \nu} \left(\delta_{\mu\nu} - \frac{1}{q^2} q_\mu q_\nu \right) S_{\mu\nu}(\mathbf{q}, t) \quad (2)$$

and

$$S_{\mu\nu}(\mathbf{q}, t) = \left(\frac{e}{m} \right)^2 \int_0^{\frac{1}{2}} d\lambda \langle P_\nu(-\mathbf{q}) P_\mu(\mathbf{q}, t + i\hbar\lambda) \rangle \quad (3)$$

is the relaxation function of the momentum density of conduction electrons. When expressed in terms of quantized Fermion field amplitudes,

$$\mathbf{P}(\mathbf{q}) = \sum \hbar(\mathbf{p} + \frac{1}{2}\mathbf{q}) \psi^+(\mathbf{p}, \sigma) \psi(\mathbf{p} + \mathbf{q}, \sigma) \quad (4)$$

where the sum extends over all possible momenta \mathbf{p} and spin directions σ . Note that in (3)

$$\mathbf{P}(\mathbf{q}, t) = e^{(i/\hbar)\mathbf{H}t} \mathbf{P}(\mathbf{q}) e^{-(i/\hbar)\mathbf{H}t} \quad (5)$$

is Heisenberg's motion with the field-free Hamiltonian H and also that $\langle \rangle$ indicates the expectation over the canonical ensemble $\exp(-\beta H)$.

For simplicity, let us take the one-dimensional case, in which our metal occupies the domain $x > 0$, the electric field $E(x)$ being parallel to the y -axis and the magnetic field $H(x)$ to the z -axis. Though the frequency ω is fixed, the field decays from the surface $x=0$ into the metal and thus contains various Fourier components with wave numbers q comparable to δ^{-1} , where δ is the skin depth. The wave number dependent conductivity (1) then implies a non-local relation between current and electric field

$$J(x) = \int_0^{\infty} dx' K(x-x') E(x') \quad (6)$$

where the kernel is related with the conductivity by

$$K(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dq_x \sigma(|q_x|, \omega) e^{iq_x x}. \quad (7)$$

In writing the generalized Ohm law in the form (6), we have tacitly assumed that the relation between current and field keeps the same form near as well as far from the surface. The assumption corresponds to the random scattering at the surface in the theory of Reuter and Sondheimer. Alternatively one may assume the specular reflection at the surface. This is equivalent to assuming

$$J(x) = \int_0^{\infty} dx' \{K(x-x') + K(x+x')\} E(x'). \quad (8)$$

Let us take this boundary condition because mathematics in this case is much simpler. Our problem is then to solve Maxwell equations supplemented by (8) under the boundary condition that the magnetic field, for instance, takes a given value $H(0)$ at the surface. The solution can be obtained by applying the Fourier transformation to

$$\frac{d^2 E(x)}{dx^2} + i \frac{4\pi\omega}{c^2} \int_{-\infty}^{\infty} dx' K(x-x') E(x') = \frac{2i\omega}{c} H(0) \delta(x), \quad (9)$$

which is assumed for $-\infty < x < \infty$. The surface impedance is then found by

$$Z = \frac{4\pi\omega}{c} \left(\frac{E(0)}{H(0)} \right) = \frac{4\omega^2}{ic} \int_{-\infty}^{\infty} dq_x \frac{1}{q_x^2 + (4\pi\omega/ic^2) \sigma(|q_x|, \omega)}. \quad (10)$$

This is the general formula valid under our assumption of specular reflection at the surface.

§ 3. The extremely anomalous skin effect

Now let us apply (10) to the extremely anomalous region. For definiteness, let us take the case of $\omega \sim 10^{12} \text{sec}^{-1}$. Then our result (19) gives $\delta \sim 10^{-5} \text{cm}$ for ordinary metals so that we are concerned with the electromagnetic wave with wave numbers $q \sim \delta^{-1} \sim 10^5 \text{cm}^{-1}$, which is much smaller than the Fermi maximum $p_F \sim 10^8 \text{cm}^{-1}$. Note also that $\hbar^2 q^2 / 2m \ll \hbar\omega \ll E_F$, where $E_F = \hbar^2 p_F^2 / 2m$ is the Fermi energy.

We of course assume that the electron mean free path l determined by scatterings by impurities and phonons is much longer than δ , and therefore ignore scatterings altogether. Thus the Hamiltonian is given by

$$H = \sum E(\mathbf{p}) \psi^+(\mathbf{p}, \sigma) \psi(\mathbf{p}, \sigma), \quad E(\mathbf{p}) = \hbar^2 p^2 / 2m. \quad (11)$$

From (2), (3), (4) and (5), it is easily found that

$$S_0(q, t) = -\frac{1}{2} \left(\frac{\hbar e}{m} \right)^2 \sum p^2 (1 - \cos^2 \theta_{pq}) F(\mathbf{p}, \mathbf{q}) \exp \left\{ \frac{i}{\hbar} (E(\mathbf{p}) - E(\mathbf{p} + \mathbf{q})) \right\}. \quad (12)$$

Here θ_{pq} is the angle between \mathbf{p} and \mathbf{q} , and

$$F(\mathbf{p}, \mathbf{q}) = - \left(\frac{f(\mathbf{p} + \mathbf{q}) - f(\mathbf{p})}{E(\mathbf{p} + \mathbf{q}) - E(\mathbf{p})} \right) \quad (13)$$

where $f(\mathbf{p})$ is the usual Fermi distribution function. The real part of the conductivity (1) is thus given by

$$\sigma'(q, \omega) = \frac{\pi \hbar}{2} \left(\frac{\hbar e}{m} \right)^2 \sum p^2 (1 - \cos^2 \theta_{pq}) F(\mathbf{p}, \mathbf{q}) \delta(E(\mathbf{p} + \mathbf{q}) - E(\mathbf{p}) - \hbar \omega). \quad (14)$$

Since we are concerned with low temperatures and $q \ll p_F$, we may replace (13) by $\delta(E(\mathbf{p}) - E_F)$. Since $\hbar^2 q^2 / 2m \ll \hbar \omega$, the energy conservation law expressed by the delta function in (14) is satisfied when $v_F q > \omega$, where v_F is the electron velocity at the Fermi surface. In this way, we obtain

$$\sigma'(q, \omega) = \frac{3\pi}{4} \left(\frac{ne^2}{m} \right) \frac{1}{v_F q} \left(1 - \left(\frac{\omega}{v_F q} \right)^2 \right) \quad (15)$$

for $v_F q > \omega$.

The imaginary part of the conductivity (1) is given by

$$\sigma''(q, \omega) = \frac{\hbar}{2} \left(\frac{\hbar e}{m} \right)^2 \sum p^2 (1 - \cos^2 \theta_{pq}) F(\mathbf{p}, \mathbf{q}) \frac{P}{\hbar \omega - E(\mathbf{p} + \mathbf{q}) + E(\mathbf{p})} \quad (16)$$

where P indicates taking the principal value. In the same way as before, (16) reduces to

$$\sigma''(q, \omega) = 3 \left(\frac{ne^2}{m} \right) \frac{1}{v_F q} \left(\frac{\omega}{v_F q} \right). \quad (17)$$

We are now ready to calculate the surface impedance (10). To simplify the calculation, however, let us ignore ω in comparison with $v_F q$. Then

$$Z = \frac{8\omega^2}{ic^2} \int_0^\infty dq_x \frac{q_x}{q_x^3 - i\delta^{-3}} = \frac{8\pi\omega\delta}{3c^2} \left(\frac{1}{\sqrt{3}} - i \right) \quad (18)$$

where

$$\delta = \left(\frac{mv_F c^2}{3\pi^2 ne^2 \omega} \right)^{1/3}. \quad (19)$$

Our result agrees with that of Reuter and Sondheimer³⁾ in the extremely anomalous region ($l\delta \gg 1$).

§ 4. Optical sum rule and Landau diamagnetism

In the previous note,¹⁾ it was shown that the conductivity (1) does not satisfy the optical sum rule, but

$$\frac{2}{\pi} \int_0^{\infty} d\omega \sigma'(q, \omega) = \frac{ne^2}{m} - c^2 q^2 \chi(q). \quad (20)$$

Here $\chi(q)$ is the diamagnetic susceptibility, which in general may depend upon the wave number q .

Now, integrating from $\omega=0$ to $\omega=v_F q$, we see that (15) does satisfy the optical sum rule without the second term on the right of (20). But this is only approximate because we have neglected q, p_F compared with unity in obtaining the conductivity (15). To be more exact, we go back to the original expression (14). Integrating this over ω , we find

$$\frac{1}{\pi} \int_{-\infty}^{\infty} d\omega \sigma'(q, \omega) = \frac{1}{2} \left(\frac{\hbar e}{m} \right)^2 \sum p^2 (1 - \cos^2 \theta_{pq}) F(\mathbf{p}, \mathbf{q}).$$

Expanding $F(\mathbf{p}, \mathbf{q})$ into the power series of q and integrating by part in momentum space, we easily obtain

$$\frac{2}{\pi} \int_0^{\infty} d\omega \sigma'(q, \omega) = \frac{ne^2}{m} - \frac{2}{3} (cq)^2 \mu_B N(0) + O(q^4). \quad (21)$$

Here μ_B is the Bohr magneton and $N(0)$ is the state density per unit volume of a single electron spectrum at the Fermi surface. Comparing (21) with our general formula (20), we obtain

$$\chi(q) = \frac{2}{3} \mu_B^2 N(0) + O(q^2). \quad (22)$$

This is the well-known Landau diamagnetism. The second term on the right of (21), being of order of $(q/p_F)^2$ compared with the first term, is an extremely small deficiency in practice. But it is still interesting to see that the deficiency is related with the Landau diamagnetic moment, because the deficiency becomes quite appreciable when the metal goes into a superconducting state. This has been discussed first by Tinkham and Ferrell.⁴⁾ For a superconductor, they have taken the observed absorption curve in the far infrared region which is approximately expressed as

$$\sigma_s'(q, \omega) = \begin{cases} 0, & \omega < \omega_c \\ \sigma'(q, \omega) (1 - (\omega_c/\omega)^2), & \omega > \omega_c. \end{cases} \quad (23)$$

Here ω_c is the critical frequency corresponding to the energy gap and σ' the normal conductivity given by (15). Inserting (23) into (20), we obtain the Pippard type diamagnetism

$$\chi_s(q) = \left(\frac{3\pi ne^2}{4mc^2} \right) \frac{1}{\xi^2 q^3}, \quad \xi = \frac{\pi v_F}{4\omega_c} \quad (24)$$

provided that $v_F q > \omega_c$.⁵⁾ This is not so surprising in view of the fact that Pippard⁵⁾ has based his equation on the non-local equation of the anomalous skin effect.

From the relation between the sum rule deficiency (23) and the diamagnetism (24), Tinkham and Ferrell have concluded that the London-Pippard diamagnetism of a superconductor is a consequence of the existence of an energy gap. But the question naturally arises why an insulator, in spite of its energy gap, does not exhibit such a large deficiency from the sum rule, nor the Meissner effect. What is peculiar for a superconductor is the fact that matrix elements of the current operator look as though electrons were still free, in spite of the energy spectrum with the gap. It is the task of the microscopic theory of superconductivity to derive this property of the current operator. The relation between the sum rule deficiency and the Meissner effect, pointed out by Tinkham and Ferrell, is rather phenomenological, in the sense that one can derive one from the other by our general formula (20), which is valid for superconductors as well as for normal conductors.

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On the Meson Mass Differences*

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In view of the recent experimental evidence indicating $m_{K^0} > m_{K^+}$, in contrast with the well established result $m_{\pi^+} > m_{\pi^0}$, the problem of the electromagnetic meson self-masses is reinvestigated. A semi-phenomenological approach is used by the introduction of a non-local effective interaction hamiltonian, gauge invariant up to the order e^2 , where new terms corresponding to one-photon and two-photon vertices are considered, to take into account the effects of the strong interactions. It is shown that the contrasting experimental result can be explained as the result of the different nature of the neutral kaons as compared with the neutral pion. Some different ways to realize the experimental results are explicitly discussed.

§ 1. Introduction

The problem of the mass difference $m_{\pi^+} - m_{\pi^0}$ has been treated by several authors¹⁾ under the assumption that charge independence holds, the mass difference between the charged and the neutral components of the pion triplet being only due to the electromagnetic interactions. The minimal electromagnetic interaction (i. e., A_μ dependent) is used, the resulting expression for the self-mass, calculated by means of perturbation theory being

$$\delta m = \frac{3\alpha}{8\pi} \frac{\Lambda^2}{m} \quad (1.1)$$

where Λ is a cut-off²⁾ ad hoc introduced to arrive at a convergent result. For $\Lambda \simeq M$, M being the nucleon mass, the experimental result

$$m_{\pi^+} - m_{\pi^0} = (4.59 \pm 0.22) \text{ Mev}^{(2)} \quad (1.2)$$

is reproduced.

Recently, Gasirowicz and Petermann³⁾ reconsidered the whole problem of the electromagnetic self-masses of mesons. They estimated the effect of the strong interactions and concluded that the following relation between the mass differences of kaons and pions would hold:

$$\frac{m_{K^+} - m_{K^0}}{m_{\pi^+} - m_{\pi^0}} = \frac{m_\pi}{m_K}, \quad (1.1')$$

* A preliminary account of the present work has been presented at the XI Annual Meeting of the Brazilian Society for the Progress of Science, SBPC, at Bahia, Brasil, July 1959.

a formula which could be derived from (1.1), assuming the same cut-off for both kinds of mesons.

Meanwhile, the recent analysis of bubble chamber experiments at Berkeley has shown that $\bar{K}^0(K^0)$ is heavier than the $K^-(K^+)$, a result which obviously contradicts the plausible relation (1.1'). The experimental results were:

$$\begin{aligned} m_{K^-} - m_{\bar{K}^0} &= -(3.7 \pm 0.7) \text{ Mev}^{(4)} \\ m_{K^+} - m_{K^0} &= -(4.8 \pm 1.1) \text{ Mev}^{(5)}. \end{aligned} \quad (1.3)$$

On theoretical grounds, this surprising result may well have deep roots. According to Pais,⁶⁾ there is the possibility that K^- and \bar{K}^0 have opposite parities and do not form an isotopic spin doublet, and so, they might have different masses.⁷⁾

We believe, however, from the fact that the mass differences are comparatively small, that their origin should be found in the electromagnetic interactions. So, before entering into more theoretical thinking, a semi-phenomenological approach seems desirable, under the hypothesis of validity of the Nishijima-Gell-Mann scheme,⁸⁾ taking into account *phenomenologically* the effects of the strong interactions. The methodology behind this approach is the same as that adopted in a series of recent papers on the electromagnetic structure of the elementary particles.⁹⁾ In the present paper we take an effective interaction hamiltonian of spin zero mesons with the electromagnetic field, non-local through the introduction of covariant form factors and satisfying the requirements of gauge-invariance up to the order e^2 . The form factors introduced correspond to one-photon and two-photon vertices. Besides the one-photon and two-photon minimal vertices

$$\begin{aligned} ie \int d^3x \int d^4y [\phi^* \partial_\mu \phi - \partial_\mu \phi^* \phi(x)] A_\mu(y) g_1(x-y) \\ + e^2 \int d^3x \int d^4y \int d^4z [\phi^* \phi(x)] A_\mu(y) g_1(x-y) A_\mu(z) g_1(x-z), \end{aligned} \quad (1.4)$$

also one-photon and two-photon vertices associated with the field strength $F_{\mu\nu}$ are considered. In particular, the one-photon $F_{\mu\nu}$ term is:*

$$\frac{ie\kappa}{m^2} \int d^3x \int d^4y [\partial_\mu \phi^* \partial_\nu \phi(x) - \partial_\nu \phi^* \partial_\mu \phi(x)] F_{\mu\nu}(y) g_2(x-y), \quad (1.5)$$

where κ is a dimensionless constant. This term attributes to the meson, in an external magnetic field, a magnetic moment. It may be considered as the mesic analogous to the Pauli term $(e\mu/2M)\bar{\psi}\sigma_{\mu\nu}\psi F_{\mu\nu}$ for nucleons.

Although the effective interaction hamiltonian considered allows us a more general treatment of the electromagnetic interactions of charged mesons, as far as

* This term, in the local limit, has been considered by Nakano, in the context of the β -formalism (unpublished). Henceforth, we will refer to it as the Nakano term. We are indebted to Professor Y. Katayama for kindly informing us about this point.

the meson mass differences are concerned, it is essential to point out the different isospinor character of pions and kaons, which reflects in a different behavior of the neutral components of these two multiplets. In fact, the neutral pion, π^0 , is a Majorana neutral, described by an hermitian operator. On the other hand, the neutral kaon K^0 , having opposite strangeness to its antiparticle \bar{K}^0 , is a pseudo-neutral,¹⁰⁾ described by a non-hermitian operator. In this way, it is possible to attribute a Nakano interaction term to the neutral kaons, but not to the neutral pion since in this last case (1.5) would vanish identically. The Nakano interaction term gives to the neutral kaons a self-mass different from zero. *Based on this assumption, it is possible to realize the experimental kaon mass difference.*

Another distinct possibility to realize the kaon mass difference, not involving the introduction of the Nakano interaction term, is to attribute to the neutral kaons the usual minimal terms (1.4). But since these particles are electrically neutral, the form factors describing their electric structure must be singular, that is, the Fourier transform of the form factor must vanish for zero momentum transfer. As the neutral kaons produce no electric current they are excluded from the charge gauge group. The gauge invariance of (1.4) for a change of gauge of the electromagnetic field, however, just requires the condition of singular form factors.*

In section 2, the effective hamiltonian is discussed with more detail and the corresponding self-masses are obtained from perturbation theory. General expressions for the self-masses, using a spectral decomposition of the form factors are given, from which several particular cases are selected. An alternative integration method is also briefly discussed.

Section 3 is devoted to the numerical discussion of the mass differences, along the line of the method used in the paper by Katayama et al.⁹⁾

Section 4 contains a brief analysis of form factors in terms of perturbation theory and baryons closed-loops. A possible interpretation of the origin of the Nakano term is there given and some estimations of the contributions of some form factors to the self-mass discussed. The conclusions of the present work are discussed in Section 5.

§ 2. The effective interaction hamiltonian and the self-masses

The effective electromagnetic interaction hamiltonian of spin zero particles, depending on covariant form factors to account for the effect of strong interactions, is, up to the second order in the elementary charge e :

* In the time of writing the present work, a letter by P. T. Mathews and J. L. Uretsky appeared in the Physical Review Letters, **3** (1959), 297, dealing with the same problem. The solution proposed for the kaon mass difference is similar to that presented here, although they used the difference of two Yukawa form factors to describe the inner structure of the neutral kaons. They reproduce the experimental result in a case, where $\sqrt{\langle r^2 \rangle}$ of the neutral kaon is $\sim 3y$, a somewhat large value.

$$H_{\text{int}} = H_1 + H_2 + H_3 \quad (2.1)$$

with

$$H_1 = ie \int d^3x \int d^4y [\phi^* \partial_\mu \phi(x) - \partial_\mu \phi^* \phi(x)] A_\mu(y) g_1(x-y) \\ + e^2 \int d^3x \int d^4y \int d^4z [\phi^* \phi(x)] A_\mu(y) g_1(x-y) A_\mu(z) g_1(x-z), \quad (2.2)$$

$$H_2 = \frac{ie\kappa}{m^2} \int d^3x \int d^4y [\partial_\mu \phi^*(x) \partial_\nu \phi(x) - \partial_\nu \phi^* \partial_\mu \phi(x)] F_{\mu\nu}(y) g_2(x-y) \\ + \frac{e^2 \kappa^2}{m^2} \int d^3x \int d^4y \int d^4z \partial_\mu [\phi^* \phi(x)] \cdot [A_\nu(z) F_{\mu\nu}(y) + F_{\mu\nu}(y) A_\nu(z)] \\ \times g_2(x-y) g_2(x-z), \quad (2.3)$$

$$H_3 = \frac{e^2}{m^2} \int d^3x \int d^4y \int d^4z \int d^4w [\phi^*(x) \phi(y)] g_3(x-y, y-z, z-w) \\ \times F_{\mu\nu}(z) F_{\mu\nu}(w) + \text{h. c.} + \dots, \quad (2.4)$$

where ϕ , A_μ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ are the meson and electromagnetic field quantities, respectively.* We have omitted here all the normal dependent terms.

The effective hamiltonian (2.1) describes:

a) the electromagnetic interaction of charged particles (charged pions or charged kaons). In this case, ϕ is a non-hermitian operator ϕ_π or ϕ_K .

The form factors are subject to the normalization

$$\int g_1(x) d^4x = 1, \quad \int g_2(x) d^4x = 1, \quad (2.5)$$

or in terms of the Fourier transform of the form-factors

$$G(k^2) = \int g(x) e^{ikx} d^4x, \\ G_1(0) = 1, \quad G_2(0) = 1; \quad (2.6)$$

b) the electromagnetic interactions of the neutral particles. As explained in the introduction, due to different isospinor character of pions and kaons, the neutral components of these two multiplets behave quite differently. In the case of the neutral pion, which is a Majorana neutral, the corresponding operator ϕ is hermitian. Then only the second term of H_1 , the second term of H_2 and the first term of H_3 survive.

On the contrary, for the neutral kaon, which is a pseudo-neutral, described by a non-hermitian ϕ , all the terms of the hamiltonian may, in principle, contribute. The normalization conditions are, nevertheless, different:

* Throughout this paper, we use natural units with $\hbar = c = 1$, the unit of length being 1 yukawa (y) = 10^{-18} cm.

$$G_1(0)=0, \quad G_2(0)=1, \quad (2.7)$$

the first condition being equivalent to the fact that the total charge of the particle is zero.

The hamiltonian terms H_1 , H_2 and H_3 are invariant under the gauge transformation*

$$\begin{aligned} \phi(x) &\rightarrow e^{ie\lambda(x)} \phi(x) \\ \phi^*(x) &\rightarrow e^{-ie\lambda(x)} \phi^*(x), \quad \square \lambda = 0. \\ A_\mu(x) &\rightarrow A_\mu + \partial_\mu \lambda \end{aligned} \quad (2.8)$$

To prove this result, we used the following relation, which reduces the non-local case, to a situation similar to the local one:

$$\int \frac{\partial \lambda}{\partial y_\mu} g(x-y) d^4 y = G(0) \frac{\partial \lambda}{\partial x_\mu}. \quad (2.9)$$

Although H_2 is gauge invariant, the first term of it, the so-called Nakano term, is not. The second term of H_2 , however, does not contribute to the self masses.

The hermiticity imposed on (2.1) requires real form factors.

The different terms of H_{int} correspond to two distinct classes of vertices: the one-photon vertices proportional to e , which may depend on A_μ or $F_{\mu\nu}$; the two-photon vertices, proportional to e^2 , depending on $A_\mu A_\mu$, $A_\nu F_{\mu\nu}$ or $F_{\mu\nu} F_{\mu\nu}$.

The electromagnetic self-masses due to the effective hamiltonian (2.1) have been calculated by means of the covariant perturbation techniques and correspond to the following types of graphs.

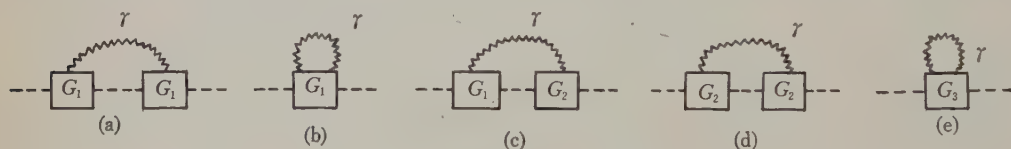


Fig. 1

We get, in terms of momentum space integrals:

$$\begin{aligned} \delta m^2 = & i \frac{e^2}{(2\pi)^4} \int \frac{d^4 k}{k^2 - i\epsilon} \frac{1}{(p-k)^2 + m^2 - i\epsilon} \left[(2p-k)_\mu G_1(k^2) \right. \\ & \left. + \frac{2\kappa}{m^2} G_2(k^2) (p k k_\mu - p_\mu k^2) \right]^2 \\ & - 4i \frac{e^2}{(2\pi)^4} \int \frac{d^4 k}{k^2 - i\epsilon} G_1^2(k^2) - \frac{6i}{(2\pi)^4} \frac{e^2}{m^2} \int d^4 k G_3(p, 0, k). \end{aligned} \quad (2.10)$$

* In the case of neutral particles (Majorana neutrals or pseudo-neutrals), in order that no electrical current be created, we may restrict the gauge transformation to: $\phi \rightarrow \phi$, $A_\mu \rightarrow A_\mu + \partial_\mu \lambda$. It may be shown that H_1 is invariant under this transformation, if and only if $G_1(0) = 0$, that is, $G_1(k^2)$ is singular.

The first bracket term of (2·10) corresponds to the graphs (a), (c), and (d), the second term to the bubble term (b), the third comes from the bubble arising from the two-photon vertex of H_3 , (e). As we already pointed out, the second term of H_2 gives no contribution to the self-masses. It is understood, as explained before, that all the terms of (2·10) may contribute for charged particles, and the same may be said for the neutral kaons (in this last case, (2·7) instead of (2·6) would hold). On the other hand, only the terms (b) and (e) may contribute to the neutral pion self-mass. It is apparent from (2·10), in the local limit, that two types of divergences occur: the well-known quadratic ones and the stronger divergences of fourth order, coming from (d) and (e).

The integration of the expression (2·10) for the self-mass, which depends on the form factors $G_i(k^2)$, has been performed in a general way introducing the spectral decomposition,

$$G_i(k^2) = \frac{1}{\pi} \int_0^\infty \frac{g_i(\eta^2)}{(k^2 + \eta^2 - i\varepsilon)^p} d\eta^2; \quad p=1, 2, \quad (2.11)$$

with a negative imaginary part $-i\varepsilon$, ($p=1$ for $i=1$ and $p=2$ for $i=2$), the integration in momentum space being done with the usual Feynman techniques.*

We restricted ourselves at first to the computation of the self-masses arising from graphs (a), (b), (c), (d), neglecting the graphs (e) which will be discussed in section 5. This means that only the usual minimal term plus the Nakano term contributes to the self-mass, which is written in the following form, (κ being the dimensionless constant of the Nakano term):

$$\delta m = \delta m_{11} + \kappa \delta m_{12} + \kappa^2 \delta m_{22}, \quad (2.12)$$

where

$$\delta m_{11} = \frac{\Gamma}{2} \frac{1}{\pi^2} \int_0^\infty \int_0^\infty g_1(l) g_1(n) dl dn \frac{1}{l-n} [l \log l - (l-4)^2 \omega_l - (l \rightarrow n)] \quad (2.13)$$

$$\delta m_{12} = \Gamma \frac{1}{\pi^2} \int_0^\infty \int_0^\infty g_1(l) g_2(n) dl dn \frac{1}{l-n} \left\{ \frac{1}{l-n} [l(6-l) \log l + l(l-4)^2 \omega_l - (l \rightarrow n)] \right. \\ \left. - 2(3-n) \log n + 2(1-n)(n-4) \omega_n - 2 \right\} \quad (2.14)$$

* We are aware of the fact that this method is not ultimately consistent with an hermitian hamiltonian, a trouble common to all hamiltonian theories with cut-off. In the dispersion relation theory, one arrives at the $i\varepsilon$ integration in a natural way. Our expression for the self-mass relative to (2·2) are the same as those obtained from the dispersion relation approach.¹¹⁾

$$\delta m_{22} = \frac{\Gamma}{2} \frac{1}{\pi^2} \int_0^\infty \int_0^\infty g_2(l) g_2(n) dl dn \frac{1}{(l-n)^2} \left\{ \frac{2}{l-n} [\Gamma^2(6-l) + \Delta_l^2 \omega_l - (l \rightarrow n)] \right. \\ \left. + [3\Delta_l \log l + 3(2-l)\Delta_l \omega_l - 2l + (l \rightarrow n)] \right\} \quad (2.15)$$

with

$$\Gamma = \frac{\alpha m}{8\pi}, \quad \alpha = \frac{e^2}{4\pi} = \frac{1}{137} \quad (2.16)$$

and

$$\Delta_l = l(l-4) \\ \omega_l = \begin{cases} \frac{2}{\sqrt{\Delta_l}} \log \frac{\sqrt{l} + \sqrt{l-4}}{2} & l > 4 \\ \frac{1}{-2} & l = 4 \\ \frac{2}{\sqrt{-\Delta_l}} \arctan \sqrt{\frac{4-l}{l}} & l < 4 \end{cases} \quad (2.17)$$

in terms of dimensionless integration variables l and n ($l = \eta^2/m^2$).

From the general expressions (2.13)–(2.15) we can derive the particular cases of Yukawa and exponential form factors, or combinations of them. The Yukawa (Y) and exponential (E) form factor correspond to the following expressions in configuration and momentum-space:

$$\left\{ \begin{array}{l} \rho_Y(r) = \frac{1}{4\pi} \lambda^2 \frac{e^{-\lambda r}}{r} \\ G_Y(k^2) = \frac{\lambda^2}{k^2 + \lambda^2} \\ \sqrt{\langle r_Y^2 \rangle} = \sqrt{\frac{6}{\lambda}} \end{array} \right\} \quad \left\{ \begin{array}{l} \rho_E(r) = \frac{\lambda^3}{8\pi} e^{-\lambda r} \\ G_E(k^2) = \left(\frac{\lambda^2}{k^2 + \lambda^2} \right)^2 \\ \sqrt{\langle r_E^2 \rangle} = \sqrt{\frac{12}{\lambda}} \end{array} \right. \quad (2.18)$$

In (2.18) we also indicated the radius of the form factors, which are defined in the sense of a root mean square radius (r.m.s.r.) by

$$\sqrt{\langle r^2 \rangle} = \sqrt{-6 \frac{1}{G(k^2)} \frac{dG}{dk^2} \Big|_{k^2=0}} \quad (2.19)$$

We also performed the integration of (2.10) following the method used by Hiida and Sawamura¹²⁾ in the n - p mass difference problem. It consists in the ansatz $G_i(x) = G_i(\mathbf{x}) \cdot \delta(x_0)$, or in momentum space $G_i(k^2) = G_i(\mathbf{k}^2)$, the integration in the k_0 plane being done using the residue theorem. Though the general formulas corresponding to (2.13)–(2.15) have been derived in this case too, we will restrict ourselves to the numerical discussion of a special case in the next section.

§ 3. Particular expressions for the self-masses. Numerical discussions

In order to study the influence of particular choices of form factors and respective values of the radius on the self-masses, we considered four different cases. We also included the case of a singular form factor to be used to describe the electrical structure of the neutral kaons.

Case A. *Y-E form factors with equal r.m.s. radii*

This case corresponds to take a Yukawa form factor for the minimal interaction term and an exponential form factor for the Nakano term, that is,

$$G_1(k^2) = G_Y(k^2), \quad G_2(k^2) = G_E(k^2). \quad (3.1)$$

To obtain the self-masses, we substitute

$$g_1(l) = \pi a \delta(l-a), \quad g_2(l) = 4\pi a^2 \delta(l-2a)$$

with

$$\langle r^2 \rangle = \frac{6}{am^2} \quad (3.2)$$

in formulas (2.13)–(2.15), the result being

$$\begin{aligned} \left\{ \begin{aligned} \delta m_{11}^A(a) &= \frac{\Gamma}{2} a [4 + a \log a - (a+2)(a-4)\omega_a] \\ \delta m_{12}^A(a) &= 4\Gamma a^2 [2 - 6 \log 2 - a \log a + (a-4)^2 \omega_a + 12(a-2)\omega_{2a}] \\ \delta m_{22}^A(a) &= 2\Gamma \frac{a^3}{a-2} [2(3-2a) + \mathcal{A}_{2a} \log 2a + 2(1-a)(\mathcal{A}_{2a}-2)\omega_{2a}]. \end{aligned} \right. \quad (3.3) \end{aligned}$$

This case represents the simplest possibility from the viewpoint of the convergence of the self-mass expression (2.10). Since no sure information exists about the radii of the form factors, we may well take them equal.

Case B. *E-E form factors with equal r.m.s. radii*

This case is similar to Case A with

$$G_1(k^2) = G_2(k^2) = G_E(k^2). \quad (3.4)$$

In this case we must put

$$g_1(l) = \pi b^2 \delta'(l-b); \quad g_2(l) = \pi b^2 \delta(l-b)$$

with

$$\langle r^2 \rangle = 12/bm^2. \quad (3.5)$$

We get

$$\begin{aligned} \left\{ \begin{aligned} \delta m_{11}^B(b) &= \Gamma \frac{b}{b-4} [(b-5) + (3b-10)\omega_b] \\ \delta m_{12}^B(b) &= -2\Gamma \frac{b^2}{b-4} [1 - 2\omega_b] \\ \delta m_{22}^B(b) &= -\Gamma \frac{b^4}{2\mathcal{A}_b} [2b - 6 - \mathcal{A}_b \log b + (b-2)(\mathcal{A}_b-2)\omega_b]. \end{aligned} \right. \quad (3.6) \end{aligned}$$

Case C. E-E form factors with different r.m.s. radii

This case is similar to Case B, with the difference that now different radii are taken for G_1 and G_2 . This introduces one more degree of freedom for the realization of the mass differences.

Case D. 2Y-E form factors with equal r.m.s. radii

Here

$$G_1(k^2) = \alpha_1 \frac{\lambda_1^2}{k^2 + \lambda_1^2} + \alpha_2 \frac{\lambda_2^2}{k^2 + \lambda_2^2}$$

$$G_2(k^2) = \left[\frac{\nu^2}{k^2 + \nu^2} \right]^2, \quad (3.7)$$

the corresponding r.m.s. radii being equal,

$$\langle r_1^2 \rangle = \langle r_2^2 \rangle \quad (3.8)$$

where

$$\langle r_1^2 \rangle = 6 \left(\frac{\alpha_1}{\lambda_1^2} + \frac{\alpha_2}{\lambda_2^2} \right). \quad (3.9)$$

Condition (2.6), $\alpha_1 + \alpha_2 = 1$, together with (3.8) gives α_1 and α_2 as a function of λ_1^2 , λ_2^2 and ν^2 .

Case E. Singular form factor

We considered only the case

$$G_1(k^2) = \frac{\varepsilon \lambda^2 k^2}{(k^2 + \lambda^2)^2}, \quad G_2(k^2) = 0 \quad (3.10)$$

which may be written as the difference of a Y-form factor and a E-form factor

$$G_1(k^2) = \varepsilon \left[\frac{\lambda^2}{k^2 + \lambda^2} - \left(\frac{\lambda^2}{k^2 + \lambda^2} \right)^2 \right]. \quad (3.11)$$

For singular form factors, definition (3.19) for the r.m.s. radius is no more valid. However, we can define a radius, with the meaning of the second moment of the charge distribution as:

$$\langle r^2 \rangle = -6 \frac{dG_1}{dk^2} \Big|_{k^2=0}. \quad (3.12)$$

In this case we substitute, according to (3.11), in (2.13)–(2.15):

$$g_1(l) = \pi a [\delta(l-a) - a\delta'(l-a)]$$

$$g_2(l) = 0$$

with

$$a = \frac{\lambda^2}{m^2}, \quad \langle r^2 \rangle = -\frac{6\varepsilon}{\lambda^2} \quad (3.13)$$

which gives for the self-mass the expression

$$\varepsilon^2 \delta m_{11}^0 = \varepsilon^2 \{ \delta m_{11}^A(a) + \delta m_{11}^B(a) - 3\Gamma a(1 - 2\omega_a) \}. \quad (3.14)$$

We may define the extension of the charge distribution as

$$\langle r_{ext}^2 \rangle = 12/\lambda^2 \quad (3.15)$$

which corresponds to the radius of the exponential part of (3.11).

For large values of a (or b), a situation which happens for pions, it is more convenient for the numerical calculations, instead of their asymptotic expansions.

For instance, in case A

$$\delta m_{11}^A(a) = \Gamma \left[\left(3a + \frac{3}{2} - \frac{8}{3} \frac{1}{a} - \dots \right) + \left(3 + \frac{4}{a} + \frac{9}{a^2} + \dots \right) \log a \right] \quad (3.16)$$

which for large a , reproduces approximately the Feynman-Petermann expression (1.1).

In Cases C and D, as well as in the Hiida-Sawamura case, we have omitted the detailed expressions for the self-masses, restricting ourselves to the presentation of the numerical results.

Numerical discussions

I) The pion mass difference

As it is well known, the pion mass difference can be realized assuming a vanishing self-mass for the neutral pion, the self-mass of the charged pions being due to the minimal interaction term H_1 . Our problem in this section is to examine the influence of the Nakano interaction term in the pion mass difference, under the same assumption that $\delta m_{\pi^0} = 0$. This was done, assuming Cases A or B, for the charged pion. The values of the Nakano constant κ which reproduce the pion mass difference are the roots of the quadratic equation (2.12)

$$\delta m_{11} + \kappa \delta m_{12} + \kappa^2 \delta m_{22} = 4.59 \text{ Mev}. \quad (3.17)$$

In Table I, we give the numerical values of the δm 's coefficients of (3.17) and the corresponding roots κ_+ and κ_- , for a common r.m.s. radius varying from 0.3 to 0.8 y .

An inspection of Table I shows that it is not possible to realize the pion mass difference for small values of $\sqrt{\langle r^2 \rangle}$ since the values of κ then are imaginary. Between 0.5 y and 0.6 y the discriminant of Eq. (3.17) vanishes and becomes afterwards positive. Also, there is in this interval, a point for which $\delta m_{11} = 4.95$ Mev. At this point,

$$\kappa_+ = 0, \quad \kappa_- = -\frac{\delta m_{11}}{\delta m_{22}}. \quad (3.18)$$

An estimation using our asymptotic formulas gives the following values of $\sqrt{\langle r^2 \rangle}$ for which (3.18) occurs:

Table I. The first row corresponds to Case A, the second to Case B, for the charged pion. In this, as well as in the other tables, the r.m.s. radii are given in $y(1y=10^{-13} \text{ cm})$ and the δm 's in Mev.

$\sqrt{\langle r^2 \rangle}$	δm_{11}	δm_{12}	δm_{22}	κ_+	κ_-
0.302	16.80	-19.63	1502	complex	complex
	11.39	-20.98	1502	"	"
0.400	9.807	-13.77	488.8	"	"
	6.712	-11.82	488.8	"	"
0.506	6.292	-8.483	190.1	"	"
	4.352	-7.267	190.1	0.059	-0.021
0.596	4.641	-6.030	98.12	0.051	0.010
	3.240	-5.155	98.12	0.15	-0.094
0.702	3.452	-4.289	50.87	0.20	-0.11
	2.436	-3.657	50.87	0.24	-0.17
0.800	2.732	-3.253	29.91	0.31	-0.20
	1.947	-2.765	29.91	0.35	-0.25

$$\begin{aligned}\sqrt{\langle r^2 \rangle} &= 0.57 y \quad (\text{Case A}) \\ \sqrt{\langle r^2 \rangle} &= 0.46 y \quad (\text{Case B}).\end{aligned}\tag{3.19}$$

The values (3.8) are in good agreement with the results of Riazuddin.¹¹⁾

One should notice that numerical values of δm 's do not differ much in the two cases A and B considered, but in both cases the κ_+ and $|\kappa_-|$ increase with $\sqrt{\langle r^2 \rangle}$. Thus, the realization of the pion mass difference is also possible with the introduction of a Nakano term, for $\sqrt{\langle r^2 \rangle} \gtrsim 0.46 y$, for real and small values of Nakano constant κ .

II) Kaon mass difference

The novelty of this case obliged us to a more careful analysis. First of all, it is relevant to point out that, under the assumption of a vanishing self-mass for the neutral kaon, it is not possible to reproduce the experimental result (1.4), even when we attribute the Nakano interaction term to the charged kaons, whatever the form factors or radii may be: the resulting κ were found all imaginary. The following result throws some light on this situation. It is possible to show, using the Hiida-Sawamura ansatz for the form factors, that the self-mass of the charged kaon is

$$\delta m = F \int_0^\infty \frac{d\rho}{\sqrt{\rho^2 + 1}} \{ [2G_1(\rho^2) - 4G_2(\rho^2)\rho^2]^2 + G_1^2(8\rho\sqrt{\rho^2 + 1} + 4\rho^2 + 4) \} > 0 \tag{3.20}$$

where $\rho = |\mathbf{k}|/m$.

It may be presumed that the same result is valid in general for the relativistic integration case,* a point which is corroborated by a detailed numerical analysis.

Two main possibilities to obtain the mass difference of kaons are now discussed.

II.1) The neutral kaon electromagnetic structure is only due to the Nakano interaction term. For the charged kaons we assume that both the minimal and the Nakano interaction term are responsible for the self-mass. We have

$$\delta m_{11} + \kappa \delta m_{12} + \kappa^2 \delta m_{22} - \kappa_0^2 \delta m_{22}^0 = -4 \text{ Mev.} \quad (3.21)$$

We always assume that the form-factor of the Nakano term is of the exponential type, with the same radius, for both K^0 and K^+ . Equation (3.21) represents then in the (κ, κ_0) plane the hyperbola

$$\kappa_0^2 - (\kappa - \tau)^2 = \sigma, \quad (3.22)$$

where

$$\tau = -\frac{\delta m_{12}}{2\delta m_{22}} > 0, \quad \sigma = \frac{\delta m_{11} - \delta m_{\text{e.p.e.r.}}}{\delta m_{22}} - \tau^2. \quad (3.23)$$

As a consequence, there exists a minimum value of $|\kappa_0|$ given by $|\kappa_0|_{\text{min}} = \sqrt{\sigma}$, which corresponds to $\kappa = \tau$. This gives a meaning to the parameters of Eq. (3.22). If we make the more simplified assumption that $\kappa^2 = \kappa_0^2$ we get for κ

$$\kappa = \frac{\delta m_{\text{e.p.e.r.}} - \delta m_{11}}{\delta m_{12}} = \xi. \quad (3.24)$$

In Table II, we present the numerical values obtained, with Cases A and B for the charged kaon.

From Table II, it is clear that the parameters τ , $\sqrt{\sigma}$, ξ increase for increasing radii. On the other hand, it is apparent that the numerical results are not much sensitive to the choice of the form factors there considered. In order to see whether the details of the structure may decrease the values of the parameters τ , $\sqrt{\sigma}$, ξ ,

* Nevertheless, $\delta m_{11} > 0$, $\delta m_{12} < 0$, $\delta m_{22} > 0$, whatever the form factor may be.

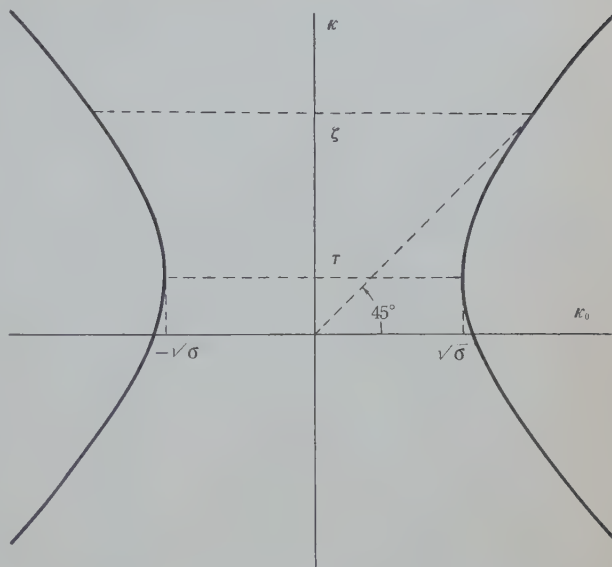


Fig. 2. Plot of (3.22) where the parameters τ , $\sqrt{\sigma}$, ξ are indicated

we considered the two cases below.

First, we investigated the effect of taking different radii as follows :

$$\langle r_2^2 \rangle^0 = \langle r_2^2 \rangle \neq \langle r_1^2 \rangle, \quad (3.25)$$

assuming Case C for the charged pions. The results are indicated in Table III.

Table II. The first row refers to Case A, the second to Case B, for the charged kaon

$\sqrt{\langle r^2 \rangle}$	δm_{11}	δm_{12}	δm_{22}	τ	$\sqrt{\sigma}$	ζ
0.301	5.872	-6.148	29.98	0.1025	0.5646	1.606
	4.310	-5.140	29.98	0.08572	0.5195	1.617
0.399	3.741	-3.272	9.620	0.1701	0.8808	2.366
	2.823	-2.756	9.620	0.1432	0.8299	2.476
0.488	2.746	-2.075	4.161	0.2493	1.249	3.251
	2.118	-1.735	4.161	0.2085	1.195	3.526
0.594	2.066	-1.320	1.833	0.3601	1.783	4.595
	1.627	-1.097	1.833	0.2992	1.726	5.129
0.691	1.679	-0.9290	0.9753	0.4762	2.366	6.113
	1.343	-0.7673	0.9753	0.3933	2.309	6.963
0.797	1.386	-0.6594	0.5299	0.6221	3.126	8.168
	1.124	-0.5417	0.5299	0.5110	3.008	9.459

Table III

$\sqrt{\langle r_1^2 \rangle}$	$\sqrt{\langle r_2^2 \rangle}$	τ	$\sqrt{\sigma}$	ζ
0.488	0.301	0.04646	0.4493	2.196
0.488	0.399	0.1122	0.7896	2.834
0.488	0.488	0.2085	1.195	3.526
0.488	0.594	0.3718	1.789	4.489
0.488	0.691	0.5700	2.439	5.502

The variation of $\sqrt{\langle r_2^2 \rangle}$, fixed $\sqrt{\langle r_1^2 \rangle} = 0.488$ y, presents some cases in which smaller values of the τ , $\sqrt{\sigma}$, ζ are obtained, as compared with the case of equal radii. The order of magnitude, however, remains the same in the range of values taken.

Finally, we considered the case of a more complicated formfactor for the minimal term of the charged kaon, as the superposition of two Yukawa form factors

(Case D), while maintaining the same assumption for the K^0 structure. We examined the effects of the variation of λ_1^2 and λ_2^2 in (3.7); however, the cases considered have shown a tendency to increase the values of the parameters τ , $\sqrt{\sigma}$, ζ .

II.2) A second possibility to reproduce the kaon mass difference is to disregard entirely the Nakano term ($\kappa = \kappa_0 = 0$), for the kaons. Assuming Case E for the neutral kaons, and a Yukawa form factor for the charged kaon with a r.m.s. radius equal to 0.488 μ , we have

$$\delta m_{11}^+ - \varepsilon^2 \delta m_{11}^0 = -4 \text{ Mev.} \quad (3.26)$$

The results are summarized in the following Table IV.

Table IV

$a = \lambda^2/m^2$	δm_{11}^0	ε	$\sqrt{\langle r_{ext}^2 \rangle}$	$\sqrt{\langle r^2 \rangle}$
6	1.298	2.280	0.5639	0.6020
10.5	2.163	1.766	0.4262	0.4005
80	12.66	0.7300	0.1544	9.328×10^{-2}
140	21.43	0.5612	0.1167	6.184×10^{-2}

Finally, we show in the following Tables V and VI the results obtained using the Hiida-Sawamura ansatz, corresponding to case B for the charged bosons. We leave to the reader the comparison with the relativistic method.

Table V. Pions

$\sqrt{\langle r^2 \rangle}$	δm_{11}	δm_{12}	δm_{22}	κ_+	κ_-
0.302	32.98	-32.64	4009	complex	complex
0.506	12.09	-11.38	508.3	"	"
0.702	6.475	-5.761	136.4	"	"

Table VI. Kaons

$\sqrt{\langle r^2 \rangle}$	δm_{11}	δm_{12}	δm_{22}	τ	$\sqrt{\sigma}$	ζ
0.309	10.03	-7.762	73.66	0.0527	0.4332	1.808
0.488	4.528	-2.796	11.24	0.1244	0.8621	3.050
0.691	2.585	-1.245	2.641	0.2357	1.561	5.289

§ 4. A perturbation analysis of the form factors

In this section, we shall briefly analyze the effective hamiltonian (2.1) by means of the usual perturbation picture in which virtual baryon pairs are the re-

levant contributions of the strong interactions. Up to the second order in the elementary charge, the terms of H_{int} may be classified into two groups: The one-photon and the two-photon terms. In the following figure, only the main contributions are indicated, where in the baryon-photon vertices of the Feynman graphs we take the contribution of the full interaction of the type $e\bar{\psi}\gamma_\mu\psi A_\mu + (e\mu_R/2M) \cdot \bar{\psi}\sigma_{\mu\nu}\psi F_{\mu\nu}$.

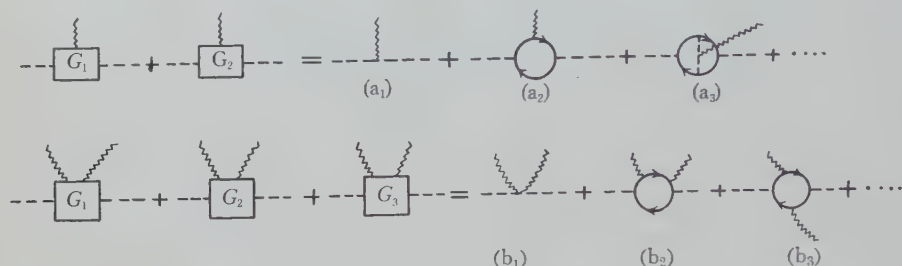


Fig. 3

It may be assumed that for a purely electromagnetic point interaction only the minimal terms contribute, the existence of terms like Nakano's being entirely due to the effects of the strong interactions. This gives us a possible model for explaining the Nakano term, if we assume that the leading graph which contributes to it is the (a_2) , where in the baryon vertex only the Pauli interaction $(e\mu_R/2M) \cdot \bar{\psi}\sigma_{\mu\nu}\psi F_{\mu\nu}$ is taken into account.* Accordingly, we estimated the value of the Nakano constant κ , assuming the meson in a weak external electromagnetic field so that the momentum transfer is small. The antisymmetry of the Pauli interaction makes the matrix element of (a_2) to converge. The main results of our estimations may be sketched as follows: 1) For neutral pions, $\kappa_0=0$, as expected from consistency reasons. 2) For charged pions dissociating virtually in nucleon pairs, a rough estimation gives

$$\kappa = \frac{1}{2\pi} \frac{g^2}{4\pi} (\mu_N - \mu_P) \frac{m^2}{M^2}. \quad (4.1)$$

The introduction of a cut-off at the nucleon mass will not alter the order of magnitude of this result. In (4.1), M is the nucleon mass, μ_N and μ_P the anomalous magnetic moments of the neutron and proton, respectively. For $g^2/4\pi \simeq 15$ we obtained $\kappa = -0.18$. The contributions from other baryon pairs have been estimated assuming global symmetry and a particular assignment of the baryon anomalous magnetic moments.** Taking into account all the baryon pairs we get a value three times larger, $\kappa = -0.54$. Comparison with the results of Table I reveals

* According to Ôno,¹³⁾ the γ_μ in the baryon-photon vertex of graph (a_2) contributes only to the G_1 form factor.

** $\mu_{\pi^+} = -\mu_{\pi^-} = 2$, $\mu_{\pi^0} = \mu_{\pi^0} = 0$, $\mu_{\pi^-} = -\mu_{\pi^0} = 2$.

that this value can be realized for a r.m.s. radius of the order of $0.8y$. 3) For kaons, a similar estimation, taking into account all the baryon pairs, gives:

$$\kappa_0 \simeq 0.13 \text{ or } 0.038, \quad \kappa_0 \simeq -\kappa$$

for $g_K^2/g_\pi^2 = 1/15$ or $1/50$, respectively.

In the present work, we have disregarded, as a working hypothesis, the contributions to the self-masses due to the H_3 interaction terms. These terms depend on three-point form factors, about which little can be said in general. In order to estimate their contribution to the self-masses, we calculated the self-energy graphs which can be derived from (b_2) and (b_3) , where in the baryon-photon vertex we assumed that only $(e\mu_R/2M_R)\sigma_{\mu\nu}q_\nu$ operates. We also assumed global symmetry and the same magnetic moment assignments above. However, the results obtained for the pion mass differences are unreasonably large as compared with the actual value of this difference. On the other hand, the corresponding results for kaons are small compared with the experimental data. This different behavior is not only due to the smaller value of g_K^2 but also to an almost complete cancellation of the contributions of the charged and neutral kaon diagrams.

§ 5. Conclusion

This paper may be considered as an attempt to analyze the electromagnetic structure of mesons, from the experimental data of their mass difference, along the lines of the method by Katayama et al. Contrarily to the situation of the nucleon electromagnetic structure where besides the information of the mass differences additional important data is available from the Stanford experiments on $e-p$ high energy scattering,¹⁴⁾ this is the only dependable information we have at present. Consequently, there is much more freedom in our quantitative analysis, relative to the choices of the meson electromagnetic form factors and their respective radii. This is the reason why only the simplest types of form factors have been here considered. However, the results obtained do not crucially depend on the particular choices made. As far as the radius of the electromagnetic structure is concerned, no definite conclusions seem to exist at present. In the case the kaon mass difference is realized by the introduction of the Nakano interaction term, with the same radius as that of the minimal term, the criterion of the smallness of the Nakano constant κ favors, in the range from 0.3 to $0.8y$ considered, the smaller values of the radius. Since our analysis gives to the pion a radius $\sqrt{\langle r^2 \rangle} \gtrsim 0.5y$ if we attribute to both pion and kaon the same radius, its value would be of the order of $0.5y$. The alternative possibility, the realization of kaon mass difference through an electric structure of the K^0 , is also compatible with the above value of the radius for the charged kaons. In this case, there exists still some freedom for the electric structure of K^0 , whose "radius" may have values from 0.06 to $0.5y$.

We have no ways to decide between the two cases indicated to realize the kaon mass difference, not to speak of the possibility by which the neutral kaon has simultaneously an electric and a magnetic structure. Such decision would require additional information on the high energy electromagnetic phenomena involving kaons, which are not available at present.

From another point of view, the semi-phenomenological approach adopted here has its reason in our inability to treat the strong interactions in a satisfactory way. The estimations of the κ value according to the model for the Nakano term suggested in the last section and the contributions to the self-mass due to the terms H_3 of the effective hamiltonian, by means of perturbation theory with baryon pairs, gives too large values in the pion cases, and too small values for the kaon case, as compared with the results of the phenomenological analysis. The pion case in this respect is worse, giving in some cases values unreasonably large, whose origin must be attributed to the inadequacy of the perturbation theory to describe effects where baryon virtual pairs are the relevant contributions.

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Second Quantization and Lorentz Invariance

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The possibility of constructing the relativistic invariant theory using the particle representation is investigated. It is shown that the Hamiltonian formalism is not adequate in this representation. The general structure of the invariant S-matrix is investigated, and some correspondences to the ordinary theory are obtained. An application to the Compton scattering is also made.

§ 1. Introduction

The problem of Lorentz invariance is treated in the conventional quantum field theory as follows:

I) Field variables are transformed as classical quantities.

II) Next, the procedure of the second quantization is performed in the new coordinate system. Then the commutation relations are retained to be invariant.

However, as Wigner¹⁾ has pointed out, the invariance of the commutation relations does not mean the existence of unitary operators which transform the field quantities as $\varphi' = U\varphi U^{-1}$. In fact these operators have never been constructed in the conventional theory. Moreover, the above procedure produces the invariance of the Feynman amplitudes, but the connection between the Feynman amplitudes and the scattering amplitudes does not seem to be Lorentz invariant.

In spite of these incompletenesses, however, the present field theory has given the correct results in a rather wide range of problems. Therefore, if the present field theory does not contain any inconsistency in itself the above procedure may be considered to be satisfactory, and the problem of Lorentz invariance need not be pursued any more. But, if one considers the local field theory to be unsatisfactory and intends to find out the possibility to construct new theories, this problem must be reinvestigated as a guiding principle, and the following two problems may become important:

I) Whether it is possible to construct any theory which can describe the creation and annihilation of particles and possesses the unitary representation of Lorentz group.

II) In the theory which does not need the unitary representation of Lorentz group, how are the Lorentz transformations treated in a consistent way?

As for the latter problem, Segal²⁾ has recently proposed an interesting attempt. This paper will be devoted to the investigation of the former problem. Then we

must be aware of the fact that there exist infinitely many inequivalent representations of the creation-annihilation operators.³⁾ We now intend to deal with the unitary representation of Lorentz group in the ring of operators generated by the creation-annihilation operators, so we must adopt the simplest one, namely the representation possessing the vacuum state. We adopt, therefore, Fock-Cook-Friedrichs' particle representation. In addition, we use Foldy's canonical form for the one-particle equation to simplify the procedure of the second quantization.

In § 2 the particle representation and Foldy's canonical form are explained briefly. In § 3 we examine the possibility of Hamiltonian formalism, and the origin of difficulties is pointed out. In § 4 the S -matrix formalism is investigated and some correspondences to the ordinary theory are shown, and an application to the Compton scattering is given in § 5. Finally in § 6 we briefly mention about the treatment of decay process.

§ 2. Particle representation and Foldy's canonical form

We first explain the particle representation. One-particle state is represented by $\varphi(\mathbf{x}, s, a)$, where " s " is spin variable and " a " is a variable which discriminates between particle and antiparticle. The latter will be called the particle variable. These variables together with coordinates will be denoted by ξ , and we write $\int d\xi$ instead of $\sum_{s,a} \int d\mathbf{x}$. It is assumed that the states of particle and antiparticle do not mix with each other, and $\varphi(\xi) \in L_\xi^{(2)}$.

The second quantized state is represented by

$$\begin{aligned} \Phi &= \{\varphi_0, \varphi_1(\xi_1), \varphi_2(\xi_1, \xi_2), \dots, \varphi_n(\xi_1, \dots, \xi_n), \dots\} \equiv \{\varphi_n(\xi_1, \dots, \xi_n)\}, \\ \|\Phi\|^2 &= \sum \|\varphi_n\|^2 < \infty, \end{aligned}$$

where φ_0 is a constant and φ_n 's are assumed to be symmetrized or antisymmetrized according to the statistics of particle. For any self-adjoint operator T on one-particle space, we define a self-adjoint operator $\mathcal{Q}(T)$ on the second quantized space by

$$\mathcal{Q}(T) \Phi = \left\{ \sum_{i=1}^n T_i \varphi_n(\xi_1, \dots, \xi_i, \dots, \xi_n) \right\}, \quad (2.1)$$

where T_i is to be understood to operate on ξ_i and further we mean $T\varphi_0 = 0$. Other operators on Φ space will be denoted like \tilde{T} . For any $\varphi(\xi) \in L_\xi^{(2)}$, the creation-annihilation operators are defined by

$$\tilde{A}^+(\varphi) \Phi = \left\{ \sqrt{n} \frac{S_y}{A_s} \varphi(\xi_n) \varphi_{n-1}(\xi_1, \dots, \xi_{n-1}) \right\} \quad (2.2.1)$$

$$\tilde{A}^-(\bar{\varphi}) \Phi = \left\{ \sqrt{n+1} \int \bar{\varphi}(\xi) \varphi_{n+1}(\xi_1, \dots, \xi_n, \xi) d\xi \right\}, \quad (2.2.2)$$

where S_y and A_s mean symmetrization and antisymmetrization respectively, and $\bar{\varphi}$ means complex conjugate of φ . Furthermore, we define the symbolic product be-

tween an operator on $L_{\xi}^{(2)}$ and the creation-annihilation operators by ;

$$T\tilde{A}^{\pm}(\varphi) = \tilde{A}^{\pm}(T\varphi). \quad (2.3)$$

Then the following relations are verified easily ;

$$\tilde{A}^{+}(\varphi)^{*} = \tilde{A}^{-}(\bar{\varphi}), \quad (2.4.1)$$

$$\left. \begin{aligned} [\tilde{A}^{-}(\bar{\varphi}'), \tilde{A}^{+}(\varphi)] &= (\varphi', \varphi) \tilde{E} \\ [\tilde{A}^{-}(\bar{\varphi}'), \tilde{A}^{-}(\bar{\varphi})] &= [\tilde{A}^{+}(\varphi'), \tilde{A}^{+}(\varphi)] = 0 \end{aligned} \right\} \text{ for bosons,} \quad (2.4.2)$$

$$\left. \begin{aligned} \{\tilde{A}^{-}(\bar{\varphi}'), \tilde{A}^{+}(\varphi)\} &= (\varphi', \varphi) \tilde{E} \\ \{\tilde{A}^{-}(\bar{\varphi}'), \tilde{A}^{-}(\bar{\varphi})\} &= \{\tilde{A}^{+}(\varphi'), \tilde{A}^{+}(\varphi)\} = 0 \end{aligned} \right\} \text{ for fermions,} \quad (2.4.3)$$

where $*$ means hermite-conjugate and \tilde{E} is the unit operator on Φ space. Let $\varphi_{\alpha}(\hat{\varepsilon})$ be any complete ortho-normal system in $L_{\xi}^{(2)}$, and $T^{(1)}$ and $T^{(2)}$ be any two self-adjoint operators on that space, then we obtain

$$\sum_{\alpha} T^{(1)} \tilde{A}^{+}(\varphi_{\alpha}) \bar{T}^{(2)} \tilde{A}^{-}(\bar{\varphi}_{\alpha}) = \Omega(T^{(1)} T^{(2)}). \quad (2.5)$$

From this equation and (2.4) we obtain

$$[\Omega(T), \tilde{A}^{+}(\varphi)] = T \tilde{A}^{+}(\varphi), \quad (2.6.1)$$

$$[\Omega(T), \tilde{A}^{-}(\bar{\varphi})] = -\bar{T} \tilde{A}^{-}(\bar{\varphi}), \quad (2.6.2)$$

$$[\Omega(T^{(1)}), \Omega(T^{(2)})] = \Omega([T^{(1)}, T^{(2)}]). \quad (2.6.3)$$

Next we summarize Foldy's canonical form. We shall use the same notation as used in Foldy's paper.⁵⁾ The state of a particle specified by a mass m and a spin s and without antiparticle is represented by a $(2s+1)$ -component wave function φ , and its wave equation is given by

$$i \frac{\partial}{\partial t} \varphi = \omega \varphi, \quad \omega = \sqrt{m^2 + p^2}. \quad (2.7)$$

The infinitesimal operators of Lorentz group are represented, on this space, by

$$\mathbf{P} = \mathbf{p} \equiv -i \nabla, \quad (2.8.1)$$

$$H = \omega, \quad (2.8.2)$$

$$\mathbf{J} = \mathbf{x} \times \mathbf{p} + \mathbf{s}, \quad (2.8.3)$$

$$\mathbf{K} = \frac{1}{2}(\mathbf{x}\omega + \omega\mathbf{x}) - (\mathbf{s} \times \mathbf{p}) / (m + \omega) - t\mathbf{p}. \quad (2.8.4)$$

For the particle with antiparticle, the wave function is $2(2s+1)$ -component, and the wave equation and the infinitesimal operators become

$$i \frac{\partial}{\partial t} \varphi = \beta \omega \varphi, \quad (2.9)$$

$$\mathbf{P} = \mathbf{p}, \quad (2.10.1)$$

$$H = \beta\omega, \quad (2.10.2)$$

$$\mathbf{J} = \mathbf{x} \times \mathbf{p} + \mathbf{s}, \quad (2.10.3)$$

$$\mathbf{K} = \frac{1}{2}\beta(\mathbf{x}\omega + \omega\mathbf{x}) - \beta(\mathbf{s} \times \mathbf{p}) / (m + \omega) - t\mathbf{p}. \quad (2.10.4)$$

For the former case, the second quantization is easily performed, using the particle representation. But, for the latter case, some considerations corresponding to the hole theory for the Dirac particle must be given before the second quantization. In consequence, we remove β 's from (2.9) and (2.10) keeping φ to be $2(2s+1)$ -component, and then perform the procedure of the second quantization. The meaning of this rule is: We learn from the classical theory the existence of the degree of freedom corresponding to the antiparticle, and, converting it to the particle of positive energy, perform the second quantization. According to the above prescription, the second quantized wave equation and infinitesimal operators are given, in both cases, by

$$i \frac{\partial}{\partial t} \Phi = \tilde{H}^0 \Phi, \quad (2.11)$$

$$\mathbf{P}^0 = \mathcal{Q}(\mathbf{p}), \quad (2.12.1)$$

$$\tilde{H}^0 = \mathcal{Q}(\omega), \quad (2.12.2)$$

$$\mathbf{J}^0 = \mathcal{Q}(\mathbf{x} \times \mathbf{p} + \mathbf{s}), \quad (2.12.3)$$

$$\tilde{\mathbf{K}}^0 = \frac{1}{2}\mathcal{Q}(\mathbf{x}\omega + \omega\mathbf{x}) - \mathcal{Q}((\mathbf{s} \times \mathbf{p}) / (m + \omega)) - t\mathcal{Q}(\mathbf{p}). \quad (2.12.4)$$

From (2.6.3) it is obvious that these operators satisfy the correct commutation relations. Thus the second quantization of free particle is completed. The main defect of this method is that we cannot obtain the relation between spin and statistics.

The improper transformations can also be handled according to Foldy's prescription, but we do not enter into details here. In Foldy's paper the particles with non-zero masses only are treated, but it turns out that the Maxwell equation can also be reduced to Foldy's canonical form with supplementary condition. This fact will be explained in the Appendix.

§ 3. Treatment of interactions. Hamiltonian formalism

In the Schrödinger picture, the interactions can be expressed by adding terms to \tilde{H}^0 and $\tilde{\mathbf{K}}^0$ only,⁶⁾ namely it is enough to put

$$\tilde{\mathbf{P}} = \tilde{\mathbf{P}}^0, \quad (3.1.1)$$

$$\tilde{H} = \tilde{H}^0 + \tilde{H}', \quad (3.1.2)$$

$$\mathbf{J} = \mathbf{J}^0, \quad (3.1.3)$$

$$\tilde{\mathbf{K}} = \tilde{\mathbf{K}}^0 + \tilde{\mathbf{K}}'. \quad (3.1.4)$$

The commutation relations to be satisfied by \tilde{H}' and $\tilde{\mathbf{K}}'$ are

$$[\tilde{P}_i, \tilde{H}'] = [\tilde{J}_i, \tilde{H}'] = 0, \quad (3.2.1)$$

$$[\tilde{P}_i, \tilde{K}_j'] = -i\delta_{ij}\tilde{H}', \quad (3.2.2)$$

$$[\tilde{J}_i, \tilde{K}_j'] = i\epsilon_{ijk}\tilde{K}_k', \quad (3.2.3)$$

$$[\tilde{H}', \tilde{K}_i^0] + [\tilde{H}', \tilde{K}_i'] + [\tilde{H}^0, \tilde{K}_i'] = 0, \quad (3.2.4)$$

$$[\tilde{K}_i', \tilde{K}_j^0] + [\tilde{K}_i', \tilde{K}_j'] + [\tilde{K}_i^0, \tilde{K}_j'] = 0. \quad (3.2.5)$$

From (3.2.2) $\tilde{H}' \neq 0$ implies $\tilde{\mathbf{K}}' \neq 0$. (3.2.4) and (3.2.5) are quadratic in the interaction terms, and this fact makes it difficult to determine \tilde{H}' and $\tilde{\mathbf{K}}'$.⁽⁶⁾ Here exists the essential difficulty in the relativistic theory.

Hereafter we deal with only the interaction of the identical neutral scalar particles to simplify the argument. We first consider the φ^3 -type interaction, and put

$$\tilde{H}' = \Sigma a_{\alpha\beta\gamma} \tilde{A}_\alpha^+ \tilde{A}_\beta^+ \tilde{A}_\gamma^- + \text{h.c.}, \quad (3.3)$$

where \tilde{A}_α^+ and \tilde{A}_γ^- represent $\tilde{A}^-(\varphi_\alpha)$ and $\tilde{A}^-(\varphi_\gamma)$ respectively. Using (3.2.1) and (2.6), we find that $a_{\alpha\beta\gamma}$ must satisfy

$$\mathbf{p}_{\alpha\delta} a_{\delta\beta\gamma} + \mathbf{p}_{\beta\delta} a_{\alpha\delta\gamma} - a_{\alpha\beta\delta} \mathbf{p}_{\delta\gamma} = 0, \quad (3.4.1)$$

$$\mathbf{J}_{\alpha\delta} a_{\delta\beta\gamma} + \mathbf{J}_{\beta\delta} a_{\alpha\delta\gamma} - a_{\alpha\beta\delta} \mathbf{J}_{\delta\gamma} = 0. \quad (3.4.2)$$

The general solution of these equations is given by

$$a_{\alpha\beta\gamma} = \int H(\mathbf{p}, \mathbf{q}) \bar{\varphi}_\alpha(\mathbf{p}) \bar{\varphi}_\beta(\mathbf{q}) \varphi_\gamma(\mathbf{p} + \mathbf{q}) d\mathbf{p} d\mathbf{q}, \quad (3.5)$$

where $H(\mathbf{p}, \mathbf{q})$ is a function of scalars which can be composed of \mathbf{p} and \mathbf{q} , and is symmetric with respect to \mathbf{p} and \mathbf{q} . (3.2.2) and (3.2.3) is satisfied if we put

$$\tilde{\mathbf{K}}' = \Sigma b_{\alpha\beta\gamma} \tilde{A}_\alpha^+ \tilde{A}_\beta^+ \tilde{A}_\gamma^- + \text{h.c.}, \quad (3.6)$$

$$b_{\alpha\beta\gamma} = \int H(\mathbf{p}, \mathbf{q}) \bar{\varphi}_\alpha(\mathbf{p}) \bar{\varphi}_\beta(\mathbf{q}) i \frac{\partial}{\partial \mathbf{p}} \varphi_\gamma(\mathbf{p} + \mathbf{q}) d\mathbf{p} d\mathbf{q} \\ + \int L(\mathbf{p}, \mathbf{q}) \bar{\varphi}_\alpha(\mathbf{p}) \bar{\varphi}_\beta(\mathbf{q}) \varphi_\gamma(\mathbf{p} + \mathbf{q}) d\mathbf{p} d\mathbf{q}, \quad (3.7)$$

where L is a sum of products of momentum vectors and scalar functions, and also symmetric with respect to \mathbf{p} and \mathbf{q} . Next we consider (3.2.4). In the present case, it means that the relations

$$[\tilde{H}', \tilde{K}_i^0] + [\tilde{H}^0, \tilde{K}_i'] = 0, \quad (3.8.1)$$

$$[\tilde{H}', \tilde{K}_i'] = 0 \quad (3.8.2)$$

must be satisfied respectively. We obtain from (3.8.1)

$$L_i(\mathbf{p}, \mathbf{q}) = i \frac{1}{\omega_p + \omega_q - \omega_{p+q}} \left[\omega_p \frac{\partial}{\partial p_i} + \omega_q \frac{\partial}{\partial q_i} + \frac{p_i}{2\omega_p} + \frac{q_i}{2\omega_q} + \frac{p_i + q_i}{2\omega_{p+q}} \right] H(\mathbf{p}, \mathbf{q}). \quad (3.9)$$

(3.8.2) contains many sorts of terms, and gives as many conditions correspondingly. For example, the requirement that the coefficient of $\tilde{A}_\alpha^+ \tilde{A}_\beta^- \tilde{A}_\gamma^- \tilde{A}_\delta^-$ must vanish gives

$$-iH(\mathbf{q}, \mathbf{q}') \frac{\partial}{\partial q_i} H(\mathbf{p}, \mathbf{q} + \mathbf{q}') - H(\mathbf{p}, \mathbf{q} + \mathbf{q}') L_i(\mathbf{q}, \mathbf{q}') + H(\mathbf{q}, \mathbf{q}') L_i(\mathbf{p}, \mathbf{q} + \mathbf{q}') = 0. \quad (3.10)$$

Putting $H(\mathbf{p}, \mathbf{q}) = (\omega_p \omega_q \omega_{p+q})^{-1/2} h(\mathbf{p}, \mathbf{q})$ and using (3.9), (3.10) becomes

$$\begin{aligned} & \frac{q_i + q'_i}{\omega_{q+q'}^2} h(\mathbf{q}, \mathbf{q}') h(\mathbf{p}, \mathbf{q} + \mathbf{q}') - h(\mathbf{q}, \mathbf{q}') \frac{\partial}{\partial q_i} h(\mathbf{p}, \mathbf{q} + \mathbf{q}') \\ & - h(\mathbf{q}, \mathbf{p} + \mathbf{q}') \frac{1}{\omega_q + \omega_{q'} - \omega_{q+q'}} \left(\omega_q \frac{\partial}{\partial q_i} + \omega_{q'} \frac{\partial}{\partial q'_i} \right) h(\mathbf{q}, \mathbf{q}') \\ & + h(\mathbf{q}, \mathbf{q}') \frac{1}{\omega_p + \omega_{q+q'} - \omega_{p+q+q'}} \left(\omega_p \frac{\partial}{\partial p_i} + \omega_{q+q'} \frac{\partial}{\partial q_i} \right) h(\mathbf{p}, \mathbf{q} + \mathbf{q}') = 0. \end{aligned} \quad (3.11)$$

It is very difficult to solve this equation, and it may be inferred that it has no solution. Consequently we cannot construct the interaction which is represented by (3.3) and (3.5).

The above result seems to imply that if \tilde{H}' contains a term which is of third order or more with respect to \tilde{A}^\pm , then it becomes to have infinitely many terms on account of (3.2.4). Therefore we next try to determine \tilde{H} and $\tilde{\mathbf{K}}$ expanding them as

$$\tilde{H} = \tilde{H}^0 + \tilde{H}^1 + \tilde{H}^2 + \dots, \quad (3.12.1)$$

$$\tilde{\mathbf{K}} = \tilde{\mathbf{K}}^0 + \tilde{\mathbf{K}}^1 + \tilde{\mathbf{K}}^2 + \dots. \quad (3.12.2)$$

The result is as follows. If we choose \tilde{H}^1 suitably, it can be written as

$$\tilde{H}^1 = i[\tilde{H}^0, \tilde{T}],$$

in case it does not contain a quadratic term with respect to \tilde{A}^\pm and a term corresponding to decay interaction*, and here \tilde{T} is a self-adjoint operator. For example, for the \tilde{H}^1 represented by (3.3) and (3.5), \tilde{T} is given by

$$\tilde{T} = \sum a'_{\alpha\beta\gamma} \tilde{A}_\alpha^+ \tilde{A}_\beta^+ \tilde{A}_\gamma^- + \text{h.c.}, \quad (3.13.1)$$

$$a'_{\alpha\beta\gamma} = -i \int \frac{H(\mathbf{p}, \mathbf{q})}{\omega_p + \omega_q - \omega_{p+q}} \bar{\varphi}_\alpha(\mathbf{p}) \bar{\varphi}_\beta(\mathbf{q}) \varphi_\gamma(\mathbf{p} + \mathbf{q}) d\mathbf{p} d\mathbf{q}, \quad (3.13.2)$$

and $\tilde{\mathbf{K}}^1$ represented by (3.6), (3.7) and (3.9) is written as $\tilde{\mathbf{K}}^1 = i[\tilde{\mathbf{K}}^0, \tilde{T}]$. Therefore, giving \tilde{H}^1 and then to construct \tilde{H}^2 , $\tilde{\mathbf{K}}^1$, etc., so as to satisfy the commutation relations is the same as expanding

* Refer to § 6.

$$\tilde{H} = \exp(-i\tilde{T})\tilde{H}^0 \exp(i\tilde{T}), \quad (3.14.1)$$

$$\tilde{K} = \exp(-i\tilde{T})\tilde{K}^0 \exp(i\tilde{T}), \quad (3.14.2)$$

and if the equation corresponding to (3.10) cannot be solved in any stage of expansion, only (3.14) is essentially a possible form of interaction.

Summarizing above considerations, we obtain the following results:

I) From (3.14) it is obvious that the spectrum of \tilde{H} is the same as that of \tilde{H}^0 , so we cannot deal with the problem of mass difference by this approach.

II) If we adopt the φ^3 -type interaction as \tilde{T} , no process occurs in perturbation calculation.

III) If \tilde{T} contains a term corresponding to a real process, for example, a term like $\tilde{A}_\alpha^+ \tilde{A}_\beta^- \tilde{A}_\gamma^- \tilde{A}_\delta^+$, \tilde{H} can describe the corresponding process, because in this case both \tilde{A}^+ and $\exp(-i\tilde{T})\tilde{A}^+ \exp(i\tilde{T})$ satisfy the condition for the creation operator of a physical particle, that is,

$$\tilde{H}\tilde{A}^+(\varphi)\Phi_0 = \omega\tilde{A}^+(\varphi)\Phi_0,$$

where Φ_0 is the vacuum state; so, considering \tilde{A}^+ to be the creation operator of a physical particle, the real process becomes to occur.

These results are very unsatisfactory, and as to the real process there exist too many arbitrarinesses. In addition, the existence of the relativistic Hamiltonian formalism does not imply the existence of the invariant S -matrix. Therefore the ordinary expression of the invariance of the S -matrix, namely $[\tilde{S}, \tilde{K}^0] = 0$, is a further requirement for the theory. Accordingly, it is more convenient to deal directly with S -matrix theory, if we are concerned only with real processes. So, in the next section we investigate the structure of the invariant S -matrix.

§ 4. Structure of S -matrix

According to the general theory of S -matrix, if Yang-Feldman's S -matrix exists for free Hamiltonian H^0 and total Hamiltonian H , the absolutely continuous parts of both operators are unitary equivalent.⁸⁾ In this respect, the results of the previous section are consistent. It has sometimes been inferred that Yang-Feldman's S -matrix does not exist for the multichannel process on account of the complexity to separate each channel.⁹⁾ However, in the case of the relativistic invariant theory, the separation of channels is physically very obvious, and we may suppose that Yang-Feldman's S -matrix does exist in our theory. We denote the invariant S -matrix which can be constructed from \tilde{H}^0 and \tilde{H} defined by (3.14), by \tilde{S}_H , and the set of all \tilde{S}_H by \mathfrak{S}_H . \tilde{S}_H satisfies

$$[\tilde{S}_H, \tilde{H}^0] = [\tilde{S}_H, \tilde{K}^0] = 0. \quad (4.1)$$

Of course, it also commutes with \tilde{P}^0 and \mathbf{J}^0 . Let the set of all unitary matrices satisfying (4.1) be \mathfrak{S} , then $\mathfrak{S} \supset \mathfrak{S}_H$. As it is difficult to deal with \mathfrak{S}_H directly, we

investigate the structure of \mathfrak{Z} instead. Still \mathfrak{Z} is more restricted than the Hamiltonian of last section because of (4.1).

It can be proved that for any $\tilde{S} \in \mathfrak{S}$ a self-adjoint operator \tilde{T} which satisfies

$$[\tilde{T}, \tilde{H}^0] = [\tilde{T}, \tilde{\mathbf{K}}^0] = 0 \quad (4.2)$$

exists and \tilde{S} is written as $\tilde{S} = \exp(i\tilde{T})$. Therefore, to investigate \mathfrak{Z} is equivalent to investigating \tilde{T} defined by (4.2).

To compute the commutation relations with $\tilde{\mathbf{K}}^0$, the operators on one-particle space defined by

$$Q^\pm \mathbf{K} = \beta \mathbf{x} \omega Q^\pm \pm i\beta_s Q^\pm - t p Q^\pm \quad (4.3)$$

are very useful. For the spinless particle Q turns out to be

$$Q_0 = \omega^{-1/2} \quad (4.4)$$

except a constant factor.

Now we consider \tilde{T} corresponding to the scattering of neutral scalar particles. Taking into account that \tilde{T} must commute with $\tilde{\mathbf{P}}^0$ and $\tilde{\mathbf{J}}^0$, we put*

$$\tilde{T} = \sum a_{\alpha\beta\gamma\delta} \tilde{A}_\alpha^+ \tilde{A}_\beta^+ \tilde{A}_\gamma^- \tilde{A}_\delta^-, \quad (4.5.1)$$

$$a_{\alpha\beta\gamma\delta} = \int F(\mathbf{p}, \mathbf{q}, \mathbf{r}) \frac{1}{\sqrt{\omega_p}} \bar{\varphi}_\alpha(\mathbf{p}) \frac{1}{\sqrt{\omega_q}} \bar{\varphi}_\beta(\mathbf{q}) \frac{1}{\sqrt{\omega_r}} \varphi_\gamma(\mathbf{r}) \\ \times \frac{1}{\sqrt{\omega_{p+q-r}}} \varphi_\delta(\mathbf{p} + \mathbf{q} - \mathbf{r}) d\mathbf{p} d\mathbf{q} d\mathbf{r}, \quad (4.5.2)$$

where $F(\mathbf{p}, \mathbf{q}, \mathbf{r})$ is a scalar function.

From (4.2) we obtain

$$(\omega_p + \omega_q - \omega_r - \omega_{p+q-r}) F(\mathbf{p}, \mathbf{q}, \mathbf{r}) = 0, \quad (4.6.1)$$

$$\left(\omega_p \frac{\partial}{\partial p_i} + \omega_q \frac{\partial}{\partial q_i} + \omega_r \frac{\partial}{\partial r_i} - \frac{p_i + q_i - r_i}{\omega_{p+q-r}} \right) F(\mathbf{p}, \mathbf{q}, \mathbf{r}) = 0. \quad (4.6.2)$$

(4.6.1) means

$$F(\mathbf{p}, \mathbf{q}, \mathbf{r}) = \delta(\omega_p + \omega_q - \omega_r - \omega_{p+q-r}) f(\mathbf{p}, \mathbf{q}, \mathbf{r}). \quad (4.7)$$

Now,

$$\left(\omega_p \frac{\partial}{\partial p_i} + \omega_q \frac{\partial}{\partial q_i} + \omega_r \frac{\partial}{\partial r_i} \right) \delta(\omega_p + \omega_q - \omega_r - \omega_{p+q-r}) \\ = \left[(p_i + q_i - r_i) - \frac{p_i + q_i - r_i}{\omega_{p+q-r}} (\omega_p + \omega_q - \omega_r) \right] \delta'(\omega_p + \omega_q - \omega_r - \omega_{p+q-r}) \\ = - \frac{p_i + q_i - r_i}{\omega_{p+q-r}} (\omega_p + \omega_q - \omega_r - \omega_{p+q-r}) \delta' = \frac{p_i + q_i - r_i}{\omega_{p+q-r}} \delta,$$

* Refer to the reason for finding (3.5).

so the relation to determine f becomes

$$\delta(\omega_p + \omega_q - \omega_r - \omega_{p+q-r}) \left(\omega_p \frac{\partial}{\partial p_i} + \omega_q \frac{\partial}{\partial q_i} + \omega_r \frac{\partial}{\partial r_i} \right) f = 0. \quad (4.8)$$

We examine some special solutions of (4.3) in order. If we adopt $f \equiv \text{const.}$, then (4.5) gives the same result as the lowest order of φ^4 interaction in the ordinary theory. Next, considering

$$\left(\omega_p \frac{\partial}{\partial p_i} + \omega_q \frac{\partial}{\partial q_i} \right) \left\{ (\omega_p + \omega_q - \omega'_{p+q}) (\omega_p + \omega_q + \omega'_{p+q}) \right\} = 0,$$

where ω' is an energy expression with different mass, a special solution of (4.8) is written as

$$f \equiv f((\omega_p + \omega_q - \omega'_{p+q})(\omega_p + \omega_q + \omega'_{p+q})).$$

Taking $f(x) = 1$, x , (4.5) gives the same result as the contribution of a Feynman diagram indicated in Fig. 1 with $\varphi^2\phi$ interaction. Similarly, adopting

$$f \equiv 1/(\omega_r - \omega_p - \omega'_{p-r})(\omega_r - \omega_p + \omega'_{p-r}),$$

the contribution of Fig. 2 is also obtained.

For other processes, the similar correspondences to the lowest order of the ordinary theory can be easily confirmed for the interaction of the scalar particles.

The above results seem to be interesting, for the interaction Hamiltonian of the ordinary theory is not only unsatisfactory in the sense mentioned in last section, but also not a well defined operator on the second quantized space, and in spite of this a part of its contributions has reappeared in the S -matrix theory treated here. This fact seems to show the reason why the ordinary theory, though containing inconsistencies in itself, gives some useful results by perturbation calculation.

Before leaving this section, we briefly mention about the extension of our method to the interaction of particles with spin. For the particles of $s = \frac{1}{2}$ and $s = 1$, the operators characterized by (4.3) are given by

$$Q_{1/2}^{\pm} = \sqrt{\frac{m + \omega}{\omega}} \mp \sqrt{\frac{1}{\omega(m + \omega)}} (\boldsymbol{\sigma} \cdot \mathbf{p}) \quad (4.9.1)$$

$$Q_1^{\pm} = \sqrt{\omega} \mp \frac{1}{\sqrt{\omega}} (\mathbf{s} \cdot \mathbf{p}) - \frac{\omega - m}{\sqrt{\omega}} R, \quad (4.9.2)$$

where R is a projection operator to the longitudinal wave and represented by

$$R = 1 - (\mathbf{s} \cdot \mathbf{p})^2 / p^2. \quad (4.10)$$

The scattering of the same particles is described taking, instead of (4.5.2),



Fig. 1



Fig. 2

$$a_{\alpha\beta\gamma\delta} = \int F(\mathbf{p}, \mathbf{q}, \mathbf{r}) (Q^+ \varphi_\alpha(\mathbf{p}), Q^- \varphi_\gamma(\mathbf{r})) \\ \times (Q^+ \varphi_\beta(\mathbf{q}), Q^- \varphi_\delta(\mathbf{p} + \mathbf{q} - \mathbf{r})) d\mathbf{p} d\mathbf{q} d\mathbf{r} \quad (4.11.1) \\ \pm (+ \leftrightarrow -),$$

or

$$a_{\alpha\beta\gamma\delta} = \int F(\mathbf{p}, \mathbf{q}, \mathbf{r}) (Q^+ \varphi_\alpha(\mathbf{p}), sQ^- \varphi_\gamma(\mathbf{r})) \cdot \\ (Q^+ \varphi_\beta(\mathbf{q}), sQ^- \varphi_\delta(\mathbf{p} + \mathbf{q} - \mathbf{r})) d\mathbf{p} d\mathbf{q} d\mathbf{r} \quad (4.11.2) \\ \pm (+ \leftrightarrow -),$$

where the inner product is to be taken with respect to the spin variable, and $(+ \leftrightarrow -)$ means the term which can be obtained from the first term interchanging $+$ and $-$ signs.

As an application of our method to the practical phenomena, the Compton scattering is treated in the next section.

§ 5. Compton scattering

According to the result of the Appendix, it is possible to handle the electromagnetic field with our method. For the particle specified by $m=0$ and $s=1$, (4.9.2) becomes

$$Q_i^\pm = \sqrt{p} (1 \mp (\mathbf{s} \cdot \mathbf{p})/p) (1 - R). \quad (5.1)$$

(5.1) contains the projection operator to the transverse wave, therefore so long as we use (5.1), only the transverse photons are treated. It is possible to deal with the longitudinal photons, but we do not mention it here.

Hereafter the spin matrices for $s=\frac{1}{2}$ and $s=1$ will be denoted by \mathbf{s} and \mathbf{S} respectively. We denote the wave function of photon by \mathbf{h} and write $(\mathbf{s}, Q^\pm \mathbf{h})$ as h^\pm . We adopt the ordinary representation for \mathbf{S} , then for an arbitrary vector \mathbf{e} , we get $[(\mathbf{s} \cdot \mathbf{e}), S_i] = -(\mathbf{s} \cdot \mathbf{S}_i \mathbf{e})$. We further use the operators defined by

$$\pm \{O^\pm, s_i\} - \left(\omega_p \frac{\partial}{\partial p_i} + \omega_q \frac{\partial}{\partial q_i} + \omega_r \frac{\partial}{\partial r_i} \right) O^\pm = 0. \quad (5.2)$$

An example is given by

$$O_{p+q}^\pm = \omega_p + \omega_q \pm (\boldsymbol{\sigma} \cdot (\mathbf{p} + \mathbf{q})).$$

To describe the Compton scattering we take

$$\tilde{T} = \Sigma a_{\alpha\beta\gamma\delta} \tilde{A}_\alpha^+ \tilde{B}_\beta^+ \tilde{A}_\gamma^- \tilde{B}_\delta^-, \\ a_{\alpha\beta\gamma\delta} = g \int \delta(\omega_p + q - \omega_r - |\mathbf{p} + \mathbf{q} - \mathbf{r}|) M_{\alpha\beta\gamma\delta}(\mathbf{p}, \mathbf{q}, \mathbf{r}) d\mathbf{p} d\mathbf{q} d\mathbf{r},$$

where \tilde{A}^\pm and \tilde{B}^\pm are the creation-annihilation operators of electrons and photons

respectively, and we use for M the following form,

$$\begin{aligned}
 M = & \{(\omega_p + q - \omega_{p+q})(\omega_p + q + \omega_{p+q})(\omega_r - q - \omega_{r-q})(\omega_r - q + \omega_{r-q})\}^{-1} \\
 & \times [(h_\beta^+ Q_{1/2}^+ \varphi_\alpha, O_{p+q}^+ h_\delta^+ Q_{1/2}^+ \varphi_\gamma) + (h_\beta^- Q_{1/2}^- \varphi_\alpha, O_{p+q}^- h_\delta^- Q_{1/2}^- \varphi_\gamma) \\
 & + m(h_\beta^+ Q_{1/2}^+ \varphi_\alpha, h_\delta^- Q_{1/2}^- \varphi_\gamma) + m(h_\beta^- Q_{1/2}^- \varphi_\alpha, h_\delta^+ Q_{1/2}^+ \varphi_\gamma) \\
 & + (h_\delta^- Q_{1/2}^+ \varphi_\alpha, O_{r-q}^+ h_\beta^- Q_{1/2}^+ \varphi_\gamma) + (h_\delta^+ Q_{1/2}^- \varphi_\alpha, O_{r-q}^- h_\beta^+ Q_{1/2}^- \varphi_\gamma) \\
 & + m(h_\delta^- Q_{1/2}^+ \varphi_\alpha, h_\beta^+ Q_{1/2}^+ \varphi_\gamma) + m(h_\delta^+ Q_{1/2}^- \varphi_\alpha, h_\beta^- Q_{1/2}^- \varphi_\gamma)]. \quad (5.3)
 \end{aligned}$$

Of course, M cannot be uniquely determined by invariance principle only, but (5.3) is one of its simplest form. In this case the correspondence to the ordinary theory cannot be obtained in the matrix element, but calculating the differential cross section, (5.3) leads to the well-known Klein-Nishina formula if we put $g = e^2/8\pi$. Thus it may be said that our method can cover the results of the ordinary theory to some extent, though not completely equivalent to it. Here one may find the possibility to step out from the local field theory, and it may be interesting to investigate whether it is possible to find out the way to determine the S -matrix uniquely according to our approach.

§ 6. Decay interaction

The decay process cannot be treated by the S -matrix formalism. Not only it is meaningless to do so, but also the operator \tilde{T} of § 4 cannot be a well defined one for the decay interaction. So we have to return to the Hamiltonian formalism. The creation-annihilation operators for particles with mass m and M will be denoted by \tilde{A}^\pm and \tilde{B}^\pm respectively, and we assume $M > 2m$. If we put

$$\tilde{H}^1 = \Sigma a_{\alpha\beta\gamma} \tilde{A}_\alpha^+ \tilde{A}_\beta^+ \tilde{B}_\gamma^- + \text{h.c.}, \quad (6.1.1)$$

$$a_{\alpha\beta\gamma} = \int h(\mathbf{p}, \mathbf{q}) (\omega_p \omega_q \omega'_{p+q})^{-1/2} \bar{\varphi}_\alpha(\mathbf{p}) \bar{\varphi}_\beta(\mathbf{q}) \varphi_\gamma(\mathbf{p} + \mathbf{q}) d\mathbf{p} d\mathbf{q}, \quad (6.1.2)$$

then the operator \tilde{T} defined by $\tilde{H}^1 = i[\tilde{H}^0, \tilde{T}]$ is represented formally by

$$\tilde{T} = \Sigma a'_{\alpha\beta\gamma} \tilde{A}_\alpha^+ \tilde{A}_\beta^+ \tilde{B}_\gamma^- + \text{h.c.}, \quad (6.2.1)$$

$$\begin{aligned}
 a'_{\alpha\beta\gamma} = & -i \int \frac{\oint}{\omega_p + \omega_q - \omega'_{p+q}} h(\mathbf{p}, \mathbf{q}) (\omega_p \omega_q \omega'_{p+q})^{-1/2} \\
 & \times \bar{\varphi}_\alpha(\mathbf{p}) \bar{\varphi}_\beta(\mathbf{q}) \varphi_\gamma(\mathbf{p} + \mathbf{q}) d\mathbf{p} d\mathbf{q}, \quad (6.2.2)
 \end{aligned}$$

where \oint means principal value. But in this case $\sum_{\alpha, \beta} |a'_{\alpha\beta\gamma}|^2$ becomes to be divergent, and the operator \tilde{T} cannot be a well defined one as mentioned before. Consequently, the transformation (3.14) loses its meaning for the decay process. However, if we expand (3.14) formally, its first order term, namely \tilde{H}^1 , is well defined for suitable $h(\mathbf{p}, \mathbf{q})$, and any process except decay does not occur by perturbation calculation, and the decay too occurs only in the lowest order. So it may be pos-

sible that the Hamiltonian defined by this formal expansion is a well defined operator. For \mathbf{K} , such reasoning is not possible, but if we demand its first order term to be well defined, then $h(\mathbf{p}, \mathbf{q})$ is restricted to, for example,

$$h(\mathbf{p}, \mathbf{q}) \equiv h(\omega_p + \omega_q - \omega_{p+q}), \quad (6.3)$$

where $h(0)$ is finite. Using (6.3), (6.1) gives the same result as the lowest order contribution with $\varphi^2\phi$ interaction in the ordinary theory. The form of h affects only the line width.

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Appendix. Reduction of the Maxwell equation to Foldy's canonical form

The Maxwell equation for the free field is

$$\varepsilon_{jkl} \frac{\partial E_l}{\partial x_k} + \frac{\partial B_j}{\partial t} = 0, \quad \varepsilon_{jkl} \frac{\partial B_l}{\partial x_k} - \frac{\partial E_j}{\partial t} = 0, \quad (\text{A} \cdot 1)$$

$$\frac{\partial E_j}{\partial x_j} = \frac{\partial B_j}{\partial x_j} = 0. \quad (\text{A} \cdot 2)$$

If we put $-i\varepsilon_{jkl} = (s_j)_{kl}$, $\psi_1 = \mathbf{E} + i\mathbf{B}$, $\psi_2 = \mathbf{E} - i\mathbf{B}$ and $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$, then (A.1) becomes

$$i \frac{\partial}{\partial t} \psi = \beta(\mathbf{s} \cdot \mathbf{p}) \psi. \quad (\text{A} \cdot 3)$$

Here s_j 's are the spin matrices for $s=1$. (A.2) is used as a supplementary condition. Using a matrix given by

$$U = \begin{pmatrix} \frac{ip p_2 - p_1 p_3}{\sqrt{2p^2(p_1^2 + p_2^2)}} & \frac{p_1}{p} & \frac{-ip p_2 - p_1 p_3}{\sqrt{2p^2(p_1^2 + p_2^2)}} \\ \frac{-ip p_1 - p_2 p_3}{\sqrt{2p^2(p_1^2 + p_2^2)}} & \frac{p_2}{p} & \frac{ip p_1 - p_2 p_3}{\sqrt{2p^2(p_1^2 + p_2^2)}} \\ \frac{p_1^2 + p_2^2}{\sqrt{2p^2(p_1^2 + p_2^2)}} & \frac{p_3}{p} & \frac{p_1^2 + p_2^2}{\sqrt{2p^2(p_1^2 + p_2^2)}} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and putting $\psi = U\varphi'$,¹⁰⁾ (A.3) becomes

$$i \frac{\partial}{\partial t} \varphi' = \begin{pmatrix} p & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -p \end{pmatrix} \otimes \beta \varphi'. \quad (\text{A} \cdot 4)$$

Further, putting $\varphi' = V\chi'$, where V is a matrix given by

$$V = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

we get

$$i \frac{\partial}{\partial t} \chi' = \begin{pmatrix} p & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p \end{pmatrix} \otimes \beta \chi'. \quad (\text{A} \cdot 5)$$

Using (A.2), we get

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & 0 \end{pmatrix} \otimes \beta \chi' = p R \otimes \beta \psi = 0,$$

so, adding this to the right-hand side of (A.5), we obtain

$$i \frac{\partial}{\partial t} \chi' = \beta p \chi'. \quad (\text{A} \cdot 6)$$

From the transformation properties of \mathbf{E} and \mathbf{B} we get for ψ

$$\begin{aligned} \mathbf{P} &= \mathbf{p}, \\ H &= \beta(\mathbf{s} \cdot \mathbf{p}) + p R \otimes \beta, \\ \mathbf{J} &= \mathbf{x} \times \mathbf{p} + \mathbf{s}. \end{aligned}$$

But for χ' , \mathbf{J} becomes to have a complex expression. So we further put $\chi = U \chi'$, and we obtain for χ

$$\begin{aligned} \mathbf{P} &= \mathbf{p}, \\ H &= \beta p, \\ \mathbf{J} &= \mathbf{x} \times \mathbf{p} + \mathbf{s}. \end{aligned}$$

Here χ turns out to be
$$\chi = \begin{pmatrix} \mathbf{E} + \frac{i}{p} \text{rot } \mathbf{B} \\ \mathbf{E} - \frac{i}{p} \text{rot } \mathbf{B} \end{pmatrix}.$$

Next we consider about the Lorentz transformations, the infinitesimal operators are expressed by

$$\mathbf{K} = \mathbf{x} H - i \beta \mathbf{s} - t \mathbf{p}, \quad H = \beta(\mathbf{s} \cdot \mathbf{p}) + p \beta R,$$

for ψ . Performing the above transformations we get for χ

$$\mathbf{K} = \beta \mathbf{x} \dot{\mathbf{p}} - \beta (\mathbf{s} \times \mathbf{p}) / p - i\beta \mathbf{p} / p - t\mathbf{p}.$$

Yet this is not a desirable expression. This is rather natural, judging from the dimension of χ . Therefore we again put $\varphi = C/\sqrt{p} \chi$, where C is a constant to be determined from the fact that the total energy is given by $(1/8\pi) \int (\mathbf{E}^2 + \mathbf{B}^2) d\mathbf{x}$ in the Maxwell theory, and consequently we get $C = 1/\sqrt{8\pi}$. For φ , \mathbf{K} becomes to have the desired expression.

Now, it was necessary to use six-component ψ in the beginning, but after the canonical form has been thus obtained, the half components of φ become redundant. Taking this fact into account, the results are summarized as follows;

$$\varphi = \frac{1}{\sqrt{8\pi p}} \left(\mathbf{E} + \frac{i}{p} \text{rot } \mathbf{B} \right), \quad (\text{A} \cdot 7)$$

$$i \frac{\partial}{\partial t} \varphi = p\varphi, \quad (\text{A} \cdot 8)$$

$$\mathbf{P} = \mathbf{p}, \quad (\text{A} \cdot 9 \cdot 1)$$

$$H = p, \quad (\text{A} \cdot 9 \cdot 2)$$

$$\mathbf{J} = \mathbf{x} \times \mathbf{p} + \mathbf{s}, \quad (\text{A} \cdot 9 \cdot 3)$$

$$\mathbf{K} = \frac{1}{2} (\mathbf{x} \dot{\mathbf{p}} + \dot{\mathbf{x}} \mathbf{p}) - (\mathbf{s} \times \mathbf{p}) / p - t\mathbf{p}, \quad (\text{A} \cdot 9 \cdot 4)$$

$$R\varphi = 0. \quad (\text{A} \cdot 10)$$

When the interaction source is present, the Maxwell equation is reduced to

$$i \frac{\partial}{\partial t} \varphi = p\varphi - \sqrt{\frac{2\pi}{p}} i \left(\mathbf{j} - \frac{\mathbf{P}}{p} \rho \right),$$

$$R\varphi = -i \sqrt{\frac{2\pi}{p}} \frac{\mathbf{P}}{p^2} \rho.$$

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Pion-Pion Interaction and Pion-Nucleon Scattering

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Pion-pion interaction is analyzed by using the dispersion relation for pion-nucleon scattering obtained by keeping the momentum transfer between an initial pion and a final nucleon constant. In order to take into account the singularity of two-pion threshold in the dispersion relation, the dispersion relation may be regarded as an integral equation for pion-nucleon scattering amplitude with the kernel of a pion-pion scattering amplitude. When this solution is compared with experiments on pion-nucleon scattering, it is found that the unknown quantity in the dispersion relation is only a pion-pion scattering amplitude. Therefore, if pion-pion amplitude is expressed in terms of an unknown parameter such as scattering length, then this value can be determined via dispersion relation. Thus we are led to the following conclusion: In the isotopic spin state $I=0$ (S -wave) of pion-pion system pion-pion interaction is attractive and the scattering length is of the order of one pion Compton wavelength, while in the isotopic spin state $I=1$ (P -wave) of pion-pion state a definite conclusion could not be obtained. Finally the possibility of explaining the momentum dependence of pion-nucleon phase shift δ_{13} related to the pion-pion interaction is briefly discussed.

§ 1. Introduction

There are some phenomena which might be considered to show the existence of the pion-pion interaction. For instance, the cross section for the multiple production of pions in pion-nucleon scattering is of the order of $(\hbar/\mu c)^2$, where μ is a pion mass. This fact suggests that the pion-pion interaction plays an important role in the multiple production of pions, since this large radius can be understood only in terms of a pion cloud.¹⁾ The role of the pion cloud in the pion-nucleon interaction was stressed already by Dyson and Takeda¹⁾ several years ago. In the problem of electromagnetic structure of nucleon too, the pion-pion interaction might be important. This was pointed out by Federbush et al.²⁾

Recently, Goebel³⁾ and Chew and Low³⁾ have suggested that one could obtain the knowledge on the pion-pion interaction from the analysis of the process $\pi + N \rightarrow N + \pi + \pi$. The result follows only from a plausible assumption about the location and residues of poles in the S matrix, but unfortunately this kind of experiment needs highly accurate measurement and the present experiment does not seem to be precise enough to determine the cross section for the pion-pion interaction.

Sato and two of us (A.T., Y.U.)^{4)*} have analyzed the effective pion-pion interaction from the dispersion relations for the pion-nucleon scattering, obtained by keeping the momentum transfer between an initial pion and a final nucleon constant. In order to estimate the contributions of the pion-pion interaction to the pion-nucleon scattering, in these dispersion relations one must, however, integrate the pion-pion scattering amplitude multiplied by a certain matrix element similar to the pion-nucleon scattering amplitude. Here "similar" means that this amplitude is not physical, i.e. the sign of the four-momentum of one pion is reversed. In Ref. I this amplitude was expressed by the one analytically continued from the scattering amplitude given by Chew et al.,⁵⁾ as was done by Federbush et al.²⁾ and Chew et al.⁶⁾ in the problem of the structure of nucleon.

When we regard the pion-nucleon scattering amplitude as the function of \hat{s} , the square of the momentum transfer between initial and final pions with reversed sign, holding another variable σ^2 fixed (see (2.7)), $\hat{s}=4\mu^2$ may become the branch point which is the threshold of the production of two pions, if the effective pion-pion interaction exists (see (2.12)). For this reason it is not clear that the previous analytic continuation from the region $\hat{s} < 4\mu^2$ to the region $\hat{s} > 4\mu^2$ (the required amplitude appear in this region) is right. As our aim is to analyze the pion-pion interaction from the pion-nucleon scattering, it is desirable to treat the pion-nucleon scattering amplitude more properly by taking into account the above singularity.

In this paper we shall regard the dispersion relations as the integral equations satisfied by the pion-nucleon scattering amplitudes where the kernel is the pion-pion scattering amplitude. If we can solve these integral equations, we shall be able to know the correct behaviour of the pion-nucleon scattering amplitude in $\hat{s} > 4\mu^2$ as well as in $\hat{s} < 4\mu^2$. This is indeed possible under some assumptions. Instead of using the expression continued analytically, we shall use these solutions as the pion-nucleon scattering amplitudes. Hereafter, we proceed with a method similar to that of Ref. I, namely we expand the pion-pion scattering amplitude in partial waves and limit these waves to *S*- and *P*-waves, for which scattering length approximation is used. In the dispersion relation all quantities other than those unknown scattering lengths can be expressed in terms of phase shifts for pion-nucleon scattering and of its coupling constant. Therefore, comparing this identity with experiment, we can estimate the measure of the scattering length of pion-pion interaction.

In § 2, dispersion relations are derived. This section is added for completeness and for explaining our notation. In § 3 the integral equations are derived and solved. In § 4 numerical results and discussions are presented. The conclusions are as follows: (1) There may exist *S*-wave pion-pion interaction with the isotopic spin $I=0$, which is attractive with the scattering length of the order of one pion Compton wavelength, (2) about the *P*-wave pion-pion interaction with $I=1$, on the other hand, we could not obtain a definite conclusion. This situation is discussed

* Hereafter referred to as Ref. I.

in § 4 in connection with the result of Ref. 1. Further, the possibility of explaining the energy dependence of pion-nucleon phase shift δ_{13} related to the pion-pion interaction is briefly discussed in § 4.

§ 2. Dispersion relation

In this section, we shall write down the dispersion relations in the representation where the momentum transfer between the incident pion and the final nucleon is kept constant. Let the four-momenta of the incident and the outgoing pions be k_1 and k_2 , respectively, while those of the initial and the final nucleon are p_1 and p_2 . Then the pion-nucleon scattering amplitude can be given in the form*

$$S = \delta_{\beta\alpha} \delta(p_1 - p_2) \delta(k_1 - k_2) - i(2\pi)^4 \delta(p_1 + k_1 - p_2 - k_2) (m^2/p_2^0 p_1^0)^{1/2} (1/4 k_2^0 k_1^0)^{1/2} \bar{u}_2 M_{\beta\alpha} u_1, \quad (2.1)$$

where the retarded matrix element $M_{\beta\alpha}$ is given by

$$(m/2k_2^0 p_1^0)^{1/2} M_{\beta\alpha} u_1 = (-i) \int d^4 z \exp[-iK \cdot z] \langle k_{2\beta} | \theta(z_0) [f(z/2), j_\alpha(-z/2)] | p_1 \rangle, \quad (2.2)$$

$f(z/2)$ and $j_\alpha(-z/2)$ are the nucleon and pion current operators, respectively, and Greek subscripts α and β are the isotopic spin indices of the incident and the outgoing pions, respectively, and $K = (p_2 + k_1)/2$. We have omitted the contribution from the equal time commutator for simplicity. Here, we define the Feynman amplitude $F_{\beta\alpha}$ for the latter purpose,

$$(m/2k_2^0 p_1^0)^{1/2} F_{\beta\alpha} u_1 = (-i) \int d^4 z \exp[-iK \cdot z] \langle k_{2\beta} | T(f(z/2), j_\alpha(-z/2)) | p_1 \rangle. \quad (2.3)$$

We separate the $M_{\beta\alpha}$ into dispersive part $(M_{\beta\alpha})^D$ and absorptive part $(M_{\beta\alpha})^A$:

$$\begin{aligned} (m/2k_2^0 p_1^0)^{1/2} (M_{\beta\alpha})^D u_1 &= (-i/2) \int d^4 z \exp[-iK \cdot z] \langle k_{2\beta} | \epsilon(z_0) [f(z/2), j_\alpha(-z/2)] | p_1 \rangle, \\ (m/2k_2^0 p_1^0)^{1/2} (M_{\beta\alpha})^A u_1 &= (-1/2) \int d^4 z \exp[-iK \cdot z] \langle k_{2\beta} | [f(z/2), j_\alpha(-z/2)] | p_1 \rangle. \end{aligned} \quad (2.4)$$

In the same way $F_{\beta\alpha}$ is written as $(F_{\beta\alpha})^D + i(F_{\beta\alpha})^A$. Then we have

$$(m/2k_2^0 p_1^0)^{1/2} (F_{\beta\alpha})^A u_1 = (-1/2) \int d^4 z \exp[-iK \cdot z] \langle k_{2\beta} | \{f(z/2), j_\alpha(-z/2)\} | p_1 \rangle. \quad (2.5)$$

* Hereafter, we set $\hbar=c=1$, and the normalization volume of system =1.

$M_{\beta\alpha}$ can be expressed by the four-invariant functions in the form

$$M_{\beta\alpha} = M^{(+)} \delta_{\beta\alpha} + [\tau_\beta, \tau_\alpha] M^{(-)}/2, \quad M^{(\pm)} = -A^{(\pm)} + i\gamma \cdot QB^{(\pm)}, \quad (2.6)$$

where, $Q = (k_1 + k_2)/2$. $(M_{\beta\alpha})^D$ and $(M_{\beta\alpha})^A$ can be expressed in the same form as in (2.6). Then from the invariance under time reversal it can be proved that the invariant functions defined for the $(M_{\beta\alpha})^D$ and $(M_{\beta\alpha})^A$ in (2.6) are the real and the imaginary parts of the invariant functions defined for the $M_{\beta\alpha}$, respectively. $M_{\beta\alpha}$ can be expressed as the function of the following two variables:*

$$\xi = -(k_2 - k_1)^2, \quad \sigma^2 = -(p_2 + k_1)^2 = -(p_1 + k_2)^2. \quad (2.7)$$

At some stages of calculation, however, it is more convenient to use variables (W^2, σ^2) rather than (ξ, σ^2) , where W is the total c.m. energy of a pion and a nucleon. W^2 is expressed by ξ and σ^2 as

$$W^2 = -\xi + \sigma^2. \quad (2.8)$$

Now we expect that the pion-nucleon scattering amplitude is analytic in the upper-half of the complex $-\xi$ or W^2 plane for fixed values of σ^2 . Then the dispersion relations will be derived in the same way as done in Ref. I. First $(M_{\beta\alpha})^A$ is divided into two parts:

$$\begin{aligned} (m/2k_2^0 p_1^0)^{1/2} (M_{\beta\alpha})_1 u_1 &= - (1/2) \int d^4 z \exp[-iK \cdot z] \langle k_{2\beta} | f(z/2) j_\alpha(-z/2) | p_1 \rangle, \\ (m/2k_2^0 p_1^0)^{1/2} (M_{\beta\alpha})_2 u_1 &= + (1/2) \int d^4 z \exp[-iK \cdot z] \langle k_{2\beta} | j_\alpha(-z/2) f(z/2) | p_1 \rangle. \end{aligned} \quad (2.9)$$

Then, from (2.5), we obtain

$$(F_{\beta\alpha})^A = (M_{\beta\alpha})_1 - (M_{\beta\alpha})_2. \quad (2.10)$$

In (2.9) we may expand the matrix element of operators into the product of the matrix element of an operator by inserting a complete set of states labeled by the quantum number n . Then, in $(M_{\beta\alpha})_1$, the term coming from the one-nucleon state becomes the "renormalized Born term," and the term coming from the one-nucleon and one-pion state becomes the imaginary part of the pion-nucleon scattering amplitude. Here contributions from the terms of higher mass spectra such as one nucleon and two pion, etc., are neglected since these neglected terms might not be important in the pion-nucleon phenomena at low energies. While in $(M_{\beta\alpha})_2$, the lowest mass state is a two-pion state. The contribution from this state can be expressed by the product of the pion-pion scattering amplitude and the unphysical pion-nucleon scattering amplitude. Here contributions coming from the states

* The relationship between these two variables and (ω, σ^2) defined in Ref. I is
 $m\omega$ (defined in Ref. I) $= (1/4)\sigma^2$ (defined here) $- (1/2)\xi$,
 σ^2 (defined in Ref. I) $= (1/4)\sigma^2$ (defined here).

higher than the four-pion state are neglected by the same reason as stated for $(M_{\beta\alpha})_1$. Dispersion relation should be subtracted once at least, since, for instance, the contribution of the equal time commutator to $M_{\beta\alpha}$ does not contain the variable ξ . In fact we subtracted the dispersion relation once. Here we introduce the notation $\delta A^{(\pm)}$ and $\delta B^{(\pm)}$ for the contributions of pion-pion scattering to the M_2 , namely

$$(M_{\beta\alpha})_2 = \delta_{\beta\alpha} [-\delta A^{(+)} + i\gamma \cdot Q \delta B^{(+)}] + [\tau_\beta, \tau_\alpha]/2 \cdot [-\delta A^{(-)} + i\gamma \cdot Q \delta B^{(-)}]. \quad (2.11)$$

Then the dispersion relations in question may be written as

$$\begin{aligned} A^{(\pm)}(\xi, \sigma^2) &= A^{(\pm)}(0, \sigma^2) - (\xi/\pi) \int_{(m+\mu)^2}^{\infty} d(W'^2) \frac{\text{Im } A^{(\pm)}(W'^2, \sigma^2)}{(W'^2 - \sigma^2)(W'^2 - \sigma^2 + \xi - i\varepsilon)} \\ &\quad - (\xi/\pi) \int_{4\mu^2}^{\infty} d\xi' \frac{\delta A^{(\pm)}(\xi', \sigma^2)}{\xi'(\xi' - \xi - i\varepsilon)}, \\ B^{(\pm)}(\xi, \sigma^2) &= B^{(\pm)}(0, \sigma^2) - \xi \frac{G^2}{(m^2 - \sigma^2)(m^2 - \sigma^2 + \xi)} \\ &\quad - (\xi/\pi) \int_{(m+\mu)^2}^{\infty} d(W'^2) \frac{\text{Im } B^{(\pm)}(W'^2, \sigma^2)}{(W'^2 - \sigma^2)(W'^2 - \sigma^2 + \xi - i\varepsilon)} - (\xi/\pi) \int_{4\mu^2}^{\infty} d\xi' \frac{\delta B^{(\pm)}(\xi', \sigma^2)}{\xi'(\xi' - \xi - i\varepsilon)}, \end{aligned} \quad (2.12)$$

where $(G^2/4\pi) = (2m/\mu)^2 f^2$, and $f^2 = 0.08$. $A^{(\pm)}$ and $B^{(\pm)}$ are regarded as the Feynman amplitudes. The explicit forms of $A^{(\pm)}(0, \sigma^2)$ and $B^{(\pm)}(0, \sigma^2)$ will be given in the next section. As they are forward pion-nucleon scattering amplitudes, they will be expressed easily by using the dispersion relations given by Goldberger, Miyazawa and Oehme.⁷⁾ Therefore, the right-hand sides, except the last terms, of these dispersion relations may be considered to be known, if we express these by using the experimental data of the pion-nucleon scattering and its coupling constant f . In this sense we introduce $A_1^{(\pm)}$ and $B_1^{(\pm)}$ corresponding to the equations of $A^{(\pm)}(\xi, \sigma^2)$ and $B^{(\pm)}(\xi, \sigma^2)$, respectively, as

$$\begin{aligned} A_1^{(\pm)}(\xi, \sigma^2) &= A^{(\pm)}(0, \sigma^2) - (\xi/\pi) \int_{(m+\mu)^2}^{\infty} d(W'^2) \frac{\text{Im } A^{(\pm)}(W'^2, \sigma^2)}{(W'^2 - \sigma^2)(W'^2 - \sigma^2 + \xi - i\varepsilon)}, \\ B_1^{(\pm)}(\xi, \sigma^2) &= B^{(\pm)}(0, \sigma^2) - \xi \frac{G^2}{(m^2 - \sigma^2)(m^2 - \sigma^2 + \xi)} \\ &\quad - (\xi/\pi) \int_{(m+\mu)^2}^{\infty} d(W'^2) \frac{\text{Im } B^{(\pm)}(W'^2, \sigma^2)}{(W'^2 - \sigma^2)(W'^2 - \sigma^2 + \xi - i\varepsilon)}. \end{aligned} \quad (2.13)$$

Then dispersion relations may be written compactly as

$$A^{(\pm)}(\xi, \sigma^2) = A_1^{(\pm)}(\xi, \sigma^2) - (\xi/\pi) \int_{4\mu^2}^{\infty} d\xi' \frac{\delta A^{(\pm)}(\xi', \sigma^2)}{\xi'(\xi' - \xi - i\varepsilon)},$$

$$B^{(\pm)}(\xi, \sigma^2) = B_1^{(\pm)}(\xi, \sigma^2) - (\xi/\pi) \int_{4\mu^2}^{\infty} d\xi' \frac{\delta B^{(\pm)}(\xi', \sigma^2)}{\xi'(\xi' - \xi - i\varepsilon)}. \quad (2.14)$$

First, we shall calculate the contribution of the pion-pion scattering to $(M_{\beta\alpha})_2$. The expression for $(M_{\beta\alpha})_2$ is

$$(m/2k_2^0 p_1^0)^{1/2} (M_{\beta\alpha})_2 u_1 = \pi \sum_{\alpha'\beta'} \int d^3 t_1 d^3 t_2 (2(2\pi)^3)^{-1} \delta(p_1 - p_2 - t_1 - t_2) \\ \times \langle k_{2\beta} | j_\alpha(0) | t_{1\alpha'}, t_{2\beta'} \rangle \langle t_{1\alpha'}, t_{2\beta'} | f(0) | p_1 \rangle, \quad (2.15)$$

where t_1 and t_2 are the four-momenta of the ingoing or outgoing pions, and Greek subscripts α' and β' are the isotopic spin indices of these pions. The integral over intermediate states should be understood to be the average of the integral over "in" states and the one over "out" states. The matrix element of $j_\alpha(0)$ in (2.15) is the pion-pion scattering amplitude. If we limit the partial waves of pion-pion scattering amplitude to the S - and P -waves, this matrix element may be written by the requirement of Lorentz invariance in the form:

$$\langle k_{2\beta} | j_\alpha(0) | t_{1\alpha'}, t_{2\beta'} \rangle = 4\pi (8k_2^0 t_1^0 t_2^0)^{-1/2} 4\xi^{1/2} ((\xi/4) - \mu^2)^{-1/2} \\ \times \{ \exp[-i\delta_s(\xi)] \sin \delta_s(\xi) (1/3) \delta_{\alpha\beta} \delta_{\alpha'\beta'} \\ + 3 \exp[-i\delta_p(\xi)] \sin \delta_p(\xi) (1/4) [(t_2 - t_1)(k_1 + k_2)] [(\xi/4) - \mu^2]^{-1} \\ \times (1/2) [\delta_{\alpha\alpha'} \delta_{\beta\beta'} - \delta_{\beta\alpha'} \delta_{\alpha\beta'}] \}, \quad (2.16)$$

where δ_s and δ_p are the S - and the P -wave pion-pion phase shifts, which correspond to the total isotopic spin state of the pion-pion system $I=0$ and $I=1$, respectively. (It should be noted that the $I=2$ state in the pion-pion system does not contribute to nucleon-antinucleon scattering.)

As the matrix element $\bar{u}(p_2) \langle t_{1\alpha'}, t_{2\beta'} | f(0) | p_1 \rangle$ is equivalent to the matrix element of the process $N(p_1) + \pi(-t_{1\alpha'}) \rightarrow \pi(t_{2\beta'}) + N(p_2)$, this may be written as

$$\langle t_{1\alpha'}, t_{2\beta'} | f(0) | p_1 \rangle = - (m/p_1^0)^{1/2} (1/4k_1^0 k_2^0)^{1/2} [-A(\xi, \sigma'^2) + i\gamma \cdot Q' B(\xi, \sigma'^2)], \quad (2.17)$$

where $\xi = -(t_1 + t_2)^2$, $\sigma'^2 = -(p_2 - t_1)^2$ and $Q' = (t_2 - t_1)/2$, corresponding to the reversed sign of the energy of an initial pion. It should be noticed that this ξ is identical with $\hat{\xi} = -(k_1 - k_2)^2$ defined previously due to the energy momentum conservation $t_1 + t_2 = k_2 - k_1$. Therefore the latter calculation becomes simple. This is the reason why we choose $\hat{\xi}$ as a variable.

Substituting (2.16) and (2.17) in (2.15), and performing the integration over the variables of the intermediate state, M_2 may be written in the form

$$(M_{\beta\alpha})_2 = \delta_{\beta\alpha} \text{Re}[\exp[-i\delta_s(\hat{\xi})] \sin \delta_s(\hat{\xi}) a^{(+)}(\hat{\xi})] \\ + ([\tau_\beta, \tau_\alpha]/2) \{ \text{Re}[\exp[-i\delta_p(\hat{\xi})] \sin \delta_p(\hat{\xi}) a^{(-)}(\hat{\xi}, \sigma^2)] \\ - i\gamma \cdot Q \text{Re}[\exp[-i\delta_p(\hat{\xi})] \sin \delta_p(\hat{\xi}) b^{(-)}(\hat{\xi})] \}, \quad (2.18)$$

where $a^{(+)}(\xi)$, $a^{(-)}(\xi, \sigma^2)$ and $b^{(-)}(\xi)$ are written as follows:

$$\begin{aligned}
 a^{(+)}(\xi) &= \int d\Omega' \left[A^{(+)}(\xi, \sigma'^2) + \frac{(m/4)}{(\xi/4) - m^2} (p_1 + p_2, t_2 - t_1) B^{(+)}(\xi, \sigma'^2) \right], \\
 b^{(-)}(\xi) &= \int d\Omega' \frac{3}{4} \frac{(t_2 - t_1, k_1 + k_2)}{(\xi/4) - \mu^2} \left[\{(\xi/2) - \sigma^2 + m^2 + \mu^2\}^2 \right. \\
 &\quad \left. - 4\{(\xi/4) - m^2\}\{(\xi/4) - \mu^2\} \right]^{-1} \\
 &\quad \times \left[(p_1 \cdot k_1 + k_2) (p_1 + p_2, t_2 - t_1) / 2 - \{(\xi/4) - m^2\} (k_1 + k_2, t_2 - t_1) \right] B^{(-)}(\xi, \sigma'^2), \\
 a^{(-)}(\xi, \sigma^2) &= \int d\Omega' \frac{3}{4} \frac{(t_2 - t_1, k_1 + k_2)}{(\xi/4) - \mu^2} \left\{ A^{(-)}(\xi, \sigma^2) + m \left[\{(\xi/2) - \sigma^2 + m^2 + \mu^2\}^2 \right. \right. \\
 &\quad \left. \left. - 4\{(\xi/4) - m^2\}\{(\xi/4) - \mu^2\} \right]^{-1} \right. \\
 &\quad \left. \times \left[(p_1 \cdot k_1 + k_2) (k_1 + k_2, t_2 - t_1) / 2 - \{(\xi/4) - \mu^2\} (p_1 + p_2, t_2 - t_1) \right] B^{(-)}(\xi, \sigma'^2) \right\},
 \end{aligned} \tag{2.19}$$

where the following abbreviation is used,

$$\int d\Omega' \equiv (\xi^{1/2}, \pi) [(\xi/4) - \mu^2]^{-1/2} \int dt_1 dt_2 \delta(p_1 - p_2 - t_1 - t_2) \delta(t_1^2 + \mu^2) \delta(t_2^2 + \mu^2) \theta(t_1^0) \theta(t_2^0).$$

Choosing the Lorentz frame characterized by the condition $\vec{p}_1 - \vec{p}_2 = 0$,* we further reduce to the following angle integrals:

$$\begin{aligned}
 a^{(+)}(\xi) &= (1/2) \int_{-1}^1 dx \left[A^{(+)}(\xi, \sigma'^2) + x \frac{((\xi/4) - \mu^2)}{p \cdot q} m B^{(+)}(\xi, \sigma'^2) \right], \\
 b^{(-)}(\xi) &= (3/4) \int_{-1}^1 dx (1 - x^2) B^{(-)}(\xi, \sigma'^2), \\
 a^{(-)}(\xi, \sigma^2) &= (3/2) \{ -(\sigma^2/2) + (\xi/4) + (m^2 + \mu^2)/2 \} a^{(-)}(\xi), \\
 a^{(-)}(\xi) &= \frac{1}{p \cdot q} \int_{-1}^1 dx x A^{(-)}(\xi, \sigma'^2) + \frac{m}{p^2} \int_{-1}^1 dx \{ (3x^2/2) - 1/2 \} B^{(-)}(\xi, \sigma'^2),
 \end{aligned} \tag{2.20}$$

where

$$p = ((\xi/4) - m^2)^{1/2}, \quad q = ((\xi/4) - \mu^2)^{1/2} \text{ and } \sigma'^2 = m^2 + \mu^2 + \xi/2 - 2p \cdot qx.$$

By identifying (2.11) and (2.18), the explicit expression of $\partial A^{(\pm)}$ and $\partial B^{(\pm)}$ may be written as

* This may be possible because of $\xi' > 4\mu^2$. See the footnote of Ref. I.

$$\begin{aligned}
\delta A^{(+)}(\xi, \sigma^2) &= -\text{Re}[\exp[-i\partial_s(\xi)] \sin \partial_s(\xi) a^{(+)}(\xi)] \\
\delta B^{(+)}(\xi, \sigma^2) &= 0 \\
\delta A^{(-)}(\xi, \sigma^2) &= -\text{Re}[\exp[-i\partial_p(\xi)] \sin \partial_p(\xi) a^{(-)}(\xi, \sigma^2)] \\
\delta B^{(-)}(\xi, \sigma^2) &= -\text{Re}[\exp[-i\partial_p(\xi)] \sin \partial_p(\xi) b^{(-)}(\xi)]. \quad (2 \cdot 21)
\end{aligned}$$

Hence if we express $\delta A^{(\pm)}$ and $\delta B^{(\pm)}$ explicitly, we can write the dispersion relations in the form

$$\begin{aligned}
A^{(+)}(\xi, \sigma^2) &= A_1^{(+)}(\xi, \sigma^2) + (\xi/\pi) \int_{4\mu^2}^{\infty} d\xi' \frac{\text{Re}[\exp[-i\partial_s(\xi')] \sin \partial_s(\xi') a^{(+)}(\xi')]}{\xi'(\xi' - \xi - i\varepsilon)}, \\
B^{(+)}(\xi, \sigma^2) &= B_1^{(+)}(\xi, \sigma^2), \quad (2 \cdot 22a)
\end{aligned}$$

$$\begin{aligned}
A^{(-)}(\xi, \sigma^2) &= A_1^{(-)}(\xi, \sigma^2) + (\xi/\pi) \int_{4\mu^2}^{\infty} d\xi' \frac{\text{Re}[\exp[-i\partial_p(\xi')] \sin \partial_p(\xi') a^{(-)}(\xi', \sigma^2)]}{\xi'(\xi' - \xi - i\varepsilon)}, \\
B^{(-)}(\xi, \sigma^2) &= B_1^{(-)}(\xi, \sigma^2) + (\xi/\pi) \int_{4\mu^2}^{\infty} d\xi' \frac{\text{Re}[\exp[-i\partial_p(\xi')] \sin \partial_p(\xi') b^{(-)}(\xi')]}{\xi'(\xi' - \xi - i\varepsilon)}. \quad (2 \cdot 22b)
\end{aligned}$$

§ 3. Integral equations

Now we shall derive the integral equations satisfied by $a^{(+)}(\xi)$, $b^{(-)}(\xi)$ and $a^{(-)}(\xi)$. We begin with $a^{(+)}(\xi)$, (2·20).

Although $A^{(+)}(\xi, \sigma'^2)$ and $B^{(+)}(\xi, \sigma'^2)$ in the expression for $a^{(+)}(\xi)$, (2·20), are unphysical scattering amplitudes, they may be obtained by continuing analytically (2·22a). Then $a^{(+)}(\xi)$ becomes

$$a^{(+)}(\xi) = a_1^{(+)}(\xi) + (\xi/\pi) \int_{4\mu^2}^{\infty} d\xi' \frac{\text{Re}[\exp[-i\partial_s(\xi')] \sin \partial_s(\xi') a^{(-)}(\xi')]}{\xi'(\xi' - \xi - i\varepsilon)}, \quad (3 \cdot 1)$$

where

$$a_1^{(+)}(\xi) = (1/2) \int_{-1}^1 dx \{ A_1^{(+)}(\xi, \sigma'^2) + x(\xi/4 - \mu^2) (pq)^{-1} m B_1^{(+)}(\xi, \sigma'^2) \}. \quad (3 \cdot 2)$$

Since $a_1^{(+)}(\xi)$ is considered to be a known function, this is an integral equations satisfied by $a^{(+)}(\xi)$, where the kernel is an S -wave pion-pion scattering amplitude.

Similar equations may be derived for $b^{(-)}(\xi)$ and $a^{(-)}(\xi)$. The integral equation satisfied by $b^{(-)}(\xi)$ is

$$b^{(-)}(\xi) = b_1^{(-)}(\xi) + (\xi/\pi) \int_{4\mu^2}^{\infty} d\xi' \frac{\text{Re}[\exp[-i\partial_p(\xi')] \sin \partial_p(\xi') b^{(-)}(\xi')]}{\xi'(\xi' - \xi - i\varepsilon)}, \quad (3 \cdot 3)$$

where

$$b_1^{(-)}(\xi) = (3/4) \int_{-1}^1 dx (1-x^2) B_1^{(-)}(\xi, \sigma'^2), \quad (3.4)$$

and the one satisfied by $a^{(-)}(\xi)$ is

$$a^{(-)}(\xi) = a_1^{(-)}(\xi) + (\xi/\pi) \int_{4\mu^2}^{\infty} d\xi' \frac{\text{Re}[\exp[-i\partial_P(\xi')] \sin \partial_P(\xi') a^{(-)}(\xi')]}{\xi'(\xi' - \xi - i\varepsilon)}, \quad (3.5)$$

where

$$a_1^{(-)}(\xi) = (1/pq) \int_{-1}^1 dx x A_1^{(-)}(\xi, \sigma'^2) + (m/p^2) \int_{-1}^1 dx [(3x^2 - 1)/2] B_1^{(-)}(\xi, \sigma'^2). \quad (3.6)$$

In general the form of these integral equations is as follows;

$$f(\xi) = f_1(\xi) + (\xi/\pi) \int_{4\mu^2}^{\infty} d\xi' \frac{\text{Re}[\exp[-i\partial(\xi')] \sin \partial(\xi') f(\xi')]}{\xi'(\xi' - \xi - i\varepsilon)}. \quad (3.7)$$

According to Omnes⁶⁾ the solution of this integral equation is given by

$$f(\xi) = \begin{cases} f_1(\xi) + \exp[\rho(\xi)] (\xi/\pi) \int_{4\mu^2}^{\infty} d\xi' \frac{\exp[-\rho(\xi')] \sin \partial(\xi') f_1(\xi')}{\xi'(\xi' - \xi)}, & \text{for } \xi < 4\mu^2 \\ f_1(\xi) + \exp[\rho(\xi)] \exp[i\partial(\xi)] (\xi/\pi) \int_{4\mu^2}^{\infty} d\xi' \frac{\exp[-\rho(\xi')] \sin \partial(\xi') f_1(\xi')}{\xi'(\xi' - \xi - i\varepsilon)}, & \text{for } \xi > 4\mu^2 \end{cases} \quad (3.8)$$

where

$$\rho(\xi) = (\xi/\pi) P \int_{4\mu^2}^{\infty} d\xi' \partial(\xi') [\xi'(\xi' - \xi)]^{-1}. \quad (3.9)$$

Here we have assumed that $f_1(\xi)$ is real and that the function in question behaves properly at infinity. In order that the method is useful, the functional form of the phase shift of the pion-pion interaction must be known. Now we have no knowledge about the behaviour of the phase shifts, but since dispersion integrals may contribute at low energies owing to the energy denominator, we may expect that the results are insensitive to the behavior of the phase shift at high energies, so we assume that the energy dependence of phase shifts is given by a scattering length approximation.

$$\tan \partial_S(\xi) = (\alpha_S/2) (\xi - 4\mu^2)^{1/2}, \quad \tan \partial_P(\xi) = (\alpha_P/2)^3 (\xi - 4\mu^2)^{3/2} \quad (3.10)$$

Then after elementary calculation, $\exp[\rho(\xi)]$, except for constant factors independent of (3.8), is given by:

(for *S*-wave)

$$\exp[\rho(\xi)] = \begin{cases} [1 + (|\alpha_s|/2)(4\mu^2 - \xi)^{1/2}]^{-\varepsilon(\alpha_s)} & \text{for } \xi < 4\mu^2, \\ [1 + (\alpha_s/2)^2(\xi - 4\mu^2)]^{-\varepsilon(\alpha_s)/2} & \text{for } \xi > 4\mu^2, \end{cases}$$

(for *P*-wave)

$$\exp[\rho(\xi)] = \begin{cases} [\{1 - (\alpha_P/2)^2(4\mu^2 - \xi)\} / \{1 - (|\alpha_P|/2)^3(4\mu^2 - \xi)^{3/2}\}]^{+\varepsilon(\alpha_P)} & \text{for } \xi < 4\mu^2, \\ [\{1 + (\alpha_P/2)^2(\xi - 4\mu^2)\} / \{1 + (\alpha_P/2)^6(\xi - 4\mu^2)^3\}^{1/2}]^{+\varepsilon(\alpha_P)} & \text{for } \xi > 4\mu^2, \end{cases} \quad (3.11)$$

where

$$\varepsilon(x) = \begin{cases} 1 & \text{for } x > 0 \\ -1 & \text{for } x < 0. \end{cases}$$

From (3.8) and (3.11), the final expressions of $A^{(+)}(\xi, \sigma^2)$, $B^{(-)}(\xi, \sigma^2)$ and $A^{(-)}(\xi, \sigma^2)$ for $\xi < 4\mu^2$ and for positive scattering lengths, for simplicity, are given by

$$A^{(+)}(\xi, \sigma^2) = A_1^{(+)}(\xi, \sigma^2) + \frac{(\alpha_s/2)}{1 + (\alpha_s/2)(4\mu^2 - \xi)^{1/2}} \frac{\xi}{\pi} \int_{4\mu^2}^{\infty} d\xi' \frac{(\xi' - 4\mu^2)^{1/2} a_1^{(+)}(\xi')}{\xi'(\xi' - \xi)},$$

$$B^{(-)}(\xi, \sigma^2) = B_1^{(-)}(\xi, \sigma^2)$$

$$+ \frac{1 - (\alpha_P/2)^2(4\mu^2 - \xi)}{1 - (\alpha_P/2)^3(4\mu^2 - \xi)^{3/2}} \frac{\xi}{\pi} \int_{4\mu^2}^{\infty} d\xi' \frac{(\alpha_P/2)^3(\xi' - 4\mu^2)^{3/2} b_1^{(-)}(\xi')}{\xi'(\xi' - \xi)[1 + (\alpha_P/2)^2(\xi' - 4\mu^2)]},$$

$$A^{(-)}(\xi, \sigma^2) = A_1^{(-)}(\xi, \sigma^2) + \frac{1 - (\alpha_P/2)^2(4\mu^2 - \xi)}{1 - (\alpha_P/2)^3(4\mu^2 - \xi)^{3/2}}$$

$$\times \frac{\xi}{\pi} \int_{4\mu^2}^{\infty} d\xi' \frac{(\alpha_P/2)^3(\xi' - 4\mu^2)^{3/2}(3/2)\{(m^2 + \mu^2)/2 + \xi/4 - \sigma^2/2\} a_1^{(-)}(\xi')}{\xi'(\xi' - \xi)[1 + (\alpha_P/2)^2(\xi' - 4\mu^2)]}.$$

(3.12)

In order to apply these relations practically, the case for $\xi < 4\mu^2$ is useful, because only this region contains the physical pion-nucleon scattering. In these relations all the quantities other than scattering lengths are expressed in terms of the experimental phase shifts of the pion-nucleon scattering and the coupling constant f^2 . Therefore we can estimate the magnitudes of the scattering lengths from these dispersion relations. Similar relations might be formally written down for negative scattering lengths.

Now we shall give explicit expressions for $A_1^{(\pm)}(\xi, \sigma^2)$, $B_1^{(\pm)}(\xi, \sigma^2)$, $a_1^{(\pm)}(\xi)$ and $b_1^{(\pm)}(\xi)$. As $A^{(\pm)}(0, \sigma^2)$ and $B^{(\pm)}(0, \sigma^2)$ in (2.12) are forward pion-nucleon scattering amplitudes at the c.m. energy $W = (\sigma^2)^{1/2}$, these may be expressed in

terms of the scattering amplitudes given by Goldberger, Miyazawa and Oehme,⁷⁾ as was stated previously. That is,

$$A^{(\pm)}(0, \sigma^2) = A^{(\pm)}(0, (m+\mu)^2) + (1/\pi) \int_{(m+\mu)^2}^{\infty} d(W'^2) \operatorname{Im} A^{(\pm)}(W'^2, 0) \\ \times \left\{ \left[1/(W'^2 - \sigma^2) \pm 1/(W'^2 + \sigma^2 - 2(m^2 + \mu^2)) \right] \right. \\ \left. - \left[1/(W'^2 - (m+\mu)^2) \pm 1/(W'^2 + (m+\mu)^2 - 2(m^2 + \mu^2)) \right] \right\},$$

and

$$B^{(\pm)}(0, \sigma^2) = B^{(\pm)}(0, (m+\mu)^2) + G^2 \left\{ \left[1/(m^2 - \sigma^2) \mp 1/(m^2 + \sigma^2 - 2(m^2 + \mu^2)) \right] \right. \\ \left. - \left[1/(m^2 - (m+\mu)^2) \mp 1/(m^2 + (m+\mu)^2 - 2(m^2 + \mu^2)) \right] \right\} \\ + (1/\pi) \int_{(m+\mu)^2}^{\infty} d(W'^2) \operatorname{Im} B^{(\pm)}(W'^2, 0) \left\{ \left[1/(W'^2 - \sigma^2) \mp 1/(W'^2 + \sigma^2 - 2(m^2 + \mu^2)) \right] \right. \\ \left. - \left[1/(W'^2 - (m+\mu)^2) \mp 1/(W'^2 + (m+\mu)^2 - 2(m^2 + \mu^2)) \right] \right\}. \quad (3 \cdot 13)$$

Introducing $\operatorname{Im} A^{(\pm)}(W^2, \xi)$ and $\operatorname{Im} B^{(\pm)}(W^2, \xi)$ and taking into account the relation

$$\cos \theta = 1 + \xi/(2q^2) = 1 + (\sigma^2 - W^2)/(2q^2), \quad (3 \cdot 14)$$

where θ and q are the scattering angle and momentum in the c.m. system, respectively, we can rewrite $A_1^{(\pm)}(\xi, \sigma^2)$ and $B_1^{(\pm)}(\xi, \sigma^2)$ in the following form,

$$A_1^{(\pm)}(\xi, \sigma^2) = A^{(\pm)}(0, (m+\mu)^2) \\ + (1/\pi) \int_{(m+\mu)^2}^{\infty} d(W'^2) \left\{ \left[\operatorname{Im} A^{(\pm)}(W'^2, \xi)/(W'^2 - \sigma^2 + \xi - i\varepsilon) \right. \right. \\ \left. \pm \operatorname{Im} A^{(\pm)}(W'^2, 0)/(W'^2 + \sigma^2 - 2(m^2 + \mu^2)) \right] \\ \left. - \operatorname{Im} A^{(\pm)}(W'^2, 0) \left[1/(W'^2 - (m+\mu)^2) \pm 1/(W'^2 + (m+\mu)^2 - 2(m^2 + \mu^2)) \right] \right\}, \\ B_1^{(\pm)}(\xi, \sigma^2) = B^{(\pm)}(0, (m+\mu)^2) \\ + G^2 \left\{ \left[1/(m^2 - \sigma^2 + \xi) \mp 1/(m^2 + \sigma^2 - 2(m^2 + \mu^2)) \right] \right. \\ \left. - \left[1/(m^2 - (m+\mu)^2) \mp 1/(m^2 + (m+\mu)^2 - 2(m^2 + \mu^2)) \right] \right\} \\ + (1/\pi) \int_{(m+\mu)^2}^{\infty} d(W'^2) \left\{ \left[\operatorname{Im} B^{(\pm)}(W'^2, \xi)/(W'^2 - \sigma^2 + \xi - i\varepsilon) \right. \right. \\ \left. \mp \operatorname{Im} B^{(\pm)}(W'^2, 0)/(W'^2 + \sigma^2 - 2(m^2 + \mu^2)) \right] \\ \left. - \operatorname{Im} B^{(\pm)}(W'^2, 0) \left[1/(W'^2 - (m+\mu)^2) \mp 1/(W'^2 + (m+\mu)^2 - 2(m^2 + \mu^2)) \right] \right\}.$$

$$\mp \text{Im} B^{(\pm)}(W'^2, 0) / (W'^2 + \sigma^2 - 2(m^2 + \mu^2)) \Big] \\ - \text{Im} B^{(\pm)}(W'^2, 0) \left[1 / (W'^2 - (m + \mu)^2) \mp 1 / (W'^2 + (m + \mu)^2 - 2(m^2 + \mu^2)) \right] \Big\} . \quad (3.15)$$

It is convenient to compare these forms with the dispersion relations given by Chew et al.⁵⁾ $\text{Im} A^{(\pm)}(W^2, \xi)$ and $\text{Im} B^{(\pm)}(W^2, \xi)$ or $\text{Im} A^{(\pm)}(W^2, \sigma^2)$ and $\text{Im} B^{(\pm)}(W^2, \sigma^2)$ are assumed to be able to expand in partial waves and particularly to dominate S - and P -wave amplitudes.

Substituting $A_1^{(\pm)}(\xi, \sigma^2)$ and $B_1^{(\pm)}(\xi, \sigma^2)$ in (3.2), (3.4) and (3.6) and performing the integration over angle x , we obtain the final expressions for $a_1^{(+)}(\xi)$, $b_1^{(-)}(\xi)$ and $a_1^{(-)}(\xi)$ given by

$$a_1^{(+)}(\xi) = A^{(+)}(0, (m + \mu)^2) \\ + (1/\pi) \int_{(m+\mu)^2}^{\infty} d(W'^2) \left\{ J_1(W'^2, \xi) \left[\text{Im} A^{(+)}(W'^2, \xi) + \text{Im} A^{(+)}(W'^2, 0) \right] \right. \\ \left. - \left[1 / (W'^2 - (m + \mu)^2) + 1 / (W'^2 - (m - \mu)^2) \right] \text{Im} A^{(+)}(W'^2, 0) \right\} \\ - (m / (4m^2 - \xi)) \left[(1/\pi) \int_{(m+\mu)^2}^{\infty} d(W'^2) \left\{ J_2(W'^2, \xi) \left[\text{Im} B^{(+)}(W'^2, \xi) \right. \right. \right. \\ \left. \left. + \text{Im} B^{(+)}(W'^2, 0) \right] \right\} + 2G^2 J_2(m^2, \xi) \right] , \\ b_1^{(-)}(\xi) = B^{(-)}(0, (m + \mu)^2) \\ + (1/\pi) \int_{(m+\mu)^2}^{\infty} d(W'^2) \left\{ J_3(W'^2, \xi) \left[\text{Im} B^{(-)}(W'^2, \xi) + \text{Im} B^{(-)}(W'^2, 0) \right] \right. \\ \left. - \text{Im} B^{(-)}(W'^2, 0) \left[1 / (W'^2 - (m + \mu)^2) + 1 / (W'^2 - (m - \mu)^2) \right] \right\} \\ + G^2 \left\{ 2J_3(m^2, \xi) - \left[1 / (m^2 - (m + \mu)^2) + 1 / (m^2 + (m + \mu)^2 - 2(m^2 + \mu^2)) \right] \right\} , \\ a_1^{(-)}(\xi) = - (1/\pi) \int_{(m+\mu)^2}^{\infty} d(W'^2) J_4(W'^2, \xi) \left[\text{Im} A^{(-)}(W'^2, \xi) + \text{Im} A^{(-)}(W'^2, 0) \right] \\ - (4m / (4m^2 - \xi)) \left\{ 2G^2 J_5(m^2, \xi) \right. \\ \left. + (1/\pi) \int_{(m+\mu)^2}^{\infty} d(W'^2) J_5(W'^2, \xi) \left[\text{Im} B^{(-)}(W'^2, \xi) + \text{Im} B^{(-)}(W'^2, 0) \right] \right\} , \quad (3.16)$$

where

$$J_1(W^2, \xi) = (2/\beta) \tan^{-1}(\beta/\alpha)$$

$$J_2(W^2, \xi) = 2 \{1 - (\alpha/\beta) \tan^{-1}(\beta/\alpha)\}$$

$$J_3(W^2, \xi) = (3/\beta) \{- (\alpha/\beta) + (1 + (\alpha/\beta)^2) \tan^{-1}(\beta/\alpha)\}$$

$$J_4(W^2, \xi) = (16/\beta^2) \{1 - (\alpha/\beta) \tan^{-1}(\beta/\alpha)\}$$

$$J_5(W^2, \xi) = (2/\beta) \{(3/\beta) - (1 + 3(\alpha/\beta)^2) \tan^{-1}(\beta/\alpha)\},$$

and

$$\alpha \equiv 2W^2 - 2(m^2 + \mu^2) + \xi$$

$$\beta \equiv (4m^2 - \xi)^{1/2} (\xi - 4\mu^2)^{1/2}.$$

$J_i(m^2, \xi)$ is obtained by exchanging α for $-2\mu^2 + \xi$ in $J_i(W^2, \xi)$. Finally we add one remark that $a_1^{(+)}(\xi)$, $b_1^{(-)}(\xi)$ and $a_1^{(-)}(\xi)$ are real.

§ 4. Numerical results and discussions

At first we shall estimate S - and P -wave scattering lengths for pion-pion scattering by comparing dispersion relations with experiment. Practically, the following two methods are used: I. Regarded as the functions of ξ and σ^2 , scattering amplitudes are differentiated with respect to ξ for fixed σ^2 . They are evaluated at $\xi=0$ and $\sigma^2 = (m + \mu)^2$, i.e. for forward pion-nucleon scattering at zero kinetic energy. II. Regarded as the functions of ξ and W^2 , scattering amplitudes are differentiated with respect to ξ for fixed W^2 . They are evaluated at $\xi=0$ and $W^2 = (m + \mu)^2$, i.e. for forward pion-nucleon scattering at zero kinetic energy. These choices enable us to simplify the principal part integration and it turns out that convergence of integrals is good. Now, in order to compare $\partial A^{(\pm)}/\partial \xi$ and $\partial B^{(\pm)}/\partial \xi$ thus obtained with experiment, one must express these $\partial A^{(\pm)}/\partial \xi$ and $\partial B^{(\pm)}/\partial \xi$ by the experimental pion-nucleon phase shifts. Unfortunately, however, these extremely depend on the D -wave pion-nucleon phase shifts in both cases, and on the P -wave effective ranges in case I. Therefore we do not know to what extent the neglect of these contributions, assumed in Ref. I, can be trusted. Though we shall discuss later on this point, we can avoid this difficulty in the following way: We begin with the combined dispersion relations $A^{(\pm)} + (W - m)B^{(\pm)}$, then the contribution from D -wave phase shifts and P -wave effective ranges are eliminated exactly; namely differentiating the identity

$$A^{(\pm)} + (W - m)B^{(\pm)} = 4\pi(2W)f_1^{(\pm)}/(E + m), \quad (4.1)$$

with respect to ξ for fixed σ^2 or for fixed W^2 corresponding to cases I or II, respectively, we obtain the following identities,

$$\begin{aligned} & (\partial/\partial \xi) A^{(\pm)}(\xi, \sigma^2) + (W - m)(\partial/\partial \xi) B^{(\pm)}(\xi, \sigma^2) \\ &= (1/2W)B^{(\pm)}(\xi, \sigma^2) - (\partial/\partial W^2)(4\pi(2W)f_1^{(\pm)}/(E + m)) \quad \text{for I,} \end{aligned} \quad (4.2)$$

$$\begin{aligned}
 &(\partial/\partial\hat{\xi})A^{(\pm)}(W^2, \hat{\xi}) + (W-m)(\partial/\partial\hat{\xi})B^{(\pm)}(W^2, \hat{\xi}) \\
 &= (\partial/\partial\hat{\xi})(4\pi(2W)f_1^{(\pm)}/(E+m)) \quad \text{for II,} \quad (4.3)
 \end{aligned}$$

where we follow the notation given by Chew et al.⁹⁾

In the real part of (4.2) or (4.3), the contribution of S -wave pion-nucleon phase shifts is estimated to be negligible in case I or is zero in case II respectively. Therefore it is sufficient to consider the contribution of P -wave pion-nucleon scattering lengths in the real part of the relation.

The numerical values of $(\partial/\partial\hat{\xi})\text{Re}A^{(\pm)} + \mu(\partial/\partial\hat{\xi})\text{Re}B^{(\pm)}$ and $(\partial/\partial\hat{\xi})A_1^{(\pm)} + \mu(\partial/\partial\hat{\xi})B_1^{(\pm)}$ are shown in Table I for both cases I and II. Here we used experimental values reported by Puppi at CERN Conference⁹⁾ for the pion-nucleon scattering lengths in $(\partial/\partial\hat{\xi})\text{Re}A^{(\pm)} + \mu(\partial/\partial\hat{\xi})\text{Re}B^{(\pm)}$, and use the empirical formula given by Anderson¹⁰⁾ for the (3, 3) phase shift and pion-nucleon coupling constant f^2 is taken as $f^2=0.08$ in $(\partial/\partial\hat{\xi})A_1^{(\pm)} + \mu(\partial/\partial\hat{\xi})B_1^{(\pm)}$.

Table I. Numerical values of $(\partial/\partial\hat{\xi})\text{Re}A^{(\pm)} + \mu(\partial/\partial\hat{\xi})\text{Re}B^{(\pm)}$ and $(\partial/\partial\hat{\xi})A_1^{(\pm)} + \mu(\partial/\partial\hat{\xi})B_1^{(\pm)}$. Here, subscript A or B indicates the contribution from $A^{(\pm)}$ or $B^{(\pm)}$ respectively. Errors are inserted. The pion-nucleon coupling constant is taken as $f^2=0.08$.

		$(\partial/\partial\xi) \operatorname{Re} A + \mu(\partial/\partial\xi) \operatorname{Re} B$ (μ^{-3})	$(\partial/\partial\xi) A_1 + \mu(\partial/\partial\xi) B_1$ (μ^{-3})	
			Born	Dispersion Integral
Charge state (+)	I	$+0.38 \pm 0.2$	-0.87	$(-0.70)_A + (1.00)_B = +0.30 \pm 0.1$
	II	$+3.10 \pm 0.4$	+1.17	$(1.00)_A + (0.01)_B = 1.01 \pm 0.1$
Charge state (-)	I	-0.51 ± 0.2	-0.87	$(0.35)_A + (-0.50)_B = -0.15 \pm 0.1$
	II	-1.97 ± 0.4	-1.17	$(-0.65)_A + (-0.07)_B = -0.72 \pm 0.1$

The error in the real part is due to inaccuracy of the pion-nucleon scattering lengths. In case I, the scattering lengths a_{11} , a_{13} , a_{31} and a_{33} should be expressed by experimental data. Because of inaccuracy of a_{13} and a_{11} , particularly, the real part may have the error of about 50% of the absolute value in case I. In case II, on the other hand, only two scattering lengths, a_{13} and a_{33} , are contained in the real part, so the error may be relatively small, i.e. about 10~20%. From these considerations, we get the values of $(\partial/\partial\hat{\xi})\text{Re}A^{(\pm)} + \mu(\partial/\partial\hat{\xi})\text{Re}B^{(\pm)}$ as shown in Table I.

The error due to inaccuracy of evaluating the dispersion integral in $\partial/\partial\hat{\xi}A_1^{(\pm)} + \mu(\partial/\partial\hat{\xi})B_1^{(\pm)}$ is assumed to be 10%, the inaccurate small phase shifts and the unknown behaviours of the scattering amplitude at high energies over the δ_{33} dominant energy region being taken into account (although, numerically, the dispersion integrals converge very rapidly under the assumption of δ_{33} dominance in both cases I and II). From these considerations we get the values of $(\partial/\partial\hat{\xi})A_1^{(\pm)} + \mu(\partial/\partial\hat{\xi})B_1^{(\pm)}$ as indicated in Table I.

Using the values in Table I, we write the numerical values of the following quantity as

$$\begin{aligned} & (\partial/\partial\xi) \operatorname{Re} A^{(+)} + \mu(\partial/\partial\xi) \operatorname{Re} B^{(+)} - (\partial/\partial\xi) A_1^{(+)} - \mu(\partial/\partial\xi) B_1^{(+)} \\ &= \begin{cases} [(0.38 \pm 0.2)_R + (0.87)_B - (0.30 \pm 0.1)_I] \mu^{-3} = (0.95 \pm 0.3) \mu^{-3} & \text{I} \\ [(3.1 \pm 0.4)_R - (1.17)_B - (1.01 \pm 0.1)_I] \mu^{-3} = (0.92 \pm 0.5) \mu^{-3}, & \text{II} \end{cases} \quad (4.4) \end{aligned}$$

$$\begin{aligned} & (\partial/\partial\xi) \operatorname{Re} A^{(-)} + \mu(\partial/\partial\xi) \operatorname{Re} B^{(-)} - (\partial/\partial\xi) A_1^{(-)} - \mu(\partial/\partial\xi) B_1^{(-)} \\ &= \begin{cases} [(-0.51 \pm 0.2)_R + (0.87)_B + (0.15 \pm 0.1)_I] \mu^{-3} = (0.51 \pm 0.3) \mu^{-3} & \text{I} \\ [(-1.97 \pm 0.4)_R + (1.17)_B + (0.72 \pm 0.1)_I] \mu^{-3} = (-0.08 \pm 0.5) \mu^{-3}, & \text{II} \end{cases} \quad (4.5) \end{aligned}$$

where the subscripts R , B and I mean the contribution from the real part, Born term and dispersion integral, respectively.

On the other hand, if we differentiated the pion-pion part of the combined dispersion relation (see (3.12)) with respect to ξ corresponding to I and II, we would find that both results are the same as far as the once-subtracted dispersion relation is correct. By virtue of the dispersion relation, therefore, the contributions from pion-pion scattering are numerically given by

$$\begin{aligned} & \frac{(\alpha_s/2)}{1 + (\alpha_s/2)2\mu} \cdot \frac{1}{\pi} \int_{4\mu^2}^{\infty} d\xi' \frac{(\xi' - 4\mu^2)^{1/2} a_1^{(+)}(\xi')}{\xi'^2} = \begin{cases} 0.95 \pm 0.3 \mu^{-3} & \text{for I,} \\ 0.92 \pm 0.5 \mu^{-3} & \text{for II,} \end{cases} \quad (4.6) \\ & \frac{1 - (\alpha_r/2)^2 4\mu^2}{1 - (\alpha_r/2)^3 (4\mu^2)^{3/2}} \cdot \frac{1}{\pi} \int_{4\mu^2}^{\infty} d\xi' \frac{(\alpha_r/2)^3 (\xi' - 4\mu^2)^{3/2} [\mu b_1^{(-)}(\xi') - (3/2) m \mu a_1^{(-)}(\xi')]}{\xi'^2 [1 + (\alpha_r/2)^3 (\xi' - 4\mu^2)]} \\ &= \begin{cases} 0.51 \pm 0.3 \mu^{-3} & \text{for I,} \\ -0.08 \pm 0.5 \mu^{-3} & \text{for II,} \end{cases} \quad (4.7) \end{aligned}$$

where the left-hand side is written for positive scattering length for simplicity. Thus the difference between I and II should be considered to be due to the errors occurring in our approach.

At first we consider S -wave ($I=0$) interaction. In Fig. 1 we plot numerical values of the left-hand side of (4.6) for particular α_s . The dotted lines show the values of the right-hand side of (4.6) for I and II. Dispersion relations for negative scattering length being not written till now, these results are added in Fig. 1. From Fig. 1, we obtain, irrespective of our choice of I or II,

$$\alpha_s \approx 1/\mu.$$

The result is interpreted as follows: Pion-pion interaction is attractive in the S -state with the scattering length of the order of one pion Compton wavelength. This result agrees qualitatively with that of Ref. I in which the contribution of D -waves for the pion-nucleon scattering is neglected.

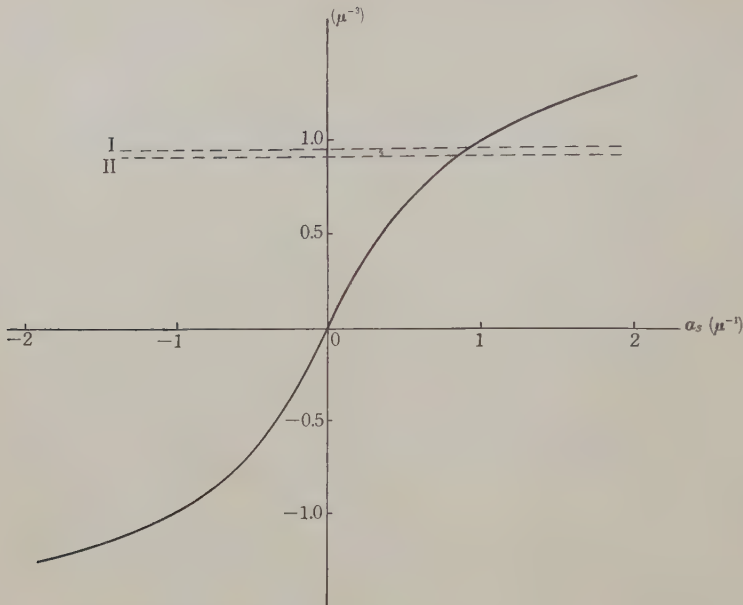


Fig. 1. Numerical values of (4.6) versus α_s .

The solid line represents the value of the left-hand side of (4.6) for any α_s , and the dotted lines represent the values of the right-hand side of (4.6) for case I and for case II.

About the P -wave pion-pion interaction, on the other hand, the situation is more complex. The right-hand side of the P -wave pion-pion dispersion relation (4.7) depends upon whether the result of case I or that of case II is taken, though the results might be considered to be consistent within the limit of errors. According to case II, the right-hand side of the dispersion relation, accordingly the pion-pion part in the left-hand side, becomes zero. In fact, if we calculate a pion-pion part of the left-hand side in the static theory, the integrand of this term becomes zero. (This is advantageous for case II, but case I is not excluded, because the pion-pion part may be able to have a small value due to the relativistic effect.) Here it should be noticed that vanishing of the pion-pion part does not always mean $\alpha_p = 0$; it means only that α_p cannot be well determined from a combined dispersion relation $(\partial/\partial\xi)A^{(-)} + (W-m)(\partial/\partial\xi)B^{(-)}$. Not only the difference between case I and case II which might be considered as an error of the right-hand is large, but also the pion-pion part in the left-hand side cannot be estimated precisely, since the integral involving $a^{(-)}(\xi)$ does not rapidly converge.

Anyway, α_p could not be well determined from the dispersion relation $(\partial/\partial\xi)A^{(-)} + (W-m)(\partial/\partial\xi)B^{(-)}$ which was combined in order to cancel D -wave pion-nucleon phase shifts. If D -wave pion-nucleon phase shifts are negligibly small or experimentally determined in future, then α_p can be estimated by using the relation for $B^{(-)}$ only of which integrals converge rapidly.

In Ref. I, under the assumption that D -wave should be neglected, α_P is estimated by using $B^{(-)}$ relation. Validity of this approximation, however, is doubtful as was mentioned previously, so we wish to add the contribution from D -wave. If we take for this the value of D -wave phase shifts estimated by Chew et al. as the first trial, then the conclusion of Ref. I does not change, i.e.

$$\alpha_P \approx 0.4 \mu^{-1}.$$

The conclusion of Ref. I that P -wave pion-pion interaction exists may be considered to be favorable, but in any case this conclusion is not definite since the adopted D -wave phase shifts are not accurate enough to be trusted. Thus we could not obtain a definite conclusion about the P -wave pion-pion interaction.

Now we should like to make a remark about the behaviour of the pion-nucleon scattering phase shift δ_{13} . According to the previous results, it may be allowed to take the value of contributions of pion-pion scattering to affect the pion-nucleon scattering amplitudes as

$$[(\partial/\partial\xi)A^{(+)}]_{\pi-\pi S}=1.0\mu^{-8}$$

$$[(\partial/\partial\xi)B^{(+)}]_{\pi-\pi S}=0$$

$$[(\partial/\partial\xi)A^{(-)}+\mu(\partial/\partial\xi)B^{(-)}]_{\pi-\pi P}=0.$$

This last choice is insensitive to the tendency of δ_{13} . Then, the contribution of pion-pion S -wave scattering to δ_{13} and δ_{33} are

$$(\delta_{13})_{\pi-\pi S}=(\delta_{33})_{\pi-\pi S}=(1/6\pi)q^3[(E+m)/(2W)][(\partial/\partial\xi)A^{(+)}]_{\pi-\pi S}\approx 0.05q^3/\mu^3.$$

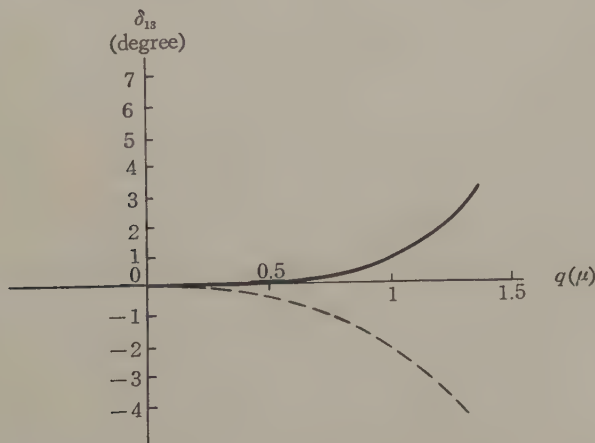


Fig. 2. Pion-nucleon scattering phase shift versus momentum

The dotted curve represents the contribution only of the Born term to δ_{13} and the solid curve represents the sum of the contribution of the Born term and that of S -wave pion-pion interaction.

The dominant contribution except that from pion-pion to the δ_{13} may be "Born term contribution" according to the Chew-Low static theory.

$$(\delta_{13})_{Born} = -(2/3)f^2 q^3/\omega\mu^2 \approx -0.05 q^3/\omega\mu^2$$

Combining these results, we obtain δ_{13} as

$$\delta_{13} \doteq 0.05(q^3/\mu^3 - q^3/\omega\mu^2) \doteq 3^\circ(q^3/\mu^3 - q^3/\omega\mu^2).$$

This rough estimate shows that $\delta_{13} \approx 0$ at low energies, and that δ_{13} increases positively as the energy of pion increases (see Fig. 2). This behaviour may be consistent with the recent experiment.

Finally, we comment on our treatment. $\text{Im}A$ and $\text{Im}B$ in the inhomogeneous term are assumed to be able to be expanded in partial waves, and the integrals involving these are assumed to converge properly. In fact, this may be open to question. But this is beyond our present consideration. Besides, our dispersion relation does not satisfy the property of crossing symmetry. On this point, further more systematic study, for instance, by Mandelstam's double representation,¹¹⁾ should be necessary.

We mention here that Frazer and Fulco¹²⁾ independently applied a similar method to ours to the problem of electromagnetic structure of nucleon.

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Letters to the Editor

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A Solution of the Combined Gravitational and Mesic Field Equations in General Relativity

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January 4, 1960

Tolman¹⁾ has pointed out the paucity of solutions of the non-linear field-equations of General Relativity :

$$\begin{aligned}
 \text{(a)} \quad & -8\pi T_1^1 = e^{-\lambda} (\nu'/r + 1/r^2) \\
 & \quad -1/r^2 + \Lambda \\
 \text{(b)} \quad & -8\pi T_2^2 = -8\pi T_3^3 \\
 & = e^{-\lambda} \left\{ \nu''/2 - \lambda' \nu'/4 \right. \\
 & \quad \left. + (\nu')^2/4 + \frac{\nu' - \lambda'}{2r} \right\} + \Lambda \\
 \text{(c)} \quad & -8\pi T_4^4 = -e^{-\lambda} (\lambda'/r - 1/r^2) \\
 & \quad -1/r^2 + \Lambda.
 \end{aligned} \tag{1}$$

The same reasons contribute to the very great difficulties in finding out the exact solutions of the combined gravitational and electromagnetic fields,—the simplest of which is the Reissner-Nordström solution. Further solutions have been attempted^{(2), (3), (4)} following Tolman's method¹⁾ of assuming suitable values of e^ν or e^λ and then fixing p and ρ . As a

further exercise of this type, we may attempt solutions of the combined gravitational and mesic fields.

In spin 0 case the mesic tensor is⁵⁾

$$\begin{aligned}
 m^{\mu\nu} &= \bar{\phi}^\mu \bar{\phi}^\nu + \frac{1}{2} g^{\mu\nu} (\mu^2 \bar{\phi}^2 - \bar{\phi}_\lambda \bar{\phi}^\lambda) \\
 \bar{\phi}_\mu &= \partial \bar{\phi} / \partial x^\mu.
 \end{aligned} \tag{2}$$

Further, in this case, we may establish a scalar potential $\bar{\phi} = 1/r \cdot e^{-kr}$.

Taking the interval as $ds^2 = e^\lambda dr^2 - \dots + e^\nu dt^2$ and considering spherical symmetry and $\partial \bar{\phi} / \partial t = 0$, we find

$$m^{\mu\nu} = \bar{\phi}^\mu \bar{\phi}^\nu + \frac{1}{2} g^{\mu\nu} [\mu^2 \bar{\phi}^2 - \bar{\phi}_\lambda \bar{\phi}^\lambda]$$

where $\bar{\phi}_\lambda = \partial \bar{\phi} / \partial r$,

for $\bar{\phi}$ depends on r only. We now introduce $\bar{\phi}^\mu = \exp \left\{ -\frac{1}{2} (\lambda + \nu) \right\} \frac{\partial \bar{\phi}}{\partial r}$

[cf. the analogous case of an electromagnetic field (6)], which, as we shall see, admits of a certain solution of the field equations.

Thus

$$\begin{aligned}
 m^{11} &= g^{44} g^{11} \dot{X}^2 + \frac{1}{2} g^{11} [\mu^2 \bar{\phi}^2 \dots] \\
 m^{22} &= m^{33} = m^{44} = \frac{1}{2} [\mu^2 \bar{\phi}^2 - (X^2) \bar{\phi}^\lambda]
 \end{aligned} \tag{3}$$

putting $X = \partial \bar{\phi} / \partial r$

$$= [(e^{-kr}/r) (-k) - e^{-kr}/r^2].$$

Thus we have for the space-time around a particle of mesic charge (within $r > 0$ to $r \sim 10^{-13}$) the field equations

$$\begin{aligned}
 -F_1^1 &= -(T_1^1 + m_1^1) = g^{44} X^2 \\
 & \quad + \frac{1}{2} [\mu^2 \bar{\phi}^2 - \bar{\phi}_\lambda \bar{\phi}^\lambda] + e^{-\lambda} (\dots) \\
 -F_2^2 &= -(T_2^2 + m_2^2)
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}(\mu^2 \bar{\phi}^2 - \bar{\phi}_\lambda \bar{\phi}^\lambda) + e^{-\lambda}(\nu''/2 \dots \\
&- F_4^4 = -(T_4^4 + m_4^4) \\
&= \frac{1}{2}(\mu^2 \bar{\phi}^2 - \bar{\phi}_\lambda \bar{\phi}^\lambda) - e^{-\lambda}(\lambda'/r \dots (4)
\end{aligned}$$

Transforming these according to Tolman¹⁾ we have finally $e^\nu = \text{Const}$

$$\begin{aligned}
(a) \quad &(e^{-\lambda} - 1)/r^2 \\
&= \int 2/r \cdot e^\nu X^2 dr \\
(b) \quad &p + m_2^2 = e^{-\lambda}/r^2 - 1/r^2 + A \quad (5) \\
(c) \quad &\rho + m_4^4 = e^{-\lambda}(\lambda'/r - 1/r^2) \\
&+ 1/r^2 - A.
\end{aligned}$$

(5a) immediately yields the value of $e^{-\lambda}$. This together with the help of (5b) fixes the value of p . Thus, by following this method, we cannot find a solution of the mesic field occurring in strictly empty space. Imagining, therefore, a nucleon in non-empty space ($p, \rho \neq 0$), we find a possible approximate solution of (3):

(considering that $1/r$ is negligible with respect to $1/r^2$ ($r \gtrsim 10^{-13}$))

$$e^\nu = \text{const}$$

$$e^\lambda = 1/(1 + e^\nu e^{kr} + (r^2)).$$

We can have positive values of ρ within a certain region. The negative value of pressure may correspond to the short range attractive force.

This solution is valid within a certain region $r = r_1$, the radius of the nucleon up to $r \sim 10^{-13}$.

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The Phase Shift Formula in the Improved WKB Method

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Recently the author¹⁾ has developed a new method of successive approximation to express the asymptotic solutions of a class of linear second order differential equations in the region containing turning points. According to this method, which is essentially a modified form of the WKB method, the phase shift formula for the radial wave function of the Schrödinger equation can be given by means of a simple procedure of iteration as follows.

When the radial Schrödinger equation is written as

$$\begin{aligned}
\frac{d^2 R}{dr^2} + \frac{2}{r} \frac{dR}{dr} + \left(k^2 - V(r) \right. \\
\left. - \frac{l(l+1)}{r^2} \right) R = 0 \quad (l \geq 1) \quad (1)
\end{aligned}$$

or

$$\begin{aligned}
\frac{d^2 y_0}{dr^2} + \left(k^2 - V(r) - \frac{l(l+1)}{r^2} \right) y_0 = 0 \\
(R = y_0/r), \quad (2)
\end{aligned}$$

then we have the phase shift formula

$$\delta_l = \delta_{l0} + \delta_{l1} + \delta_{l2} + \dots, \quad (3)$$

where

$$\begin{aligned}
\delta_{ln} &= l^{1/2} (l + \frac{1}{2})^{1/2} \pi / 2 - A_n, \\
A_n &= \int_{\beta_n}^{\infty} \{ k - (k^2 - l(l+1)/\rho^2) \\
&- V_n(\rho) \}^{1/2} d\rho + k\beta_n,
\end{aligned}$$

$$V_n(r_n) = \phi_n^{-1/4} (d^2 \phi_n^{1/4} / dr_n^2),$$

$$V_0(r) = V(r),$$

$$\phi_n = \frac{k^2 - l(l+1)/r_{n-1}^2 - V_{n-1}(r_{n-1})}{k^2 - l(l+1)/r_n^2}, \quad (4)$$

β_n is the zero of $k^2 - l(l+1)/\rho^2 - V_n(\rho)$ as a function of ρ , and r_n is given by

$$\int_{\alpha}^{r_n} \left(k^2 - \frac{l(l+1)}{\rho^2} \right)^{1/2} d\rho = \int_{\beta_{n-1}}^{r_{n-1}} \left(k^2 - \frac{l(l+1)}{\rho^2} - V_{n-1}(\rho) \right)^{1/2} d\rho, \quad (5)$$

$$(\alpha^2 = l(l+1)/k^2, r_0 = r).$$

For the deduction of the above formula, let us consider the equation of the $n-1$ -th approximation,

$$\frac{d^2 y_{n-1}}{dr_{n-1}^2} + \left(k^2 - \frac{l(l+1)}{r_{n-1}^2} - V_{n-1}(r_{n-1}) \right) y_{n-1} = 0. \quad (6)$$

Introducing a transformation

$$y_{n-1} = \phi_n^{-1/4} y_n \quad (7)$$

accompanied by (5), we have

$$\frac{d^2 y_n}{dr_n^2} + \left(k^2 - \frac{l(l+1)}{r_n^2} - V_n(r_n) \right) y_n = 0, \quad (8)$$

which is of the same form as (6). And in many practical cases $V_n(r_n)$ is bounded everywhere and $V_n/V_{n-1} = O(k^{-2})$, as is easily verified. Hence neglecting $V_n(r_n)$ in (8), we have the approximate solution of (6) as

$$y_{n-1} = \text{const.} \phi_n^{-1/4} r_n j_l(kr_n),$$

where j_l is the spherical Bessel function

of order l . For the large values of r_n , we have

$$y_{n-1} \sim \text{const.} \cos(kr_n - (l+1)\pi/2) \\ kr_n \sim kr_{n-1} - A_{n-1} + l^{1/2}(l+1)^{1/2}\pi/2.$$

Now (6) is a transformed form of the fundamental equation given by the repeated use of similar procedure. Hence the phase shift formula (3) is derived by returning to the first independent variable r .

Even if $l=0$, we can take α and β_n as zero when the potential is finite at the origin. In other cases we should adopt other suitable transformations instead of (5).

The first approximation δ_{l0} corresponds to the usual WKB phase shift formula derived by Langer's method,²⁾ which can also be refined by the use of the modified WKB method for a simple turning point (see § 2 of reference 1) and gives the following formula,

$$\delta_l = \delta'_{l0} + \delta'_{l1} + \delta'_{l2} + \dots, \quad (9)$$

where

$$\delta'_{ln} = A'_n - A'_{n-1}$$

$$A'_0 = \int_{\gamma}^{\infty} \left\{ k^2 - (l + \frac{1}{2})^2 / \rho^2 - V(\rho) \right\}^{1/2} - k \} d\rho - k\gamma,$$

$$A'_n = \int_{\gamma_n}^{\infty} \left\{ (\tau - Q_n(\tau))^{1/2} - \tau^{1/2} \right\} d\tau - \frac{2}{3} \tau_n^{3/2} \quad (n=1, 2, \dots),$$

$$Q_n(t_n) = \phi_n^{-1/4} (d^2 \phi_n^{1/4} / dt_n^2) \quad (n=1, 2, \dots),$$

$$\phi_n(t_n) = t_n^{-1} \{ t_{n-1} - Q_{n-1}(t_{n-1}) \} \quad (n=2, 3, \dots),$$

$$\frac{2}{3} t_n^{3/2} = \int_{\tau_{n-1}}^{\tau_{n-1}} (\tau - Q_{n-1}(\tau)) d\tau$$

$$(n=2, 3, \dots),$$

$$\varphi_1(t_1) = t_1^{-1} \{k^2 r^2 - r^2 V(r) - (l + \frac{1}{2})^2\},$$

$$\frac{2}{3} t_1^{3/2} = \int_{\gamma}^{\rho} \left\{ k^2 - V(\rho) - \frac{(l + \frac{1}{2})^2}{\rho^2} \right\} d\rho,$$

γ_n and γ are the zeros of $\tau - Q_n(\tau)$ and $k^2 - V(\rho) - (l + \frac{1}{2})^2/\rho^2$, respectively, and Δ'_n is the value corresponding to Δ'_n in the case in which $V(\rho) = 0$. In (9) δ_{l0} is the well-known WKB phase shift. Imai's connection formula³⁾ also gives the correct result to the order of δ'_{l1} .

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Nucleon Structure and BeV Interactions

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In previous notes,¹⁾ estimations of the most effective values of collision parameters in the several BeV pion-nucleon and nucleon-nucleon collisions have been presented by assuming that the elastic scatterings are shadow effects of the inelastic collisions. The obtained values are almost constant at least up to 6 BeV, and the value 0.8×10^{-13} cm seems to have some internal connections with the

charge distributions of the nucleon detected by the high energy electron scattering experiments at the Stanford University.²⁾

In this note, we shall attempt to extract some information about the special extension of the nucleon from the BeV pion-nucleon scattering experiments. In order to get the notion of the special extension, we shall rewrite the inelastic cross section formula

$$\sigma_{inel} = \pi \lambda^2 \sum_l (2l+1) |M(l)|^2$$

$$= \pi \sum_l [(l+1)^2 \lambda^2 - l^2 \lambda^2] |M(l)|^2 \quad (1)$$

in somewhat classical form:

$$\sigma_{inel} = \pi \sum_l |M(l)|^2 \Delta(l^2 \lambda^2)$$

$$\rightarrow \pi \int_0^{\infty} dy^2 |M(y)|^2 = 2\pi \int_0^{\infty} y dy |M(y)|^2 \quad (2)$$

by using "collision parameter" $y = \lambda l$.

In this formula, the square of the matrix element $|M(y)|^2 = |M(l)|^2$ may be regarded as the probability since the inelastic process takes place when the incident particle hits the nucleon at distance y from its center. Therefore, if one assumes a spherical distribution $F(r)$ of the probability per unit volume, then $|M(y)|^2$ may be written as

$$|M(y)|^2 = \int_{-\infty}^{+\infty} dz F(\sqrt{y^2 + z^2}) \quad (3)$$

where z denotes the coordinate parallel to the axis of the collision. Inserting (3) into (1), we have

$$\sigma_{inel} = \pi \lambda^2 \sum_l (2l+1) \int_{-\infty}^{+\infty} dz F(\sqrt{\lambda^2 l^2 + z^2}), \quad (4)$$

which gives the connection between

partial wave amplitudes and the form factor $F(r)$ of the nucleon as a target in BeV collisions.

Now, by assuming possible types of the form factor, we calculate the partial wave cross sections of 1.4 BeV π^-p scattering³⁾ and compare them with the previously obtained phenomenological results.¹⁾

(A) *Gaussian form factor*, $F(r) = F(0)e^{-(r/a)^2}$

In this case, σ_{inel} and the most probable collision parameters are given by

$$\sigma_{inel} = a\pi^{3/2}\lambda^2 F(0) \sum_l (2l+1) \times \exp\{-(l\lambda^2/a)^2\} \quad (\text{A} \cdot 1)$$

and

$$\langle d \rangle = \int_0^\infty y^2 dy \exp\{-(y/a)^2\} / \int_0^\infty y dy \times \exp\{-(y/a)^2\} \frac{\sqrt{\pi}}{2} a. \quad (\text{A} \cdot 2)$$

Therefore, the parameters $F(0)$ and a are calculated from the experimental values of σ_{inel} and $\langle d \rangle \approx 0.8 \times 10^{-13}$ cm, the results are $a/\lambda = 3.5$ and $a\pi^{3/2}\lambda^2 F(0) = 1.6$ mb. Then, the partial wave cross sections are calculated by

$$\sigma_l^{inel} = (1.6 \text{ mb}) \times (2l+1) \exp\{-(l/3.5)^2\}. \quad (\text{A} \cdot 3)$$

The results of calculations are illustrated in Fig. 1. They reproduce the qualitative features of the phenomenological results; however, the fit seems to be incomplete. So we have to examine the next possibility.

(B) *Exponential form factor*, $F(r) = F(0)e^{-r/a}$

In this case, similar treatments lead to

$$\sigma_{inel} = 2\pi\lambda^2 a F(0) \sum_l (2l+1) \times \left(\frac{l\lambda}{a}\right) K_1\left(\frac{l\lambda}{a}\right) \quad (\text{B} \cdot 1)$$

where $K_1(x) \equiv \int_0^\infty dz \exp(-x\sqrt{1+z^2})$ is the well-known modified Bessel function. The two parameters, $F(0)$ and a , are determined as before, and we have $a/\lambda = 1.5$ and $2\pi\lambda^2 a F(0) = 2.4$ mb. Then the partial wave cross sections are given by

$$\sigma_l^{inel} = (2.4 \text{ mb}) (2l+1) \left(\frac{l}{1.5}\right) K_1\left(\frac{l}{1.5}\right) \quad (\text{B} \cdot 2)$$

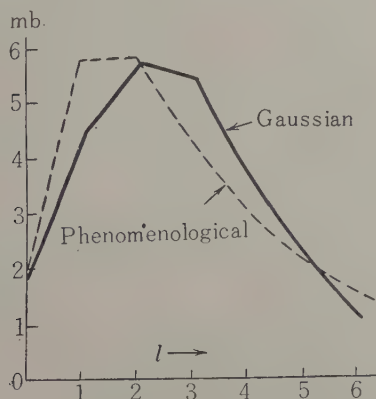


Fig. 1

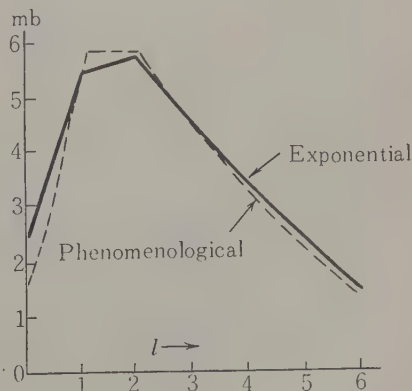


Fig. 2

and the results of calculations are given in Fig. 2. The agreements are much improved.

In view of many uncertainties involved, these results may not be taken too serious in their details. However, the above results seem to indicate the existence of rigid (energy independent) form factor.

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A Subjective Generalization of Statistical Mechanics

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In this note, we generalize an approach to statistical mechanics recently proposed by E. T. Jaynes on the basis of information theory,¹⁾ so as to include non-equilibrium phenomena. The basic postulate of this approach is to identify the phase space density describing the state of the system with the one obtained by the least-biased guess compatible with whatever information we have concerning the state of the system.

According to the information theory, this can be done by maximizing the quantity,

$$S^* = - \int \rho \ln \rho dx, \quad (1)$$

under the condition that ρ satisfies certain constraints besides the normalization condition which represent all the information we have, where ρ is the phase space density and

$$\int \dots dx$$

is the integral over the whole phase space. For instance, given the average total energy of the system, we get a canonical distribution. For ergodic systems, this approach reproduces all of the equilibrium and local equilibrium statistical mechanics, provided our information about the state of the system refers to a single instant of time.

This viewpoint immediately suggests a possible generalization. Namely, we can take information which refer to different instants of times, or to one or more time intervals. To illustrate the method, consider the case in which we know the average values of a macroscopic variable $Q(t)$ at the times $t=t_1, t_2, \dots, t_r$ denoted by $\bar{Q}_1, \bar{Q}_2, \dots, \bar{Q}_r$ respectively. Then, the condition under which maximizing S^* is written as (besides the normalization condition)

$$\int \rho Q(t_k) dx = \bar{Q}_k, \quad k=1, 2, \dots, r. \quad (2)$$

Applying the Lagrange multiplier method, the least-biased guess for the phase space density is obtained in the following form,

$$\rho = \exp[\mu + \sum_{k=1}^r \lambda_k Q(t_k)], \quad (3)$$

where μ and λ 's are obtained as functions of $\bar{Q}_1, \bar{Q}_2, \dots, \bar{Q}_r$ and t_1, t_2, \dots, t_r by inserting Eq. (3) into Eq. (2) and the

normalization condition for ρ , and solving the resulting equations.

The phase space density thus obtained does not depend on time except on the fixed ones, t_1, t_2, \dots, t_r , at which the average values of Q are known. Therefore, it neither satisfies Liouville's equation, nor describes the time development of the system. Rather, ρ should be regarded as representing a kind of history of the system at the times t_1, t_2, \dots, t_r . In order to find a relationship between our phase space density ρ and the usual one $\rho(t)$ satisfying Liouville's equation, consider the case in which we are interested in the average value of an arbitrary variable R at a time $t > 0$, given the average values of Q at the earlier instants of times t_1, t_2, \dots, t_r which are all taken to be negative. Here every dynamical variable is referred to the time zero unless specified explicitly. Supposing there exists some phase space density $\rho(t)$ satisfying Liouville's equation which describes the time development of the system at least for $t > 0$, the average value of R , $\langle R(t) \rangle$, can be expressed in either way: in terms of ρ and the time-dependent variable $R(t)$, or in terms of $R=R(0)$ and the time-dependent phase space density $\rho(t)$. Thus we get

$$\langle R(t) \rangle = \int \rho R(t) dx = \int \rho(t) R(0) dx. \quad (4)$$

Putting $t=0$ in the above expression, we have

$$\langle R(0) \rangle = \int \rho R(0) dx = \int \rho(0) R(0) dx. \quad (5)$$

Since the variable $R(0)$ is arbitrary in

the above expression, we obtain the desired relationship,

$$\rho = \rho(0). \quad (6)$$

Thus, ρ gives the initial condition at the time zero [which $\rho(t)$ must satisfy in order to be compatible with the results of previous observations. Our description in terms of ρ and $R(t)$ may be compared with the Heisenberg representation in quantum mechanics, whereas the ordinary description in terms of $\rho(t)$ and R corresponds to the Schrödinger representation.

We tried this method to linear dissipative phenomena, and a formulation without the assumption of initial local equilibrium seems to be possible. Also an entropy concept can be formulated using this method, as will be seen in the subsequent article. A full account of this note will appear elsewhere.

The author wishes to express his sincere thanks to Professor Tomoyasu Tanaka for his continual guidance and help.

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An Entropy Concept in Statistical Mechanics

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In the Gibbsian statistical mechanics, it is customary to define entropy in

terms of a "coarse-grained" phase space density which is formed from a "fine-grained" one by somewhat artificial procedures.¹⁾ This definition was criticized by J. M. Blatt who considered the spin echo case, in which the entropy defined in the above manner can decrease by reversing the direction of external magnetic field, i. e. by reversing the Larmor precession of spins.

The purpose of this note is to propose an alternative and apparently more satisfactory definition of entropy. For simplicity, let us consider an isolated system which can be in an external field, and which is not necessarily in thermal equilibrium. Consider a macroscopic observer I who started observing the system at some instant of time t_1 . He wants to get as much information as possible about the state of the system at the time t_1 . For this purpose he can carry out observations any number of times, but only after the time t_1 . Thus, to him arises the problem of getting the most reasonable guess about the state of the system from all the data he got after the time t_1 . This can be done on the basis of the information theory,²⁾ by maximizing the quantity,

$$S^* = - \int \rho \ln \rho dx \quad (1)$$

under the condition that ρ be compatible with all the data he got. Here the same notation was used as in the preceding article. The values of S^* and ρ thus obtained will be denoted by $S(t_1)$ and $P(t_1)$ respectively. Next consider the second observer II, who started making observations at some later time

t_2 . Following the same procedure as the first observer did, he will get $S(t_2)$ and $P(t_2)$ for S^* and ρ . Here we shall assume that the observations are "homogeneous" in time, in the sense that after the time t_2 , both observers make exactly the same observations. Considering another observer of the similar nature who started at an arbitrary time t , we can define a function $S(t)$ and a phase space density $P(t)$. We propose to identify this function $S(t)$ with the entropy of the system at the time t . The non-decreasing nature of this entropy is obvious from the fact that $S(t)$ was obtained as the conditional maximum, its conditions decreasing as the time goes on.

Next, we shall show that our definition of entropy is correct for equilibrium and local equilibrium states, as it ought to be. For simplicity, suppose our observer is capable of measuring the average values of a macroscopic variable Q only, and moreover, equilibrium or local equilibrium states are specified by giving average values of Q . Consider the system in an equilibrium or local equilibrium state with $Q = \tilde{Q}(0)$ at the time zero. According to the equilibrium statistical mechanics, entropy of this state can be obtained by maximizing S^* under the single condition,

$$\int \rho Q(0) dx = \tilde{Q}(0), \quad (2)$$

$\tilde{Q}(t)$ being the average value of Q at the time t . As the time goes on, $\tilde{Q}(t)$ is expressed as follows by using the same phase space density ρ just obtained,

$$\int \rho Q(t) dx = \tilde{Q}(t), \quad t > 0, \quad (3)$$

where $Q(t)$ satisfies the equations of motion of the isolated system. On the other hand, according to our definition of entropy, Eq. (2) and Eq. (3) just constitute the conditions under which to maximize S^* in order to obtain entropy at the time zero, the condition Eq. (3) being redundant in this case. This shows the equivalence of our entropy and the one defined in the equilibrium statistical mechanics.

So far, no rigorous mathematical proof that our entropy actually increases with the time has been obtained. Nevertheless, if one thinks about the fact that in defining entropy, only a part of the information concerning the state of the system, namely, that concerning the future behavior, has been taken into account, it is natural to expect that our entropy also increases with the time for changing isolated systems. In particular, for those systems with very short memories, our entropy at the time t can be obtained by observations during a very short time interval after the time t , which is almost an instant for a macroscopic observer, since the subsequent observations in future will not add any new information to affect the value of S^* . This shows that in this case our entropy reduces to the one

defined by making macroscopic observations at each instant of time (i. e. coarse-graining in the usual sense). Also, this is the only non-equilibrium case, although most usual, in which the ordinary coarse-graining procedure is valid. Furthermore, an entropy production rate can be expressed as a product of current and force as in the thermodynamics of irreversible processes.

Finally, returning to the spin echo example, Blatt's criticism does not apply to our definition of entropy provided we extend our method slightly by allowing our observer to reverse the direction of magnetic field at any time after he started making observations, because in future he can recover the lost information during the past due to the "apparently" random precession of spins.

Full details of this note will be published elsewhere. The author would like to express his sincere thanks to Professor Tomoyasu Tanaka for his continual guidance and helpful discussions.

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Energy Loss and Radiation of a Gyrating Charged Particle in a Magnetic Field

— Non-Ionized Medium —

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The Fourier series expansions are used to obtain the expressions for the components of the electromagnetic field at an arbitrary point of observation and for the total energy loss of a gyrating charged particle in a non-ionized medium having a uniform magnetic field. For a non-relativistic particle, it is shown that the total energy loss is split into the collision loss, whose formula is found to be the familiar one for linear motion, and the loss due to cyclotron radiations. The relative magnitude of the latter to the former is less than $(\omega_0/\omega_p)^2$, where ω_0 is the cyclotron frequency and $\omega_p^2 = 4\pi n_e e^2/m_e$ where n_e and m_e are the density and mass of electrons in the medium. In the relativistic case, we get the explicit formula of the polarization loss, depending upon the external magnetic field, and of the losses due to the Čerenkov and synchrotron radiations. The spectral and angular distributions of these two radiations are discussed.

§ 1. Introduction

The theoretical study of the energy loss and radiation of a charged particle passing through a medium, especially a plasma, having a magnetic field is important with respect to some astrophysical problems. As is well known, a plasma in a magnetic field has a tensor dielectric constant and behaves like an optically-active anisotropic medium.¹⁾ Therefore, the treatment of a magnetoplasma is much more complicated than that of a non-ionized medium. Studies on the energy dissipation of a charged particle moving parallel to the magnetic field were made by several authors.²⁾ But for the motion perpendicular to the magnetic field the previous research has been limited to the radiation loss³⁾ and the calculation of the collision loss has not yet been made. This can be said for the non-ionized medium as well as for the magnetoactive plasma. Although in this paper we shall confine ourselves to the former case, our present work may be a useful step towards the latter investigation.

A charged particle moving uniformly loses its energy by interacting with the medium through which it passes. This loss can be divided into two parts;⁴⁾ namely, one is due to close collisions, which are binary, and the other due to

* The main part of this work was carried out at Osaka City University.

distant collisions consists of the polarization loss and the loss due to Čerenkov radiation. For a spiralling particle, however, the cyclotron or synchrotron radiations become an additional cause of the energy dissipations.

The motion of a charged particle in the direction of the magnetic field is not affected by this field, so that we can use the familiar formula of the energy loss for linear motion. Hence it is sufficient to consider only the circular motion perpendicular to the magnetic field. We may assume the distortion of the orbit to be negligible, the velocity of the particle being large.

First, in § 2, the potentials of the electromagnetic field produced by a gyrating charged particle will be derived. The calculation is rather tedious and the previous ones are limited to the value in the wave zone. Here the value at an arbitrary point of observation is exactly evaluated. On account of the circular motion, the quantities concerning the field are resolved into harmonic components by the Fourier series expansion.⁵⁾ In § 3 the general formula of the total energy loss, including the radiation loss, per unit time is derived from the work of the reaction force of the medium on the particle.⁶⁾ This expression is not of a convenient form and it is impossible to get numerical values unless adequate approximations are used.

In the non-relativistic case, we can use some approximations with high accuracy and it is shown in § 4 that the total energy loss is the sum of the collision loss and the loss due to cyclotron radiations. As is expected from the fact that the Larmor radius is extremely larger than the adiabatic limit of the impact parameter, the value of the collision loss is the same one as for the linear motion.

For the relativistic case, applying the asymptotic formulas of the Bessel functions with large order and large argument, it is found that the polarization loss depends on the magnitude of the external magnetic field and the emission of the Čerenkov radiation may be possible in addition to the synchrotron radiation (§ 5). And the spectral and angular distributions of these two radiations will be discussed. In the Appendix, another treatment of the same problem by the method of the Poynting vector is presented.

§ 2. The electromagnetic field of a gyrating charged particle

If only the radiation emitted by the particle is considered, it is sufficient to know the electromagnetic field in the wave zone and it can be done easily. However, as we shall deal with collisions, we must find the value of the field near the orbit of the particle.

Let us consider a charged particle with mass m_0 , charge q and velocity \mathbf{v}_0 gyrating in a uniform magnetic field H_{ex} which is directed along the z axis. The angular velocity ω_0 (i.e. the Larmor frequency) of the particle is $(qH_{ex}/m_0c) \times \sqrt{1 - v_0^2/c^2}$ and the radius of the orbit, $r_0 = v_0/\omega_0$. The components of the radius vector \mathbf{r}_0 and of the velocity \mathbf{v}_0 of the particle are in the cylindrical coordinate (r, φ, z)

$$\mathbf{r}_0 = (r_0, \varphi_0 = \omega_0 t, 0), \quad \mathbf{v}_0 = (0, v_0, 0).$$

The field quantities, such as the scalar and vector potentials, ϕ and \mathbf{A} , are analysed into the Fourier series respectively

$$\phi = \sum_{n=-\infty}^{\infty} \phi_n e^{-in\omega_0 t}, \quad \mathbf{A} = \sum_{n=-\infty}^{\infty} \mathbf{A}_n e^{-in\omega_0 t}. \quad (2.1)$$

The charge and current densities of the particle are

$$\left. \begin{aligned} \rho &= q\delta(\mathbf{r} - \mathbf{r}_0) = \sum_{n=-\infty}^{\infty} \rho_n e^{-in\omega_0 t}, \\ \mathbf{j} &= \rho \mathbf{v}_0 = \sum_{n=-\infty}^{\infty} \mathbf{j}_n e^{-in\omega_0 t}, \end{aligned} \right\} \quad (2.2)$$

where the δ function may be expressed by the cylindrical coordinate as

$$\delta(\mathbf{r} - \mathbf{r}_0) = \delta(r - r_0) \delta(\varphi - \omega_0 t) \delta(z) / r. \quad (2.3)$$

From the Maxwell equations ϕ_n and \mathbf{A}_n obey the next equations:

$$\Delta \phi_n + \frac{n^2 \omega_0^2 \epsilon_n}{c^2} \phi_n = -\frac{4\pi}{\epsilon_n} \rho_n, \quad (2.4)$$

$$\Delta \mathbf{A}_n + \frac{n^2 \omega_0^2 \epsilon_n}{c^2} \mathbf{A}_n = -\frac{4\pi}{c} \mathbf{j}_n, \quad (2.5)$$

where ϵ_n is the dielectric constant of the medium and a function of frequency $n\omega_0$. ρ_n and \mathbf{j}_n are easily obtained as

$$\rho_n = \frac{q\omega_0}{2\pi} \int_{-\pi/\omega_0}^{\pi/\omega_0} \delta(\mathbf{r} - \mathbf{r}_0) e^{in\omega_0 t} dt = \frac{q}{2\pi} \frac{\delta(r - r_0)}{r} \delta(z) e^{in\varphi}, \quad (2.6)$$

$$\mathbf{j}_n = \mathbf{v}_0 \rho_n. \quad (2.7)$$

The solutions of (2.4) and (2.5) are expressed as follows,

$$\phi_n(\mathbf{r}) = \int \frac{\exp[in\omega_0 \sqrt{\epsilon_n d/c}] \rho_n(\mathbf{r}')}{\epsilon_n d} d\mathbf{r}', \quad (2.8)$$

$$\mathbf{A}_n(\mathbf{r}) = \int \frac{\exp[in\omega_0 \sqrt{\epsilon_n d/c}] \mathbf{j}_n(\mathbf{r}')}{cd} d\mathbf{r}', \quad (2.9)$$

where $d = |\mathbf{r} - \mathbf{r}'|$.

Integrations of (2.8) and (2.9) with respect to \mathbf{r}' and \mathbf{z}' give us

$$\phi_n = \frac{qe^{in\varphi}}{2\pi\epsilon_n} \int_{-\pi}^{\pi} \frac{\exp i[k_n \sqrt{R^2 + r_0^2 - 2r_0 R \sin \theta \cos \chi} + n\chi]}{\sqrt{R^2 + r_0^2 - 2r_0 R \sin \theta \cos \chi}} d\chi, \quad (2.10)$$

$$A_{nr} = -\frac{qv_0 e^{in\varphi}}{2\pi c} \int_{-\pi}^{\pi} \frac{\exp i[k_n \sqrt{R^2 + r_0^2 - 2r_0 R \sin \theta \cos \chi} + n\chi]}{\sqrt{R^2 + r_0^2 - 2r_0 R \sin \theta \cos \chi}} \sin \chi d\chi, \quad (2.11)$$

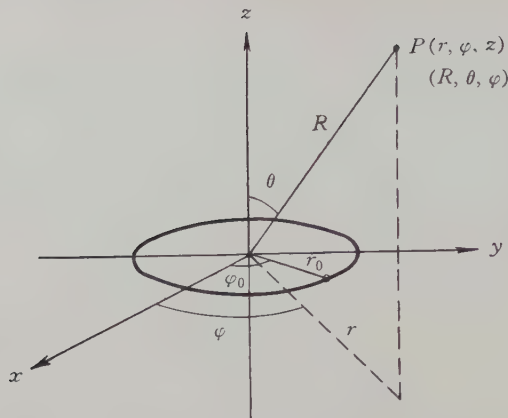


Fig. 1. Geometry of the source and field points

$$A_{nz} = \frac{qv_0 e^{in\varphi}}{2\pi c} \int_{-\pi}^{\pi} \frac{\exp i[k_n \sqrt{R^2 + r_0^2 - 2r_0 R \sin \theta \cos \chi} + n\chi]}{\sqrt{R^2 + r_0^2 - 2r_0 R \sin \theta \cos \chi}} \cos \chi d\chi \quad (2.12)$$

$$A_{nz} = 0, \quad (2.13)$$

where $\chi = \varphi - \varphi'$, $R^2 = r^2 + z^2$, $r = R \sin \theta$ and $k_n = n\omega_0 \sqrt{\epsilon_n}/c$ is the wave number.

If we consider only the wave zone, namely $R \gg r_0$, the above integrands are easily approximated (see the Appendix).^{5),7)} But as we need to consider the case of $R \sim r_0$, it is necessary to use the expansion:

$$\begin{aligned} & \frac{\exp[ik_n \sqrt{R^2 + r_0^2 - 2r_0 R \sin \theta \cos \chi}]}{\sqrt{R^2 + r_0^2 - 2r_0 R \sin \theta \cos \chi}} \\ &= ik_n \sum_{m=0}^{\infty} (2m+1) P_m(\sin \theta \cos \chi) \begin{cases} j_m(k_n r_0) h_m^{(1)}(k_n R) & R > r_0 \\ j_m(k_n R) h_m^{(1)}(k_n r_0) & R < r_0, \end{cases} \end{aligned} \quad (2.14)$$

where j_m and $h_m^{(1)}$ are respectively the spherical Bessel function and the spherical Hankel function of the first kind and of order m .

Substituting (2.14) in (2.10), (2.11) and (2.12), and integrating over the angle χ , we obtain

$$\phi_n = \frac{4\pi i k_n q e^{in\varphi}}{\epsilon_n} \sum_{m=|n|}^{\infty} Y_{m,n}(\theta, 0) Y_{m,n}\left(\frac{\pi}{2}, 0\right) \begin{cases} j_m(k_n r_0) h_m^{(1)}(k_n R) & R > r_0 \\ j_m(k_n R) h_m^{(1)}(k_n r_0) & R < r_0, \end{cases} \quad (2.15)$$

$$\begin{aligned} A_{nz} &= \pm \frac{2\pi q v_0 k_n e^{in\varphi}}{c} \\ &\times \begin{cases} 0, & \text{for } n=0 \\ \left[\sum_{m=|n|-1}^{\infty} Y_{m,|n|-1}(\theta, 0) Y_{m,|n|-1}\left(\frac{\pi}{2}, 0\right) \begin{cases} j_m(k_n r_0) h_m^{(1)}(k_n R) & R > r_0 \\ j_m(k_n R) h_m^{(1)}(k_n r_0) & R < r_0 \end{cases} \right. \\ \left. - \sum_{m=|n|+1}^{\infty} Y_{m,|n|+1}(\theta, 0) Y_{m,|n|+1}\left(\frac{\pi}{2}, 0\right) \begin{cases} j_m(k_n r_0) h_m^{(1)}(k_n R) & R > r_0 \\ j_m(k_n R) h_m^{(1)}(k_n r_0) & R < r_0 \end{cases} \right], & \text{for } |n| \geq 1 \end{cases} \end{aligned} \quad (2.16)$$

$$A_{n\varphi} = \frac{2\pi i q v_0 k_n e^{in\varphi}}{c} \times \begin{cases} 2 \sum_{m=1}^{\infty} Y_{m1}(\theta, 0) Y_{m1}\left(\frac{\pi}{2}, 0\right) j_m(0) h_m^{(1)}(0), & \text{for } n=0 \\ \left[\sum_{m=|n|-1}^{\infty} Y_{m, |n|-1}(\theta, 0) Y_{m, |n|-1}\left(\frac{\pi}{2}, 0\right) \begin{cases} j_m(k_n r_0) h_m^{(1)}(k_n R) & R > r_0 \\ j_m(k_n R) h_m^{(1)}(k_n r_0) & R < r_0 \end{cases} \right. \\ \left. + \sum_{m=|n|+1}^{\infty} Y_{m, |n|+1}(\theta, 0) Y_{m, |n|+1}\left(\frac{\pi}{2}, 0\right) \begin{cases} j_m(k_n r_0) h_m^{(1)}(k_n R) & R > r_0 \\ j_m(k_n R) h_m^{(1)}(k_n r_0) & R < r_0 \end{cases} \right], \\ \text{for } |n| \geq 1. \end{cases} \quad (2.17)$$

From $\phi_n(\mathbf{r})$ and $A_n(\mathbf{r})$ just obtained above, the electromagnetic field $\mathbf{E}(\mathbf{r})$ and $\mathbf{H}(\mathbf{r})$, at an arbitrary point of observation \mathbf{r} , may be easily calculated.

§ 3. General formula of energy loss

There are some ways to estimate the energy loss arising from distant collisions. Among these methods, the use of the Poynting vector is not convenient but intricate especially in our case, because we must evaluate the energy flux out of the toroidal surface having the circular orbit of the particle as axis. The simplest way to calculate the energy loss is, as usual, to get the work done by the reaction force of the medium on the particle.⁽⁶⁾ This method gives us the energy loss per unit time as follows,

$$-\frac{dW}{dt} = -q(\mathbf{v}_0 \mathbf{E})_{r=v_0 t} = -q v_0 E_{\varphi}(r_0, \omega_0 t, 0). \quad (3.1)$$

Really this formula represents the total energy loss per unit time, including the contribution of close collisions. However, it must be noticed that whether Eq. (3.1) includes the close collision loss or not depends on the choice of the upper limit of order m , which corresponds to the minimum impact parameter (see the next section).

It is obvious from (3.1) that we have only to know E_{φ} whose harmonic component is given by

$$E_{n\varphi} = -\frac{1}{R \sin \theta} \frac{\partial \phi_n}{\partial \varphi} + \frac{i k_n}{\sqrt{\epsilon_n}} A_{n\varphi}. \quad (3.2)$$

Substituting (2.15) and (2.17) into (3.2), we have

$$E_{n\varphi}(R, \theta, \varphi) = \frac{4\pi n k_n q e^{in\varphi}}{\epsilon_n R \sin \theta} \sum_{m=|n|}^{\infty} Y_{mn}(\theta, 0) Y_{mn}\left(\frac{\pi}{2}, 0\right) \begin{cases} j_m(k_n r_0) h_m^{(1)}(k_n R) & R > r_0 \\ j_m(k_n R) h_m^{(1)}(k_n r_0) & R < r_0 \end{cases} \\ - \frac{2\pi q v_0 k_n^2 e^{in\varphi}}{c \sqrt{\epsilon_n}} \left[\sum_{m=|n|-1}^{\infty} Y_{m, |n|-1}(\theta, 0) Y_{m, |n|-1}\left(\frac{\pi}{2}, 0\right) \begin{cases} j_m(k_n r_0) h_m^{(1)}(k_n R) & R > r_0 \\ j_m(k_n R) h_m^{(1)}(k_n r_0) & R < r_0 \end{cases} \right]$$

$$+ \sum_{m=|n|+1}^{\infty} Y_{m, |n|+1}(\theta, 0) Y_{m, |n|-1}\left(\frac{\pi}{2}, 0\right) \begin{cases} j_m(k_n r_0) h_m^{(1)}(k_n R) & R > r_0 \\ j_m(k_n R) h_m^{(1)}(k_n r_0) & R < r_0 \end{cases} \quad (3.3)$$

Evidently E_{φ} is written as

$$E_{\varphi} = \sum_{n=1}^{\infty} (E_{n\varphi} e^{-in\omega_0 t} + E_{-n\varphi} e^{in\omega_0 t}), \quad (3.4)$$

where $E_{-n\varphi}$ is the complex conjugate of $E_{n\varphi}$ and easily obtained from (3.3).

Using the above relations, finally we get the energy loss of the particle moving in a circle as follows,

$$\begin{aligned} -\frac{dW}{dt} = & -\frac{8\pi q^2 c \beta_0^2}{r_0^2} \sum_{n=1}^{\infty} n^2 \left[\operatorname{Re} \left\{ \frac{1}{\sqrt{\epsilon_n}} \sum_{m=n}^{\infty} Y_{mn}^2 \left(\frac{\pi}{2}, 0 \right) j_m(n\beta_0 \sqrt{\epsilon_n}) h_m^{(1)}(n\beta_0 \sqrt{\epsilon_n}) \right\} \right. \\ & - \frac{1}{2} \beta_0^2 \operatorname{Re} \left\{ \sqrt{\epsilon_n} \sum_{m=n-1}^{\infty} Y_{m, n-1}^2 \left(\frac{\pi}{2}, 0 \right) j_m(n\beta_0 \sqrt{\epsilon_n}) h_m^{(1)}(n\beta_0 \sqrt{\epsilon_n}) \right. \\ & \left. \left. + \sqrt{\epsilon_n} \sum_{m=n+1}^{\infty} Y_{m, n+1}^2 \left(\frac{\pi}{2}, 0 \right) j_m(n\beta_0 \sqrt{\epsilon_n}) h_m^{(1)}(n\beta_0 \sqrt{\epsilon_n}) \right\} \right], \quad (3.5) \end{aligned}$$

where $\beta_0 = v_0/c$ and a relation $k_n r_0 = n\beta_0 \sqrt{\epsilon_n}$ is used.

The spherical harmonics $Y_{mn}(\pi/2, 0)$ is zero at $m-n=\text{odd}$, and when $m-n=2l$ is even its square is

$$Y_{mn}^2 \left(\frac{\pi}{2}, 0 \right) = \frac{2n+4l+1}{4\pi} \frac{(2l)!(2n+2l)!}{2^{2n+4l} l!^2 (n+l)!^2}, \quad (l: \text{integer}).$$

Further, there is the following relation,

$$h_m^{(1)}(z) = j_m(z) + in_m(z),$$

in which $n_m(z)$ is the spherical Neumann function of order m .

In Eq. (3.5), the contributions from the terms with j_m^2 and $n_m j_m$ correspond to the radiation and collision losses respectively, where the collision loss means the polarization loss plus the binary collision loss. Thus we shall divide the total loss as follows,

$$-\frac{dW}{dt} = \left(-\frac{dW}{dt} \right)_{\text{coll}} + \left(-\frac{dW}{dt} \right)_{\text{rad}}. \quad (3.6)$$

From Eq. (3.5) one cannot say anything about further details about the energy loss. Thus, in the following sections we shall derive the explicit formulas, applying the asymptotic formulas for the Bessel functions.

§ 4. Non-relativistic case

For the non-relativistic velocity of the particle ($\beta_0 \ll 1$), the next asymptotic formula for the Bessel functions may be used.^{(8), (9)} That is, for $n\beta_0 |\sqrt{\epsilon_n}| \ll 2m-1$ and $\beta_0 |\sqrt{\epsilon_n}| \ll 2e^{-1}$, we have

$$j_m(n\beta_0\sqrt{\epsilon_n}) \cong \frac{(n\beta_0\sqrt{\epsilon_n})^m}{(2m+1)!!}, \quad n_m(n\beta_0\sqrt{\epsilon_n}) \cong -\frac{(2m-1)!!}{(n\beta_0\sqrt{\epsilon_n})^{m+1}}. \quad (4.1)$$

1) Collision loss

Using Eq. (4.1), the series of terms containing $j_m n_m$ in Eq. (3.5) are reduced to

$$\left. \begin{aligned} \sum_{m=n}^{\infty} Y_{mn}^2 \left(\frac{\pi}{2}, 0 \right) j_m(n\beta_0\sqrt{\epsilon_n}) n_m(n\beta_0\sqrt{\epsilon_n}) &= -\frac{1}{4\pi n\beta_0\sqrt{\epsilon_n}} \sum_{l=0}^{\infty} \frac{(2l)!(2n+2l)!}{2^{2n+4l} l!^2 (n+l)!^2}, \\ \sum_{m=n-1}^{\infty} Y_{m,n-1}^2 \left(\frac{\pi}{2}, 0 \right) j_m(n\beta_0\sqrt{\epsilon_n}) n_m(n\beta_0\sqrt{\epsilon_n}) &= -\frac{1}{4\pi n\beta_0\sqrt{\epsilon_n}} \sum_{l=0}^{\infty} \frac{(2l)!(2n+2l-2)!}{2^{2n+4l-2} l!^2 (n+l-1)!^2}, \\ \sum_{m=n+1}^{\infty} Y_{m,n+1}^2 \left(\frac{\pi}{2}, 0 \right) j_m(n\beta_0\sqrt{\epsilon_n}) n_m(n\beta_0\sqrt{\epsilon_n}) &= -\frac{1}{4\pi n\beta_0\sqrt{\epsilon_n}} \sum_{l=0}^{\infty} \frac{(2l)!(2n+2l+2)!}{2^{2n+4l+2} l!^2 (n+l+1)!^2}. \end{aligned} \right\} \quad (4.2)$$

The Stirling formula may be applied to the right-hand side of (4.2) and then we have, for example,

$$\sum_{l=0}^{\infty} \frac{(2l)!(2n+2l)!}{2^{2n+4l} l!^2 (n+l)!^2} \sim \frac{1}{\pi} \sum_{l=0}^{\infty} \frac{1}{\sqrt{l(n+l)}}. \quad (4.3)$$

The summation may be further replaced by an integral. Although these infinite series or integrals diverge, actually there exists an upper limit of order m or of the integral variable. This fact is based on the following reason.

$Y_{mn}(\theta, \varphi)$ is the eigenfunction of the angular momentum operator, and m and n are the azimuthal and magnetic quantum numbers, respectively, of a spherical wave. The virtual photon accompanying by the particle whose angular dependence will be given by $Y_{mn}(\theta, \varphi)$ has an angular momentum $n\hbar$ along the z axis,¹⁰⁾ and has a frequency $n\omega_0$ as a result of the circular motion of the particle with an angular velocity ω_0 . On the other hand, a maximum frequency of the virtual photon exists and is given by v_0/p_{min} , where p_{min} is the lower limit of the impact parameter.* Accordingly, the maximum value of m , which is designated by M , is given by

$$M\omega_0 = v_0/p_{min} \quad (4.4)$$

with

$$p_{min} = \begin{cases} qe/m_e v_0^2 & \text{if } qe/\hbar v_0 > 1 \\ \hbar \sqrt{1-\beta_0^2}/m_e v_0 & \text{if } qe/\hbar v_0 < 1, \end{cases}$$

where $-e$ and m_e are the charge and mass of the electron.

Finally, we obtain

* If we take as p_{min} atomic radius in (4.4), (3.5) represents only the loss due to distant collisions.

$$\text{each expression of (4.2)} = -\frac{1}{4\pi^2} \frac{1}{n\beta_0\sqrt{\epsilon_n}} \ln\left(\frac{M}{n}\right).$$

Then the formula of the collision loss may be shown to be reduced to

$$-\frac{dW}{dt}\Big|_{coll} = -\frac{2q^2v_0}{\pi r_0^2} \sum_{n=1}^{\infty} n \ln\left(\frac{M}{n}\right) \text{Im}\left(\frac{1}{\epsilon_n} - \beta_0^2\right) = \frac{2q^2v_0}{\pi r_0^2} \sum_{n=1}^{\infty} \frac{\epsilon_{2n}}{|\epsilon_n|^2} n \ln\left(\frac{M}{n}\right), \quad (4.5)$$

where ϵ_{2n} is the imaginary part of ϵ_n and $\epsilon_n = \epsilon_{1n} + i\epsilon_{2n}$. The summation over n in Eq. (4.5) has poles at the resonant frequencies ω_j of $n\epsilon_{2n}/|\epsilon_n|^2$ and the contributions of terms with small n 's are completely negligible. In order to compare with the case of the linear motion, we shall replace the sum by the integral over frequency ω as follows,

$$-\frac{dW}{dt}\Big|_{coll} = \frac{2q^2}{\pi v_0} \int_0^{\infty} \frac{\epsilon_{2\omega}}{|\epsilon_{\omega}|^2} \ln\left(\frac{M\omega_0}{\omega}\right) \omega d\omega = \frac{4\pi n_e q^2 e^2}{m_e v_0} \sum_j f_j \ln\left(\frac{M\omega_0}{\omega_j}\right), \quad (4.6)$$

where ω_j and f_j are respectively the frequency and the strength of the j -th oscillator of the atomic electrons, and where n_e is the electron density of the medium.

When we substitute the value of M defined by (4.4) in the energy loss formula (4.6), it agrees with the well-known one for the linear motion. After all, in the non-relativistic case it can be said that the collision loss for the circular motion may be approximated by the one for the linear motion with sufficient accuracy. This should be expected from the fact that the Larmor radius r_0 is extremely larger than the adiabatic limit of the impact parameter $p_{max} = v_0/\omega_j$, because $\omega_0 \ll \omega_j$, however strong the magnetic field may be.

II) Radiation loss

Using the asymptotic form (4.1), the summation over m of the terms containing j_m^2 is reduced, for instance, to

$$\sum_{m=n}^{\infty} Y_{mn}^2\left(\frac{\pi}{2}, 0\right) j_m^2(n\beta_0\sqrt{\epsilon_n}) = \frac{1}{4\pi} \sum_{l=0}^{\infty} \frac{(2n+4l+1)(2l)!(2n+2l)!}{2^{2n+4l} l!^2 (n+l)!^2} \frac{(n\beta_0\sqrt{\epsilon_n})^{2n+4l}}{(2n+4l+1)!!^2},$$

where the right series decreases very rapidly with l , and by retaining only the first term we have

$$\sum_{m=n}^{\infty} Y_{mn}^2\left(\frac{\pi}{2}, 0\right) j_m^2(n\beta_0\sqrt{\epsilon_n}) = \frac{1}{4\pi} \frac{(n\beta_0\sqrt{\epsilon_n})^{2n}}{(2n+1)!}.$$

Similarly, we get

$$\sum_{m=n-1}^{\infty} Y_{m,n-1}^2\left(\frac{\pi}{2}, 0\right) j_m^2(n\beta_0\sqrt{\epsilon_n}) = \frac{1}{4\pi} \frac{(n\beta_0\sqrt{\epsilon_n})^{2n-2}}{(2n-1)!}$$

and

$$\sum_{m=n+1}^{\infty} Y_{m,n+1}^2\left(\frac{\pi}{2}, 0\right) j_m^2(n\beta_0\sqrt{\epsilon_n}) = \frac{1}{4\pi} \frac{(n\beta_0\sqrt{\epsilon_n})^{2n+2}}{(2n+3)!},$$

in which the last one will be omitted, because it is negligibly small compared with the other two.

Thus the contribution from the terms of j_m^2 in (3.5) give us

$$\begin{aligned} -\frac{dW}{dt}\bigg|_{rad} &= \frac{2q^2 c \beta_0^2}{r_0^2} \sum_{n=1}^{\infty} \frac{n(n+1)}{(2n+1)!} \operatorname{Re} \left\{ \frac{(n \beta_0 \sqrt{\epsilon_n})^{2n}}{\sqrt{\epsilon_n}} \right\} \\ &= \frac{2q^2 v_0}{r_0^2} \sum_{n=1}^{\infty} \frac{n+1}{(2n+1)!} (n \beta_0)^{2n+1} \epsilon_{1n}^{n-1/2}, \end{aligned} \quad (4.7)$$

where the effect of ϵ_{2n} has been neglected. If we put $\epsilon_{1n}=1$ in (4.7), we obtain the well-known formula for the cyclotron radiation in vacuum.

III) Partition of energy loss

It is readily shown from (4.6) and (4.7) that the radiation loss is less than a fraction $(\omega_0/\omega_p)^2$ of the collision loss, where $\omega_p^2 = 4\pi n_e e^2/m_e$. In a dense medium, $(\omega_0/\omega_p)^2$ is ordinarily much less than unity and the contribution from radiations to the stopping power may be negligible. As $(\omega_0/\omega_p)^2$ is proportional to H_{ex}^2/n_e , however, the radiation loss in a rarefied gas having a strong magnetic field is not always negligible compared with the collision loss.

The ratio of the collision loss to the radiation loss, designated by \mathcal{R} , is as follows:

$$\mathcal{R} = \frac{6\pi n_e m_0^2 c^2}{m_e Z^2 H_{ex}^2} \frac{1-\beta_0^2}{\beta_0^3} \ln \left\{ \frac{m_e c^2 \beta_0^2}{\hbar \bar{\omega}_j \sqrt{1-\beta_0^2}} \right\}, \quad (4.8)$$

where $q=Ze$ and $\bar{\omega}_j = \sum f_j \ln \omega_j$. For the radiation loss, there has been used the next familiar value in vacuum,

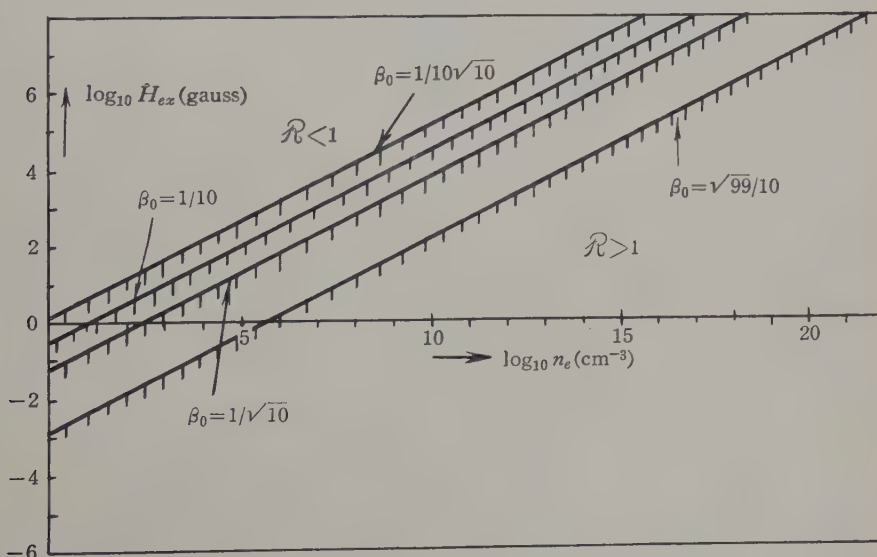


Fig. 2. The lines of $\mathcal{R}=1$ for an electron gyrating in a hydrogen gas, drawn for some values of β_0 . The regions under the lines correspond to $\mathcal{R}>1$.

$$-\left. \frac{dW}{dt} \right|_{rad} = \frac{2\omega_0^2 q^2 \beta_0^2}{3c(1-\beta_0^2)^2}. \quad (4.9)$$

Fig. 2 shows the region of $\mathcal{R} > 1$ in the $n_e H_{ec}$ plane for an electron gyrating in a hydrogen gas. Eq. (4.8) is not valid for the ultra-relativistic case, because of the H_{ec} dependence of the collision loss (see the next section).

§ 5. Relativistic case

In the relativistic case, we must use the following asymptotic formulas for the spherical Bessel functions with large order and large argument.^{(8), (9)}

Case 1) If $m + \frac{1}{2} > n_i \beta_0 |\sqrt{\epsilon_n}|$, we have

$$\left. \begin{aligned} j_m(n\beta_0\sqrt{\epsilon_n}) &\simeq \frac{\exp[(m+\frac{1}{2})(\tanh\alpha_m - \alpha_m)]}{\sqrt{2n\beta_0\sqrt{\epsilon_n}} \sqrt{(2m+1)\tanh\alpha_m}}, \\ n_m(n_i\beta_0\sqrt{\epsilon_n}) &\simeq -\frac{2\exp[(m+\frac{1}{2})(\alpha_m - \tanh\alpha_m)]}{\sqrt{2n\beta_0\sqrt{\epsilon_n}} \sqrt{(2m+1)\tanh\alpha_m}}, \end{aligned} \right\} \quad (5.1)$$

where

$$\tanh^2 \alpha_m = 1 - \left(\frac{n_i \beta_0 \sqrt{\epsilon_n}}{m + \frac{1}{2}} \right)^2.$$

Case 2) If $m + \frac{1}{2} < n_i \beta_0 |\sqrt{\epsilon_n}|$, we have

$$\left. \begin{aligned} j_m(n\beta_0\sqrt{\epsilon_n}) &\simeq \frac{1}{\sqrt{n_i\beta_0\sqrt{\epsilon_n}}} \frac{\cos[(m+\frac{1}{2})(\tan\gamma_m - \gamma_m) - \pi/4]}{\sqrt{(m+\frac{1}{2})\tan\gamma_m}}, \\ n_m(n_i\beta_0\sqrt{\epsilon_n}) &\simeq -\frac{1}{\sqrt{n_i\beta_0\sqrt{\epsilon_n}}} \frac{\sin[(m+\frac{1}{2})(\tan\gamma_m - \gamma_m) - \pi/4]}{\sqrt{(m+\frac{1}{2})\tan\gamma_m}}, \end{aligned} \right\} \quad (5.2)$$

where

$$\tan^2 \gamma_m = \left(\frac{n_i \beta_0 \sqrt{\epsilon_n}}{m + \frac{1}{2}} \right)^2 - 1.$$

I) Collision loss

Substituting (5.1) in the cross term $j_m n_m$ of (3.5), we obtain the polarization loss, including the close collision loss, in the following way.

$$-\left. \frac{dW}{dt} \right|_{coll} = \frac{2q^2 c \beta_0^2}{\pi r_0^2} \sum_{n=1}^{\infty} n \operatorname{Re} \frac{i}{\epsilon_n} \sum_{l=0}^{l_{max}} \frac{1}{\sqrt{l(l+n)} \sqrt{1 - \left(\frac{n_i \beta_0 \sqrt{\epsilon_n}}{n + 2l + \frac{1}{2}} \right)^2}}, \quad (5.3)$$

where we have used (5.1) even for small n because the contributions from the terms with small n 's are completely negligible. The summation over l is limited to a certain upper limit l_{max} given by (4.4) and its imaginary part is negligibly small. If $\beta_0 |\sqrt{\epsilon_n}| \ll 1$, we get (4.5) from (5.3) again.

Considering that $\beta_0|\sqrt{\epsilon_n}|$ is nearly equal to unity in the ultra-relativistic case, and replacing the sum by the integral, the sum over l is reduced to

$$\sum_l \dots \simeq \frac{1}{2} \int_{l_{mac}}^{l_{mac}} \frac{n+2l}{l(l+n)} dl = \frac{1}{2} \ln \left(\frac{M^2}{4n} - \frac{n}{4} \right) = \frac{1}{2} \ln \left(\frac{m_e^2 v_0^4}{4\hbar^2 \omega \omega_0 (1-\beta_0^2)} - \frac{\omega}{4\omega_0} \right),$$

where (4.4) has been used. Then the collision loss per unit time in the ultra-relativistic case is found to be

$$-\frac{dW}{dt} \Big|_{coll} = \frac{4\pi n_e e^2 q^2}{m_e v_0} \sum_j f_j \ln \left(\frac{m_e^2 v_0^4}{4\hbar^2 \omega_j \omega_0 (1-\beta_0^2)} - \frac{\omega_j}{4\omega_0} \right)^{1/2}. \quad (5.4)$$

The second term in the logarithmic factor is negligible compared with the first term. Here it should be noted that (5.4) depends on the external magnetic field, in contrast with the case of motion parallel to the magnetic field, and that it is not applicable to the limiting case $H_{ex} \rightarrow 0$.

II) Synchrotron radiation

When $m+1/2 > n\beta_0|\sqrt{\epsilon_n}|$, the contributions of the square terms $j_m^2(n\beta_0\sqrt{\epsilon_n})$ in (3.5) give us the energy loss by synchrotron radiations. First, we must apply the recurrence formula and then use the asymptotic one for (3.5).^{*} Here it should be noticed that if $m \gg n$, viz. $\alpha_m \gg 1$, $j_m^2(n\beta_0\sqrt{\epsilon_n})$ is nearly equal to zero; and thus we have only to consider the case of $\alpha_m < 1$. Therefore we have

$$\tanh \alpha_m - \alpha_m \simeq -\frac{1}{3} \alpha_m^3 \simeq -\frac{1}{3} \left\{ 1 - \left(\frac{n\beta_0\sqrt{\epsilon_n}}{m+\frac{1}{2}} \right)^2 \right\}^{3/2}.$$

Approximately we get the following

$$\begin{aligned} & \frac{1}{2} \beta_0^2 \sqrt{\epsilon_n} \left[\sum_{m=n-1} Y_{m,n-1}^2 \left(\frac{\pi}{2}, 0 \right) j_m^2(n\beta_0\sqrt{\epsilon_n}) + \sum_{m=n+1} Y_{m,n+1}^2 \left(\frac{\pi}{2}, 0 \right) j_m^2(n\beta_0\sqrt{\epsilon_n}) \right] \\ & - \frac{1}{\sqrt{\epsilon_n}} \sum_{m=n} Y_{mn}^2 \left(\frac{\pi}{2}, 0 \right) j_m^2(n\beta_0\sqrt{\epsilon_n}) \simeq \frac{1}{2} \beta_0 \sqrt{\epsilon_n} Y_{n-1,n-1}^2 \left(\frac{\pi}{2}, 0 \right) j_{n-1}^2(n\beta_0\sqrt{\epsilon_n}) \\ & \simeq \frac{\beta_0 n^{-3/2}}{16\pi\sqrt{\pi}} \frac{\exp[-(2n/3)(1-\beta_0^2\epsilon_n)^{3/2}]}{\sqrt{1-\beta_0^2\epsilon_n}}, \end{aligned} \quad (5.5)$$

where the remaining terms are found to be almost cancelled, using the recurrence formula and adequate approximations.

As we can neglect the imaginary part of ϵ_n in (5.5), we obtain the loss due to the synchrotron radiation as follows,

$$-\frac{dW}{dt} \Big|_{syn} = \frac{qc\beta_0^3}{2\sqrt{\pi} r_0^2} \sum_{n=1}^{\infty} \frac{\sqrt{n}}{\sqrt{1-\beta_0^2\epsilon_{1n}}} \exp \left[-\frac{2n}{3} (1-\beta_0^2\epsilon_{1n})^{3/2} \right]. \quad (5.6)$$

For small n , the spectrum represented by (5.6) is not a good approximation. If

^{*} If we directly apply (5.1) for the j_m^2 term in (3.5), we are misled to the wrong result (energy gain) on account of incorrectness of (5.1).

we neglect the dispersion and replace ϵ_{1n} by a certain average ϵ , we have

$$(5.6) = \frac{q^2 c \beta_0^3}{2\sqrt{\pi} r_0^2 \sqrt{1 - \beta_0^2 \epsilon}} \sum_{n=1}^{\infty} \sqrt{n} e^{-n/n_0},$$

where n_0 is defined by

$$\frac{2n_0}{3} (1 - \beta_0^2 \epsilon)^{3/2} = 1.$$

If we put $y = n/n_0$ and approximate the sum by the integral, we obtain

$$-\frac{dW}{dt} = \frac{3\sqrt{3} q^2 c \beta_0^3}{4\sqrt{2\pi} r_0^2 (1 - \beta_0^2 \epsilon)^{11/4}} \int_0^{\infty} y e^{-y} dy = \frac{3\sqrt{3} q^2 c \beta_0^3}{8\sqrt{2} r_0^2 (1 - \beta_0^2 \epsilon)^{11/4}}. \quad (5.7)$$

Putting ϵ equal to unity, the above expressions are a little different from the ordinary ones in vacuum given by the method of the Poynting vector (see the Appendix or reference 7)). One of the reasons is due to the poor approximations to the lower harmonics.

III) Čerenkov radiation

When the condition $\beta_0 |\sqrt{\epsilon_n}| > 1$ is satisfied, it is also possible for a gyrating charged particle to emit the Čerenkov radiation, provided the wavelength is much smaller than the radius of the orbit.

As it is sufficient to consider only the case of $\gamma_m < 1$, we can write as

$$\tan \gamma_m - \gamma_m \simeq \frac{1}{3} \gamma_m^3$$

and thus from (5.1)

$$j_m^2(n\beta_0\sqrt{\epsilon_n}) \simeq \frac{2}{2m+1} \frac{\cos^2 \left\{ \frac{2m+1}{6} \gamma_m^3 - \frac{\pi}{4} \right\}}{n\beta_0\sqrt{\epsilon_n} \tan \gamma_m}.$$

Therefore we have

$$\begin{aligned} \sum_{m=n} Y_{mn}^2 \left(\frac{\pi}{2}, 0 \right) j_m^2(n\beta_0\sqrt{\epsilon_n}) &\simeq \frac{1}{2\pi n\beta_0\sqrt{\epsilon_n}} f(n), \\ \sum_{m=n-1} Y_{m,n-1}^2 \left(\frac{\pi}{2}, 0 \right) j_m^2(n\beta_0\sqrt{\epsilon_n}) &\simeq \frac{1}{2\pi n\beta_0\sqrt{\epsilon_n}} f(n-1), \\ \sum_{m=n+1} Y_{m,n+1}^2 \left(\frac{\pi}{2}, 0 \right) j_m^2(n\beta_0\sqrt{\epsilon_n}) &\simeq \frac{1}{2\pi n\beta_0\sqrt{\epsilon_n}} f(n+1), \end{aligned}$$

where the function $f(n)$ is

$$f(n) = \frac{1}{\pi} \sum_l \frac{1}{\sqrt{l(l+n)}} \frac{\cos^2 \left(\frac{2n+4l+1}{6} \gamma_{n+2l}^3 - \frac{\pi}{4} \right)}{\sqrt{\left(\frac{n\beta_0\sqrt{\epsilon_n}}{n+2l+\frac{1}{2}} \right)^2 - 1}} \quad (5.8)$$

and

$$f(n) \simeq f(n-1) \simeq f(n+1) \quad (\text{for large } n),$$

where the upper limit of l is determined by the condition $n\beta_0|\sqrt{\epsilon_n}| > n+2l+1/2$.

Thus the energy loss due to the Čerenkov radiation is expressed as

$$-\frac{dW}{dt}\Big|_{\check{C}er} = \frac{4q^2v_0}{r_0^2} \sum_{\beta_0|\sqrt{\epsilon_n}|>1} \operatorname{Re} \left\{ n \left(\beta_0^2 - \frac{1}{\epsilon_n} \right) f(n) \right\}, \quad (5.9)$$

where the sum over n is limited to the case of $\beta_0|\sqrt{\epsilon_n}| > 1$ and the imaginary part of $f(n)$ is small enough to be omitted. If we put $\cos^2(\dots)$ equal to unity in $f(n)$, and considering $\beta_0^2|\sqrt{\epsilon_n}| \gtrsim 1$, $f(n)$ is reduced to

$$f(n) \simeq \frac{1}{\pi} \sum_l \left(1 + \frac{l+\frac{1}{2}}{n+l} \right) \frac{1}{\sqrt{2l(n\beta_0\sqrt{\epsilon_n} - n - 2l - \frac{1}{2})}} \simeq \frac{1}{4},$$

where the second term in brackets is neglected and the sum is replaced by the integral.

After all, we get the following expression,

$$-\frac{dW}{dt}\Big|_{\check{C}er} = \frac{q^2v_0}{r_0^2} \sum_{\beta_0|\sqrt{\epsilon_n}|>1} n \left(\beta_0^2 - \frac{\epsilon_{1n}}{|\epsilon_n|^2} \right) (1 - g(n)), \quad (5.10)$$

where $g(n)$ represents the effect of interference and is given by

$$g(n) = \frac{4}{\pi} \sum_l \frac{\sin^2 \left(\frac{2n+4l+1}{6} \gamma_{n+2l}^3 - \frac{\pi}{4} \right)}{\sqrt{l(l+n)} \sqrt{\left(\frac{n\beta_0\sqrt{\epsilon_n}}{n+2l+\frac{1}{2}} \right)^2 - 1}}.$$

Replacing the sum over n by the integral, we have

$$-\frac{dW}{dt}\Big|_{\check{C}er} = \frac{q^2}{v_0} \int \left(\beta_0^2 - \frac{\epsilon_{1\omega}}{|\epsilon_\omega|^2} \right) (1 - g(\omega/\omega_0)) \omega d\omega, \quad (5.11)$$

where $g(\omega/\omega_0)$ is much less than unity for high frequencies. On account of the poor approximation, however, the low frequency part of the spectrum given by (5.10) or (5.11) is not valid. As is mentioned before, the Čerenkov radiation with rather low frequency will be weak or impossible, because of the destructive effect of interference (see the Appendix).

IV) Angular distribution of radiation

As will be shown in the Appendix, most of the synchrotron radiation is concentrated within an angle $\theta \simeq \pm \sqrt{1 - \beta_0^2 \epsilon}$, where $\theta = \pi/2 - \theta$. On the other hand, the angular distribution of the Čerenkov radiation is almost like a δ function at $\theta = \cos^{-1}(1/\beta_0\sqrt{\epsilon})$. Fig. 3 and Fig. 4 schematically represent the angular distributions of these two radiations.

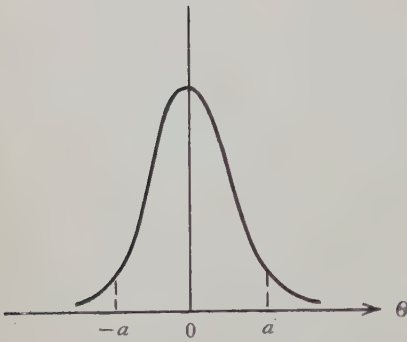


Fig. 3. The angular distribution of the synchrotron radiation. $a \simeq \sqrt{1 - \beta_0^2 \epsilon}$.

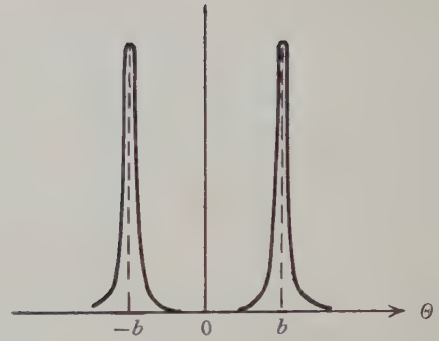


Fig. 4. The angular distribution of the Čerenkov radiation. $b \simeq \cos^{-1}(1/\beta_0 \sqrt{\epsilon})$.

In a dense medium, the two angular distributions may rather be of great difference; in a rarefied gas, however, they may practically be indistinguishable.

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Appendix

The next treatment follows the method of reference 7). The fields of a gy-
rating charged particle in the wave zone are given from (2.2) as

$$\left. \begin{aligned} -D_\varphi = H_\theta &= \frac{q\beta_0^2}{r_0 R} \sum_{n=-\infty}^{\infty} n \sqrt{\epsilon_n} e^{-in\gamma} J_n'(n\beta_0 \sqrt{\epsilon_n} \sin \theta), \\ D_\theta = H_\varphi &= -\frac{iq\beta_0}{r_0 R} \cot \theta \sum_{n=-\infty}^{\infty} n e^{-in\gamma} J_n(n\beta_0 \sqrt{\epsilon_n} \sin \theta), \end{aligned} \right\} \quad (\text{A} \cdot 1)$$

where

$$\gamma = \omega_0 t + \frac{\pi}{2} - \varphi - \frac{\omega_0 \sqrt{\epsilon_n}}{c} R.$$

If we neglect the absorption, namely ϵ_n is real, we get

$$\left. \begin{aligned} -D_\varphi = H_\theta &= \frac{2q\beta_0^2}{r_0 R} \sum_{n=1}^{\infty} n \sqrt{\epsilon_n} J_n'(n\beta_0 \sqrt{\epsilon_n} \sin \theta) \cos n\gamma, \\ D_\theta = H_\varphi &= -\frac{2q\beta_0}{r_0 R} \cot \theta \sum_{n=1}^{\infty} n J_n(n\beta_0 \sqrt{\epsilon_n} \sin \theta) \sin n\gamma. \end{aligned} \right\} \quad (\text{A} \cdot 1)'$$

Thus the energy flow per unit time in a solid angle $d\Omega$ is reduced to

$$\left. \begin{aligned} dW &= \sum_{n=1}^{\infty} dW_n \\ dW_n &= \frac{q^2 n^2 \beta_0^2 c}{2\pi r_0^2 \sqrt{\epsilon_n}} [\cot^2 \theta J_n^2(n\beta_0 \sqrt{\epsilon_n} \sin \theta) + \beta_0^2 \epsilon_n J_n'^2(n\beta_0 \sqrt{\epsilon_n} \sin \theta)] d\Omega, \end{aligned} \right\} \quad (\text{A} \cdot 2)$$

where the time average over a period of motion has been taken. Therefore the radiation energy of a frequency $n\omega_0$ per unit time is

$$W_n = \frac{q^2 \beta_0^2 c n}{r_0^2 \epsilon_n} \left[2\beta_0^2 \epsilon_n J_{2n}'(2n\beta_0 \sqrt{\epsilon_n}) + (\beta_0^2 \epsilon_n - 1) \int_0^{2n\beta_0 \sqrt{\epsilon_n}} J_{2n}(x) dx \right]. \quad (\text{A} \cdot 3)$$

1. Synchrotron radiation ($\beta_0^2 \epsilon_n < 1$)

Neglecting the dispersion, and replacing ϵ_n by a certain average ϵ , the following spectrum is obtained:

if $\omega < \omega_c$,

$$W_n dn = \frac{3\sqrt{3}}{2^{4/3}\pi} \frac{q^2 c \beta_0}{r_0^2 \epsilon} \frac{\Gamma(2/3)}{(1 - \beta_0^2 \epsilon)^2} y^{1/3} dy,$$

if $\omega > \omega_c$,

$$W_n dn = \frac{3\sqrt{3}}{4\sqrt{2}\pi} \frac{q^2 c \beta_0}{r_0^2 \epsilon} \frac{1}{(1 - \beta_0^2 \epsilon)^2} y^{1/2} e^{-y} dy,$$

(A·4)

where $\omega_c = n_c \omega_0$, $y = n/n_c$ and $n_c = 3/2 \cdot (1 - \beta_0^2)^{-3/2}$.

2. Čerenkov radiation ($\beta_0^2 \epsilon_n > 1$)

Using the recurrence formula, and considering the relation

$$\int_0^{\infty} J_\nu(x) dx = 1,$$

(A·3) may be rewritten as follows,

$$\begin{aligned} W_n &= \frac{q^2 c \beta_0^2 n}{r_0^2 \epsilon_n} (\beta_0^2 \epsilon_n - 1) \left[1 - \int_{2n\beta_0 \sqrt{\epsilon_n}}^{\infty} J_{2n}(x) dx + \frac{2\beta_0 \sqrt{\epsilon_n}}{\beta_0^2 \epsilon_n - 1} \{ J_{2n}(2n\beta_0 \sqrt{\epsilon_n}) \right. \\ &\quad \left. - \beta_0 \sqrt{\epsilon_n} J_{2n+1}(2n\beta_0 \sqrt{\epsilon_n}) \} \right]. \end{aligned}$$

For large n , being approximately

$$J_{2n}(2n\beta_0 \sqrt{\epsilon_n}) \simeq J_{2n+1}(2n\beta_0 \sqrt{\epsilon_n}),$$

we have

$$W_n = \frac{q^2 c \beta_0^2 n}{r_0^2 \epsilon_n} (\beta_0^2 \epsilon_n - 1) \left[1 - \int_{2n\beta_0 \sqrt{\epsilon_n}}^{\infty} J_{2n}(x) dx - \beta_0 \sqrt{\epsilon_n} J_{2n}(2n\beta_0 \sqrt{\epsilon_n}) \right], \quad (\text{A} \cdot 5)$$

where we have used an approximation

$$2(1 - \beta_0 \sqrt{\epsilon_n}) \simeq 1 - \beta_0^2 \epsilon_n.$$

The second and third terms in brackets represent the destructive effect of interference and may be neglected in the case of $n \gg 1$. Hence, when the wavelength is much smaller than the radius of the orbit, the same spectrum as for the linear motion is obtained.

3. Angular distribution

Neglecting the dispersion, and putting ϵ_n equal to ϵ , the radiation energy emitted in a solid angle $d\Omega$ per unit time is given by

$$dW = \frac{\beta_0^2 \epsilon}{4(1 - \beta_0^2 \epsilon \sin^2 \theta)^{7/2}} \left[1 + \cos^2 \theta - \frac{\beta_0^2 \epsilon}{4} (1 + 3\beta_0^2 \epsilon) \sin^4 \theta \right] d\Omega. \quad (\text{A} \cdot 6)$$

Thus it is necessary that

$$1 - \beta_0^2 \epsilon \sin^2 \theta \geq 0.$$

a) Čerenkov radiation

If we put $\theta = \pi/2 + \theta$, we get the condition

$$\cos \theta \leq \frac{1}{\beta_0 \sqrt{\epsilon}}.$$

The maximum intensity is evidently lying at an angle given by

$$\cos \theta = \frac{1}{\beta_0 \sqrt{\epsilon}}, \quad (\text{A} \cdot 7)$$

which is nothing but the Čerenkov relation. As is easily seen from (A.6), the angular distribution is almost like a δ function.

b) Synchrotron radiation

The most part of the radiation is emitted within an angle

$$\theta \simeq \sqrt{1 - \beta_0^2 \epsilon}. \quad (\text{A} \cdot 8)$$

Thus the angular distributions of these two radiations may be schematically shown by Fig. 3 and Fig. 4.

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On a Non-local Electromagnetic Model for Electron and Muon Masses*

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In a phenomenological way, a non-local electromagnetic interaction with a Pauli term is assumed in order to explain the whole masses of electron and muon. Qualitative discussions are devoted to the properties of form factors on the assumption of similar internal structures for both particles.

§ 1. Introduction

No remarkable difference of properties between electron and muon, except their large mass difference, has yet been discovered experimentally. In spite of suggestion that the large mass difference would come from the different internal structures of electron and muon, we could not get any definite conclusion from the present available experimental data.¹⁾

Concerning the origin of their mass-difference itself, many possibilities have been presented up to now. Schwinger,²⁾ for instance, has raised the possibility of explaining the high muon mass through a strong interaction of this particle with an unobserved iso-singlet σ -meson. Similar ideas have been repeated by other authors.^{3), 4)} Recently, Marx and Nagy⁵⁾ assumed that the muon has a moderate strong interaction with the kaon doublet, realizing then the muon mass from the electron's. Since there would be no way of detecting this new particle or coupling, they might be equivalent to the introduction of a different structure of the muon from that of the electron.

A possibility that the electron mass has some electromagnetic origin has also been considered frequently. It is well known that we cannot explain the whole masses of fermions entirely by the field reaction of their minimal interactions with the electromagnetic field, as the basic equation of the particles is invariant under the Touschek's transformation.⁶⁾ The only possibility is to introduce some violation term under this transformation. An explanation of the electron mass has already been presented by us and Taketani.⁷⁾ The interaction term thus introduced would make the system unrenormalizable, because it should have a coupling constant with the dimensions of length.^{***8)} An advantage of introducing the unrenormalizable

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*** We take natural units: $\hbar=c=1$, and $1\text{ y (yukawa)}=10^{-13}\text{ cm}$.

interaction is that it may give a sufficiently large effect of the high-energy region where the internal structure of the particle reveals itself.⁹⁾ The reactive mass, then, is given by

$$\delta m \sim \text{const} \cdot e \cdot f \cdot A^{-(\gamma+1)}, \quad (1.1)$$

where f is the coupling constant of the unrenormalized interaction with the dimension of $[L^\gamma]$ ($\gamma > 0$) and A is the cutoff energy. Therefore, we have a possibility of explaining the electron and muon masses through this interaction by suitably adjusting their internal structures.

There may be some evidence for believing the existence of such unrenormalizable interaction. The present discrepancies between the theoretical value of the anomalous magnetic moment and the experimental value are*

$$|\mu_{exp} - \mu_{theor}| \lesssim (10^{-5} \sim 10^{-6}) \cdot e/2m_e \quad \text{for electron,} \quad (1.2)$$

$$|\mu_{exp} - \mu_{theor}| \lesssim (10^{-8} \sim 10^{-4}) \cdot e/2m_\mu \simeq (10^{-5} \sim 10^{-6}) \cdot e/2m_e \quad \text{for muon.} \quad (1.3)$$

This may suggest that we have a possibility of assuming a *universal Pauli term* with a magnitude of $\kappa(e/2m_e) = (10^{-5} \sim 10^{-6}) \cdot e/2m_e$ for both particles beside the usual minimal interaction as

$$-L_{int} \sim e\bar{\psi}\gamma^\mu\psi A_\mu - \frac{e\kappa}{4m_e}\bar{\psi}\sigma^{\mu\nu}\psi \cdot F_{\mu\nu} \quad (1.4)$$

where ψ , $\bar{\psi}$ and A_μ , $F_{\mu\nu}$ are the field strengths of electron (or muon) and electromagnetic fields respectively. We take the hermitian representation for the Dirac matrices γ^μ and $\sigma^{\mu\nu} = i(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)/2$, and $g^{00}=1$, $g^{11}=g^{22}=g^{33}=-1$ for the space-time metric.

If we take a universal (local) interaction for electron and muon, the difference between both particles could come only from their different internal structures, which reveal themselves through the non-locality of the above interactions. Then our aim is to analyse the form factors through the interactions**

* The experimental values for the electron magnetic moment are

$$\mu_{exp}^{(e)} = 1.001146 \pm 0.000012, 1.001165 \pm 0.000011, 1.001167 \pm 0.000005^{10)}$$

in electron Bohr magneton. For the muon we quote the result of the determination of Garwin et al.,¹¹⁾

$$\mu_{exp}^{(\mu)} = 1.0020 \pm 0.0005$$

in muon Bohr magneton. The theoretical value of Sommerfeld¹²⁾ up to fourth order is

$$\mu_{theor} = 1.0011596$$

in lepton magneton. Then we have

$$\mu_{exp}^{(e)} - \mu_{theor}^{(e)} = -0.000014 \pm 0.000012, 0.0000054 \pm 0.000011, 0.0000074 \pm 0.000005$$

for electron and

$$\mu_{exp}^{(\mu)} - \mu_{theor}^{(\mu)} = 0.00018 \pm 0.0005 \quad \text{for muon.}$$

** Though the way of constructing the hamiltonian is not yet justified, the resultant hamiltonian consists of infinite series by using a usual method. The exact hamiltonian up to the second order needs the terms of higher order.¹³⁾

$$\begin{aligned}
-\bar{L}_{int} = & \int d^4x \int d^4y \left[e\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(y)F_1(x-y) \right. \\
& \left. - \frac{e\kappa}{4m_e}\bar{\psi}(x)\sigma^{\mu\nu}\psi(x)F_{\mu\nu}(y)F_2(x-y) \right] \quad (1.5)
\end{aligned}$$

in connection with the electron and muon masses. As a consistent non-local field theory has not yet been established, our analysis is to be taken tentatively.

In § 2 we get the general restrictions of the form factors so as to realize a positive reactive mass for the electron (or muon). The discussions rest on the second-order perturbation theory. Then, in § 3 we consider the problem of realizing the electron and muon masses and raise some possible examples for the form factors.

§ 2. Electromagnetic self-mass and the general restrictions on the form factors

By calculating the second-order self-mass, we get the result that only the mixed vertices (minimal and Pauli interactions) do contribute in the limit of a zero lepton bare mass. Then, in the assumption of a zero bare mass, we obtain the following expression for the self-mass δm :

$$\delta m = -\frac{e^2}{4\pi} \frac{3i\kappa}{4\pi^3 m_e} \int d^4q \frac{F_1(q^2)F_2(q^2)}{q^2 + i\epsilon}, \quad \epsilon = 0_+, \quad (2.1)$$

where

$$F_i(q^2) = \int d^4x e^{-iqx} F_i(x), \quad i=1, 2. \quad (2.2)$$

In order to get a finite self-mass, the form factors must be complex functions so as to cancel the divergent integrals, the simplest structures being

$$F_i(q^2) = \int dM^2 \frac{g_i(M^2)}{M^2 - q^2 - i\epsilon}, \quad (2.3)$$

where the spectral functions must become zero at the limit $M^2 \rightarrow \infty$,

$$\lim_{M^2 \rightarrow \infty} g_i(M^2) \rightarrow 0. \quad (2.4)$$

The first requirement which corresponds to the well-known Feynman cutoff⁽¹⁴⁾ may destroy the usual concept of an hermitian hamiltonian. But without this requirement, we get a quadratic divergent result regardless of the shape of form factors used. The second one means that we have not a δ -like charge at the center of the particle.

Using the spectral representation (2.3), we have

$$\delta m = -\frac{e^2}{4\pi} \frac{3}{4\pi} \frac{\kappa}{m_e} \Delta \simeq \mp 2 \cdot (10^{-2} \sim 10^{-3}) \frac{\Delta}{m_e}, \quad \kappa \gtrless 0, \quad (2.5)$$

$$\mathcal{A} = \int dM^2 \int dM'^2 g_1(M^2) g_2(M'^2) \frac{\log(M^2/M'^2)}{M^2 - M'^2}. \quad (2.6)$$

Then the details and the differences of the internal structures are reduced to the functional \mathcal{A} .

Since the self-mass must be positive from our standpoint of a zero bare mass, the quantity \mathcal{A} must be negative (positive) in accordance with $\kappa > 0$ ($\kappa < 0$). As the present experimental data are not definite enough to determine the sign of the discrepancy κ , we must consider both possibilities.

i) *Case of $\kappa < 0$.* There is no further restriction of the form factors. This case can be realized by simply assuming the form factors as

$$g_1(M^2) \geq 0, \quad g_2(M^2) \geq 0 \quad \text{for all } M^2. \quad (2.7)^*$$

It is sufficient to take only simple Yukawa form factors for both interactions.

ii) *Case of $\kappa > 0$.* This case may be the more probable one, but we cannot get a positive self-mass without the assumption that at least one of the form factors is indefinite in sign. Since the argument is symmetric for both form factors, we restrict one of the form factors, say $g_2(M^2)$, to be positive definite

$$g_2(M^2) \geq 0 \quad \text{for all } M^2. \quad (2.8)$$

Then the other form factor must have the following form :

$$\begin{aligned} g_1(M^2) &= g_1^+(M^2) - g_1^-(M^2), \\ g_1^-(M^2) &\neq 0 \quad \text{for some value of } M^2. \end{aligned} \quad (2.9)$$

The low-energy data, if it is possible, can restrict the form factors through the relations

$$\int dM^2 \frac{g_1(M^2)}{M^2} = 1, \quad \int dM^2 \frac{g_2(M^2)}{M^2} = \frac{1}{6} \langle r_i^2 \rangle, \quad (2.10)$$

where $\langle r_i^2 \rangle$ means the mean square radius (m.s.r.). At present we have no information concerning its magnitude or sign. The m.s.r. $\langle r_2^2 \rangle$ is positive from our assumption, but the sign of $\langle r_1^2 \rangle$ can be taken to be positive or negative depending on future information. (If experiments are to tell us that $\langle r_2^2 \rangle$ is negative, the role of g_1 and g_2 in the following discussion should be reversed.) Since the knowledge of the m.s.r. may determine the behavior of the spectral functions at the low mass values, we must consider the two cases of positive and negative $\langle r_1^2 \rangle$.

If the m.s.r. $\langle r_1^2 \rangle$ is positive, which is probable, we can guess that the positive part of the spectral function should be predominant at the low mass value

$$g_1(M^2) \sim g_1^+(M^2) \quad \text{for small } M^2. \quad (2.11)$$

* We take the case of a positive mean square radius, as otherwise the argument becomes more complicated as in the next case.

On the contrary, if the m.s.r. $\langle r_1^2 \rangle$ is negative, though it is less probable, we can imagine that the negative part should be predominant at the low mass value and the positive part might play an important role for the moderate mass values

$$g_1(M^2) \sim \begin{cases} -g_1^-(M^2) & \text{for small } M^2, \\ g_1^+(M^2) & \text{for moderate values of } M^2. \end{cases} \quad (2.12)$$

On the other hand, as the behavior of this spectral function for very large values of M^2 might determine the function A , (2.6) could be approximated by

$$A \sim \int_{M^2 \gg 0} dM^2 (g_1^+(M^2) - g_1^-(M^2)) \frac{\log(M^2/\Lambda^2)}{M^2 - \Lambda^2}, \quad (2.13)$$

where Λ^2 is the effective mass value which comes from the form factor $g_2(M^2)$. Then, in order to get the right sign for the self-mass, the spectral function $g_1(M^2)$ should be

$$g_1(M^2) \sim -g_1^-(M^2) \quad \text{for large } M^2. \quad (2.14)$$

With this requirement, we get

$$0 > A \gtrsim - \int_{M^2 \gg 0} dM^2 \frac{\log(M^2/\Lambda^2)}{M^2 - \Lambda^2} g_1^-(M^2) \gtrsim - \frac{1}{\Lambda^2} \int_{M^2 \gg 0} dM^2 g_1^-(M^2), \quad (2.15)$$

which can determine the lower limit of the strength of the form factor at high energy:¹⁵⁾

$$F_1(q^2) \sim -\frac{1}{q^2} \int dM^2 g_1^-(M^2). \quad (2.16)$$

Combining the requirements (2.11) or (2.12) with (2.14), we can conclude as follows: *If the m.s.r. $\langle r_1^2 \rangle$ is positive, the spectral function $g_1(M^2)$ should change its sign odd times. If the m.s.r. $\langle r_1^2 \rangle$ is negative, the spectral function $g_1(M^2)$ should change its sign even times.*

The simplest examples of spectral function are illustrated in Figs. 1a and 1b. It is sufficient to take the two-Yukawa combination shape for the former, while for the latter the three-Yukawa combination shape is at least necessary.

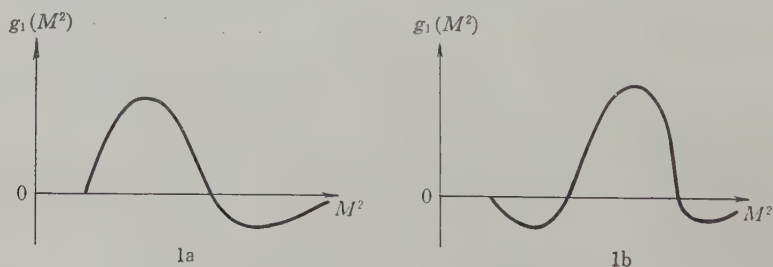


Fig. 1. The simplest examples of the spectral function

1a: the case of $\langle r_1^2 \rangle > 0$

1b: the case of $\langle r_1^2 \rangle < 0$

§ 3. Electron and muon masses

We discuss the possibilities of the model of form factors in order to get the electron and muon masses. Whatever the form factors might be, they must give the following orders for the functional \mathcal{A} :

$$\begin{aligned} \mathcal{A}^{(e)} &\simeq \mp 0.5 \cdot (10^8 \sim 10^9) m_e^2 & \text{for electron,} \\ \mathcal{A}^{(\mu)} &\simeq \mp (10^{10} \sim 10^{11}) m_e^2 & \text{for muon,} \end{aligned} \quad \kappa \geq 0. \quad (3.1)$$

Though there may be many different possibilities of constructing independent models for the electron and muon, the discussions are hereafter restricted to the cases with some similarity in both particles' structures.

The simplest case of this kind is the one in which the structures of both particles can be related with each other by a scale transformation

$$g_i^{(e)}(M^2) = g_i^{(\mu)}(\lambda M^2). \quad (3.2)$$

We then get the relations

$$\langle r_i^2 \rangle^{(e)} = \lambda \langle r_i^2 \rangle^{(\mu)}, \quad (3.3)$$

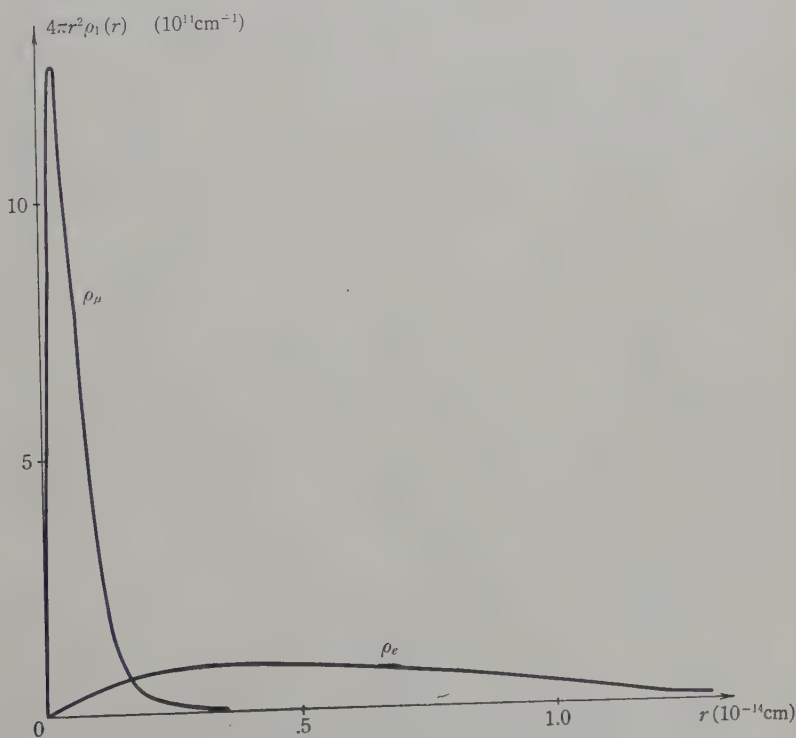


Fig. 2. The charge distribution for the case of $\kappa < 0$
 ρ_e : the electron's distribution with $\langle r^2 \rangle = (0.1y)^2$
 ρ_μ : the muon's distribution with $\langle r^2 \rangle = (0.6 \cdot 10^{-2}y)^2$

$$\Delta^{(e)} = \lambda^{-1} \Delta^{(\mu)} \quad (3.4)$$

from (2.10) and (2.6). If we take κ as universal, we must have $\lambda \simeq 200$ in order to get the electron-muon mass difference. This means that the m.s.r. of the electron is far larger than that of the muon, similar to the classical situation.

If we take the assumption of (2.7) for the case of $\kappa < 0$, the m.s.r. of the electron must be

$$\langle r^2 \rangle^{(e)} \sim (0.1y)^2 \quad (3.5)$$

with the approximation

$$\langle r_1^2 \rangle^{(e)} \simeq \langle r_2^2 \rangle^{(e)}, \quad \Delta^{(e)} \simeq \frac{6}{\langle r^2 \rangle^{(e)}}. \quad (3.6)$$

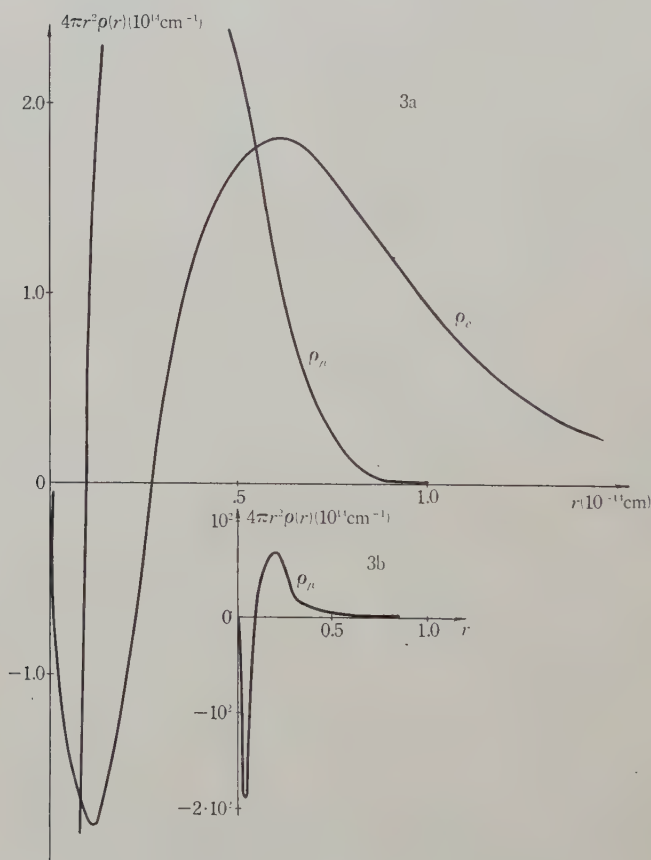


Fig. 3. The charge distribution for the case of $\kappa > 0$
 ρ_e : the electron's distribution with $\langle r^2 \rangle = (0.1y)^2$, $\alpha = \beta = 0.25$
 ρ_μ : the muon's distribution with $\langle r^2 \rangle = (0.1y)^2$, $\alpha = \beta = 0.02$;
the whole shape is repeated in Fig. 3b.

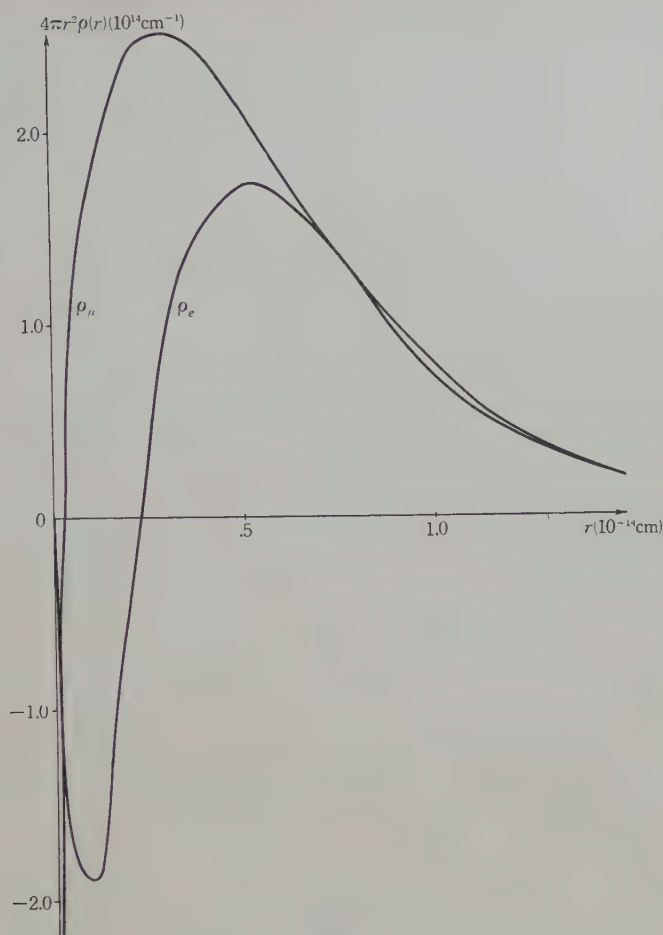


Fig. 4. The charge distribution for the case of $\kappa > 0$
 ρ_e : the electron's distribution with $\langle r^2 \rangle = (0.1y)^2$,
 $\alpha = 0.50$, $\beta = 0.08$;
 ρ_{μ} : the muon's distribution with $\langle r^2 \rangle = (0.1y)^2$,
 $\alpha = 0.50$, $\beta = 10^{-23}$.

The m.s.r. of the muon is then far smaller,

$$\langle r^2 \rangle^{(\mu)} \sim (0.6 \cdot 10^{-2} y)^2, \quad (3.7)$$

in virtue of the above argument.

Although the larger m.s.r. of the electron than the muon's one is not inconsistent with the present information, we should also consider the possibility of assuming the different internal features for the structures with the fixed m.s.r.'s (including the case of $\langle r^2 \rangle^{(e)} \simeq \langle r^2 \rangle^{(\mu)}$) and general shape. We illustrate an example for the case of $\kappa > 0$.

For the case of $\langle r_1^2 \rangle > 0$, we approximate the spectral form factors as

$$g_2(M^2) = \frac{6}{\langle r_2^2 \rangle} \delta \left[M^2 - \frac{6}{\langle r_2^2 \rangle} \right], \quad (3.8)$$

$$g_1(M^2) = \frac{6}{\langle r_1^2 \rangle} \frac{1}{\alpha - \beta} \left[\frac{1 - \beta}{\alpha} \delta \left[M^2 - \frac{6}{\alpha \langle r_1^2 \rangle} \right] - \frac{1 - \alpha}{\beta} \delta \left[M^2 - \frac{6}{\beta \langle r_1^2 \rangle} \right] \right], \quad (3.9)$$

where α and β are dimensionless constants with the restriction $1 \geq \alpha \geq \beta \geq 0$. The corresponding form factors in the energy-momentum space are given by

$$F_2(q^2) = \frac{1}{1 - \frac{\langle r_2^2 \rangle}{6} q^2}, \quad (3.10)$$

$$F_1(q^2) = \frac{1}{\alpha - \beta} \left(\frac{1 - \beta}{1 - \frac{\alpha \langle r_1^2 \rangle}{6} q^2} - \frac{1 - \alpha}{1 - \frac{\beta \langle r_1^2 \rangle}{6} q^2} \right). \quad (3.11)$$

The function \mathcal{A} becomes then

$$\mathcal{A} = -\frac{6}{\langle r^2 \rangle} \frac{1}{\alpha - \beta} \left[\frac{1 - \beta}{1 - \alpha} \log \alpha - \frac{1 - \alpha}{1 - \beta} \log \beta \right], \quad (3.12)$$

with the assumption of $\langle r_1^2 \rangle = \langle r_2^2 \rangle \equiv \langle r^2 \rangle$.

If we take

$$\langle r^2 \rangle = (0.1y)^2$$

for the electron and muon, their masses are realized by the following sets of values:

$$\text{i) } \alpha^{(e)} \simeq \beta^{(e)} \sim 0.25, \quad \alpha^{(\mu)} \simeq \beta^{(\mu)} \sim 0.02, \quad (3.13)$$

$$\text{ii) } \alpha^{(e)} \simeq \alpha^{(\mu)} \sim 0.50, \quad \beta^{(e)} \sim 0.08, \quad \beta^{(\mu)} \sim O(10^{-23}). \quad (3.14)$$

For the first case, the assumption of $\alpha^{(e)} \simeq \beta^{(e)}$ is taken so as to get a unique solution, while the assumption of $\alpha^{(e)} \simeq \alpha^{(\mu)}$ made in the second case does not give a unique answer. The larger the value of α is, the smaller the corresponding values of $\beta^{(e)}$ and $\beta^{(\mu)}$.

For the case of $\langle r_1^2 \rangle < 0$, the situation is so ambiguous that it does not seem worthwhile to discuss it here.

Some examples of the charge distribution are given in Figs. 2, 3 and 4. The distributions raised in Figs. 2 and 3 are very different for the electron and muon, while the one in Fig. 4 is similar for both particles except for the extremely internal parts. The decision will be given by future experiments up to the region of $0.1y$.

§ 4. Discussions and conclusions

The considerations here presented depend on future information concerning the lepton structures. At present, we know neither of the value of the m.s.r. nor of the difference between the theoretical and experimental values of the anomalous

magnetic moments. Since the arguments sensitively depend on these data, we tentatively consider every possible general case, i.e. the cases in which the m.s.r. would be positive and negative and the discrepancy of the anomalous moment would also be positive and negative. But the main effort was concentrated in the case of a positive m.s.r. and $\mu_{exp} - \mu_{theor} > 0$, because it seemed most probable from the data now available.

If we accept this case, it is reasonable to presume that the internal structure has, near the center, a behavior opposite to the one which is shown in the outer region. For instance, the charge distribution near the center turns out to be negative, which is very similar to the electromagnetic structure of the nucleon proposed elsewhere.¹⁵⁾

While the electron mass could be explained by the reasonable value of the m.s.r. $\langle r^2 \rangle \sim (0.1\gamma)^2$, the explanation of muon mass is still curious from this argument. Though we could explain it by assuming a strong form factor, why this should be the case is still an open question. Much more information about the muon structure should be accumulated, until we can get a definite conclusion.

We started with the assumption of the existence of the discrepancy $\mu_{exp} - \mu_{theor}$, but we could also take the unrenormalizable interaction for the case $\mu_{exp} = \mu_{theor}$. However, the answer will then be less reliable, so we omitted to consider about this case. If the spreading of the leptons could come from the weak interactions, the electromagnetic interaction would also be singular as the weak interactions.

Concerning the phenomenological aspect of the lepton structure, some knowledge of it will be obtained from high energy data. With the development of the experimental techniques, it is to be expected that the scattering experiments of identical leptons, for instance, will reveal their structures, avoiding the type of inconveniences involved, e.g. in the Stanford experiment, where it is difficult to separate the structure effects due to either nucleon or electron.

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Excitations in a High Density Electron Gas.* I

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A new systematical method which enables us to describe an electron gas in terms of bosons is developed. This boson corresponds to an "exciton", *i.e.* a pair of an electron outside the Fermi sphere and a hole inside. The formalism is particularly suitable to the system at high density, as suggested by Sawada's discussion of the same system. As a straightforward application, the effect of electron exchange on the plasma frequency is calculated. The result coincides with that of a Hartree-Fock treatment.

§ 1. Introduction

In his reformulation of Sawada's theory¹⁾ on the Coulomb interaction in a dense electron gas, Wentzel²⁾ used operators corresponding to processes where an electron with a momentum inside the Fermi sphere is excited to a momentum state outside, or *vice versa*. These operators were treated as boson creation and annihilation operators, which is justifiable at high density limit. Also a substitute Hamiltonian was adopted that gives the same commutators, or in other words, the same equation of motion of the aforesaid operators, as the basic relations of Sawada's argument. In view of Sawada's success in formulating the high-density problem in terms of this "boson" avoiding the perturbation expansion of Gell-Mann and Brueckner,³⁾ the boson (excited electron plus hole), which we shall loosely call an "exciton", can be an effective concept, particularly at high density of electrons. It is the purpose of this paper to present a systematical formulation of the electron gas problem in terms of the excitons.

In order to simplify the presentation, we shall consider a system of N spinless fermions, which shall be called electrons in the following. Let us assume that they are confined in a box of volume Ω with periodic boundaries. The Hamiltonian of the system is such that the number of the fermions should be conserved.

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As a reference state we shall take the state in which the N electrons occupy the lowest N momentum states (orbitals), *i.e.* the states with momenta smaller than p_F are occupied and the other states are vacant. Let us denote this reference state by Φ_0 .

Now any state of the system can be written in the form

$$\prod a_P^* a_p \Phi_0, \quad (1)$$

where a and a^* are the fermion annihilation and creation operators and it is understood that p always stands for a momentum vector inside the Fermi sphere, while P always stands for one outside. In order to define these base states uniquely, we shall consider all the momentum vectors to be arranged in a certain order, and understand that the annihilation or creation operators in the expression (1) are arranged according to this prescribed order. Then the state above can be considered as a certain assembly of excitons, each exciton being a definite pair of a hole with momentum p and an electron with momentum P . This pairing is unique thanks to the prescription above. We are going to describe the fermion system as an ensemble of excitons of this type. Let us first note that excitons are of Bose type in a certain sense because

$$[a_P^* a_p, a_{P'}^* a_{p'}] = 0. \quad (2)$$

§ 2. Transformation

Let us introduce here Bose operators C_p^P and C_p^{*P} which have two indices defined in the same region as above and obey the standard commutation relations:

$$\begin{aligned} [C_p^P, C_{p'}^{*P'}] &= \delta_{pp'} \delta_{PP'}, \\ [C_p^P, C_{p'}^{P'}] &= [C_p^{*P}, C_{p'}^{*P'}] = 0. \end{aligned} \quad (3)$$

The vacuum state of the bosons will be denoted by Ψ_0 .

Define the operator U in the product space of the fermions and the bosons by

$$\hat{U} = A \cdot \exp(\sum \sum C_p^{*P} a_p^* a_P) \cdot I, \quad (4)$$

where A and I are the projection operators to Φ_0 and Ψ_0 , respectively. Then one can easily show that

$$U \prod a_P^* a_p \Phi_0 \Psi_0 = \sum_P (-)^P \prod C_p^{*P} \Psi_0, \quad (5)$$

where P is a permutation of the lower indices of C^* 's. This theorem shows that the operator U can effectuate a one-to-one correspondence between the fermion states and a certain sub-space of the boson states. Before accomplishing that, we have to introduce an ordering operator in accordance with the prescription stated in Section 1: Let O be an operator which operates on the indices of C or C^* and, by definition, its eigenvalue is one if the pairing of the indices P and p is in compliance with the prescription, and zero otherwise. Then the

product operator OU just effectuates the transformation of $\Pi a_p^* a_p \Phi_0$ to $\Pi C_p^{*P} \Psi_0$ with the same pairs of indices. It may be noted that the ordering operator defined above is a projection operator :

$$O^2 = O. \quad (6)$$

Inversely, the hermitian conjugate of U ,

$$\tilde{U} = \Gamma \cdot \exp(\sum \sum C_p^P a_p^* a_p) \cdot A, \quad (7)$$

transforms any boson state back to a corresponding fermion state,

$$\tilde{U} \Pi C_p^{*P} \Psi_0 \Phi_0 = \Pi a_p^* a_p \Phi_0 \Psi_0. \quad (8)$$

Hence, the transformations OU and $\tilde{U}O$ complete the one-to-one correspondence of the exciton states in the fermion representation and in the boson representation. It is evident that $\tilde{U}OU$ is unity in the fermion state space.

§ 3. The Hamiltonian in the boson representation

Let the Hamiltonian of the fermion system be \mathcal{H} . Then the equation of motion for any fermion state Φ is

$$i \frac{\partial}{\partial t} \Phi = \mathcal{H} \Phi. \quad (9)$$

Now according to the last statement in Section 2, this equation may be written in the form :

$$i \frac{\partial}{\partial t} \Phi \Psi_0 = \mathcal{H} \tilde{U} O U \Phi \Psi_0, \quad (10)$$

where we assumed that the energy of the state Ψ_0 is zero. Hereby we see that we can investigate the dynamics of the original fermion system in the boson representation if we take $OU\mathcal{H}\tilde{U}O$ as its Hamiltonian. In other words, we can transcribe the dynamics of the fermion system into that of the boson system, where, however, we have to limit ourselves to the subspace $O\mathcal{F}$ and hence the Hamiltonian $U\mathcal{H}\tilde{U}$ should also be cut in such a way that it connects only the states in this subspace.

Now let us proceed to work out the corresponding boson Hamiltonian assuming that the Hamiltonian of the fermion system is given by

$$\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2 \quad (11)$$

where

$$\mathcal{H}_1 = \sum \epsilon_\lambda a_\lambda^* a_\lambda$$

and

$$\mathcal{H}_2 = \frac{1}{2} \sum \sum \sum V_\lambda a_{\kappa+\lambda}^* a_\mu^* a_{\mu+\lambda} a_\kappa.$$

The general principle of the calculation which we make in the following may be

stated as follows. Assume first that the resulting Hamiltonian is expanded in powers of C and C^* operators. Then we can find its coefficients by evaluating the expectation values of the multiple commutators of $U\mathcal{H}\tilde{U}$ with C and C^* , in the product state $\Phi_0\tilde{\Psi}_0$.

The first term, which is a constant, is just the expectation value of $U\mathcal{H}\tilde{U}$ itself. We shall denote the expectation value of any operator Q in the state $\Phi_0\tilde{\Psi}_0$ by $\langle Q \rangle$. The result is

$$\langle U\mathcal{H}\tilde{U} \rangle = \langle \mathcal{H} \rangle = \sum^F \epsilon_p + \frac{1}{2} \sum_{\neq}^F \sum (V_0 - V_{p-p'}). \quad (12)$$

The coefficient of C_p^P is given by $\langle [U\mathcal{H}\tilde{U}, C_p^{*P}] \rangle$, which is equal to

$$\langle [U\mathcal{H}\tilde{U}, C_p^{*P}] \rangle = \langle [\mathcal{H}, a_p^* a_p] \rangle = 0, \quad (13)$$

because of the translation invariance of the Hamiltonian. The hermitian property of $U\mathcal{H}\tilde{U}$ ensures that the coefficient of C_p^{*P} also vanishes. Following a similar procedure, we can get the coefficients of the next higher terms. We shall not write down the details of the calculation, which is straightforward, but just list the results:

$$\begin{aligned} \langle [C_{p_1}^{P_1}, [U\mathcal{H}\tilde{U}, C_p^{*P}]] \rangle &= \langle [a_{p_1}^* a_{P_1}, [\mathcal{H}, a_p^* a_p]] \rangle \\ &= \{(\epsilon_P - \epsilon_p) + \sum_{p'} (V_{p-p'} - V_{P-p'})\} \delta_{p_1 p} \delta_{P_1 P} + (V_{P-p} - V_{P_1-P}) \delta_{p+p', P+P'}, \end{aligned} \quad (14)$$

and

$$\langle [[U\mathcal{H}\tilde{U}, C_p^{*P}], C_{p'}^{*P'}] \rangle = \langle [[\mathcal{H}, a_p^* a_p], a_{p'}^* a_{P'}] \rangle = (V_{P-p} - V_{P'-p}) \delta_{p+p', P+P'}. \quad (15)$$

Concerning these two formulas, there are some points to be noted. First there appears no term with V_0 . For example, the sum $\sum_{p'} V_{p-p'}$ should be understood to exclude the term with $p=p'$. The reason is the operator $\sum \sum a_k^* a_\mu^* a_\mu a_k$, which is associated with V_0 in the expression for \mathcal{H}_2 , always commutes with $a_p^* a_p$. Secondly, the expression of the type (15) always vanishes when $P'=P$ and/or $p'=p$. This is just a consequence of the Pauli principle.

Leaving the calculation of higher terms to a later section, let us write down the exciton Hamiltonian to the second order. To this we shall introduce an abbreviation for the exchange energy, defined by

$$A(\lambda) = \sum^F V_{\lambda-p'}, \quad (16)$$

where we understand that if λ lies inside the Fermi sphere we have to omit the term with $p'=\lambda$. We shall also use the following notation:

$$\omega_p^P = \epsilon_P - \epsilon_p. \quad (17)$$

Then it is easy to see the relevant Hamiltonian is given by

$$\begin{aligned}
U\mathcal{H}\widetilde{U} = & \sum \epsilon_p + \frac{1}{2} V_0 N(N-1) - \frac{1}{2} \sum A(p) + \sum \sum \{ \omega_p^P + A(p) - A(P) \} C_p^{*P} C_p^P \\
& + \sum \sum \sum \{ V_{P-p} - V_{p_1-p} \} C_{p_1}^{*p_1+P-p} C_p^P + \frac{1}{2} \sum \sum \sum \{ V_{P-p} - V_{P-p'} \} \\
& \times (C_{p'}^{p'+p-P} C_p^P + \text{complex conj.}).
\end{aligned} \tag{18}$$

To obtain the Hamiltonian for the excitons we remember that we still have to take the ordering operator into account. This may be done conveniently in each actual application we meet. We shall discuss an example in the next section. Here, however, we shall give some remarks concerning new features appearing in the Hamiltonian above.

The expression in the first line of Eq. (18) is just the expectation value of the Hamiltonian for the reference state ϕ_0 . The first sum is the Fermi energy, the next the average "Coulomb" interaction and the last sum is the exchange energy. Therefore, by definition, the remainder represents the so-called correlation energy in a certain approximation. The present correlation Hamiltonian, as a matter of course, has the same structure in the main as that of Wentzel.²⁾ However, in the present Hamiltonian there appear several new terms which were neglected by Sawada and Wentzel because of their less importance at high density. Namely, $\{A(p) - A(P)\}$'s in the second line, $-V_{p_1-p}$'s in the third and $-V_{P-p'}$'s in the last line. These are all originated from the exchange effect. For example, the term $-V_{p_1-p} C_{p_1}^{*p_1+P-p} C_p^P$ corresponds to the following process. One electron with momentum P makes a transition to the state $p_1 + P - p$, while another electron makes a transition from p_1 to p . The appropriate matrix element is V_{p_1-p} . This type of transition was consistently neglected in Sawada's and Wentzel's calculations. However, in the present formulation we do not discard any process in the mid-way, sticking consistently to the exciton picture. Hence, we describe the process above in such a way, as the creation and annihilation operators show, that one electron is excited from p_1 to $p_1 + P - p$ and another is deexcited from P to p , the final state being physically the same as above. The phase, however, is different from the former by π , because the electrons in the states p and $p_1 + P - p$ have been exchanged. This is the reason for the appearance of the minus sign.

In short and formally stated, the present Hamiltonian has a structure consistent with the Fermi statistics of the original particles. Besides, it is obtained straightforwardly by a mathematical apparatus. (In fact, one can supplement Wentzel's Hamiltonian in accordance with the Fermi statistics and get to the result (18). This procedure, however, is apparently limited in its applicability.)

§ 4. The plasma oscillation

In this section we shall investigate the effect of the exchange terms on the plasma frequency. Sawada has already argued that the effect is small. However, as a specific example of application, we shall here derive a detailed expression for the plasma frequency of electrons in a neutralizing positive charge sea, including

the aforesaid exchange effect. Since the present author has not been able to find out any method to diagonalize the Hamiltonian (18), we are going to restrict ourselves to the discussion of the plasma oscillation and resort to the perturbation method.

The equation of motion for the annihilation operator of an exciton with momentum transfer q is, according to Eq. (18),

$$i\dot{C}_{p,q} = \tilde{\omega}_{p,q} C_{p,q} + \sum_{p'} (V_q - V_{p-p'}) C_{p',q} + \sum_{p_1}^0 (V_q - V_{p_1+p+q}) C_{-p_1,-q}^*, \quad (19)$$

where we have used the slightly altered notations defined as follows:

$$C_{p,q} \equiv C_p^{p+q}$$

and

$$\tilde{\omega}_{p,q} \equiv \omega_p^{p+q} + A(p) - A(p+q); \quad (20)$$

and the summation denoted by \sum with a superscript "0" means that one has to sum only those terms with the indices that are compatible with the prescription stated in Section 1. Hence in the last sum there appear only such p_1 's that the combination (p, q) and $(-p_1, -q)$ is in compliance with the ordering prescription. This may be conveniently done in the following way: Given a certain q , one may prescribe the order of the momenta according to the magnitude of $p \cdot q$ or $P \cdot q$. Correspondingly we shall divide the momentum space relevant to Eq. (19) into the five regions indicated in Fig. 1. Then in the second line of the equation, p belongs to the region $r_0 + r$, $p+q$ to R while $-p_1$ to $r_1 + r_0$ and $-p_1 - q$ to R_1 . Hence in the summation we have only to be careful when both p and $-p_1$ run in the region r_0 . Here, and everywhere in the following, the summation sign \sum^0 of Eq. (19) will just mean that one should sum the terms in compliance with the condition $p \cdot q > -p_1 \cdot q$ or $(p + p_1) \cdot q > 0$.

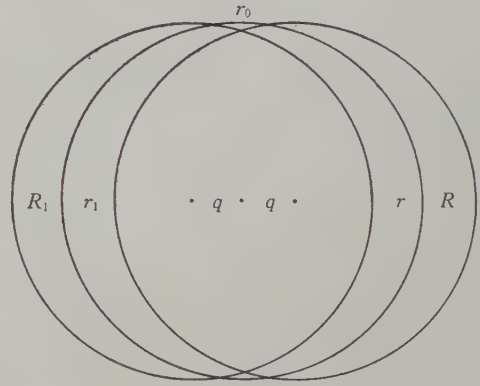


Fig. 1.

Then the equation above can be rewritten in the following way:

$$i\dot{C}_{p,q} = \tilde{\omega}_{p,q} C_{p,q} + V_q \sum (C_{p',q} + C_{-p',-q}^*) - \sum V_{p-p'} C_{p',q} - V_q \sum_{(p+p') \cdot q < 0} C_{-p',-q}^* - \sum_{(p+p') \cdot q > 0} V_{p+p'+q} C_{-p',-q}^*. \quad (21)$$

The complex conjugate equation is

$$i\dot{C}_{-p,-q}^* = -\tilde{\omega}_{p,q} C_{-p,-q}^* - V_q \sum (C_{p',q} + C_{-p',-q}^*) + \sum V_{p-p'} C_{-p',-q}^* + V_q \sum C_{p',q} + \sum V_{p+p'+q} C_{p',q}. \quad (21')$$

To get the eigen-frequency, let us regard C or C^* as non-quantized amplitudes. Since we are treating progressive waves, we must set these expressions equal to $\nu C_{p,q}$ or $\nu C_{-p,-q}^*$, respectively, where ν is the eigen-frequency of the wave. The first lines in the equations are just those terms which were discussed in Sawada's or Wentzel's papers. As mentioned before, we shall treat the remainder as perturbations.

First the unperturbed amplitudes are immediately given by these equations as follows:

$$C_{p,q} = \frac{V_q}{\nu - \tilde{\omega}_{p,q}} C_\nu \quad (22)$$

and

$$C_{-p,-q}^* = \frac{-V_q}{\nu + \tilde{\omega}_{p,q}} C_\nu, \quad (22')$$

where

$$C_\nu = \sum_p (C_{p,q} + C_{-p,-q}^*). \quad (23)$$

In these expressions C_ν is actually a non-perturbed amplitude. However, as long as the first perturbational approximation is intended, it makes no difference whether we take it to be nonperturbed or perturbed. Hence we shall regard it as the perturbed amplitude. Inserting these expressions into the perturbing terms in Eq. (21) and Eq. (21'), *i.e.* into the second lines, and solving them for $C_{p,q}$ and $C_{-p,-q}^*$, and then inserting these results back into the defining equation for C_ν , (23), we get

$$\begin{aligned} \frac{1}{2V_q} = & \sum_p \frac{\tilde{\omega}_{p,q}}{\nu^2 - \tilde{\omega}_{p,q}^2} - \sum_p \sum_{p'} \frac{V_{p-p'} (\nu^2 + \tilde{\omega} \tilde{\omega}')}{(\nu^2 - \tilde{\omega}^2) (\nu^2 - \tilde{\omega}'^2)} + \sum_{(p+p'), q < 0} \frac{V_q (\nu^2 - \tilde{\omega} \tilde{\omega}')}{(\nu^2 - \tilde{\omega}^2) (\nu^2 - \tilde{\omega}'^2)} \\ & + \sum_{(p+p'), q > 0} \frac{V_{p+p'+q} (\nu^2 - \tilde{\omega} \tilde{\omega}')}{(\nu^2 - \tilde{\omega}^2) (\nu^2 - \tilde{\omega}'^2)}. \end{aligned} \quad (24)$$

Since we are treating the plasma oscillation, we can assume that

$$\nu \gg \tilde{\omega}$$

and simplify the secular equation for ν by expansion. We are going to get the dispersion formula up to the second order in q .

The first terms in the expansion give

$$\nu^2 = 2V_q \left\{ \sum_{p'}^q \omega_{p,q} + \sum_{(p+p'), q < 0}^q \sum_{p'}^q (V_q - V_{p+p'+q}) \right\}. \quad (25)$$

Here the summation sign \sum^q denotes that one should sum over those p 's inside a beret-shaped region which is bounded by the surfaces of the Fermi sphere and of the sphere obtained by displacing it by the amount $-q$. We have simplified the expression by using the equality:

$$\sum_p^q \sum_{p'}^q V_{p-p'} - \sum_p^q \sum_{p'}^q V_{p-p'} = \sum_p^q \sum_{p'}^q V_{p+q-p'} - \sum_p^q \sum_{p'}^q V_{p+q+p'}. \quad (26)$$

By using a similar equality, one can easily see that the next terms in the expansion contribute

$$\begin{aligned} \Delta\nu^2 = \frac{2V_q}{\nu^2} \left\{ \sum \omega_{p,q}^3 + \frac{1}{2} \sum_p \sum_{p'}^q [(\omega - \omega')^2 V_{p-p'} - (\omega + \omega')^2 V_{p+p'+q}] \right. \\ \left. + \sum_{(p+p') \cdot q < 0}^q \sum_{p'}^q (V_q - V_{p+p'+q}) (\omega^2 + \omega'^2 - \omega\omega') \right\}. \end{aligned} \quad (27)$$

It is readily seen that the last sum in Eq. (27) is higher by q^2 than the last one in Eq. (25).

Let us evaluate the expressions for ν^2 and $\Delta\nu^2$ to the order q^2 in the case of spinless electrons in a positive charge sea. For this case

$$V_\lambda = \frac{4\pi e^3}{\lambda^2} \Omega^{-1} \quad (28)$$

and

$$\omega_{p,q} = \frac{p \cdot q}{m} + \frac{q^2}{2m}. \quad (29)$$

We shall give the values of the separate sums in Appendix 1. The final result is

$$\nu^2 = \frac{2e^2 p_F^3}{3\pi m} + \frac{e^4}{16\pi^2} q^2, \quad (30)$$

and with this value of ν^2 inserted,

$$\Delta\nu^2 = \frac{3}{5} \frac{p_F^2}{m^2} q^2 - \frac{1}{5\pi} \frac{e^2 p_F}{m} q^2. \quad (31)$$

Transcribing these results in terms of the usual notations:

$$\begin{aligned} p_F^3 &= 6\pi^2 n, \\ \frac{1}{n} &= \frac{4\pi}{3} r_s^3 a_B^3, \end{aligned}$$

and

$$a_B = (e^2 m)^{-1}, \quad (32)$$

where n is the number density of the electrons and a_B is the Bohr radius, we obtain as the required dispersion formula for the plasma oscillation

$$\nu^2 = \frac{4\pi e^2 n}{m} + \frac{3}{5} \frac{p_F^2}{m^2} q^2 \left\{ 1 - \frac{1}{3\pi} \left(\frac{2}{9\pi} \right)^{1/3} r_s + \frac{5}{48\pi^2} \left(\frac{2}{9\pi} \right)^{2/3} r_s^2 \right\}. \quad (33)$$

It is to be noted here that a correction to the non-dispersive term, which would otherwise contain a term proportional down to r_s , has been cancelled out by virtue of the relation (26). It may also be noted that the last term in the dispersion coefficient, which is proportional to e^4 or the correction factor proportional to r_s^2 , has arisen from our taking account of the Pauli principle in the case

of doubly annihilation or creation processes in the region r_0 .

The first and the second terms in the dispersion coefficient, (31), may also be written in the form:

$$q^2 \left\{ \left\langle \left(\frac{p}{m} \right)^2 \right\rangle + \frac{4}{15} \frac{\epsilon_x}{m} \right\}, \quad (34)^*$$

where the average is to be taken over the Fermi sphere and ϵ_x is the exchange energy per electron in the Fermi ground state, *i.e.*

$$\epsilon_x = -\frac{3}{4\pi} e^2 p_F. \quad (35)$$

This, including the result about the non-dispersive term, is just what one obtains on the basis of Hartree-Fock treatment of the plasma oscillation. We shall give the analysis in the Appendix 2.

§ 5. Interaction between the excitons and concluding remarks

One can extend the calculation to obtain the interaction Hamiltonian in a similar way. The three exciton interaction can be obtained by evaluating the expression

$$\langle [C_{p_1}^{I_1}, [[U\mathcal{H}\widetilde{U}, C_{p'}^{*I'}], C_{p'}^{*I'}]] \rangle;$$

which, transformed into the fermion representation, is equal to

$$\langle [a_{p_1}^* a_{p_1}, [[\mathcal{H}, a_{p'}^* a_{p'}], a_{p'}^* a_{p'}]] \rangle.$$

The evaluation of this latter expression is straightforward, and one thus obtains

$$\begin{aligned} & \langle [C_{p_1}^{I_1}, [[U\mathcal{H}\widetilde{U}, C_{p'}^{*I'}], C_{p'}^{*I'}]] \rangle \\ &= (V_{I'-p'} - V_{I'-p}) \delta_{p_1, p} \delta_{p'+I_1, p+I'} + (V_{I'-p} - V_{I'-p'}) \delta_{p_1, p'} \delta_{p'+I_1, p+I'} \\ &+ (V_{I'-p'} - V_{I'-p}) \delta_{I_1, I'} \delta_{p_1+I, p+p'} + (V_{I'-p} - V_{I'-p'}) \delta_{I_1, I'} \delta_{p_1+I', p+p'}. \end{aligned} \quad (36)$$

Here one may note that

$$[[\mathcal{H}_1, a_p^* a_p], a_{p'}^* a_{p'}] = 0, \quad (37)$$

so that \mathcal{H}_1 contributes only to the two-exciton Hamiltonian. Hence, the three-exciton interaction is given by

$$\begin{aligned} & \sum \sum (V_{I'-p} - V_{I'-p'}) \{ C_{p'}^{*I'+I-p} C_{p'}^{I'} C_p^{I'} + C_p^{*I'} C_{p'}^{*I'} C_{p'}^{I'+I-p} \} \\ & - \sum \sum (V_{I'-p} - V_{I'-p'}) \{ C_{p'+p-I}^{*I'} C_{p'}^{I'} C_p^{I'} + C_p^{*I'} C_{p'}^{*I'} C_{p'+p-I}^{I'} \}. \end{aligned} \quad (38)$$

The four-exciton interaction is rather heavy to work out, but the calculation goes in quite a similar straightforward way. However, we shall refrain from

* The author learned later that DuBois obtained the same answer through a different analysis. D. F. DuBois, thesis, California Institute of Technology, 1958.

writing it down here. Instead we shall only note that in the above expansion of the fermion Hamiltonian on the basis of exciton picture the interaction Hamiltonian will involve evaluation of a finite number of coefficients, since the multiple commutator of \mathcal{H}_2 with various $a_p^* a_p$'s vanishes beyond the fourth.

We shall defer the investigation of the interactions to a later paper.

We showed once⁴⁾ that any system of interacting Bose particles can be completely described in terms of Bogoliubov excitations.⁵⁾ There appeared interactions involving up to six excitations. The formulation was analogous to the spin wave theory of Dyson.⁶⁾ The same procedure was followed for the fermion case first, obtaining essentially the same Hamiltonian. It is, however, not hermitian and, besides, it is not straightforward to take full account of the Pauli principle. The utilization of the transformation (4) is certainly superior in this respect. The present method may have a more versatile applicability.

We presented here only one simple application and laid a stress on the formal aspect of the method. However, the investigation of the diamagnetism and the Froehlich interaction of metals on the basis of this formalism is interesting and, in fact, the work on these lines is under way in our laboratory.

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Appendix 1

We shall list here the various sums involved in Eqs. (25) and (27) :

$$\sum_q^q \omega_{p,q} = \frac{\Omega}{(2\pi)^3} 2\pi \frac{1}{m} \frac{1}{3} p_F^3 q^2 \quad (\text{A} \cdot 1)$$

$$\sum_q^q \omega_{p,q}^3 = \frac{\Omega}{(2\pi)^3} 2\pi \frac{1}{m^3} \frac{1}{5} p_F^5 q^4 + \dots \quad (\text{A} \cdot 2)$$

$$\sum_{(p+p') \cdot q < 0}^q \sum_q^q (V_q - V_{p+p'+q}) \approx \sum \sum V_q = \frac{4\pi e^2}{\Omega} \left[\frac{\Omega}{(2\pi)^3} \right]^2 \frac{\pi^2}{8} q^4 + \dots \quad (\text{A} \cdot 3)$$

$$\begin{aligned} & \frac{1}{2} \sum_q^q \sum_{p'}^q [(\omega_{p,q} - \omega_{p',q})^2 V_{p-p'} - (\omega_{p,q} + \omega_{p',q})^2 V_{p+p'+q}] \\ &= -\frac{4}{15} \frac{4\pi e^2}{\Omega} \frac{q^2}{m^2} \left[\frac{\Omega}{(2\pi)^3} 2\pi \frac{1}{2} p_F^2 q \right]^2 + \dots \end{aligned} \quad (\text{A} \cdot 4)$$

Appendix 2

Here we shall give a brief discussion based on Hartree-Fock's self-consistent field.

First transform the density matrix defined by the orthogonal orbitals equal in number to the number density of the electrons, to the "Wigner representation":

$$\rho(q, p) = \int \left(q + \frac{1}{2} x \middle| \rho \middle| q - \frac{1}{2} x \right) e^{-i x p / \hbar} dx, \quad (\text{A} \cdot 5)$$

with

$$(q' | \rho | q'') = \sum_r^n (q' | r) (r | q''). \quad (\text{A} \cdot 5')$$

This obeys the equation of motion of the type,

$$\begin{aligned} \frac{\partial}{\partial t} \rho(q, p) &= \frac{2}{\hbar} \sin \frac{\hbar}{2} \left[\frac{\partial}{\partial p} \frac{\partial}{\partial q} - \frac{\partial}{\partial p} \frac{\partial}{\partial q} \right] \mathcal{H}(q, p) \rho(q, p) \\ &\approx \frac{\partial \rho}{\partial p} \frac{\partial \mathcal{H}}{\partial q} - \frac{\partial \mathcal{H}}{\partial p} \frac{\partial \rho}{\partial q}, \end{aligned} \quad (\text{A} \cdot 6)$$

with the Hamiltonian

$$\mathcal{H}(q, p) = \frac{p^2}{2m} + e^2 \int \frac{dq'}{|q - q'|} \int \rho(q', p') \frac{dp'}{(2\pi\hbar)^3} - e^2 \int \frac{4\pi\hbar^2}{|p - p'|^2} \rho(q, p') \frac{dp'}{(2\pi\hbar)^3}. \quad (\text{A} \cdot 6')$$

Let us assume that

$$\rho(\mathbf{q}, \mathbf{p}, t) = \rho_0(p) + \rho_1(\mathbf{q}, \mathbf{p}, t) \quad (\text{A} \cdot 7)$$

and linearize the equation. Here the assumed positive charge background cancels the Coulomb field due to the homogeneous distribution of the electrons $\rho_0(p)$.

$$\begin{aligned} \frac{\partial}{\partial t} \rho_1(q, p, t) &= e^2 \left(\int \frac{\partial}{\partial q} \frac{dq'}{|q - q'|} \int \rho_1(q', p', t) \frac{dp'}{(2\pi)^3} \right. \\ &\quad \left. - \int \frac{4\pi}{|p - p'|^2} \frac{\partial}{\partial q} \rho_1(q, p', t) \frac{dp'}{(2\pi)^3} \right) \frac{\partial \rho_0}{\partial p} \\ &\quad - \left(\frac{p}{m} - e^2 \int \frac{4\pi}{|p - p'|^2} \frac{\partial}{\partial p'} \rho_0(p') \frac{dp'}{(2\pi)^3} \right) \frac{\partial \rho_1}{\partial q}. \end{aligned} \quad (\text{A} \cdot 8)$$

Putting the fluctuating part of the Wigner distribution function in the form:

$$\rho_1 = g(p) e^{i(kq - vt)}, \quad (\text{A} \cdot 9)$$

we obtain the eigen-value equation. The same perturbational method as in the text gives as the dispersion formula

$$1 = -\frac{4\pi e^2}{mk^2} \int \frac{(k \cdot p)^2}{\nu^2 - (k \cdot p/m)^2} \frac{1}{p} \frac{d\rho_0}{dp} \frac{d^3 p}{(2\pi)^3} \\ + \frac{4\pi e^2}{k^2} \frac{2\pi e^2}{m^2} \iint \frac{k \cdot p}{(\nu - k \cdot p/m)^2} \frac{1}{p} \frac{d\rho_0}{dp} \frac{k \cdot p'}{(\nu - k \cdot p'/m)^2} \frac{1}{p'} \frac{d\rho_0}{dp'} \frac{(k \cdot (p-p'))^2}{|p-p'|^2} \frac{d^3 p}{(2\pi)^3} \frac{d^3 p'}{(2\pi)^3}. \quad (\text{A} \cdot 10)$$

Assuming that $\nu^2 \gg (k \cdot p/m)^2$ for the plasma oscillation and expanding in power of k^2 , we can reduce Eq. (A. 10) to

$$\nu^2 = \frac{4\pi e^2 n}{m} + k^2 \left\langle \left(\frac{p}{m} \right)^2 \right\rangle \\ + k^2 \cdot \frac{2\pi e^2}{nmk^4} \iint \frac{(k \cdot p)(k \cdot p')(k \cdot (p-p'))^2}{|p-p'|^2} \frac{1}{p} \frac{d\rho_0}{dp} \frac{1}{p'} \frac{d\rho_0}{dp'} \frac{d^3 p}{(2\pi)^3} \frac{d^3 p'}{(2\pi)^3}. \quad (\text{A} \cdot 11)$$

The last integral, divided by the factor k^2 , can be shown to be equal to

$$\frac{4\pi e^2}{nm} \frac{1}{15} \iint \frac{1}{pp'} \log \left| \frac{p-p'}{p+p'} \right| \rho_0 \rho_0' \frac{d^3 p}{(2\pi)^3} \frac{d^3 p'}{(2\pi)^3}. \quad (\text{A} \cdot 12)$$

Hence Eq. (A. 11) can be written in the form

$$\nu^2 = \frac{4\pi e^2 n}{m} + k^2 \left\langle \left(\frac{p}{m} \right)^2 \right\rangle + \frac{4}{15} \frac{\epsilon_x}{m}, \quad (\text{A} \cdot 13)$$

where ϵ_x is the exchange energy per electron in the state ϕ_0 . Ferrell⁸⁾ gave a result of the same form except for a difference of the factor 1 instead of 4/15. The same result as Eq. (A. 13) was given by Silin.⁹⁾

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Excitations in a High Density Electron Gas. II

—Diamagnetism—

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As another application of the method of describing a high density fermion gas in terms of electron-hole pairs, diamagnetism of the gas is investigated. The following two corrections to Landau diamagnetism obtained come from the screened interaction of exchange type. It is intended to lay a stress on the effectiveness of the physical concept of electron-hole pair excitation.

§ 1. Introduction

In a previous paper¹⁾ under the same title we presented a general formalism which describes a system of fermions in terms of bosons of a certain type. The intention was to point out the importance of Sawada's²⁾ or Wentzel's³⁾ exciton, *i.e.* electron-hole pair as a basic concept in dealing with the electron gas at high density. The formalism described there is such that it enables us to investigate the behaviour of the gas entirely on the basis of the pair picture. This is considered particularly effective when one tries to study the properties of the gas down from the high density limit. However, the only one application presented there was a derivation of the dispersion relation of the plasma oscillation modified by exchange effect of the electrons. To clarify the point further, we shall present another application in this paper. The subject which we are going to deal with is diamagnetism of the electron gas.

This line of investigation has already been tried by Wentzel,³⁾ who obtained Landau diamagnetism while he took only the Coulomb part of the interaction into account. Hence our present task is to investigate the effect of the exchange part. The effect will be, as a matter of course, of higher order in the mean distance of the electrons. However, the point is to show that the Hamiltonian of the excitons is appropriate to the description of the gas in a successive way from the high density limit.

§ 2. Unperturbed Hamiltonian and its diagonalization

The "exciton" representation of the Hamiltonian of an electron gas in a positive charge background is given by Eq. (18) of the paper I to the second order

in creation or annihilation operators. This can be written in the following form:

$$\mathcal{H} = \sum_q \mathcal{H}_q$$

$$\mathcal{H}_q = 1/2 \cdot \sum_p \sum_{p'} (B_{pp'}^{(q)} \pi_{p,q} \pi_{-p',-q} + A_{pp'}^{(q)} \varphi_{p,q} \varphi_{-p',-q}), \quad (2.1)$$

φ 's and π 's are coordinates and conjugate momenta of the oscillators:

$$\varphi_{p,q} = \frac{1}{\sqrt{2}} (c_{p,q} + c_{-p,-q}^*) = \varphi_{-p,-q}^*$$

$$\pi_{p,q} = \frac{i}{\sqrt{2}} (c_{p,q}^* - c_{-p,-q}) = \pi_{-p,-q}^* \quad (2.2)$$

and the coefficients are given by

$$A_{pp'}^{(q)} = A_{-p,-p'}^{(-q)} = \tilde{\omega}_{p,q} \delta_{pp'} + (V_q - V_{p-p'}) + (V_q - V_{p+q+p'}),$$

$$B_{pp'}^{(q)} = B_{-p,-p'}^{(-q)} = \tilde{\omega}_{p,q} \delta_{pp'} + V_{p+q+p'} - V_{p-p'}, \quad (2.3)$$

where

$$\tilde{\omega}_{p,q} = \omega_{p,q} + \gamma_{p,q} \quad (2.4)$$

with

$$\gamma_{p,q} = \sum_{p'}^{\leq p_F} (V_{p-p'} - V_{p+q-p'})$$

and V_q is a q Fourier component of the electronic interaction: $V_q = 4\pi e^2 / \Omega q^2$. Refer to the paper I for detailed description of the notation.

φ 's and π 's satisfy the standard commutation relations; in particular,

$$[\varphi_{p,q}, \pi_{p',q'}] = i \delta_{pp'} \delta_{qq'}, \quad (2.5)$$

and, according to Eq. (1), their equations of motion are given by

$$\frac{d\varphi_{p,q}}{dt} = \sum_{p'} B_{pp'}^{(q)} \pi_{-p',-q}$$

$$\frac{d\pi_{-p,-q}}{dt} = \sum_{p'} A_{pp'}^{(q)} \varphi_{p',q}. \quad (2.6)$$

Let us consider the corresponding eigen-value problem:

$$-i\nu \hat{\epsilon}_{p,q}^\nu = \sum_{p'} B_{pp'}^{(q)} \gamma_{p',q}^\nu$$

$$+ i\nu \gamma_{p,q}^\nu = \sum_{p'} A_{pp'}^{(q)} \hat{\epsilon}_{p',q}^\nu. \quad (2.7)$$

This set gives a secular equation of the following form:

$$\begin{vmatrix} B & -i\nu \\ +i\nu & A \end{vmatrix} = 0, \quad (2.8)$$

from the form of which one can see that $-\nu$ is also an eigen-value if ν is one.

However, we shall agree to consider only the positive eigen-values, the number of which is the same as that of p 's in that the beret shaped region of the Fermi sphere which can be moved out of the sphere by a displacement q .

From Eq. (7) and the symmetric properties of the matrices A and B , one can derive, with an appropriate normalization, the relations:

$$\sum_p (\xi_{p,q}^\nu \gamma_{p,q}^{\nu'*} - \xi_{p,q}^{\nu'*} \gamma_{p,q}^\nu) = i\delta_{\nu,\nu'} \quad (2.9)$$

and

$$\sum_p (\xi_{p,q}^\nu \gamma_{p,q}^{\nu'} - \xi_{p,q}^{\nu'} \gamma_{p,q}^\nu) = 0. \quad (2.10)$$

Introduce then a new set of creation annihilation operators by

$$\begin{aligned} b_q^\nu &= \frac{1}{i} \sum_p (\gamma_{p,q}^{\nu*} \varphi_{p,q} - \xi_{p,q}^{\nu*} \pi_{-p,-q}) \\ b_q^{\nu*} &= -\frac{1}{i} \sum_p (\gamma_{p,q}^\nu \varphi_{-p,-q} - \xi_{p,q}^\nu \pi_{p,q}). \end{aligned} \quad (2.11)$$

Here we shall assume that $\xi_{p,q}^\nu = \xi_{-p,-q}^\nu$, $\gamma_{p,q}^\nu = \gamma_{-p,-q}^\nu$. Then the relations (9) and (10) ensure that

$$[b_q^\nu, b_{q'}^{\nu'*}] = \delta_{qq'} \delta_{\nu\nu'}. \quad (2.12)$$

The inverse relations to Eq. (11) are

$$\begin{aligned} \varphi_{p,q} &= \sum_\nu (\xi_{p,q}^\nu b_q^\nu + \xi_{p,q}^{\nu*} b_{-q}^{\nu*}), \\ \pi_{p,q} &= \sum_\nu (\gamma_{p,q}^\nu b_{-q}^\nu + \gamma_{p,q}^{\nu*} b_q^{\nu*}), \end{aligned} \quad (2.13)$$

from which one can derive

$$\sum_\nu (\xi_{p,q}^\nu \gamma_{p',q}^{\nu*} - \xi_{p,q}^{\nu*} \gamma_{p',q}^\nu) = i\delta_{pp'}, \quad (2.14)$$

$$\sum_\nu (\xi_{p,q}^\nu \xi_{p',q}^{\nu*} - \xi_{p,q}^{\nu*} \xi_{p',q}^\nu) = 0, \quad (2.15)$$

and

$$\sum_\nu (\gamma_{p,q}^\nu \gamma_{p',q}^{\nu*} - \gamma_{p,q}^{\nu*} \gamma_{p',q}^\nu) = 0. \quad (2.16)$$

One can easily see that this new set of creation annihilation operators brings the Hamiltonian (1) to a diagonal form:

$$\mathcal{H}_q + \mathcal{H}_{-q} = \sum_\nu \nu + \sum_\nu \nu (b_q^{\nu*} b_q^\nu + b_{-q}^{\nu*} b_{-q}^\nu). \quad (2.17)$$

§ 3. The Zeeman energy

Let the vector potential of the magnetic field be

$$A(x) = A_k e^{-ik \cdot x} + A_{-k} e^{ik \cdot x}, \quad \text{with } A_{-k} = A_k^* \quad (3.1)$$

so that the strength of the field is given by

$$H(x) = -ik \times A_k e^{-ik \cdot x} + ik \times A_{-k} e^{ik \cdot x}. \quad (3.2)$$

Then the Zeeman energy, or the magnetic energy, is $\mathcal{H}' + \mathcal{H}''$ with

$$\mathcal{H}' = \frac{e}{2mc} \left\{ \left(\sum_{\kappa} (2\kappa - k) a_{\kappa-k}^* a_{\kappa} \right) A_k + \left(\sum_{\kappa} (2\kappa + k) a_{\kappa+k}^* a_{\kappa} \right) A_{-k} \right\} \quad (3.3)$$

$$\mathcal{H}'' = \frac{e^2 n}{mc^2} A_k \cdot A_{-k} \quad (3.4)$$

where we retained only a constant term in \mathcal{H}'' , the magnetic energy of second order in the vector potential. The electronic charge and mass are denoted by $-e$ and m , respectively, and c is the light velocity. The number density of the electrons is n .

Transforming \mathcal{H}' into the "exciton" representation by means of U and \tilde{U} , Eqs. (4) and (7) of the paper I gives

$$\mathcal{H}' = -\frac{1}{c} (j_k A_k + j_{-k} A_{-k})$$

with

$$j_k = -\frac{e}{m} \sum_p^{(k)} \left(p + \frac{k}{2} \right) (c_{p,k} - c_{p,-k}^*), \quad (3.5)$$

we retained here only terms linear in the "exciton" creation or annihilation operators. This approximation is consistent with our Hamiltonian (1) truncated beyond the third order in c or c^* 's, as will be seen from the following argument.

The fluctuating number density of the electrons,

$$\rho(x) = \sum_{\kappa} \sum_{\kappa'} a_{\kappa}^* a_{\kappa'} e^{-i(\kappa - \kappa') \cdot x},$$

is represented, after transformation by means of U and \tilde{U} , to the first order in c or c^* by

$$\rho(x) = \sum_{p,q} (c_{p,q} e^{iq \cdot x} + c_{p,q}^* e^{-iq \cdot x}).$$

The Hamiltonian (2.1) then causes motion of $\rho(x)$ according to

$$\dot{\rho}(x) = \frac{1}{i} \sum_q e^{iq \cdot x} \sum_p^{(q)} \sum_{p'}^{(q)} B_{pp'}^{(q)} (c_{p,q} - c_{p,-q}^*),$$

where we have neglected the effect of the ordering operator, which will be justifiable because of the smallness of the overlapping region r , specified in the paper I. The right-hand side of this equation can be further simplified thanks to the equality:

$$\sum_{p'}^{(q)} B_{pp'}^{(q)} = v_{p,q}, \quad (3.6)$$

which is always valid for any V_q that is a function of the absolute magnitude of q . Thus one obtains a continuity equation,

$$(-e)\dot{\rho}(x) = -\nabla \cdot j(x),$$

with the following expression for the current density:

$$j(x) = \sum_q j_q e^{iq \cdot x},$$

with the expression (3.5) for j_q .

Now that the expression for the magnetic energy has been determined, we shall proceed to calculate the magnetic susceptibility. To do this, we have to find out a canonical transformation which will cancel out \mathcal{H}' , which is linear in the vector potential. This is easy enough, however, because the unperturbed Hamiltonian has been brought to a diagonal form (2.17). Define a hermitic operator by

$$S = -\frac{e}{\sqrt{2}mc} \sum_p \sum_v (2p+k) \frac{1}{v} \left\{ \gamma_{p,k}^v (b_k^v A_k - b_{-k}^v A_{-k}) + \gamma_{p,k}^{v*} (b_k^{v*} A_{-k} - b_{-k}^{v*} A_k) \right\}. \quad (3.7)$$

Then, the canonical transformation $\exp(iS)$ just fulfills the requirements, as can readily be seen. Thus the transformed Hamiltonian, to the second order in A , is just the unperturbed Hamiltonian plus a constant, which is given by $\mathcal{H}'' + (1/2)[iS, \mathcal{H}']$, or explicitly,

$$\begin{aligned} \frac{1}{2}[iS, \mathcal{H}'] = & -\frac{e^2}{m^2 c^2} \sum_p \sum_{p'} \sum_v \left(p + \frac{k}{2} \right) \left(p' + \frac{k}{2} \right) \\ & \times \frac{1}{v} (\gamma_{p,k}^v \gamma_{p',k}^{v*} + \gamma_{p,k}^{v*} \gamma_{p',k}^v) : (A_k A_{-k} + A_{-k} A_k). \end{aligned} \quad (3.8)$$

The expression involving the transformation coefficients in this equation, however, has a neat relation with the matrix B :

$$\sum_{p'} B_{pp'} \left(\sum_v \frac{1}{v} (\gamma_{p',q}^v \gamma_{p',q}^{v*} + \gamma_{p',q}^{v*} \gamma_{p',q}^v) \right) = \delta_{pp'}. \quad (3.9)$$

One can prove this by multiplying the first of Eq. (2.7) by γ_j^* and adding the resultant to its complex conjugate. The relation (2.9), then, brings about the result (3.9).

Thus, the final expression for the magnetic energy can be written in the following compact form:

$$\frac{1}{2}[iS, \mathcal{H}'] = -\frac{e^2}{m^2 c^2} \sum_p \sum_{p'} \left(p + \frac{k}{2} \right) \left(p' + \frac{k}{2} \right) (B^{-1})_{pp'} : (A_k A_{-k} + A_{-k} A_k). \quad (3.10)$$

Before evaluating the coefficient, or the susceptibility, we shall prove that the result will be gauge-invariant. The k Fourier component of the current density can be derived from the expression for the magnetic energy by taking variation of the vector potential. Thus,

$$\langle j_k \rangle = K(k) \cdot A_{-k} \quad (3.11)$$

with

$$K(k) = -c \left\{ \frac{e^2}{mc^2} n - \frac{e^2}{2m^2 c^2} \sum_p^{(k)} \sum_{p'}^{(k)} (2p+k) (2p+k') (B^{-1})_{pp'} \right\}.$$

Since the tensor K is clearly symmetric, the requirement of gauge-invariance is equivalent to that of continuity of the current. However, we shall explicitly show here that $k \cdot K(k)$ vanishes, at least, up to the order of k^3 . First we have the equality:

$$\sum_{p'}^{(k)} (B^{-1})_{pp'} \omega_{p'k} = 1. \quad (3.12)$$

This can be obtained from the defining equation

$$1 = \sum_{p'}^{(k)} \sum_{p''}^{(k)} (B^{-1})_{pp'} B_{p'p''}$$

by making use of the equality (3.6). The expression $k \cdot K(k)$, then, reduces to

$$k \cdot K(k) = -\frac{e^2}{mc} \left\{ nk - \sum_p^{(k)} (2p+k) \right\},$$

which may now be evaluated by an elementary integration. The result is

$$k \cdot K(k) = -\frac{e^2}{mc} \left\{ nk - \frac{\Omega}{(2\pi)^2} \left(\frac{2}{3} p_F^3 + \frac{1}{4} k^3 + \dots \right) k \right\}.$$

The part of contribution linear in k from the paramagnetic current just cancels that from the London diamagnetic current because

$$n = \frac{\Omega}{(2\pi)^3} \frac{4\pi}{3} p_F^3$$

and thus the remainder is of order k^4 , as required for the gauge-invariance of the susceptibility for homogeneous field limit, *i.e.* $k \rightarrow 0$.

Now we shall proceed to evaluate the susceptibility. First one has to know the matrix B^{-1} explicitly. Writing B as a sum of the diagonal kinetic energy part ω and of the potential energy part W , which is proportional to e^2 , one can get B^{-1} in a series form:

$$B^{-1} = (\omega + W)^{-1} = \omega^{-1} (1 + W\omega^{-1})^{-1} = \omega^{-1} - \omega^{-1} W \omega^{-1} + \dots \quad (3.13)^*$$

* If one writes for B^{-1}

$$B^{-1} = \omega^{-1} - \omega^{-1} \mathcal{Q} \omega^{-1},$$

in terms of an effective potential part \mathcal{Q} , then one easily sees that \mathcal{Q} should satisfy the equation $(1 + W\omega^{-1})\mathcal{Q} = W$.

We have not succeeded to solve this equation, nor even to work out the quantitative characteristics of $\mathcal{Q}_{pp'}$ when $p \rightarrow p'$. However, it can be argued that the solution $\mathcal{Q}_{pp'}$ may have lost its singularity, which the original $W_{pp'}$ had at $p = p'$. This is the weakest point in our calculation which gives a trace of ambiguity to the final result. Instead of going into this problem, we shall believe in the common statement that the screening of the Coulomb potential arises from virtual excitation of the plasma oscillation. See the next section.

Hence to the order e^2 ,

$$(B^{-1})_{pp'} = \frac{1}{\omega_{pk}} \delta_{pp'} - \frac{1}{\omega_{pk}} \sum_{p''}^{\leq p_F} (V_{p-p''} - V_{p+k+p''}) \delta_{pp''} + \frac{1}{\omega_{pk} \omega_{p'k}} (V_{p-p'} - V_{p+k+p'}) . \quad (3.14)$$

With this approximation of B^{-1} one can readily go into the formula (3.10) and get

$$K(k) = - \frac{e^2}{mc} n + \frac{2e^2}{m^2 c} \left\{ \sum_p^{(k)} \frac{p_x^2}{\omega_{pk}} + \frac{1}{2} \sum_p^{(k)} \sum_{p'}^{(k)} \left(\frac{p_x}{\omega_{pk}} - \frac{p'_x}{\omega_{p'k}} \right)^2 (V_{p+k+p'} - V_{p-p'}) \right\} . \quad (3.15)$$

Here we assumed that k is in the z -direction, and A , the vector potential, is in the x -direction.

Evaluation of this last sum is rather tedious, but very elementary. Assuming the $V_{p-p'}$ is of the form

$$V_{p-p'} = \frac{4\pi e^2}{\Omega} \frac{1}{|p-p'|^2 + \varepsilon^2} , \quad (3.16)$$

where we have introduced a parameter for the screening effect for the sake of convenience for the following discussions, we shall just write down the result:

$$\frac{1}{2} \sum_p^{(k)} \sum_{p'}^{(k)} \left(\frac{p_x}{\omega_{pk}} - \frac{p'_x}{\omega_{p'k}} \right)^2 (V_{p+k+p'} - V_{p-p'}) = q^2 \frac{e^2 m^2 \Omega}{\pi^3} \frac{1}{18} \left\{ -2 + \ln 2 - \ln \frac{\varepsilon}{p_F} \right\} . \quad (3.17)$$

This expression for the susceptibility clearly diverges for $\varepsilon=0$, *i.e.* for the bare Coulomb interaction, so long as we assume the first approximation (3.15). Note that $V_{p+k+p'}$ in Eq. (3.15) makes no trouble with its bare Coulomb form. Thus for the main correction to the Landau diamagnetism, one waits for evaluation of the screening constant ε . We shall do this in the next section.

§ 4. The screening effect

The screening of the exchange interaction (3.16), which was found essential for the evaluation of the main correction to the Landau diamagnetism, will be tried in this section. As mentioned in connection with Eq. (3.13), there is probably another screening effect which has never been studied before, but we shall not go into this investigation. Instead of this, we shall show how the well-known polarization screening is incorporated in our formalism. The perturbation which gives rise to this effect is nothing but that three-exciton Hamiltonian, given as Eq. (3.8) in the paper I:

$$\begin{aligned} \mathcal{H}_3 = & \sum \sum (V_{P-p} - V_{P'-p}) \{ c_{p'}^{*P'+P-p} c_{p'}^{P'} c_p^P + c_p^{*P} c_{p'}^{*P'} c_{p'+p-P}^{P'+P-p} \} \\ & - \sum \sum (V_{P-p} - V_{P'-p'}) \{ c_{p'}^{*P'+p-P} c_{p'}^{P'} c_p^P + c_p^{*P} c_{p'}^{*P'} c_{p'+p-P}^{P'+P-p} \} . \end{aligned} \quad (4.1)$$

This is an especially satisfactory feature of our formalism, because this part is the next term to the main harmonic one (2.1) in our series expansion of our exciton Hamiltonian.

It was first shown by Bohm and Pines⁴⁾ that electrons are organized by the long range part of the Coulomb interactions to perform collective oscillations with long wavelength but high frequency so that long wavelength density fluctuations are strongly inhibited. This was found to mean, in particular, that the individual electrons, with correspondingly reduced number of degrees of freedom, are interacting via screened Coulomb forces with a range of the order of the inter-electronic distance.

Hubbard⁵⁾ regarded the electron gas as a dielectric medium and thought of the electrons as interacting with one another like particles in this medium, their interaction being therefore screened. From this viewpoint the plasma oscillations are thought of as being the polarization waves in the medium.

In order to get this polarization correction, we only have to eliminate the three-exciton Hamiltonian (4.1) by a canonical transformation $\exp(iS)$ with

$$iS = -i \lim_{\varepsilon \rightarrow 0} \int_0^{\infty} \exp(-\varepsilon t) \exp(i\mathcal{H}t) \mathcal{H}_3 \exp(-i\mathcal{H}t) dt. \quad (4.2)$$

Then the transformed Hamiltonian to the second order in \mathcal{H}_3 is given by $\mathcal{H}_{\text{eff}} = \mathcal{H} + (1/2)[iS, \mathcal{H}_3]$. This is to involve a modified coefficient $V_{p-p'}$ of the type (3.16). Its exact calculation, however, is very troublesome because it involves various minor processes other than the relevant polarization. Before clarifying what we have to do for retaining only this process, we shall analyze the structure of \mathcal{H}_3 .

If one drops the exchange coefficients, one easily sees that the three-exciton Hamiltonian (4.1) can be written in the form:

$$\mathcal{H}_3 = \sum_q V_q \mathcal{Q}_q \sum_p (c_{p,q} + c_{-p,-q}^*)$$

with

$$\mathcal{Q}_q = \sum_{p'} \sum_{q'} (c_{p',q'+q}^* - c_{p'-q,q'+q}^*) c_{p',q'} = \mathcal{Q}_{-q}^*. \quad (4.3)$$

As is apparent from the attached indices, \mathcal{Q}_q , here is an operator which causes scattering of electrons or holes with a momentum change q , while the other factor is the q -Fourier component of the density fluctuation.

Consider the motion of the operators due to the unperturbed Hamiltonian (2.1). While, as is well known, the density fluctuation of long wavelength, *i.e.* of small q , involves a high frequency (namely, the plasma frequency), the operator \mathcal{Q}_q changes with time comparatively slowly. This can be seen, for example, by calculating the time variation of \mathcal{Q}_q :

$$i \frac{d}{dt} \mathcal{Q}_q = [\mathcal{Q}_q, \mathcal{H}] = - \sum_{p'} \sum_{q'} \left\{ \left(\frac{p' \cdot q}{m} + \frac{q' \cdot q}{m} + \frac{q^2}{2m} \right) c_{p',q'+q}^* c_{p',q'} \right.$$

$$-\left(\frac{\mathbf{p}' \cdot \mathbf{q}}{m} - \frac{q^2}{2m}\right) c_{\mathbf{p}'-\mathbf{q}, \mathbf{q}'+\mathbf{q}}^* c_{\mathbf{p}', \mathbf{q}'} \Big\}. \quad (4.4)$$

If one assumes that q and q' be small in comparison with p' , which is of the order of the Fermi momentum, one can see that the frequencies involved are much smaller than the plasma frequency. On this basis we are allowed to eliminate the fluctuating density, treating the operators \mathcal{Q}_q constant with time.

Then, by writing the density fluctuation in terms of b -operators according to (2.13), one can readily perform the integration in Eq. (4.2) and thus gets

$$iS = - \sum_q V_q \mathcal{Q}_q \sqrt{2} \sum_p \sum_{\nu} \frac{1}{\nu} (\xi_{p,q}^{\nu} b_q^{\nu} - \xi_{p,q}^{\nu*} b_{-q}^{\nu*}). \quad (4.5)$$

This canonical transformation gives, with only the commutators pertaining to the density fluctuations retained in compliance with the above statement,

$$\frac{1}{2} [iS, \mathcal{H}_s] = - \sum_q V_q^2 \mathcal{Q}_q \mathcal{Q}_{-q} \sum_p \sum_{p'} \sum_{\nu} \frac{1}{\nu} (\xi_{p,q}^{\nu} \xi_{p',q}^{\nu*} + \xi_{p,q}^{\nu*} \xi_{p',q}^{\nu}). \quad (4.6)$$

Rearranging the operators $\mathcal{Q}_q \mathcal{Q}_{-q}$ into normal products and retaining binary terms in c -operators give

$$\mathcal{Q}_q \mathcal{Q}_{-q} = \sum_{p'} \sum_{q'} (I_{p', q'-q} + I_{p'+q, q'-q}) c_{p', q'}^* c_{p', q'} - \sum_{p'} \sum_{q'} c_{p'-q, q'}^* c_{p', q'} - \sum_{p'} \sum_{q'} c_{p'+q, q'}^* c_{p', q'}, \quad (4.7)$$

where $I_{p,q}$ is defined as follows:

$$I_{p,q} = \begin{cases} 1, & \text{if } |p| < p_F \text{ and } |p+q| > p_F \\ 0, & \text{otherwise.} \end{cases}$$

Now, if one neglects the exchange coefficients in the harmonic Hamiltonian (2.1), one can derive the following relation from the second equation of (2.7):

$$\sum_p \xi_{p,q}^{\nu} = \frac{\sum_p \frac{i\nu}{\omega_{p,q}} \eta_{p,q}^{\nu}}{1 + 2V_2 \sum_p \frac{1}{\omega_{p,q}}}. \quad (4.8)$$

Then the relation (2.14) gives

$$\sum_p \sum_{p'} \sum_{\nu} \frac{1}{\nu} (\xi_{p,q}^{\nu} \xi_{p',q}^{\nu*} + \xi_{p,q}^{\nu*} \xi_{p',q}^{\nu}) = \frac{\sum_p \frac{1}{\omega_{p,q}}}{1 + 2V_q \sum_p \frac{1}{\omega_{p,q}}}. \quad (4.9)$$

Taking this result, Eq. (4.7) with (4.9), into account, we finally obtain an effective Hamiltonian with the same form as (2.1), but with renormalized coefficients. These are all free from the singularity met with in the bare Coulomb interaction. Thus the $V_{p-p'}$ which is appearing explicitly in (2.3) is replaced by $\mathcal{V}_{p-p'}$, where the renormalized coefficient \mathcal{V}_q is given by

$$\mathcal{V}_q = \frac{V_q}{1 + 2V_q \sum_p \frac{1}{\omega_{p,q}}} = \frac{4\pi e^2}{\Omega} \frac{1}{q^2 + \frac{2}{\pi} e^2 m p_F} \quad (4.10)$$

which is of the same form as (3.16). The $\gamma_{p,q}$ is also modified to the extent

$$\gamma_{p,q} \rightarrow \frac{1}{2} \left\{ \sum_{p''}^{\geq p_F} \mathcal{V}_{p''-p-q} + \sum_{p''}^{\leq p_F} \mathcal{V}_{p''-p} - \sum_{p''}^{\leq p_F} V_{p''-p-q} - \sum_{p''}^{\geq p_F} V_{p''-p} \right\}.$$

As one may recognize, the resulting Hamiltonian does not strictly meet the gauge-invariance argument of Section 3. To dispose of this lack of invariance formally, one has also to replace the $V_{p'+q+p}$'s, those appearing explicitly and those involved in $\gamma_{p,q}$ as well, by renormalized ones of the type (4.10). This somewhat arbitrary replacement will be allowed because, to the present approximation, the matter with which we are concerned is the extent of cutoff of the singularity of $V_{p-p'}$ involved in Eq. (3.15). That is, the cutoff value of $V_{p+q+p'}$ does not affect the result within our approximation. Hence, we shall refrain from giving any detailed analysis of this point.

Now that we have got the appropriate value of the parameter ε , we can write down the expression for the susceptibility according to (3.11) and (3.17). The result may conveniently be written down as its ratio to the Landau susceptibility:

$$\frac{\chi}{\chi_0} = 1 - \frac{\alpha r_s}{6\pi} \left(-4 + \ln \frac{2\pi}{\alpha r_s} \right) \quad (4.11)$$

with

$$\alpha = (2/9\pi)^{1/3}$$

where χ_0 , the Landau susceptibility, is given by

$$\chi_0 = - \frac{1}{12\pi^2} \frac{1}{\alpha r_s} \frac{1}{m^2 c^2 a_B^2}.$$

This result may be compared with that of the collective description of electron interactions, mentioned in Pines' review article:

$$\frac{\chi}{\chi_0} = 1 - \frac{\alpha r_s}{6\pi} \left\{ -4 + \ln \frac{2\pi}{\alpha r_s} + \ln \frac{2\alpha r_s}{\pi (p_0/p_F)^2} \right\},$$

where p_0 is the cutoff momentum for plasma oscillation. Formally, therefore, our result corresponds to the choice

$$p_0/p_F = \sqrt{2\alpha/\pi} r_s^{1/2} \approx 0.518 r_s^{1/2}.$$

This should be compared with the value $0.353 r_s^{1/2}$, which was

* In this self-energy $\gamma_{p,q}$ we have thus taken into account only the plasmon exciting processes which is connected with the scattering of electrons or holes (Feynman diagram Fig. 1). This is evoked by the three-"exciton" interaction.

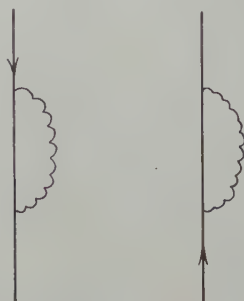


Fig. 1.

obtained by approximately minimizing the long-range correlation energy with respect to p_0 .

§ 5. Concluding remarks

Our primary result is Eq. (3.11) for $\langle j_k \rangle$, which is general enough so long as we limit ourselves to the harmonic approximation for the Hamiltonian (2.1). The point in this connection is that only the exchange coefficients come into the expression. However, we have not succeeded yet in evaluating the inverse matrix B^{-1} , although it can be well expected that this may give a finite expression for the susceptibility.

We left this unsolved and turned to base our discussion on the approximation (3.14) for B^{-1} . The ensuing integral expression (3.15) is, in point of fact, an average over the Fermi ground state of the operator expression for j_k which is obtained after the two canonical transformations to eliminate the magnetic energy term linear in A and then to eliminate also the resulting cross energy involving the Coulomb interaction and A .⁷⁾ This, however, gives a diverging result for the static susceptibility.

It is readily suggested by the expression (3.15) that we must take the screening effect into account to get the susceptibility finite. It was shown in Section 4 that this type of screening arises partly from the three-“exciton” interaction. The final result is (4.11) for the susceptibility, which corresponds to that of the collective description of electron interaction with the cutoff momentum somewhat larger than that of Pines. Formally, therefore, the result is just that approximation stated in the preceding paragraph but with the screening Coulombic interaction.

It should, however, be noted again that this result is based on the approximation (3.14) for B^{-1} . Hence it cannot be said anyway that (4.11) were final. We are intending to analyze the problem along the line mentioned in the footnote for (3.13).

Finally, we have not justified formally that replacement of $V_{p+q+p'}$ by the screened one on the present elementary approach. Before doing this, we shall have to solve that problem of B^{-1} , which may be possible to give a finite result within the simple gauge-invariant scheme of Section 3. This will settle the effectiveness of the simple “exciton” picture, because any need of more and more “exciton” interaction will invalidate the effectiveness.

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Pion-Pion Interaction and Pion Production in Pion-Nucleon Collision

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The evidence for the π - π interaction is investigated by considering the angular distribution of nucleon in pion production by pion-nucleon collision. It is shown that it will be quite difficult to understand the sharp forward angular distribution of nucleon without considering π - π interaction. We can also estimate the strength of the π - π interaction assuming the interaction Lagrangian density $\lambda(\phi_\alpha\phi_\alpha)^2$, and get the value $|\lambda|/\sqrt{4\pi} \sim 4$.

§ 1. Introduction

Since the π - π interaction was proposed by Kovacs¹⁾ to account for the angular correlation between final pions in pion production by pion-nucleon collision, many authors^{2),3)} have made their efforts to investigate this interaction. And it was pointed out by D. Ito et al.^{3),4)} that the strong π - π interaction proposed by Kovacs could reproduce almost all of the kinematical features of experiment except the branching ratio $R = (\pi^- + p \rightarrow \pi^+ + \pi^- + n) / (\pi^- + p \rightarrow \pi^- + \pi^0 + p)$.

However, no conclusive evidences for the existence of π - π interaction seem to have appeared up to the present. Recently, Chew⁵⁾ proposed a new method for the determination of the pion-nucleon coupling constant from nucleon-nucleon scattering data by taking advantage of unphysical pole resulting from one pion exchange diagram. Taylor,⁶⁾ in the same manner, considered the associated production of K -meson and hyperon in pion-nucleon collision and discussed the relative parity between charged and neutral K -mesons. Taylor et al.⁶⁾ also determined the pion-nucleon coupling constant from pion photoproduction data in the same way.

We consider, in this paper, the angular distribution of final nucleon in pion production by pion-nucleon collision,

$$\pi + N \rightarrow \pi + \pi + N, \quad (1)$$

in order to investigate the existence of π - π interaction (one pion exchange diagram) with use of the same method as what Chew^{5),6)} et al. have used.

In Section 2, the expression of angular distribution of nucleon for the process (1) is given. In Section 3 we compare this expression with experimental data and discuss the existence of π - π interaction. Also in Section 3 the strength of π - π interaction is estimated by assuming the interaction Lagrangian density $\lambda(\phi_\alpha\phi_\alpha)^2$.

§ 2. Angular distribution of nucleon

The most general form of the production amplitude for the process (1) may be written as

$$M(x, \cos \theta) = K(x) \left[\frac{g\lambda}{2pp'} \frac{\bar{u}(p') \gamma_5 u(p)}{\alpha_0 - \cos \theta} + T(x, \cos \theta) \right] \quad (2)$$

where

$$\alpha_0 \equiv \frac{1}{pp'} \left[p_0 p'_0 - m^2 + \frac{p^2}{2} \right],$$

$K(x)$ is a function depending on the energy of incident pion, and g, λ are parameters representing the strength of $NN\pi$ and of $\pi\pi$ interactions, respectively. Isotopic spin variables are not explicitly written. θ is the angle between incoming and outgoing nucleon in *c.m.s.*, x is other variables irrelevant in the present consideration, and p, p' are respectively the momentum of initial and final nucleons. The first term on the right-hand side of Eq. (2) comes from the $\pi\pi$ interaction and the second term represents the contribution from the ordinary Yukawa interaction. In Fig. 1, we show Feynman diagrams representing the process (1).

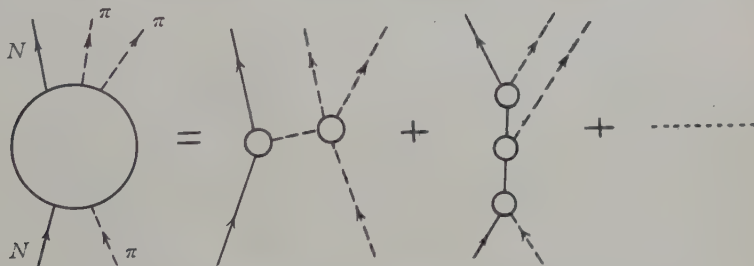


Fig. 1. Feynman diagrams representing the process (1)
Solid line and dashed line refer to nucleon and pion, respectively.
The first diagram represents pion production via $\pi\pi$ interaction
and above the second one represents pion production via ordinary
Yukawa interactions.

Lowest order perturbational calculation reveals us that the main term of $T(x, \cos \theta)$ has the following form:

$$T(x, \cos \theta) \propto \left[\frac{(kq_1)}{(pk)(p'q_1)} + \frac{(kq_2)}{(pk)(p'q_2)} + \frac{(q_1q_2)}{(pq_1)(p'q_2)} + (p \leftrightarrow p') \right] \bar{u}(p') \gamma_5 u(p) \quad (3)$$

where k, q_1, q_2 are momenta of incident and outgoing two pions. Therefore, the angular distribution of nucleon multiplied by $(\alpha_0 - \cos \theta)^2$ is written as follows:

$$\begin{aligned} (\alpha_0 - \cos \theta)^2 \frac{d\sigma}{d\Omega} = & g^2 \lambda^2 A(p, \langle p' \rangle) (\beta_0 - \cos \theta) + \lambda g^4 B(p, \langle p' \rangle) (\beta_0 - \cos \theta) (\alpha_0 - \cos \theta) \\ & + g^6 C(p, \langle p' \rangle) (\beta_0 - \cos \theta) (\alpha_0 - \cos \theta)^2 \end{aligned} \quad (4)$$

where $\beta_0 \equiv \alpha_0 - \mu^2/2pp'$, $\langle p' \rangle$ is the average momentum of final nucleon, and the integration over the variables of final pions is already carried out. Strictly speaking, $B(p, \langle p' \rangle)$ and $C(p, \langle p' \rangle)$ also depend on $\cos \theta$, but as their angular dependence would not be so large we neglect it hereafter.

According to the experimental results at 1.4 Bev pion lab. kinetic energy, obtained by Eisberg et al.,⁷⁾ the momentum distribution of final nucleon is localized fairly well around the average momentum ~ 0.5 Bev/ c in *c.m.s.* By making use of the average momentum $\langle p' \rangle = 0.51$ Bev/ c , we obtain $\alpha_0 = 1.07$ and $\beta_0 = 1.04$.

The second and third terms of the right-hand side of Eq. (4) are vanishingly small near the unphysical pole $\alpha_0 = \cos \theta$ of the angular distribution of nucleon. And if the $\pi\pi$ interaction really exists, then the first term of Eq. (4) gives the dominant contribution in this region and $(\alpha_0 - \cos \theta)^2 d\sigma/d\Omega$ should be linear with respect to $\cos \theta$.

§ 3. Comparison with experiment and discussions

The most available experiment on pion production in pion-nucleon collision at present is the one at 1.4 Bev pion lab. kinetic energy performed by Eisberg et al.⁷⁾ In Fig. 2, we show the angular distribution of nucleon in *c.m.s.* from their

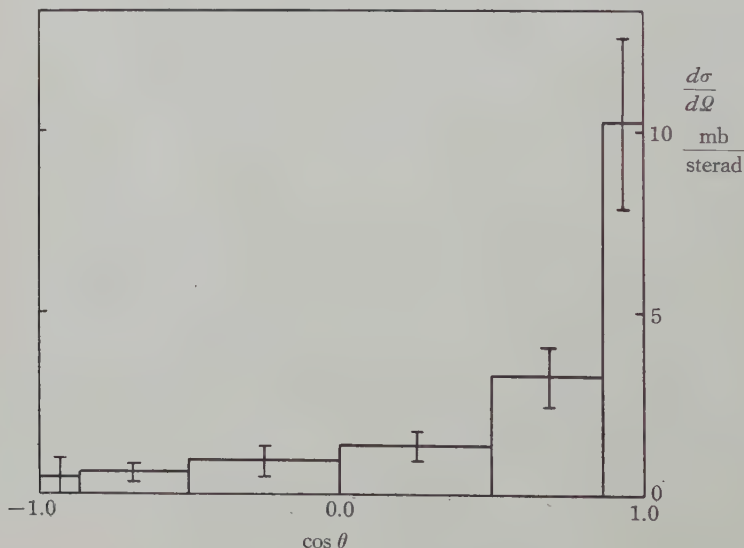


Fig. 2. Center-of-mass differential angular distribution of nucleon⁷⁾ in the process (1)

experiment. It seems to have a very sharp peak in the forward region. In Fig. 3, we show $(\alpha_0 - \cos \theta)^2 (d\sigma/d\Omega)_{\text{exp.}}$ deduced from Fig. 2, which seems to prefer the expected linear dependence on $\cos \theta$.

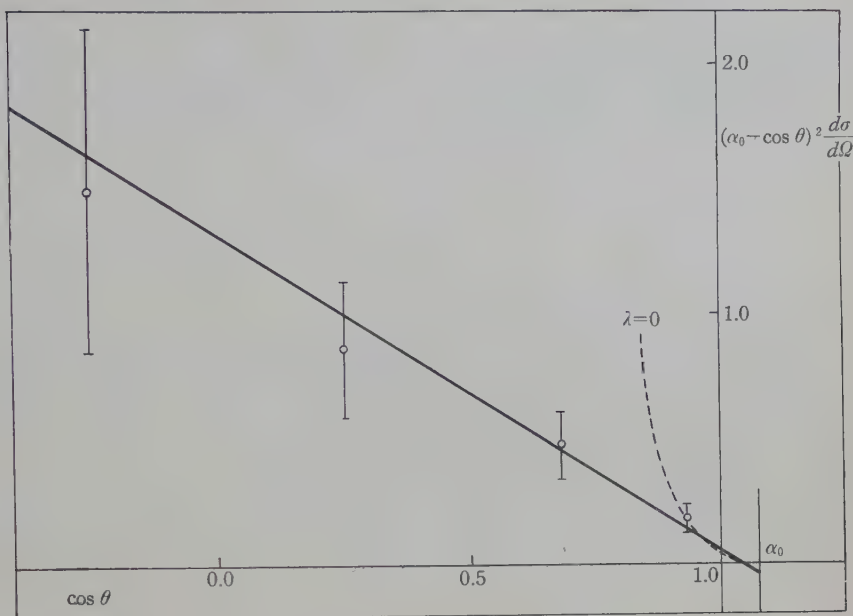


Fig. 3. $(\alpha_0 - \cos \theta)^2 d\sigma/d\Omega$ as function of $\cos \theta$
The solid line refers to $A(\beta_0 - \cos \theta)$ and dashed curve refers to
"no π - π interactions".

In fact, if there were no π - π interactions, $\lambda=0$, the first and second terms in Eq. (4) would, of course, vanish and only the third term would survive. Then $(\alpha_0 - \cos \theta)^2 \cdot d\sigma/d\Omega$ will show cubic dependence on $\cos \theta$ passing through the abscissa at $\cos \theta = \beta_0$ and making quadratic contact at $\cos \theta = \alpha_0$. Anyhow, one can hardly draw such curves as covering all experimental data, so the dotted curve in Fig. 3 shows one example of cubic function in the forward region. Even if we take into account the angular dependence of $C(p, \langle p' \rangle)$, we shall not be able to draw a curve fitting for all experimental data in the forward region, for perturbational calculations based on the ordinary Yukawa interaction reveal only the backward preference of nucleon.⁴⁾ We conjecture that the qualitative situation will not be largely altered even if higher order corrections are introduced.

At present, it may be quite premature to give any definite conclusions on account of the experimental uncertainties. However, the above analysis clearly shows that it will be quite difficult to understand the very sharp forward angular distribution of nucleon without considering the π - π interaction.

More reliable and detailed experiments of the process (1) in these energy regions are now in progress at Berkeley, so we expect to be able to obtain conclusive evidences for the existence of the π - π interaction in the near future. By knowing the value of the left-hand side of Eq. (4) at $\cos \theta = \alpha_0$, we may estimate the strength of π - π interaction coupling constant. Although we have ambiguity due to the fact $1 < \beta_0 = 1.04 < \alpha_0 = 1.07$, we get the value $|\lambda|/\sqrt{4\pi} \sim 4$, assuming the inter-

action Lagrangian density $\lambda(\phi_\alpha\phi_\alpha)^2$ and $g^2/4\pi=15$. This is not inconsistent with the values previously reported.

The author wishes to express his sincere thanks to Professor D. Ito for his kind guidance and encouragement and to Professor H. Tanaka for valuable discussions.

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On Multipole Model of Baryon-Pion Interactions

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The multipole model of baryon-pion interactions proposed by one of the present authors is developed in such a way that the quantitative comparison with experimental results is possible. In this model, we assume that the baryons correspond to the definite internal states of a non-local entity, and the strong and the weak interactions of baryons and pion fields are regarded as their monopole and dipole interactions, respectively. By assuming the internal wave functions of the extended baryons to be one to one admixture of the symmetric and the antisymmetric parts with respect to the simultaneous reflections both in the isobaric and the Minkowski spaces, we are able to show that the weak decay interactions derived from our model are equivalent to the $|\Delta I|=1/2$ global symmetric interactions, which are recently deduced by d'Espagnat and Prentki through their phenomenological considerations.

§ 1. Introduction

In the previous two papers, one of the present authors (D. I.) has proposed the multipole model of elementary particles.¹⁾ In this model, we have assumed that the baryons are different states of a structured particle, and the strong and the weak interactions of the baryons and pion fields are regarded as the monopole and the dipole interactions between the pion field and the non-local particle. On the basis of these assumptions, we are able to get a unified insight into the reason why the weak interactions are "weak", why they are of vector (—axial vector) type, and why the parity violation does occur only in decay interactions, and so on.

In the previous papers, however, the discussions were so qualitative that we could not hope to get a complete reproduction of the experimental results for weak interactions in all their quantitative details, especially in the selection rules in the charge space.

It is the purpose of the present note to develop the model in more detail, and to deduce the interaction Lagrangian which is able to reproduce the precise values of the charge branching ratios and the asymmetry parameters of pion-angular distributions in hyperon decays.

§ 2. Multipole expansion

In our model, the baryon-pion interactions are described by an interaction Lagrangian of a non-local baryon field $\psi(x, z)$ and the pion field $\phi_i(x)$ with the following form:

$$-\mathcal{L}_{int} = g \sum_{i=1}^3 \int d^3x \int d^4z \bar{\psi}(x, z) T_i \psi(x, z) \cdot \phi_i(x+z), \quad (1)$$

where g is a coupling constant, and T_i is a vector operator in the charge space operating on the internal wave function of the extended baryon, with the center of mass at x_μ , which interacts with the pion field at the point $x_\mu + z_\mu$. When the wavelength of the pion field is longer than the extension of the baryon which is assumed to be of the order of 10^{-10} cm, the interaction (1) may be expanded into multipole terms,

$$-\mathcal{L}_{int} = g \sum_i \int d^3x \int d^4z \bar{\psi}(x, z) T_i \psi(x, z) \left[\phi_i(x) + z_\mu \frac{\partial \phi_i(x)}{\partial x_\mu} + \dots \right]. \quad (2)$$

In our model, the strong interactions are described by the monopole term, while the weak interactions are described by the dipole term in the above expansion.

Now, let us assume that the baryons are all spin 1/2 particles and correspond to the definite internal states $u_{n\alpha}(z)^{A\dot{B}}$ of our extended particle, where n specifies the kind of baryons, α is an ordinary Dirac spinor index and A and B are spinor indices in the charge space. We will assume, according to d'Espagnat et al.,²⁾ that the charge space is a four-dimensional Euclidian space, and charge states are characterized as the eigenstates of \mathbf{I}^+ and \mathbf{I}^- . The spinor index A is related to the representation of \mathbf{I}^+ , while \dot{B} is related to that of \mathbf{I}^- . (The dot has no principal importance here.) Now, we expand $\psi(x, z)$ into its internal eigenstates,

$$\psi(x, z) = \sum_{n\beta A\dot{B}} B_{n\beta}(x)_{A\dot{B}} u_{n\beta}(z)^{A\dot{B}}, \quad (3)$$

$$\bar{\psi}(x, z) = \sum_{n\beta A\dot{B}} \bar{B}_{n\beta}(x)_{A\dot{B}} \bar{u}_{n\beta}(z)^{A\dot{B}}.$$

The coefficients $B(x)$'s describe the center-of-mass motion and may be related to usual baryon fields by

$$(B_{1,\beta}(x)_{A\dot{B}}) \equiv \begin{bmatrix} \bar{\Sigma}^0(x) & p(x) \\ \bar{\Sigma}^-(x) & n(x) \end{bmatrix}_{\beta} \equiv (B_j^k)_{\beta}, \quad (4.1)$$

$$(B_{2,\beta}(x)_{A\dot{B}}) \equiv \begin{bmatrix} Z^0(x) & \Sigma^+(x) \\ \bar{\Sigma}^-(x) & Y^0(x) \end{bmatrix}_{\beta} \equiv (\Sigma_j^k)_{\beta}. \quad (4.2)$$

By introducing (3) into (2), we have

$$\begin{aligned} -\mathcal{L}_{int} &= \int d^3x \left[g \cdot \bar{B}_{m\alpha}(x)^{A\dot{B}} \left(\int d^4z \bar{u}_{m\alpha}(z)_{A\dot{B}} T_i u_{n\beta}(z)^{c\dot{D}} \right) \times B_{n\beta}(x)_{c\dot{D}} \phi_i(x) \right. \\ &\quad \left. + g \cdot \bar{B}_{m\alpha}(x)^{A\dot{B}} \left(\int d^4z \bar{u}_{m\alpha}(z)_{A\dot{B}} T_i z_\mu \times u_{n\beta}(z)^{c\dot{D}} \right) B_{n\beta}(x)_{c\dot{D}} \frac{\partial \phi_i(x)}{\partial x_\mu} + \dots \right] \\ &= -\mathcal{L}_{S.I.} - \mathcal{L}_{W.I.} \end{aligned} \quad (5)$$

Note that we take $\overline{B}_{m\alpha}(x)^{A\dot{B}} \equiv B_{m\beta}^+(x)_{A\dot{B}} \gamma_{4,\beta\alpha}$.

As was discussed in the previous papers, the parity conservation and the charge independence in the strong interactions are guaranteed, if the internal wave functions satisfy the following orthonormal relations:

$$\int d^4z \bar{u}_{m\alpha}(z)_{A\dot{B}} T_i u_{n\beta}(z)^{C\dot{D}} = \delta_{mn}^* \cdot \partial_{\dot{B}}^{\dot{D}} (\gamma_5)_{\alpha\beta} (\tau_i)_A^C. \quad (6)$$

Then, the first term in (5) turns out to be the well-known global symmetric form of the strong interactions:

$$-\mathcal{L}_{S.I.} = g \int d^3x \overline{B}_{m,}(x)^{A\dot{B}} \gamma_5(\tau_i)_A^C B_{m,}(x)_{C\dot{B}} \phi_i(x). \quad (7)$$

§ 3. Reflection properties of internal structure

In the previous paper, we have simply assumed that the internal density distribution $\bar{u}_\alpha(z)u_\beta(z)$ is not identical with its mirror image, just as in the case of the electron density distributions in optically active molecules.¹⁾ On account of this assumption, the dipole moment of the internal density distribution turns out to be a linear combination of the vector and the pseudovector parts:

$$(1/l) \int d^4z \bar{u}_\alpha(z) z_\mu u_\beta(z) \approx a(\dot{\gamma}_\mu + r \cdot \gamma_\mu \gamma_5)_{\alpha\beta}, \quad (8)$$

(where $l \approx 10^{-20}$ cm), and the parity violation is interpreted as an effect due to the optically active structure of elementary particles. Now we shall supplement these assumptions by considering the reflection properties in the charge space, and assume that *the internal density distributions are not symmetrical with respect to simultaneous inversion of the internal space coordinates z and the charge coordinates.*

The inversion in the charge space is described by so-called "antilinear geometric transformations of charge spinors"³⁾

$$\phi_A \rightarrow \phi_A' = s_A^{\dot{B}} \phi_{\dot{B}}, \quad \phi^{\dot{A}} \rightarrow \phi^{\dot{A}'} = \bar{s}_B^{\dot{A}} \phi^{\dot{B}}, \quad (9.1)$$

$$\phi_{\dot{A}} \rightarrow \phi_{\dot{A}'} = \bar{s}_A^{\dot{B}} \phi_{\dot{B}}, \quad \phi^{\dot{A}} \rightarrow \phi^{\dot{A}'} = s_B^{\dot{A}} \phi^{\dot{B}},$$

$$\bar{s}_C^{\dot{B}} s_A^{\dot{C}} = \delta_A^{\dot{B}}, \quad s_C^{\dot{B}} \bar{s}_A^{\dot{C}} = \delta_A^{\dot{B}}. \quad (9.2)$$

By making use of these rules of transformations, we split the internal density matrices into the symmetric and antisymmetric parts with respect to the simultaneous inversion. Then, the second term in (5), $\mathcal{L}_{W.I.}$, is written as

$$-\mathcal{L}_{W.I.} = \int d^3x g \overline{B}_{m\alpha}(x)^{A\dot{B}} B_{n\beta}(x)_{C\dot{D}} \frac{\partial \phi_i(x)}{\partial x_\mu}$$

$$\times \left[\frac{1}{2} \int dz \{ \bar{u}_{m\alpha}(z)_{A\dot{B}} T_i z_\mu u_{n\beta}(z)^{C\dot{D}} - s_A^{\dot{B}} \bar{s}_B^{\dot{A}} \gamma_{4,\alpha\alpha'} \right]$$

$$\begin{aligned}
& \times \bar{u}_{m\alpha'}(-\mathbf{z}, z_0)_{F\dot{E}} T_i z_\mu u_{n\beta'}(-\mathbf{z}, z_0)^{H\dot{G}} \gamma_{4,\beta'\beta} \bar{S}_G^{\dot{C}} S_H^{\dot{D}} \} \\
& + \frac{1}{2} \int d\mathbf{z} \{ \bar{u}_{m\alpha}(z)_{A\dot{B}} T_i z_\mu u_{n\beta}(z)^{C\dot{D}} + s_A^{\dot{E}} \bar{s}_B^{\dot{F}} \gamma_{4,\alpha\alpha'} \\
& \times \bar{u}_{m\alpha'}(-\mathbf{z}, z_0)_{F\dot{E}} T_i z_\mu u_{n\beta'}(-\mathbf{z}, z_0)^{H\dot{G}} \gamma_{4,\beta'\beta} \bar{S}_G^{\dot{C}} S_H^{\dot{D}} \} \Big] \\
& \equiv -\mathcal{L}_{W.I.}^{(A)} - \mathcal{L}_{W.I.}^{(S)}. \quad (10)
\end{aligned}$$

It is convenient for later purposes to rewrite the antisymmetric part $-\mathcal{L}_{W.I.}^{(A)}$ of (10) in the following way,

$$\begin{aligned}
-\mathcal{L}_{W.I.}^{(A)} &= g \int d^3x \bar{B}_{m\rho}(x)^{A\dot{B}} \gamma_{4,\rho\alpha} \frac{\partial \phi_i(x)}{\partial x_\mu} \frac{1}{2} \int d\mathbf{z} \{ \gamma_{4,\alpha\alpha'} \\
& \times \bar{u}_{m\alpha'}(z)_{A\dot{B}} T_i z_\mu u_{n\beta'}(z)^{C\dot{D}} \gamma_{4,\beta'\beta} - s_A^{\dot{E}} \bar{s}_B^{\dot{F}} \bar{u}_{m\alpha}(-\mathbf{z}, z_0)_{F\dot{E}} \\
& \times T_i z_\mu u_{n\beta}(-\mathbf{z}, z_0)^{H\dot{G}} \bar{S}_G^{\dot{C}} S_H^{\dot{D}} \} \gamma_{4,\beta\sigma} B_{n\sigma}(x)^{C\dot{D}} \\
& = -g \int d^3x \bar{B}_{m\alpha}(-\mathbf{x}, x_0)^{A\dot{B}} \frac{\partial \phi_i(-\mathbf{x}, x_0)}{\partial x_\mu} \frac{1}{2} \int d\mathbf{z} \{ \gamma_{4,\alpha\alpha'} \\
& \times \bar{u}_{m\beta'}(-\mathbf{z}, z_0)_{A\dot{B}} T_i g_{\mu\nu} z_\nu u_{n\beta'}(-\mathbf{z}, z_0)^{C\dot{D}} \gamma_{4,\beta'\beta} - s_A^{\dot{E}} \bar{s}_B^{\dot{F}} \\
& \times \bar{u}_{m\alpha}(z)_{F\dot{E}} T_i g_{\mu\nu} z_\nu u_{n\beta}(z)^{H\dot{G}} \bar{S}_G^{\dot{C}} S_H^{\dot{D}} \} B_{n\beta}(-\mathbf{x}, x_0)^{C\dot{D}} \\
& = -g \int d^3x \bar{B}_{m\alpha}(x)^{A\dot{B}} \frac{\partial \phi_i(x)}{\partial x_\mu} \frac{1}{2} \int d\mathbf{z} \{ \gamma_{4,\alpha\alpha'} \bar{u}_{m\alpha'}(-\mathbf{z}, z_0)_{A\dot{B}} \\
& \times T_i z_\mu u_{n\beta'}(-\mathbf{z}, z_0)^{C\dot{D}} \gamma_{4,\beta'\beta} - s_A^{\dot{E}} \bar{s}_B^{\dot{F}} \bar{u}_{m\alpha}(z)_{F\dot{E}} T_i z_\mu \\
& \times u_{n\beta}(z)^{H\dot{G}} \bar{S}_G^{\dot{C}} S_H^{\dot{D}} \} B_{n\beta}(x)^{C\dot{D}}, \quad (11)
\end{aligned}$$

where

$$(g_{\mu\nu}) \equiv \begin{bmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & +1 \end{bmatrix}. \quad (12)$$

By making use of (11), Eq. (10) is rewritten as

$$\begin{aligned}
-\mathcal{L}_{W.I.}^{(A)} &= \int d^3x g \bar{B}_{m\alpha}(x)^{A\dot{B}} \frac{1}{2} \int d\mathbf{z} \{ \bar{u}_{m\alpha}(z)_{A\dot{B}} T_i z_\mu u_{n\beta}(z)^{C\dot{D}} \\
& - \gamma_{4,\alpha\alpha'} \bar{u}_{m\alpha'}(-\mathbf{z}, z_0)_{A\dot{B}} T_i z_\mu u_{n\beta'}(-\mathbf{z}, z_0)^{C\dot{D}} \gamma_{4,\beta'\beta} \} B_{n\beta}(x)^{C\dot{D}} \frac{\partial \phi_i(x)}{\partial x_\mu} \\
& + \int d^3x g \bar{B}_{m\alpha}(x)^{A\dot{B}} s_A^{\dot{E}} \bar{s}_B^{\dot{F}} \frac{1}{2} \int d\mathbf{z} \{ \bar{u}_{m\alpha}(z)_{F\dot{E}} T_i z_\mu \\
& \times u_{n\beta}(z)^{H\dot{G}} + \gamma_{4,\alpha\alpha'} \bar{u}_{m\alpha'}(-\mathbf{z}, z_0)_{F\dot{E}} T_i z_\mu u_{n\beta'}(-\mathbf{z}, z_0)^{H\dot{G}} \\
& \times \gamma_{4,\beta'\beta} \} \bar{S}_G^{\dot{C}} S_H^{\dot{D}} B_{n\beta}(x)^{C\dot{D}} \frac{\partial \phi_i(x)}{\partial x_\mu}. \quad (13)
\end{aligned}$$

Now, we shall examine the reflection properties of the quantities

$$\bar{B}_{m\alpha}(\dot{x})^{A\dot{B}}\Gamma^{\mp}(m, n; i; \mu)_{\alpha\beta; A\dot{B}}{}^{C\dot{D}}B_{n\beta}(x)_{C\dot{D}}, \quad (14)$$

appearing in Eq. (13), where we have put

$$\begin{aligned} \Gamma^{\mp}(m, n; i; \mu)_{\alpha\beta; A\dot{B}}{}^{C\dot{D}} &\equiv \frac{1}{2l} \int dz \{ \bar{u}_{m\alpha}(z)_{A\dot{B}} T_i z_{\mu} u_{n\beta}(z)^{C\dot{D}} \\ &\quad \mp \gamma_{4, \alpha\alpha'} \bar{u}_{m\alpha'}(-\mathbf{z}, z_0)_{A\dot{B}} T_i z_{\mu} u_{n\beta'}(-\mathbf{z}, z_0)^{C\dot{D}} \gamma_{4, \beta'\beta} \}. \end{aligned} \quad (15)$$

By the space reflection $\mathbf{x} \rightarrow -\mathbf{x}$, (14) is transformed into

$$\begin{aligned} \bar{B}_{m\alpha}(x)^{A\dot{B}} \gamma_{4, \alpha\alpha'} \Gamma^{\mp}(m, n; i; \mu)_{\alpha'\beta'; A\dot{B}}{}^{C\dot{D}} \gamma_{4, \beta'\beta} B_{n\beta}(x)_{C\dot{D}} \\ = \mp g_{\mu\lambda} \bar{B}_{m\alpha}(x)^{A\dot{B}} \Gamma^{\mp}(m, n; i; \lambda)_{\alpha\beta; A\dot{B}}{}^{C\dot{D}} B_{n\beta}(x)_{C\dot{D}}. \end{aligned}$$

It is seen that the operators Γ^{\mp} have the same reflection properties as those of the operators $\gamma_5 \gamma_{\mu}$ and γ_{μ} :

$$\begin{aligned} \gamma_5 \gamma_{\mu} &\rightarrow -g_{\mu\nu} \gamma_5 \gamma_{\nu}, \\ \gamma_{\mu} &\rightarrow g_{\mu\nu} \gamma_{\nu}. \end{aligned} \quad (16)$$

Therefore, we may put

$$\Gamma^{-} = \{ \rho_{mn}(\tau_i)_{A\dot{B}}{}^C \delta_{\dot{B}}^{\dot{D}} + \rho'_{mn}(\tau_i^T)_{\dot{B}}{}^{\dot{D}} \delta_A^C \} (\gamma_5 \gamma_{\mu})_{\alpha\beta}. \quad (17)$$

$$\Gamma^{+} = \{ \rho \cdot \rho_{mn}(\tau_i)_{A\dot{B}}{}^C \delta_{\dot{B}}^{\dot{D}} + \rho' \cdot \rho'_{mn}(\tau_i^T)_{\dot{B}}{}^{\dot{D}} \delta_A^C \} (\gamma_{\mu})_{\alpha\beta}. \quad (18)$$

Taking into account the relation

$$s_A^{\dot{B}}(\tau_i)_{\dot{B}}{}^D S_D^{\dot{C}} = -(\tau_i^T)_A{}^C,$$

and inserting (17) and (18) into (13), we can get the following expression:

$$\begin{aligned} -\mathcal{L}_{W.I.} &= gl \int d^3x \left[\rho_{mn} \{ \bar{B}_{m\alpha}(x)^{A\dot{B}} (\gamma_5 \gamma_{\mu})_{\alpha\beta} (\tau_i)_A{}^C B_{n\beta}(x)_{C\dot{B}} \right. \\ &\quad - \rho \bar{B}_{m\alpha}(x)^{A\dot{B}} (\gamma_{\mu})_{\alpha\beta} (\tau_i^T)_{\dot{B}}{}^{\dot{C}} B_{n\beta}(x)_{A\dot{C}} \} \frac{\partial \phi_i(x)}{\partial x_{\mu}} \\ &\quad + \rho'_{mn} \{ \bar{B}_{m\alpha}(x)^{A\dot{B}} (\gamma_5 \gamma_{\mu})_{\alpha\beta} (\tau_i^T)_{\dot{B}}{}^{\dot{C}} B_{n\beta}(x)_{A\dot{C}} \\ &\quad \left. - \rho' \bar{B}_{m\alpha}(x)^{A\dot{B}} (\gamma_{\mu})_{\alpha\beta} (\tau_i)_A{}^C B_{n\beta}(x)_{C\dot{B}} \} \frac{\partial \phi_i(x)}{\partial x_{\mu}} \right]. \end{aligned} \quad (19)$$

In our model, the strangeness selection rule of weak interactions is described by the internal selection rules $\int \bar{u}_m z_{\mu} u_n dz \sim \delta_{m, n \mp 1}$, as was mentioned in the previous papers. Further, if we choose $\rho'_{mn} = 0$ and $\rho^2 = 1$, the expression $\mathcal{L}_{W.I.}$ obtained above reduces to the $|AI| = 1/2$ global symmetric interactions given by d'Espagnat and Prentki⁽⁴⁾:

$$\begin{aligned} -\mathcal{L}_{W.I.} &= f \int d^3x \left\{ \bar{\Sigma}_k^j(x) \gamma_5 \gamma_{\mu} \tau_{kn} B_n^j(x) \frac{\partial \phi(x)}{\partial x_{\mu}} \right. \\ &\quad \left. - \rho \cdot \bar{\Sigma}_k^j(x) \gamma_{\mu} \tau_{jn}^T B_k^n(x) \frac{\partial \phi(x)}{\partial x_{\mu}} \right\} + H. C., \end{aligned} \quad (20)$$

where we have put $f \equiv gl \cdot \rho_{21}$. The explicit forms corresponding to the decays of Σ^\pm and Λ particles are written down as

$$\begin{aligned} -\mathcal{L}_{W.I.}(\Sigma^\pm \rightarrow N\pi) &= f[\sqrt{2}\bar{\Sigma}^+(\gamma_5 + \rho)\gamma_\mu p \cdot \partial_\mu \pi^0 \\ &\quad + \sqrt{2}\bar{\Sigma}^\pm \gamma_5 \gamma_\mu n \cdot \partial_\mu \pi^\pm - \rho\sqrt{2}\bar{\Sigma}^\pm \gamma_\mu n \cdot \partial_\mu \pi^\mp] + H.C., \\ -\mathcal{L}_{W.I.}(\Lambda^0 \rightarrow N\pi) &= f[\bar{\Lambda}(\gamma_5 - \rho)\gamma_\mu p \cdot \partial_\mu \pi^- \\ &\quad - (1/\sqrt{2})\bar{\Lambda}(\gamma_5 - \rho)\gamma_\mu n \cdot \partial_\mu \pi^0] + H.C. \end{aligned}$$

As far as the effect of renormalization due to the strong interaction is small, it is clear that the consistent results with experimental facts are obtained (see ref. 4)) from these Lagrangians.

§ 4. Final remarks

On the basis of the multipole model proposed by one of the authors, we have shown that we are able to conclude that the Lagrangians correctly describing the experimental results on baryons and pion fields are derived from the assumed properties of the internal structure of the non-local field, especially from the properties with respect to the simultaneous inversion of the ordinary and the charge spaces. It is, of course, not desirable to make assumptions on such an unverified small region, and it is very much expected to reach the same results without any reference to such a small region. In spite of these objections, however, it seems to be very suggestive that the parity non-conservation and the $| \Delta I | = 1/2$ selection rule are intimately connected with the combined inversion of the charge and the Minkowski spaces in such a small region.

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On the Non-local Boundary Condition in Quantum Field Theory

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The non-local boundary condition introduced by Bogolyubov et al. is useful for eliminating the difficulty of negative probability concerning the indefinite metric in Hilbert space. Their theory is developed in the S -matrix formalism and does not give a causal description of the physical state. A causal description is given here in the form of an ordinary Schrödinger equation. The Hamiltonian is not hermitian in general, though it gives the unitary S -matrix. Such a characteristic situation is discussed by using a simple model.

§ 1. Introduction and summary

An indefinite metric in Hilbert space was first introduced by Dirac,¹⁾ and has since been discussed by various authors. Especially, Heisenberg²⁾ emphasized that an indefinite metric would play an important role in constructing the convergent field theory in connection with the quantization of the non-linear wave equation. It seems to be interesting to use an indefinite metric in further development of field theory; however, the well-known difficulty of negative probability associated with an indefinite metric is very cumbersome and has not yet been solved satisfactorily.

Recently Bogolyubov et al.³⁾ pointed out that a non-local boundary condition is available to overcome this difficulty without concerning any specific type of interaction. In their theory, the Hilbert space of a whole system is composed of two parts, Hilbert space I and II in the same way as was done by Heisenberg, and a boundary condition between the state vectors at infinite past and future is imposed, in order to make the S -matrix for Hilbert space I unitary. Essentially, this is an S -matrix formalism proposed by Heisenberg, and its relation to the Hamilton formalism is not clear, except that the Hamiltonian in the total Hilbert space is given in the same form as the usual theory. The success of constructing the unitary S -matrix for the physical state comes from that the unitary condition on the transformation function for the finite time is replaced by a weaker one, i.e., a non-local boundary condition.

This method may be understood as a phenomenological approach to the more fundamental theory, by which this condition may automatically be satisfied. From this viewpoint, some considerations are given in the present paper to develop the theory related more closely to the ordinary formalism. In § 2, a Schrödinger equation is given, which describes the causal development of the state vector in

Hilbert space I and is free from the difficulty of negative probability. The non-local boundary condition has been taken into account in a somewhat similar way as was done by Bogolyubov in the case of classical fields. But in our case, the Hamiltonian is given explicitly only in a form of the power series expansion. Moreover, it is not only non-local, but also non-hermitian generally as to be expected from our *weaker* condition. Then there arises a problem to what extent this formalism could be accepted as a permissible and natural extension of the current field theory. This problem is discussed in § 3 by using a simple model of the interaction.

§ 2. The Schrödinger equation in Hilbert space I

We assume that the state vector ϕ in the total Hilbert space is composed of two parts ϕ_1 and ϕ_2 , which belong to Hilbert space I and II, respectively,

$$\phi = \phi_1 + \phi_2. \quad (2.1)$$

The Schrödinger equation for ϕ is given in the interaction representation as

$$i \frac{d\phi(t)}{dt} = H(t) \phi(t), \quad (2.2)$$

where $H(t)$ is an interaction Hamiltonian which comprises creation and annihilation operators of various particles. We assume that $H(t)$ is hermitian.*

We shall distinguish two kinds of particles, one of which corresponds to the physically observed particles, while the other corresponds to what can never be observed but appear only in "intermediate states", the latter being called internal particles (fields) in this paper. In the representation in which the particle number is diagonal, states containing an arbitrary number of internal particles construct ϕ_2 , and the others ϕ_1 . The non-local boundary condition is stated as

$$\phi_2(+\infty) + \phi_2(-\infty) = 0. \quad (2.3)$$

We shall explicitly write down the integral equation which follows from (2.2) and satisfies the condition (2.3). Using a projection operator P defined by

$$\phi_1 = P\phi, \quad \phi_2 = (1-P)\phi,$$

one obtains two equations by operating P and $(1-P)$ on (2.2),

$$i \frac{d\phi_1(t)}{dt} = PH(t)[\phi_1(t) + \phi_2(t)], \quad (2.4)$$

$$i \frac{d\phi_2(t)}{dt} = (1-P)H(t)[\phi_1(t) + \phi_2(t)]. \quad (2.5)$$

* The terminology "hermitian" for the total Hilbert space is used here in accordance with the definition given by Gupta.⁴⁾

Let us define the function $\eta^{t_0}(t_1, t)$ by

$$\eta^{t_0}(t_1, t) = \frac{1}{2}[\epsilon(t_1 - t) - \epsilon(t_0 - t)], \quad (2.6)$$

where $\epsilon(t_1 - t)$ equals to $+1$ or -1 according as $t_1 > t$ or $t_1 < t$, respectively. It follows from (2.6) that

$$\eta^{t_0}(t_1, t) = -\eta^{t_1}(t_0, t), \quad \eta^{t_0}(t_0, t) = 0.$$

Then, (2.4) may be integrated as

$$\phi_1(t_1) = \phi_1(t_0) + \frac{1}{i} \int \eta^{t_0}(t_1, t) PH(t) [\phi_1(t) + \phi_2(t)] dt, \quad (2.7)$$

where the time t_1 may be either past or future to t_0 . The condition (2.3) is applied to (2.5). The integral equation which satisfies (2.3) and (2.5) is, however, not uniquely determined as will be discussed at the end of this section and in the Appendix. A typical form may be written as

$$\phi_2(t_1) = \frac{1}{2i} \int \epsilon(t_1 - t) (1 - P) H(t) [\phi_1(t) + \phi_2(t)] dt. \quad (2.8)$$

Our purpose is to get the transformation function for ϕ_1 defined by

$$\phi_1(t_1) = U(t_1, t_0) \phi_1(t_0), \quad (2.9)$$

which may subsequently give the Hamiltonian of the Schrödinger equation in the form

$$i \frac{d\phi_1(t)}{dt} = H_1(t) \phi_1(t). \quad (2.10)$$

It should be noted that the quantity which is given as an initial condition at the time t_0 is $\phi_1(t_0)$ only, while ϕ_2 cannot be given arbitrarily at any time, being really determined uniquely by ϕ_1 in the form

$$\phi_2(t_1) = V(t_1, t_0) \phi_1(t_0). \quad (2.11)$$

U and V may not be given in any closed form, but they are given from (2.7) and (2.8) by power series expansions as follows:

$$\begin{aligned} U(t_1, t_0) = & 1 + \frac{1}{i} \int \eta^{t_0}(t_1, t) PH(t) P dt \\ & + \left(\frac{1}{i}\right)^2 \iint \eta^{t_0}(t_1, t) \eta^{t_0}(t, t') PH(t) PH(t') P dt dt' \\ & + \left(\frac{1}{i}\right) \left(\frac{1}{2i}\right) \iint \eta^{t_0}(t_1, t) \epsilon(t - t') PH(t) (1 - P) H(t') P dt dt' + O(g^3), \end{aligned} \quad (2.12)$$

$$V(t_1, t_0) = \frac{1}{2i} \int \epsilon(t_1 - t) (1 - P) H(t) P dt$$

$$\begin{aligned}
& + \left(\frac{1}{i}\right) \left(\frac{1}{2i}\right) \iint \epsilon(t_1 - t) \eta^{t_0}(t, t') (1 - P) H(t) P H(t') P dt dt' \\
& + \left(\frac{1}{2i}\right)^2 \iint \epsilon(t_1 - t) \epsilon(t - t') (1 - P) H(t) (1 - P) H(t') P dt dt' + O(g^3), \quad (2.13)
\end{aligned}$$

where g is a coupling constant involved linearly in $H(t)$. It is easily seen that U in (2.12) is not unitary.

The Hamiltonian $H_1(t)$ appearing in (2.10) is connected with U as

$$U(t + \delta t, t) = 1 + \frac{1}{i} H_1(t) \delta t, \quad (2.14)$$

where $\delta t \rightarrow 0$ is anticipated. $H_1(t)$ is thus given explicitly by making t_1 approach to t_0 in (2.12), taking up the terms of first order of δt , and putting $t_1 = t_0 = t$ at the last step, as follows:

$$H_1(t) = P H(t) P + \left(\frac{1}{2i}\right) \int \epsilon(t - t') P H(t) (1 - P) H(t') P dt' + O(g^3). \quad (2.15)$$

By using the contraction method, $H_1(t)$ is brought to the form not explicitly involving internal fields. In such a form, the internal field may rather be considered as the *structure* of interaction than as the usual operator.

Now, we shall return to Eq. (2.8). It is noticed that the quantities which obey (2.3) may be added to the right-hand side in various ways without contradiction to (2.7). By adding such "redundant" terms, one may obtain different Hamiltonians than (2.14) by following the similar procedure given above. Such a situation will be made clearer in the Appendix, where the general type of Hamiltonian which gives the same S -matrix as ours, is given by another method.

§ 3. The norm of state vector

The non-unitary character of the transformation function shown above contradicts with the conservation law of probability in quantum mechanics. We shall investigate such a situation in this section. In $H_1(t)$ the non-hermitian part appears only in the second order term, which will be denoted by $H_1^{(2)}(t)$, if the higher order terms are ignored. One may decompose $H_1^{(2)}(t)$ into the hermitian part $W_1(t)$ and anti-hermitian part $W_2(t)$ as follows:

$$H_1^{(2)}(t) = W_1(t) + W_2(t),$$

$$W_{1,2}(t) = -\frac{1}{4i} \int \epsilon(t - t') P [H(t) (1 - P) H(t') \mp H(t') (1 - P) H(t)] P dt', \quad (3.1)$$

where the sign $-$ or $+$ corresponds to $W_1(t)$ or $W_2(t)$, respectively. Then, up to the second order, the norm of $\phi_1(t_1)$ is given by

$$(\phi_1(t_1), \phi_1(t_1)) = \left(\phi_1(t_0), \left[1 + \frac{2}{i} \int_{t_0}^{t_1} W_2(t) dt \right] \phi_1(t_0) \right). \quad (3.2)$$

On the other hand, it holds that

$$(\phi(t), \phi(t)) = \text{constant}. \quad (3.3)$$

Although (3.3) is guaranteed from the beginning, we give a direct proof to make the situation clearer. From (2.13) one gets

$$\begin{aligned} & (\phi_2(t_1), \phi_2(t_1)) - (\phi_2(t_0), \phi_2(t_0)) \\ &= \left(\phi_1(t_0), \left[\frac{1}{4} \iint \left\{ \epsilon(t_1 - t) \epsilon(t_1 - t') - \epsilon(t_0 - t) \epsilon(t_0 - t') \right\} \right. \right. \\ & \quad \left. \left. \times PH(t) (1 - P) H(t') P dt dt' \right] \phi_1(t_0) \right). \end{aligned} \quad (3.4)$$

Using the identity

$$\epsilon(t_1 - t) \epsilon(t_1 - t') - \epsilon(t_0 - t) \epsilon(t_0 - t') = 2[\gamma^{t_0}(t_1, t) - \gamma^{t_0}(t_1, t')] \epsilon(t - t'),$$

(3.4) reduces to

$$(\phi_2(t_1), \phi_2(t_1)) - (\phi_2(t_0), \phi_2(t_0)) = \left(\phi_1(t_0), \left(\frac{-2}{i} \right) \int_{t_0}^{t_1} W_2(t) dt \phi_1(t_0) \right). \quad (3.5)$$

Comparing (3.5) with (3.2), one gets (3.3) immediately. (3.3) is, however, nothing to do with the conservation of probability. The interpretation of probability is given only to the norm of ϕ_1 , but not to ϕ_2 , the latter having no physical meaning directly.

We now investigate the expression (3.2) by using a model. Consider an interaction between a charged scalar field $\phi(x)$ with mass m "nucleon", and two neutral scalar fields, i.e. $A_1(x)$ with mass μ "meson" and $A_2(x)$ with mass κ "internal meson". The Hamiltonian is assumed to be

$$H(t) = g \int \phi^*(x) \phi(x) A(x) d^3x, \quad (3.6)$$

where

$$A(x) = A_1(x) + A_2(x),$$

with the commutation relations

$$\begin{aligned} i[\phi(x), \phi^*(x')] &= D^m(x - x'), \\ i[A_1(x), A_1(x')] &= D^\mu(x - x'), \\ i[A_2(x), A_2(x')] &= -D^\kappa(x - x'). \end{aligned} \quad (3.7)$$

A state in which the occupation number of internal mesons in any frequency is different from zero belongs to Hilbert space II.

To calculate (3.2), we assume that the initial state $\phi_1(t_0)$ is normalized to unity and is given by the wave packet in momentum space as follows:

$$\phi_1(t_0) = \sum_{\mathbf{k}} f(\mathbf{k} - \mathbf{k}_0) \phi(\mathbf{k}, -\mathbf{k}), \quad (3.8)$$

where $f(\mathbf{k} - \mathbf{k}_0)$ is large near $\mathbf{k} = \mathbf{k}_0$, and $\phi(\mathbf{k}, -\mathbf{k})$ denotes an eigenstate of the free Hamiltonian with two nucleons of momentum \mathbf{k} and $-\mathbf{k}$ and with the same charge. Using the relation

$$\langle PA(x)(1-P)A(x')P \rangle_0 = -D_+^*(x-x'),$$

one may rewrite $W_2(t)$ as follows:

$$\begin{aligned} W_2(t) = & \frac{-g^2}{4i} \iint \epsilon(t-t') \left\{ N \left[\phi^*(x) \phi(x) \phi^*(x') \phi(x') \right] D_1^*(x-x') \right. \\ & + N \left[\phi^*(x) \phi(x') + \phi(x) \phi^*(x') \right] \\ & \times \left[D_+^m(x-x') D_+^{*m}(x-x') + D_-^m(x-x') D_-^{*m}(x-x') \right] \\ & \left. + D_+^m(x-x') D_+^{*m}(x-x') D_+^*(x-x') + D_-^m(x-x') D_-^{*m}(x-x') D_-^*(x-x') \right\} \\ & \times d^3\mathbf{x} (dx'), \end{aligned} \quad (3.9)$$

where $D_{\pm}(x)$ are the positive and negative frequency parts, $D = i(D_+ - D_-)$, $D_1 = D_+ + D_-$, and N means a normal product. By the Fourier decomposition one finds that only the first term survives. By performing straightforward calculations (3.2) reduces to

$$\begin{aligned} (\phi_1(t_1), \phi_1(t_1)) = & 1 + \frac{ig^2}{4V} \sum_{\mathbf{k}, \mathbf{k}'} \frac{g^*(\mathbf{k}') g(\mathbf{k})}{E' E [(\mathbf{k}' - \mathbf{k})^2 + \kappa^2]^{1/2} [(\mathbf{k}' - \mathbf{k})^2 + \kappa^2 - (E' - E)^2]} \\ & \times \sin[(E' - E)t_1] \exp[i(E' - E)t_1], \end{aligned} \quad (3.10)$$

where $g(\mathbf{k}) = f(\mathbf{k} - \mathbf{k}_0) + f(-\mathbf{k} - \mathbf{k}_0)$, $E = \sqrt{\mathbf{k}^2 + m^2}$, $E' = \sqrt{\mathbf{k}'^2 + m^2}$, V is quantization volume, and t_0 is put equal to zero for simplicity.

The factor $g^*(\mathbf{k}') g(\mathbf{k}) \exp[i(E' - E)t_1]$ in the integrand shows that the time width $t_1 \approx \Delta t$ in which the norm is different from 1 is related to the energy width ΔE of the packet as

$$\Delta t \approx 1/\Delta E, \quad (3.11)$$

which is the same form as the uncertainty relation in quantum theory. On the other hand, if ΔE is small, the second term of (3.10) also becomes small due to the factor $\sin[(E' - E)t_1]$. In the limiting case $\Delta E = 0$, i.e. $\phi_1(t_0) = \phi(\mathbf{k}, -\mathbf{k})$, this term vanishes. Thus the restriction on the time width is stronger than (3.11).

The uncertainty relation in quantum theory appears for the expectation values of operators in any state, the norm of which can be put equal to unity in the ordinary formalism. In our case, the uncertainty character concerns also to the norm of state vector in such a way that it may not simply be separated from the uncertainty for the expectation values. It seems from the above consideration,

though it is too simplified, that this non-conservative character of norm does not immediately introduce an inconsistency into quantum theory; however, the more complete physical interpretation is beyond the scope of the present paper.

Appendix

In this appendix, another method to define the Hamiltonian in Hilbert space I is given. This is simpler than that given in § 2, though the physical meaning is less clear.

In the total Hilbert space the reaction matrix K is obtained from the Schrödinger equation (2.2), in the same way as the ordinary formalism, as follows:⁵⁾

$$K = \frac{1}{2} \int G(t) dt, \quad (\text{A} \cdot 1)$$

where

$$G(t) = H(t) \left[1 - \frac{i}{2} \int \epsilon(t-t') G(t') dt' \right]. \quad (\text{A} \cdot 2)$$

The S -matrix is related to the K -matrix as

$$S = \frac{1 - iK}{1 + iK}.$$

The above statement is also true in Hilbert space I, in which the Hamiltonian involves the integrations in terms of extra parameters such as in (2.15). We denote K and G in Hilbert space I which are obtained from $H_1(t)$ by K_1 and G_1 , respectively. K_1 and K are connected by the relation⁸⁾

$$K_1 = PKP. \quad (\text{A} \cdot 3)$$

One finds from (A.1) that (A.3) is satisfied if

$$G_1(t) = PG(t)P. \quad (\text{A} \cdot 4)$$

Then, $H_1(t)$ is obtained in terms of $G_1(t)$ in the following way:

$$H_1(t) = G_1(t) \left[1 - \frac{i}{2} \int \epsilon(t-t') G_1(t') dt' \right]^{-1}. \quad (\text{A} \cdot 5)$$

(A.4) is, however, not uniquely determined from (A.3), which is also satisfied even if one adds to the right-hand side of (A.4) any quantity which vanishes after performing the time integrations. Thus one obtains $H_1(t)$ in infinitely many ways, which give different transformation functions for the finite time. Such a situation corresponds to the arbitrariness in Eq. (2.8).

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Theory of Classical Fluids : Hyper-Netted Chain Approximation. III

—A New Integral Equation for the Pair Distribution Function—

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The hyper-netted chain approximation proposed in I is reformulated. The formulae in this approximation are rederived for the free energy, chemical potential and pair distribution function. The pair distribution function in our approximation is shown to satisfy a new integral equation, which is compared with the Yvon, Born and Green one. Variational principle for our approximation is given.

Errata for I and II are given as appendix.

§ 1. Introduction

The HNC (hyper-netted chain) approximation is proposed in I,¹⁾ so that we may theoretically approach to the behaviors of fluids of not too small densities by starting with the Mayer virial expansion formulae. In this approximation, we try to consider all the graphs which can be summed up by means of sequences of the Fourier transformation. These graphs are considered to be those which can be reduced to a line or a ring by a sequence of 'identifications'.* In I, we have grouped the graphs by the 'times of identifications'* by which a graph is reduced to a line or a ring. The author regrets the fact that he committed errors in the calculations in I and II,²⁾ in which some graphs which should have been included were neglected. The errata for I and II will be given in the Appendix. The correct results were reported in the preliminary report.³⁾

In this paper, we will reformulate the hyper-netted chain approximation in a more refined form than in I.

* The terminology will follow that in I. Some important words to be explained are as follows:

A 'propagation' is a part of graph which is connected to another part of the graph by just two points. The corresponding factor is called 'propagator'.

A 'junction' is a point where three or more lines meet.

Two or more propagations with the same endpoints are called 'identifiable' if they have no junctions midway. 'Identification' is the process of replacing two or more identifiable propagations by a line.

'Times of identification' are counted by carrying identifications on every identifiable parts on a graph at the same time.

The notation is somewhat changed. $f^{(0)}(r)$, $F^{(0)}(k)$, $f(r)$, $\mathfrak{F}(k)$, $\mathfrak{h}(r)$, and $\mathfrak{G}(k)$ in I are written as $b(r)$, $B(k)$, $z(r)$, $Z(k)$, $z_S(r)$ and $Z_S(k)$, respectively, in this paper.

The Mayer virial expansion formulae in which we are interested are as follows:

1a. The free energy A :⁴⁾

$$\frac{A}{VkT} = \rho \ln \left(\frac{h^3}{2\pi m kT} \right)^{3/2} \frac{\rho}{e} + \frac{A'}{VkT}, \quad (1)$$

$$-\frac{A'}{kT} = \frac{\rho^2}{2} \iint d\mathbf{r}_1 d\mathbf{r}_2 b_{12} + \sum_{n=3}^{\infty} \frac{\rho^n}{n!} \iint \cdots \int d\mathbf{r}_1 \cdots d\mathbf{r}_n \sum_{\substack{n \geq i > j \geq 1}} \prod b_{ij}. \quad (2)$$

Sum over all products which are more than singly connected.

1b. The chemical potential μ :*

$$\frac{\mu}{kT} = \ln \left(\frac{h^3}{2\pi m kT} \right)^{3/2} \rho + \frac{\mu'}{kT}, \quad (3)$$

$$-\frac{\mu'}{kT} = \rho \int d\mathbf{r}_2 b_{12} + \sum_{k=2}^{\infty} \frac{\rho^k}{k!} \iint \cdots \int d\mathbf{r}_2 \cdots d\mathbf{r}_{k+1} \sum_{\substack{k+1 \geq i > j \geq 2 \\ k+1 \geq i \geq 2}} \prod b_{ij} b_{i1}. \quad (4)$$

Sum over all products for which each particle of the set $\{\mathbf{r}_2, \dots, \mathbf{r}_{k+1}\}$ is connected to \mathbf{r}_1 by at least two independent paths and also the particles $\{\mathbf{r}_2, \dots, \mathbf{r}_{k+1}\}$ are connected with each other independently of \mathbf{r}_1 .

1c. The pair distribution function $g(r)$ or the potential of average force $\ln g(r)$:⁵⁾

$$\ln g(r) = -\frac{\phi(r)}{kT} + w(r), \quad (5)$$

$$w(r_{12}) = \sum_{m=1}^{\infty} \frac{\rho^m}{m!} \iint \cdots \int d\mathbf{r}_3 \cdots d\mathbf{r}_{m+2} \sum_{\substack{m+2 \geq i > j \geq 3 \\ m+2 \geq i \geq 3 \\ 2 \geq i \geq 1}} \prod b_{ij} b_{k\kappa}, \quad (6)$$

Sum over all products for which each particle of the set $\{\mathbf{r}_3, \dots, \mathbf{r}_{m+2}\}$ is connected \mathbf{r}_1 and \mathbf{r}_2 by an independent path and also the particles $\{\mathbf{r}_3, \dots, \mathbf{r}_{m+2}\}$ are connected with each other independently of \mathbf{r}_1 and \mathbf{r}_2 .

where

$$b_{ij} = b(r_{ij}) = \exp \{-\phi(r_{ij})/kT\} - 1,$$

$\phi(r)$ being the intermolecular potential.

Mayer⁴⁾ introduced graphs to illustrate the integrals in the sums of the right-hand side of (2), (4) and (6). In our development, it is convenient to use the graphical representation of the integrands and integral of (2), (4) and (6) systematically. So we use the graphs as the symbols for products or integrals. We start with definition of the symbolical representations in § 2 and go forth with the thorough use of the symbolical methods.

In § 3 the hyper-netted chains which satisfy a set of equations are defined. In § 4 the formulae for the free energy, chemical potential and pair distribution function in terms of the hyper-netted chains are obtained in the HNC approxima-

* $-\mu'/kT$ is equal to $-\ln \gamma = \sum_{k=1}^{\infty} \beta_k \rho^k$ in Mayer's textbook.⁴⁾

tion. The expansion formulae for the free energy, chemical potential and pair distribution function in terms of the hyper-netted chains are given in § 5. The variational principle for the free energy in the HNC approximation is given in § 6. By using this variational principle, we calculate in § 7 the expressions for the pressure and internal energy in the HNC approximation. The set of equations for the pair distribution function in the HNC approximation is compared with that of Yvon, Born and Green in § 8.

Discussions are given only for a one-component system.

§ 2. Definitions of graphs

2a. Product $\prod_{n \geq i > j \geq 1} b_{ij}$ is expressed by a graph which consists of n white circles with subscripts 1, 2, \dots , n ; $\bigcirc_1, \bigcirc_2, \dots, \bigcirc_n$; and lines connecting \bigcirc_i and \bigcirc_j corresponding to the appearance of factors b_{ij} in the product, where the line is attached by letter B , specifying that the factor is $b(r)$. In case where another factor appears, a corresponding letter will be attached. For instance,

$$b_{12} \equiv \begin{array}{c} B \\ \text{---} \bigcirc_1 \text{---} \bigcirc_2 \\ 1 \quad 2 \end{array}, \quad b_{12}b_{23} \equiv \begin{array}{c} B \quad B \\ \text{---} \bigcirc_1 \text{---} \bigcirc_2 \text{---} \bigcirc_3 \\ 1 \quad 2 \quad 3 \end{array},$$

$$b_{12}b_{13}b_{23} \equiv \begin{array}{c} \bigcirc_1 \\ B \quad B \\ \text{---} \quad \text{---} \\ B \\ \bigcirc_2 \quad \bigcirc_3 \\ 2 \quad 3 \end{array}$$

and

$$z_{12} \equiv \begin{array}{c} Z \\ \text{---} \bigcirc_1 \text{---} \bigcirc_2 \\ 1 \quad 2 \end{array}, \quad b_{12}z_{12} \equiv \begin{array}{c} B \\ \text{---} \bigcirc_1 \text{---} \bigcirc_2 \\ Z \end{array},$$

and so on.

2b. In order to express the integrals in (2), (4) and (6), we introduce a symbol expressing

$$\rho^m \iint \dots \int d\mathbf{r}_{n-m+1} \dots d\mathbf{r}_n \prod_{n \geq i > j \geq 1} b_{ij}.$$

It is the graph obtained from the graph expressing $\prod_{n \geq i > j \geq 1} b_{ij}$ by replacing the white circles $\bigcirc_{n-m+1}, \dots, \bigcirc_n$ by the black circles $\bullet_{n-m+1}, \dots, \bullet_n$: a black circle means that an integration is to be taken over the coordinate. For instance,

$$\rho \int d\mathbf{r}_1 b_{12} \equiv \begin{array}{c} B \\ \text{---} \bullet_1 \text{---} \bigcirc_2 \\ 1 \quad 2 \end{array}, \quad \rho^2 \iint d\mathbf{r}_1 d\mathbf{r}_2 b_{12} \equiv \begin{array}{c} B \\ \text{---} \bullet_1 \text{---} \bullet_2 \\ 1 \quad 2 \end{array},$$

$$\rho^2 \iint d\mathbf{r}_2 d\mathbf{r}_3 b_{12}b_{23}b_{34} \equiv \begin{array}{c} B \quad B \quad B \\ \text{---} \bigcirc_1 \text{---} \bullet_2 \text{---} \bullet_3 \text{---} \bigcirc_4 \\ 1 \quad 2 \quad 3 \quad 4 \end{array},$$

and so on.

Now, we write the formulae (2), (4) and (6) as follows:

$$-A'/kT = \frac{1}{2} \begin{array}{c} B \\ \bullet_1 \text{---} \bullet_2 \end{array} + \sum_{n=3}^{\infty} \frac{1}{n!} \{ \text{Sum of all the topologically different} \\ \text{graphs which consist of } \bullet_1, \bullet_2, \dots, \bullet_n \text{ and some } \text{---} B \text{---} \text{ and} \\ \text{which are more than singly connected.} \} \quad (2')$$

$$-\mu'/kT = \begin{array}{c} B \\ \circ_1 \text{---} \bullet_2 \end{array} + \sum_{k=2}^{\infty} \frac{1}{k!} \{ \text{Sum of all the topologically different} \\ \text{graphs which consist of } \circ_1, \bullet_2, \bullet_3, \dots, \bullet_{k+1} \text{ and some } \text{---} B \text{---} \\ \text{and for which each black circle is connected to } \circ_1 \text{ by at least} \\ \text{two independent paths and the black circles are connected to} \\ \text{each other independently of } \circ_1. \} \quad (4')$$

$$w(r_{12}) = \sum_{m=1}^{\infty} \frac{1}{m!} \{ \text{Sum of all the topologically different graphs which} \\ \text{consist of } \circ_1, \circ_2, \bullet_3, \bullet_4, \dots, \bullet_{m+2} \text{ and some } \text{---} B \text{---} \text{ and in} \\ \text{which each black circle is connected to } \circ_1 \text{ and } \circ_2 \text{ by an inde-} \\ \text{pendent path and the black circles are connected to each other} \\ \text{independently of } \circ_1 \text{ and } \circ_2; \text{ where } \text{---} B \text{---} \text{ directly connecting} \\ \circ_1 \text{ and } \circ_2 \text{ is not allowed.} \} \quad (6')$$

2c. In the sums in (2'), (4') and (6') appear many graphs which are different only in the numbers attached to the black circles: they are, of course, of the same contributions to (2'), (4') and (6'); e.g.,

$$\begin{array}{c} 2 \quad B \quad 3 \\ \bullet \quad \text{---} \bullet \\ B \quad \text{---} B \\ \circ \quad \text{---} \bullet \\ 1 \quad B \quad 4 \end{array} = \begin{array}{c} 3 \quad B \quad 2 \\ \bullet \quad \text{---} \bullet \\ B \quad \text{---} B \\ \circ \quad \text{---} \bullet \\ 1 \quad B \quad 4 \end{array}.$$

We now define the graph with black circles without subscripts. It represents the sum of the graphs, with black circles with subscripts, which are topologically different from each other only in the numbering of the black circles, divided by the factorial of the number of the black circles. So that, e.g.,

$$\begin{array}{c} B \\ \bullet \quad \text{---} \bullet \\ B \quad \text{---} B \\ \circ \quad \text{---} \bullet \\ 1 \quad B \end{array} = \frac{1}{3!} \left\{ \begin{array}{c} 2 \quad B \quad 3 \\ \bullet \quad \text{---} \bullet \\ B \quad \text{---} B \\ \circ \quad \text{---} \bullet \\ 1 \quad B \quad 4 \end{array} + \begin{array}{c} 2 \quad B \quad 4 \\ \bullet \quad \text{---} \bullet \\ B \quad \text{---} B \\ \circ \quad \text{---} \bullet \\ 1 \quad B \quad 3 \end{array} + \begin{array}{c} 3 \quad B \quad 2 \\ \bullet \quad \text{---} \bullet \\ B \quad \text{---} B \\ \circ \quad \text{---} \bullet \\ 1 \quad B \quad 4 \end{array} \right\} \\ = \frac{1}{2} \begin{array}{c} 2 \quad B \quad 3 \\ \bullet \quad \text{---} \bullet \\ B \quad \text{---} B \\ \circ \quad \text{---} \bullet \\ 1 \quad B \quad 4 \end{array}.$$

Then, (2'), (4') and (6') are written in terms of these graphs as

$$-A'/kT = \bullet \xrightarrow{B} \bullet + \text{Sum of all the topologically different graphs which consist of three or more black circles without subscript and some } \xrightarrow{B} \text{ and which are more than singly connected.} \quad (2'')$$

$$-\mu'/kT = \circ \xrightarrow{B} \bullet + \text{Sum of all the topologically different graphs which consist of one white circle and two or more black circles without subscript and some } \xrightarrow{B} \text{ and for which each black circle is connected to the white circle by at least two independent paths and the black circles are connected to each other independently of the white circle.} \quad (4'')$$

$$w(r_{12}) = \text{Sum of all the topologically different graphs which consist of two white circles } \circ_1 \text{ and } \circ_2 \text{ and one or more black circles and some } \xrightarrow{B} \text{ and for which each black circle is connected to } \circ_1 \text{ and } \circ_2 \text{ by an independent path and the black circles are connected to each other independently of } \circ_1 \text{ and } \circ_2, \text{ where } \xrightarrow{B} \text{ directly connecting } \circ_1 \text{ and } \circ_2 \text{ is not allowed.} \quad (6'')$$

Given a graph with black circles with subscript, the number of corresponding graphs topologically different only in the numbering of the black circles is equal to the factorial of the number of black circles in the graph divided by the number of symmetry of the corresponding graph with black circles without subscript. As the consequence, a graph with black circles without subscript expresses an integral of the type:

$$\frac{\rho^m}{\text{The number of symmetry of the graph with black circles without subscript.}} \cdot \iint \dots \int d\mathbf{r}_{n-m+1} \dots d\mathbf{r}_n \prod_{n \geq i > j \geq 1} b_{ij}.$$

That is, e.g.,

$$\begin{array}{c} \begin{array}{ccc} \begin{array}{c} \bullet \xrightarrow{B} \bullet \\ | \quad | \\ \bullet \xrightarrow{B} \bullet \end{array} & = & \frac{1}{2 \times 4} \begin{array}{c} 2 \quad B \quad 3 \\ \bullet \quad \quad \bullet \\ | \quad | \\ 1 \quad B \quad 4 \end{array} \end{array} \quad \begin{array}{ccc} \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ | \quad | \\ \bullet \quad \bullet \end{array} & = & \frac{1}{2 \times 3!} \begin{array}{c} 3 \\ \bullet \quad \bullet \quad \bullet \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ 1 \quad \bullet \quad 5 \quad \bullet \quad 2 \\ | \quad | \quad | \quad | \\ 6 \quad B \quad 7 \end{array} \end{array} \\[10pt] \begin{array}{ccc} \begin{array}{c} \circ \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} & = & \frac{1}{2} \begin{array}{c} 3 \\ \bullet \quad \bullet \quad \bullet \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ 1 \quad \bullet \quad 4 \quad \bullet \quad 2 \end{array} \end{array} \end{array}$$

and so on.

§ 3. The hyper-netted chains

The hyper-netted chain approximation is introduced as the approximation in which all those graphs are considered, which are reducible to a line by a sequence of identifications. Then, these graphs are considered to be constructed of propagations which are reducible to a line by identifications. We group these propagations into two classes, which will be called 'hyper-netted chain Z ' and 'hyper-netted chain Z_s '. To define these, we define an ' s -point' as a point which must be passed to go from an end of a propagation to another end.

The hyper-netted chain Z is the total of the propagations which are reducible to a line by identifications and which have no s -point. The corresponding propagators are $z(r)$ and $Z(k)$.*

The hyper-netted chain Z_s is the total of the propagations which are reducible to a line by identifications and which have at least one s -point. The corresponding propagators are $z_s(r)$ and $Z_s(k)$.

According to the definitions, the structure of Z and Z_s are of the form :

$$\begin{array}{c} \text{---} Z_s \text{---} \\ \text{1} \quad \text{2} \end{array} = \begin{array}{c} \text{---} Z \text{---} \bullet \text{---} Z \text{---} \\ \text{1} \quad \quad \quad \text{2} \end{array} + \begin{array}{c} \text{---} Z \text{---} \bullet \text{---} Z \text{---} \text{---} Z \text{---} \\ \text{1} \quad \quad \quad \quad \quad \text{2} \end{array} + \dots, \quad (7f)$$

where we have grouped the graphs in Z_s together according to the number of s -points, and

$$\begin{array}{c} \text{1} \\ \text{---} Z \text{---} \\ \text{2} \end{array} = \begin{array}{c} \text{1} \\ \text{---} B \text{---} \\ \text{2} \end{array} + \begin{array}{c} \text{1} \\ \text{---} B \text{---} \text{---} Z_s \text{---} \\ \text{2} \end{array} + \begin{array}{c} \text{1} \\ \text{---} B \text{---} \text{---} Z_s \text{---} Z_s \text{---} \\ \text{2} \end{array} + \dots \\ + \begin{array}{c} \text{1} \\ \text{---} Z_s \text{---} \text{---} Z_s \text{---} \\ \text{2} \end{array} + \begin{array}{c} \text{1} \\ \text{---} Z_s \text{---} \text{---} Z_s \text{---} Z_s \text{---} \\ \text{2} \end{array} + \dots \quad (8f)$$

where we have grouped the graphs in Z together according to the number of propagations in which the propagation separates when we erase the endpoints. $Z_{\text{C2}}^{\text{O1}}$ is the sum of all topologically different graphs of the type of (8f) and each graph is equal to a graph with black circles with subscript divided by the number

* $Z(k)$ and $Z_s(k)$ are Fourier transforms of $z(r)$ and $z_s(r)$, respectively. That is, e. g.,

$$Z(k) = \int d\mathbf{r} z(r) \exp(i\mathbf{k} \cdot \mathbf{r}).$$

of symmetry. The number of symmetry is equal to the product of the numbers of symmetry of each edge and the number of symmetry of the graph as a whole. Then (8f) can be written as

$$\begin{aligned}
 Z &= \left\{ \begin{array}{c} 1 \\ \text{---} \\ 2 \end{array} \right\} + 1 \left\{ \sum_{n=1}^{\infty} \sum_{Z_S' < Z_S'' < \dots < Z_S^{(l)} **} \frac{1}{n!} \left\{ \begin{array}{c} 1 \\ \text{---} \\ 2 \end{array} \right\}^{n'} \cdot \frac{1}{n''!} \left\{ \begin{array}{c} 1 \\ \text{---} \\ 2 \end{array} \right\}^{n''} \dots \frac{1}{n^{(l)}!} \left\{ \begin{array}{c} 1 \\ \text{---} \\ 2 \end{array} \right\}^{n^{(l)}}} \right. \\
 &\quad \left. + B \left\{ \begin{array}{c} 1 \\ \text{---} \\ 2 \end{array} \right\} - Z_S \right\} \\
 &= \left\{ \begin{array}{c} 1 \\ \text{---} \\ 2 \end{array} \right\} + 1 \left\{ \sum_{n=1}^{\infty} \frac{1}{n!} \left\{ \sum_{Z_S'} \left\{ \begin{array}{c} 1 \\ \text{---} \\ 2 \end{array} \right\}^{n'} + B \left\{ \begin{array}{c} 1 \\ \text{---} \\ 2 \end{array} \right\} - Z_S \right\}^n \right. \right. \\
 &\quad \left. \left. + B \left\{ \begin{array}{c} 1 \\ \text{---} \\ 2 \end{array} \right\} - Z_S \right\} \right\} \quad (8f')
 \end{aligned}$$

so that

$$z(r) = \{b(r) + 1\} e^{z_S(r)} - 1 - z_S(r). \quad (8)$$

In the graphical equation (8f'), $Z_S', Z_S'', \dots, Z_S^{(l)}$ means a graph belonging to the hyper-netted chain Z_S . Note here that the inverse of the number of symmetry of

each edge is included in the respective $\left\{ \begin{array}{c} 1 \\ \text{---} \\ 2 \end{array} \right\}^{n'}$. On the other hand, (7f) is in its

explicit form,

$$Z_S(k) = \frac{\rho Z(k)^2}{1 - \rho Z(k)}. \quad (7)$$

This set of equations, (7) and (8), determines $z(r)$ and $z_S(r)$.

§ 4. Formulae in the HNC approximation

We calculate the sums of the graphs which are reducible to a line or a ring by identifications, in (2''), (4'') and (6''). We consider first (6'') and then (4'') and (2'').

4a. The potential of average force

The total of the graphs which connect \mathbf{r}_1 and \mathbf{r}_2 and which can be reduced to a line by identifications are Z and Z_S . In (6''), we have only to consider the

* The inequalities mean that the sum is taken over all those sets $\{Z_S', Z_S'', \dots, Z_S^{(l)}\}$ for which $Z_S', \dots, Z_S^{(l)}$ are different from each other, in the way as to consider every set only once.

graphs in which the black circles are connected independently of \mathbf{r}_1 and \mathbf{r}_2 . The graphs in Z are not of such nature. The graphs to be considered are those of Z_s and we get

$$w(r)_{\text{HNC}} = z_s(r). \quad (9)$$

Now, we note that the set of equations (7) and (8) is just the set of equations which determines the pair distribution function in the HNC approximation.

4b. The chemical potential

After a sequence of identifications except the point \mathbf{r}_1 , any graph which is reducible to a line or a ring is reduced to one of the graphs in Fig. 1.

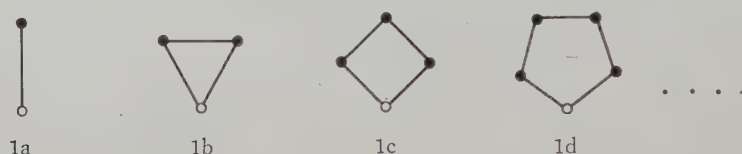


Fig. 1.

The graphs which are reduced to the form of Figs. 1a, 1b, 1c, ... by identifications have the structure of Figs. 2a, 2b, 2c, ..., respectively. (While the converse is not true.) Then, the contribution to be calculated is the sum of the contri-

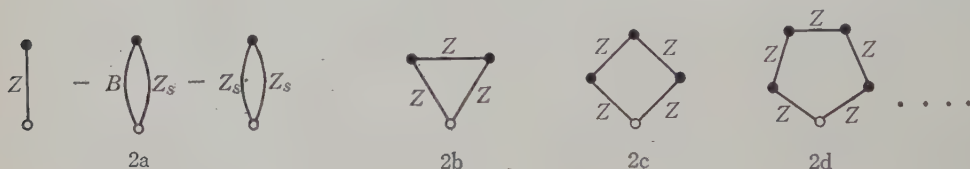


Fig. 2

butions of graphs in Figs. 2a, 2b, 2c, ..., subtracted by the contributions of the graphs which are not reduced to Figs. 1a, 1b, ... among those of Figs. 2a, 2b, ..., respectively. These graphs to be subtracted in Fig. 2a are those in which the times of identification to reduce to a line are larger for one of Z_s constructing Z than for the other Z_s (cf. (8f)), i.e.,

$$\sum_{Z'} \sum_{Z_s}^{(h+s)} Z' \left(\text{graph with two black dots and two open circles connected by two arcs} \right) Z'_s - B \left(\text{graph with two black dots and two open circles connected by two arcs} \right) Z_s.$$

Here, $\sum_{Z_s}^{(h+s)}$ means that the sum over is taken over Z'_s for which the times of identification to reduce to a line is the same or larger than Z' in the first sum; $Z' \left| \begin{smallmatrix} \bullet \\ \circ \end{smallmatrix} \right|$ means a graph belonging to the set $Z \left| \begin{smallmatrix} \bullet \\ \circ \end{smallmatrix} \right|$. Such graphs in Figs. 2b, 2c, ..., are those in which the times of identification to reduce to a line is larger for an

edge Z attached to \mathbf{r}_1 than the other edges of Z_s , that is

$$\sum_{Z_{S'}} \sum_{Z'}^{(h)} Z' \bigcirc Z_s \equiv \sum_{Z'} \sum_{Z_{S'}}^{(l)} Z' \bigcirc Z_s.$$

Here, (l) on $\sum_{Z_{S'}}^{(l)}$ means the lower; cf. above $\sum_{Z_{S'}}^{(h+s)}$. As the consequence, the contribution to be subtracted is

$$\sum_{Z'} \sum_{Z_{S'}} Z' \bigcirc Z_s - B \bigcirc Z_s = Z \bigcirc Z_s - B \bigcirc Z_s.$$

Then, the contribution to be calculated is

$$\begin{aligned} \left(-\frac{\mu'}{kT}\right)_{HNC} = & Z \bigcirc - B \bigcirc Z_s - Z_s \bigcirc Z_s + Z \triangle Z + Z \diamond Z + Z \text{ (pentagon) } Z + \dots \\ & - Z \bigcirc Z_s + B \bigcirc Z_s \end{aligned} \quad (10 f)$$

or explicitly

$$\begin{aligned} \left(-\frac{\mu'}{kT}\right)_{HNC} = & \int d\mathbf{r} \left\{ \rho z(r) - \rho b(r) z_s(r) - \frac{\rho}{2} z_s(r)^2 \right\} + \frac{1}{2} \left\{ z_s(0) - \rho \int d\mathbf{r} z(r)^2 \right\} \\ & - \int d\mathbf{r} \left\{ \rho z(r) z_s(r) - \rho b(r) z_s(r) \right\} \\ = & \rho Z(0) + \frac{1}{2} z_s(0) - \frac{\rho}{2} \int d\mathbf{r} \{ z(r) + z_s(r) \}^2. \end{aligned} \quad (10)$$

4c. The free energy

After a sequence of identifications, any of the graphs to be considered is reduced to one of the forms in Fig. 3. The graphs which reduce to the form of Figs.

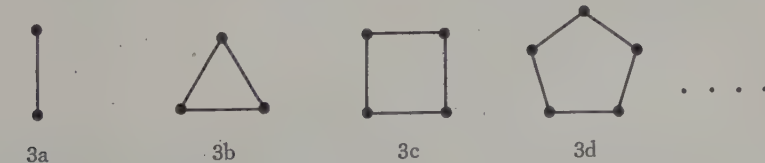


Fig. 3.

3a, 3b, 3c, ..., by identifications have the structure of Figs. 4a, 4b, 4c, ..., respectively. (While the converse is not true again.) Then, the contribution to be calculated is the sum of the contributions of Figs. 4a, 4b, 4c, ..., subtracted by the

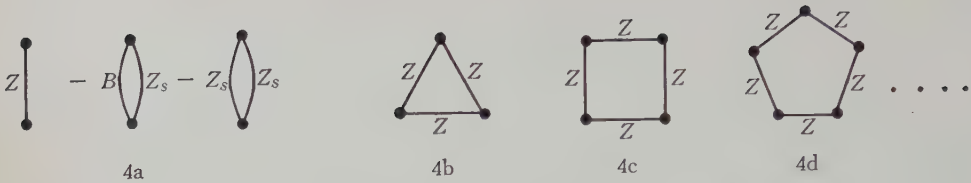


Fig. 4.

contributions of the graphs which do not reduce to Figs. 3a, 3b, 3c, ..., among those of Figs. 4a, 4b, 4c, ..., respectively. These graphs in Fig. 4a are those in which the times of identification to reduce to a line is larger for one of Z_s constructing Z than for the other Z_s (cf. Eq. (8f)), that is,

$$\sum_{Z'} \sum_{Z_s'}^{(h+s)} Z' \left(Z_s' - Z_s \right) B,$$

where $Z' \bigcirc$ means a graph belonging to $Z \bigcirc$. These graphs in Figs. 4b, 4c, ..., are those in which the times of identification to reduce to a line is larger for an edge Z than the other edges. That is,

$$\sum_{Z_s'} \sum_{Z'}^{(k)} Z' \left(Z_s' \right) = \sum_{Z'} \sum_{Z_s'}^{(l)} Z' \left(Z_s' \right).$$

Then, the contribution to be subtracted is from the graph:

$$\sum_{Z'} \sum_{Z_s'} Z' \left(Z_s' - Z_s \right) B = Z \left(Z_s - Z_s \right) B.$$

As the consequence, the contribution to be calculated is

$$\begin{aligned} \left(-\frac{A'}{kT} \right)_{HNC} = & Z \left(Z_s - Z_s \right) B + Z \left(Z_s - Z_s \right) B + Z \left(Z_s - Z_s \right) B + \dots \\ & - Z \left(Z_s \right) B + B \left(Z_s \right) \end{aligned} \quad (11 f)$$

or explicitly

$$\begin{aligned} \left(-\frac{A'}{V kT} \right)_{HNC} = & \frac{\rho^2}{2} Z(0) + \frac{1}{2V} \sum_k \left\{ -\ln(1 - \rho Z(k)) - \rho Z(k) - \frac{1}{2} \rho^2 Z(k)^2 \right\} \\ & - \frac{\rho^2}{4} \int d\mathbf{r} z_s(r)^2 - \frac{\rho^2}{2} \int d\mathbf{r} z(r) z_s(r) \\ = & \frac{\rho^2}{2} Z(0) + \frac{1}{2V} \sum_k \left\{ -\ln(1 - \rho Z(k)) - \rho Z(k) - \frac{\rho^2}{2} [Z(k) + Z_s(k)]^2 \right\}. \quad (11) \end{aligned}$$

§ 5. Expansion formulae in terms of the hyper-netted chains*

In the previous section, we have summed up the contribution of the graphs which can be reduced to a line or a ring by identifications. Here we turn to the graphs which cannot be reduced to a line or a ring. These contributions will be collected according to the graphs of junctions which are obtained after possible identifications. Then, the expansion formulae (2), (4) and (6) reduce to the form:

$$-\frac{A'}{VkT} = -\left(\frac{A'}{VkT}\right)_{HNC} - \frac{A''}{VkT}, \quad (12)$$

$$-\frac{\mu'}{kT} = -\left(\frac{\mu'}{kT}\right)_{HNC} - \frac{\mu''}{kT}, \quad (13)$$

$$w(r) = w(r)_{HNC} + w''(r), \quad (14)$$

where $-A''/VkT$, $-\mu''/kT$ and $w''(r)$ are equal to the right-hand sides of (2), (4) and (6) respectively if we replace all the b_{ij} there by $z(r_{ij}) + z_s(r_{ij})$ and if we add a restriction about the sum by that in the graphs all the coordinates are junctions except for 1 and 2 for $w''(r_{12})$ and 1 for μ'' and there are no identifiable parts.

§ 6. Variational principle for the HNC approximation

The formulae for the free energy in the HNC approximation (11) is written by the use of (8), as

$$\begin{aligned} -\left(\frac{A'}{NkT}\right)_{HNC} = & \frac{\rho}{2} \int d\mathbf{r} \left\{ [b(r) + 1] e^{z_s(r)} - 1 - z_s(r) \right\} \\ & + \frac{1}{2\rho V} \sum_k \left\{ -\ln(1 - \rho Z(k)) - \rho Z(k) - \frac{\rho^2}{2} [Z(k) + Z_s(k)]^2 \right\}. \end{aligned} \quad (15)$$

Now it is seen that the set of equations (7) and (8) which determines $z(r)$ and $z_s(r)$ is obtained by the variational principle which makes the value of the right-hand side of (15) stationary; the variation about $z(r)$ and $z_s(r)$ gives us

$$\begin{aligned} \delta \left(-\frac{A'}{NkT} \right)_{HNC} = & \frac{\rho}{2} \int d\mathbf{r} \delta z_s(r) \left\{ [b(r) + 1] e^{z_s(r)} - 1 - z_s(r) - z(r) \right\} \\ & + \frac{\rho}{2V} \sum_k \delta Z(k) \left\{ \frac{\rho Z(k)^2}{1 - \rho Z(k)} - Z_s(k) \right\}. \end{aligned} \quad (16)$$

The condition of extremum is (7) and (8).

If we have some suitable approximate functional forms with several parameters for $z(r)$ and $z_s(r)$ in such a form as the integrations in (15) are easily performed, we will be able to get an approximation for the free energy in the HNC approxi-

* Cf. I, § 4; II, § 5 and II, Appendix II.

mation by introducing these in $z(r)$ and $z_s(r)$ in the right-hand side and determining the parameters in the way to make it stationary.

§ 7. Equation of state

Pressure is obtained from the knowledges of the free energy and the chemical potential :

$$\frac{p}{kT} = \frac{\rho\mu}{kT} - \frac{A}{VkT} = \rho + \frac{p'}{kT} \quad (17)$$

$$\frac{p'}{kT} = \frac{\rho\mu'}{kT} - \frac{A'}{VkT} = \left(\frac{p'}{kT}\right)_{HNC} + \frac{\rho\mu''}{kT} - \frac{A''}{VkT} \quad (17')$$

$$\begin{aligned} \left(\frac{p'}{kT}\right)_{HNC} &= \left(\frac{\rho\mu'}{kT}\right)_{HNC} - \left(\frac{A'}{VkT}\right)_{HNC} \\ &= -\frac{\rho^2}{2}Z(0) + \frac{1}{2V} \sum_k \left\{ -\ln(1-\rho Z(k)) - \rho[Z(k) + Z_s(k)] \right. \\ &\quad \left. + \frac{\rho^2}{2}[Z(k) + Z_s(k)]^2 \right\}. \end{aligned} \quad (18)$$

This formula is easily obtained also by directly differentiating the formula for the free energy (15) with respect to V with the consideration that the right-hand side of (16) is zero and so the V dependences of $z(r)$ and $z_s(r)$ need not be considered.

On the other hand, the pressure is calculated by the equation

$$p' = -\frac{\rho^2}{6} \int d\mathbf{r} g(r) \frac{d\phi(r)}{dr} r. \quad (19)$$

Then we can calculate the pressure by approximating $g(r)$ in this formula by $g(r)_{HNC}$ given by (5) and (9) :

$$p'_{HNC} = -\frac{\rho^2}{6} \int d\mathbf{r} r \frac{d\phi(r)}{dr} e^{-\phi(r)/kT + z_S(r)}. \quad (20)$$

The direct confirmation of the identity of (18) and (20) is obtained by integrating (18) partially.*

* (18) with $Z(0)$ replaced by $\int d\mathbf{r} z(r)$ has the form

$$\int_0^\infty 4\pi r^2 dr \sigma(r) + \frac{1}{8\pi^3} \int_0^\infty 4\pi k^2 dk \tau(k).$$

This is partially integrated as

$$\frac{4\pi}{3} r^3 \sigma(r) \Big|_{r=0}^\infty - \frac{4\pi}{3} \int_0^\infty r^3 dr \frac{d\sigma(r)}{dr} + \frac{1}{6\pi^2} k^3 \tau(k) \Big|_{k=0}^\infty - \frac{1}{6\pi^2} \int_0^\infty k^3 dk \frac{d\tau(k)}{dk}.$$

This partial integration applied to (18) leads to (20).

Another proof of the identity of (18) and (20) is obtained by confirming that

$$\frac{\rho^2}{2}g(r)_{HNC} = -\frac{\rho^2}{2}[b(r)+1]e^{z_S(r)} = \frac{\partial[A/V]_{HNC}}{\partial\phi(r)}, \quad (21)$$

which is easily proved by using (15) A' and noticing that the variations with respect to $z(r)$ and $z_S(r)$ need not be considered. Then, that (19) with $g(r)$ substituted by $g(r)_{HNC}$ is equal to $\partial A'_{HNC}/\partial V$ is the consequence of Hiroike's work.⁶⁾

Now, we have two formulae for the pressure, then we have two equivalent formulae also for the free energy and the chemical potential, for instance,

$$\begin{aligned} \left(-\frac{A'}{VkT}\right)_{HNC} &= \left(\frac{p'}{kT}\right)_{HNC} - \left(\frac{\rho p'}{kT}\right)_{HNC} \\ &= \left(\frac{p'}{kT}\right)_{HNC} + \rho^2 Z(0) + \frac{\rho}{2}z_S(0) - \frac{\rho^2}{2} \int d\mathbf{r} [z(r) + z_S(r)]^2; \end{aligned} \quad (22)$$

p'_{HNC}/kT is to be substituted either by (18) or by (20) :

The formula for the internal energy is obtained as

$$\begin{aligned} \frac{E}{N} &= -\frac{3}{2}kT + \frac{E'}{N} \\ \left(\frac{E'}{N}\right)_{HNC} &= \frac{\partial}{\partial(1/kT)} \left(\frac{A'}{NkT}\right)_{HNC} = - \int d\mathbf{r} \frac{\partial b(r)}{\partial(1/kT)} e^{z_S(r)} = \int d\mathbf{r} \phi(r) e^{-\phi(r)/kT + z_S(r)} \\ &= \int d\mathbf{r} \phi(r) [g(r)]_{HNC}, \end{aligned} \quad (23)$$

which is the one to be obtained.

§ 8. The relation with the Born and Green integral equation

We will compare our set of equations (7) and (8) with the Born and Green integral equation for the pair distribution function.

As stated in IIIa,³⁾ our set of equations reduces to

$$Z_S(k) = \frac{\rho \varepsilon^2 B(k)^2}{1 - \rho \varepsilon B(k)}, \quad (24)$$

if we neglect the second and higher powers of $z_S(r)$ in (8) and approximate $z_S(r)$ by a suitable mean value

$$\varepsilon \equiv 1 + \overline{z_S(r)}. \quad (25)$$

This is just the linearized Born and Green integral equation.⁷⁾

Now, to make comparison of our set of equations with the non-linearized Born-Green integral equation, we will rewrite (7) and (8) : (7) is rewritten as

$$Z_S(k) = \rho Z(k) \{Z(k) + Z_S(k)\}. \quad (26)$$

This is in the coordinate representation

$$z_s(r_{12}) = \rho \int d\mathbf{r}_3 z(r_{13}) \{z(r_{32}) + z_s(r_{32})\}. \quad (27)$$

Using the bipolar coordinate and introducing (8), we have

$$\begin{aligned} rz_s(r) &= 2\pi\rho \int_0^\infty s ds \{b(r) e^{z_s(r)} + e^{z_s(r)} - 1 - z_s(r)\} \\ &\quad \times \int_{-s}^s (r+t) dt \{b(|r+t|) e^{z_s(|r+t|)} + e^{z_s(|r+t|)} - 1\}. \end{aligned} \quad (28)$$

By the partial integration, it is

$$\begin{aligned} rz_s(r) &= \pi\rho \int_0^\infty ds \left\{ \frac{db(s)}{ds} e^{z_s(s)} + \frac{dz_s(s)}{ds} [(b(s) + 1) e^{z_s(s)} - 1] \right\} \\ &\quad \times \int_{-s}^s (t^2 + s^2) (r+t) dt \{ (b(|r+t|) + 1) e^{z_s(|r+t|)} - 1 \}. \end{aligned} \quad (29)$$

This is the Born-Green integral equation⁷⁾ if we neglect

$$\frac{dz_s(s)}{ds} [(b(s) + 1) e^{z_s(s)} - 1] = \frac{dz_s(s)}{ds} [z(s) + z_s(s)] = \frac{dz_s(s)}{ds} [g(s) - 1]_{HNC}$$

compared with

$$\frac{db(s)}{ds} e^{z_s(s)} = -\frac{1}{kT} \frac{d\phi(s)}{ds} [g(s)]_{HNC}.$$

At low densities, the variation of $z_s(r)$ is expected to be small compared with that of $-\phi(s)/kT$ and this neglect will be allowed. However, the situation will be different at high densities where the neglect will not be justified and the author expects that our set of equations gives a better result in this region.*

§ 9. Conclusion

It has been shown that the free energy and the chemical potential in the HNC approximation are functional of the pair distribution function in that approximation and the latter satisfies a new integral equation, in which more graphs are considered correctly than in the case of Yvon, Born and Green, hence it is expected to give better results for the pair distribution function.* Moreover, our integral equation has an analytically simpler form and seems to be suited for numerical computations.

Acknowledgement

The author wishes to express his sincere thanks to Dr. K. Hiroike for his helpful discussions.

* If we consider the expansion formula for $dg(r)/dr$, we find that more graphs are taken account of in our formula than in the Born-Green one. On this point, we are arguing that our formula is better than that of Born and Green.

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- 1) T. Morita, Prog. Theor. Phys. **20** (1958), 920, to be referred to as I.
- 2) T. Morita, Prog. Theor. Phys. **21** (1959), 361, to be referred to as II.
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- 6) K. Hiroike, J. Phys. Soc. Japan **12** (1957), 864.
- 7) H. S. Green, *The Molecular Theory of Fluids* (North-Holland Publishing Co., 1952), Chap. III, § 6.

Errata

Theory of Classical Fluids: Hyper-Netted Chain Approximation. I and II

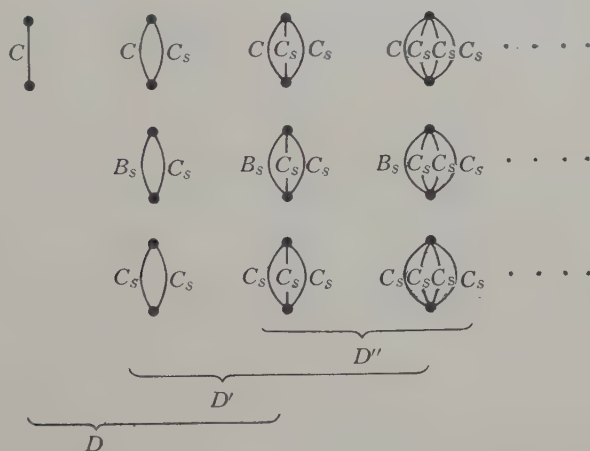
Tohru MORITA

 Prog. Theor. Phys. **20** (1958), 920; **21** (1959), 361.

As is briefly reported in the preliminary report,³⁾ the author erroneously neglected in I and II some graphs which were to be included in the HNC approximation. The corrections to be made when we include these graphs are as follows:

Errata for Part I

Fig. 11 is to be replaced by


 Fig. 11. D , D' , D'' —watermelons of C

Eq. (21), the propagation by means of D , is to be replaced by

$$f^{(2)}(\mathbf{r}) = \{f^{(1)}(\mathbf{r}) + h^{(0)}(\mathbf{r}) + 1\} \exp \{ \Delta h^{(1)}(\mathbf{r}) \} - 1 - h^{(1)}(\mathbf{r}).$$

Eq. (17), the contribution of D'' to $-A'/NkT$, is to be replaced by

$$\left(-\frac{A'}{NkT} \right)_{D''} = \frac{\rho}{2} \int d\mathbf{r} \left\{ [f^{(1)}(\mathbf{r}) + h^{(0)}(\mathbf{r}) + 1] [e^{\Delta h^{(1)}(\mathbf{r})} - 1 - \Delta h^{(1)}(\mathbf{r})] - \frac{1}{2} [\Delta h^{(1)}(\mathbf{r})]^2 \right\}.$$

The last term of (17'), $-\frac{\rho}{4} \int d\mathbf{r} [\Delta h^{(1)}(\mathbf{r})]^2$, is to be replaced by

$$-\frac{\rho}{2} \int d\mathbf{r} \left\{ h^{(0)}(\mathbf{r}) \Delta h^{(1)}(\mathbf{r}) + \frac{1}{2} [\Delta h^{(1)}(\mathbf{r})]^2 \right\} \quad \text{or} \quad -\frac{\rho}{4} \int d\mathbf{r} [h^{(1)}(\mathbf{r})^2 - h^{(0)}(\mathbf{r})^2].$$

The last integral of (24), $-\frac{\rho}{4} \int d\mathbf{r} [\Delta h^{(n-1)}(\mathbf{r})]^2$, is to be replaced by

$$-\frac{\rho}{4} \int d\mathbf{r} \{ [h^{(n-1)}(\mathbf{r})]^2 - [h^{(n-2)}(\mathbf{r})]^2 \}.$$

The last row of Eq. (27), $-\frac{\rho}{2} \int d\mathbf{r} f^{(n)}(\mathbf{r}) f^{(n-1)}(\mathbf{r}) - \frac{\rho}{4} \int d\mathbf{r} \sum_{s=0}^{n-1} [\Delta h^{(s)}(\mathbf{r})]^2$, is to be replaced by

$$-\frac{\rho}{2} \int d\mathbf{r} f^{(n)}(\mathbf{r}) h^{(n-1)}(\mathbf{r}) - \frac{\rho}{4} \int d\mathbf{r} h^{(n-1)}(\mathbf{r})^2.$$

The last term of Eq. (28), $-\frac{\rho}{4} \int d\mathbf{r} \sum_{s=0}^n [\Delta h^{(s)}(\mathbf{r})]^2$, is to be replaced by

$$-\frac{\rho}{4} \int d\mathbf{r} h^{(n)}(\mathbf{r})^2.$$

The last term of Eq. (29), $-\frac{\rho}{4} \int d\mathbf{r} \sum_{s=0}^{\infty} [\Delta h^{(s)}(\mathbf{r})]^2$, is to be replaced by

$$-\frac{\rho}{4} \int d\mathbf{r} h(\mathbf{r})^2.$$

The second rows of formulae (30) and (30') are to be replaced by

$$f^{(n+1)}(\mathbf{r}) = [f^{(n)}(\mathbf{r}) + h^{(n-1)}(\mathbf{r}) + 1] \exp\{\Delta h^{(n)}(\mathbf{r})\} - 1 - h^{(n)}(\mathbf{r})$$

and

$$f^{(n+1)}(r) = [f^{(n)}(r) + h^{(n-1)}(r) + 1] \exp\{\Delta h^{(n)}(r)\} - 1 - h^{(n)}(r),$$

respectively.

The last term of Eq. (29'), $\pi\rho \int_0^{\infty} r^2 dr \sum_{s=0}^{\infty} [\Delta h^{(s)}(r)]^2$, is to be replaced by

$$\pi\rho \int_0^{\infty} r^2 dr h(r)^2.$$

The restrictions for the products which are to be summed in Eqs. (36), (37) and the corresponding formula for $\ln g(r_{12})$ in p. 934 are to be added by that 'which have no identifiable parts'.

The second row of Eq. (48) is to be replaced by

$$f^{(n+1)}(r) = [f^{(n)}(r) + h^{(n-1)}(r) + 1] \exp\{\Delta h^{(n)}(r)\} - 1 - h^{(n)}(r), \quad n \geq 1.$$

The set of graphs which gives the d , d' and d'' in Fig. 16 is to be replaced by that which is obtained from Fig. 11 given above by replacing the large letters B and C by the small letters b and c .

Errata for Part II

In 7-8 rows from the bottom of p. 365, the sentence:

"the watermelons of C are represented by Fig. 4 if we replace the letters B and C by C and D respectively."

is to be replaced by

"the watermelons of C are represented by Fig. 5'."

and Fig. 5' is to be drawn which is equal to Fig. 11 of Part I given just above, if we have added Greek letters ν and ν' on upper and lower vertices respectively of each graph of Fig. 11.

The last term of (17), $-\frac{1}{2} \int d\mathbf{r} [\Delta h_{\nu\nu'}^{(1)}(\mathbf{r})]^2$, is to be replaced by

$$-\int d\mathbf{r} h_{\nu\nu'}^{(0)}(\mathbf{r}) \Delta h_{\nu\nu'}^{(1)}(\mathbf{r}) - \frac{1}{2} \int d\mathbf{r} [\Delta h_{\nu\nu'}^{(1)}(\mathbf{r})]^2 \quad \text{or} \quad -\frac{1}{2} \int d\mathbf{r} [h_{\nu\nu'}^{(1)}(\mathbf{r})^2 - h_{\nu\nu'}^{(0)}(\mathbf{r})^2].$$

The last term of (18), $-\frac{1}{2} \int d\mathbf{r} [\Delta h_{\nu\nu'}^{(n-1)}(\mathbf{r})]^2$, is to be replaced by

$$-\frac{1}{2} \int d\mathbf{r} [h_{vv'}^{(n-1)}(\mathbf{r})^2 - h_{vv'}^{(n-2)}(\mathbf{r})^2].$$

The last term of (21), $-\frac{1}{2} \int d\mathbf{r} \sum_{s=0}^{n-1} [\Delta h_{vv'}^{(s)}(\mathbf{r})]^2$, is to be replaced by

$$-\frac{1}{2} \int d\mathbf{r} h_{vv'}^{(n-1)}(\mathbf{r})^2.$$

The last term of (22), $-\frac{1}{2} \int d\mathbf{r} \sum_{s=0}^n [\Delta h_{vv'}^{(s)}(\mathbf{r})]^2$, is to be replaced by

$$-\frac{1}{2} \int d\mathbf{r} h_{vv'}^{(n)}(\mathbf{r})^2.$$

The last term of (23), $-\frac{1}{2} \int d\mathbf{r} \sum_{s=0}^{\infty} [\Delta h_{vv'}^{(s)}(\mathbf{r})]^2$, is to be replaced by

$$-\frac{1}{2} \int d\mathbf{r} h_{vv'}(\mathbf{r})^2.$$

The second row of (24) is to be replaced by

$$f_{vv'}^{(n+1)}(\mathbf{r}) = [f_{vv'}^{(n)}(\mathbf{r}) + h_{vv'}^{(n-1)}(\mathbf{r}) + 1] \exp[\Delta h_{vv'}^{(n)}(\mathbf{r})] - 1 - h_{vv'}^{(n)}(\mathbf{r}).$$

To the restriction for the products which are to be summed in Eqs. (44), (45) and (A·10) is added 'which have no identifiable parts'.

The second row of (50) is to be replaced by

$$f_{vv'}^{(n+1)}(r) = [f_{vv'}^{(n)}(r) + h_{vv'}^{(n-1)}(r) + 1] \exp[\Delta h_{vv'}^{(n)}(r)] - 1 - h_{vv'}^{(n)}(r), \quad n \geq 1.$$

K^+-K^0 Mass Difference

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In order to explain the K^0 - K^+ mass difference by means of electromagnetic interaction, the form factors in the Pauli term and in the electromagnetic-polarizability-term for meson are investigated. The magnitudes of contributions from these terms are estimated for the case of the exponential form factor and compared with that from nucleons.

§ 1. Introduction

Recently it has been found that the mass of K^0 -meson is about 4.5 Mev heavier than that of the K^+ -meson.¹⁾ However, it may ordinarily be thought that the charged kaon will be heavier, because the conservation law of angular momentum allows the spinless boson to emit only longitudinal photon (Coulomb interaction) in self-energy interaction. Therefore, contrary to the case of pion in which the charged pion has a larger mass, in order to derive the K^0 - K^+ mass difference by means of only the electromagnetic interaction, we shall need to introduce some strange electromagnetic form factors or interactions. Thus we may expect that the mass difference of kaon may, to some extent, exhibit the character of kaon structure and furthermore that of boson.

In this note we shall phenomenologically discuss some possibilities of solving this problem. In § 2, we shall introduce an effective electromagnetic interaction Hamiltonian of boson which is gauge invariant and non-local, and which consists of the current term, the Pauli term and the electromagnetic polarizability-term. In § 3, we shall calculate the mass difference by relativistic perturbation theory up to the second order using the above Hamiltonian. In the first place the restrictions on the form factors without the polarizability-term will be investigated. Next we shall determine the magnitudes of the Pauli term and the polarizability-term in the case of the exponential form factor so as to make the result fit the experimental value. Section 4 will be devoted to field-theoretical discussion on the order of the magnitudes of these interaction strength.

§ 2. Effective Hamiltonian

Let us consider a non-local gauge transformation of the following type:²⁾

$$A_\mu(x) \rightarrow A_\mu(x) + \frac{\partial}{\partial x_\mu} \Lambda(\tilde{x}),$$

$$\phi(x) \rightarrow \exp \left\{ -ie \int f(x-x') A(x') d^4 x' \right\} \phi(x), \quad (2.1)$$

$$\phi^*(x) \rightarrow \phi^*(x) \cdot \exp \left\{ +ie \int f(x-x') A(x') d^4 x' \right\},$$

where $A_\mu(x)$ is the electromagnetic field operator and $\phi(x)$ the meson field operator which is an iso-vector for the pion and an iso-spinor for the kaon. The form factor is a matrix in the iso-space and is rewritten in terms of the scalar functions $f^{(\pi)}(x-x')$, $f_s^{(K)}(x-x')$ and $f_v^{(K)}(x-x')$ as follows:

$$\begin{aligned} f(x-x') &= T_3 f^{(\pi)}(x-x') & (\text{for } \pi) \\ &= \mathbf{1} f_s^{(K)}(x-x') + \tau_3 f_v^{(K)}(x-x') & (\text{for } K), \end{aligned} \quad (2.2)$$

with

$$T_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

By assuming the invariance of the Lagrange density under the gauge transformation Eq. (2.1), the following expression is the most general one of the Hamiltonian in discussing the electromagnetic interaction which contains the derivatives of A up to the 1st order:*

$$\begin{aligned} H_{int}(x) &= ie (\phi^*(x) \partial_\mu \phi(x) - \partial_\mu \phi^*(x) \phi(x)) \int f(x-x') A_\mu(x') d^4 x' \\ &+ e^2 \phi^*(x) \phi(x) \left\{ \int f(x-x') A_\mu(x') d^4 x' \right\} \left\{ \int f(x-x'') A_\mu(x'') d^4 x'' \right\} \\ &+ \frac{e^2 \mu}{m^2} \phi^*(x) \phi(x) \iint g(x-x', x-x'') F_{\mu\nu}(x') F_{\mu\nu}(x'') d^4 x' d^4 x'', \end{aligned} \quad (2.3)$$

with $F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$ where m is the meson mass and $g(x-x', x-x'')$ is another matrix form factor relating to three points. The first term in Eq. (2.3) is the usual current term (c.f. Fig. 1 (a)), the second term is the A^2 -term which is necessary for the guarantee of gauge invariance (c.f. Fig. 1 (b)).

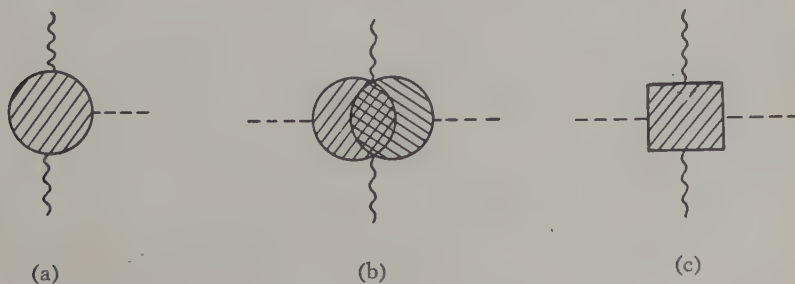


Fig. 1.

* We have dropped the normal dependent terms and we shall use the natural unit ($\hbar=c=1$).

We can choose a suitable type of the form factor $f(x-x')$ so as to make these two terms include the effect of the anomalous magnetic moment. The last term represent the electromagnetic polarizability whose coefficient $e^2\mu/m^2$ denotes its strength (c.f. Fig. 1 (c)). It is remarkable that the first term vanishes for the neutral pion, but remains non-vanishing for the kaon, because the field operator of the former is hermite while that of the latter is non-hermite.

The form factor of the first two terms is assumed to be of the form

$$f(x-x') \equiv f_1(x-x') + \partial_\nu \partial'_\nu f_2(x-x') = f_1(x-x') - \partial_\nu \partial'_\nu f_2(x-x'). \quad (2.4)$$

In the above expression, $f_1(x-x')$ is the usual charge form factor and the f_2 part contains the Pauli term for the boson³⁾ which is customarily derived from the baryon loop.

The Pauli term can be derived in the present case in the following manner :

$$\begin{aligned} & (\partial_\mu \phi^*(x) \cdot \phi(x) - \phi^*(x) \partial_\mu \phi(x)) \int \partial'_\nu \partial'_\nu f_2(x-x') A_\mu(x') d^4 x' \\ &= -\partial_\nu (\partial_\mu \phi^*(x) \cdot \phi(x) - \phi^*(x) \partial_\mu \phi(x)) \int f_2(x-x') \partial'_\nu A_\mu(x') d^4 x' \\ &= \partial_\mu \phi^*(x) \cdot \partial_\nu \phi(x) \int f_2(x-x') F_{\mu\nu}(x') d^4 x' \\ &+ (\phi^*(x) \cdot \partial_\mu \partial_\nu \phi(x) - \partial_\mu \partial_\nu \phi^*(x) \cdot \phi(x)) \int f_2(x-x') \partial'_\mu A_\nu(x') d^4 x'. \end{aligned} \quad (2.5)$$

Eq. (2.5) shows that the A^2 -term in Eq. (2.3) cannot be reduced to the last one in the same expression Eq. (2.3). Thus it is seen that we can introduce three kinds of form factors f_1 , f_2 and g .

§ 3. Mass difference

In the present stage of physics, nobody knows how to construct the unitary S -matrix up to an arbitrary order in case of non-local interaction.* However, our aim is not to discuss the existence of such a unitary S -matrix but to give an explanation for the mass difference of the kaon. Accordingly, we shall tentatively assume that the ordinary perturbation method is also applicable to the present case of non-local interaction, without making any profound consideration on this difficult problem.

We shall calculate the mass difference by the perturbation theory up to the second order of e , making use of the Hamiltonian given in § 2, and shall determine the magnitudes and the types of the form factors in Eq. (2.3) from the observed value.

In momentum representation, we put

* It has been proved that an S -matrix of non-local interaction satisfies the unitarity condition up to the second order approximation.⁵⁾

$$\left. \begin{aligned} f(x-x') &\rightarrow F(k^2), \\ g(x-x', x-x'') &\equiv g(x-x') \cdot g(x-x'') \rightarrow G(k'^2) \cdot G(k''^2), \end{aligned} \right\} \quad (3.1)$$

and especially for the kaon

$$\begin{aligned} F(k^2) &= \mathbf{1} F^{(S)}(k^2) + \tau_3 F^{(V)}(k^2), \\ G(k'^2) \cdot G(k''^2) &= \mathbf{1} G^{(S)}(k'^2) \cdot G^{(S)}(k''^2) + \tau_3 G^{(V)}(k'^2) \cdot G^{(V)}(k''^2), \end{aligned} \quad (3.2)$$

where we assume that the form factor, $g(x-x', x-x'')$, is separable into two parts.

Now the mass difference, $\delta m \equiv m^{(\pm)} - m^{(0)}$, is given as follows.

$$\begin{aligned} \delta m &= \frac{1}{2m} \frac{\pi^2 e^2 m^2}{(2\pi)^4} M, \\ M &= -\frac{1}{m^2} \int \frac{(2p-k)^2 \{F(k^2)\}^2 d^4 k}{k^2 \{(p-k)^2 + m^2\}} + \frac{4}{m^2} \int \frac{\{F(k^2)\}^2 d^4 k}{k^2} - \frac{6\mu}{m^4} \int \frac{k^2 [G'(k^2)]^2}{k^2} d^4 k \\ &\quad \text{(for } \pi) \quad (3.3) \\ &= -\frac{4}{m^2} \int \frac{(2p-k)^2 F^{(S)}(k^2) F^{(V)}(k^2) d^4 k}{k^2 \{(p-k)^2 + m^2\}} + \frac{16}{m^2} \int \frac{F^{(S)}(k^2) F^{(V)}(k^2)}{k^2} d^4 k \\ &\quad - \frac{12\mu}{m^4} \int \frac{k^2 [G_1^{(V)}(k^2)]^2}{k^2} d^4 k \quad \text{(for } K) \quad (3.3') \end{aligned}$$

where p is a meson four momentum and G' denotes the difference of form factors of the charged and neutral pions. Discussion will be separately done in two cases:

- (a) The case where the polarizability term is disregarded, ($\mu=0$)
- (b) The case where it is taken into account. ($\mu \neq 0$)

(a) ($\mu=0$)

$F(k^2)$ will be as follows:

$$F(k^2) = F_1(k^2) + \kappa \frac{k^2}{\Lambda^2} F_2(k^2), \quad (3.4)$$

with

$$\begin{aligned} F_i(k^2) &= T_3 F_i(k^2), & \text{(for } \pi) \\ F_i(k^2) &= \mathbf{1} F_i^{(S)}(k^2) + \tau_3 F_i^{(V)}(k^2) & \text{(for } K). \end{aligned} \quad (3.4')$$

The second term in Eq. (3.4) involves the Pauli term as shown in Eq. (2.6). κ is a parameter and Λ is the cutoff momentum. Feynmann's cut* is adopted for $F_1(k^2)$ and $F_2(k^2)$:

$$F_i(k^2) = \frac{\Lambda^{2n}}{(k^2 + \Lambda^2)^n} \quad (n \geq 2), \quad (3.5)$$

* Attention must be paid to that the form factor of this type satisfies the macroscopic causality.⁴⁾

with an appropriate value of κ we can realize the experimental pion mass difference. In the case of kaon, however, any real value of κ cannot give the observed mass difference for such a simple type for various values of Λ . At a glance, it seems possible to give a correct mass difference by choosing the form factor of the type:

$$F^{(S)}(k^2)F^{(V)}(k^2) < 0. \quad (3.6)$$

But as far as we take Eq. (3.5) for the form factor, this choice is not allowable owing to the restriction on the normalization condition:

$$\begin{aligned} \int f^{(S)}(\vec{x}) d^3x &= F^{(S)}(k^2)|_{k=0} = \frac{1}{2}, \\ \int f^{(V)}(\vec{x}) d^3x &= F^{(V)}(k^2)|_{k=0} = \frac{1}{2}. \end{aligned} \quad (3.7)$$

As a modification of Feynmann's cut, we shall take the following ones:

$$F^{(S)}(k^2), F^{(V)}(k^2) = \frac{1}{2} \int \frac{\lambda^4 f(\lambda) d\lambda}{(k^2 + \lambda^2)^2}, \quad (3.8)$$

$$F^{(S)}(k^2, \Lambda^2), F^{(V)}(k^2, \Lambda^2) = \sum_{n=2}^{\infty} a_n \frac{\Lambda^{2n}}{(k^2 + \Lambda^2)^n}. \quad (3.9)$$

In the above expression, it may be possible in general to choose such a value of a_n or such a type of $f(\lambda)$ that the observed mass difference may be reproduced. For example, we get the result using Eq. (3.8) with

$$f(\lambda) = \delta(\lambda - \Lambda) + c\delta(\lambda - \Lambda'),$$

by taking $\Lambda \simeq$ (baryon mass) and $\Lambda' \simeq$ (pion mass) with a suitable value of c . Although there are infinite possibilities for reasonable choices of form factors, it is clear that the charge distribution of the neutral kaon is concentrated near $r \simeq 0$ and cannot have a simple form such as the exponential type. These results have also been shown by a few authors.⁶⁾

(b) ($\mu \neq 0$)

If we introduce the polarizability-term, it is possible, of course, in the case of kaon to get a consistent result even for a simple form factor by moderating the parameter μ . As it is not interesting to treat various form factors, we will try only the exponential type with some assumption as follows:

$$\left. \begin{aligned} F(k^2) &= \Lambda^4 / (k^2 + \Lambda^2)^2, & (\text{for } \pi) \\ F_1(k^2) &= F_2(k^2), \quad F^{(S)}(k^2) = F^{(V)}(k^2) = \frac{\Lambda^4}{2(k^2 + \Lambda^2)^2}, & (\text{for } K) \\ G(k^2) &= \frac{\Lambda^4}{(k^2 + \Lambda^2)^2}, & (\text{for } \pi) \\ G^{(V)}(k^2) &= \frac{\Lambda^4}{2(k^2 + \Lambda^2)^2}, & (\text{for } K) \end{aligned} \right\} \quad (3.10)$$

Then the following relation between κ and μ is obtained from the mass difference, for example, for pion

$$66\kappa^2 + 68\kappa - 45 - 2400\mu = 0, \quad \text{for } A/m = 7 \quad (3.11)$$

and for kaon

$$8\kappa^2 + 9\kappa + 42 - 67\mu = 0, \quad \text{for } A/m = 2.4. \quad (3.11')$$

If the cutoff momentum is taken to be nearly equal to the baryon mass, the following statement can be made insensitively to the magnitude of the cutoff momentum; namely μ has the maximum value -0.03 for pion and the minimum value 0.5 for kaon corresponding to the κ value 0.5 for pion and kaon.

§ 4. Discussion

In the previous section we have phenomenologically analyzed the magnitude of three kinds of interaction terms, i.e. the current, the Pauli and the polarizability term. In the present section we shall discuss these results from the standpoint of field theory.

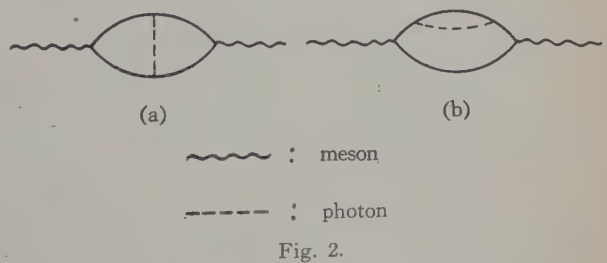
The coefficient of the Pauli term may be of the order of magnitude in meson-mass unit

$$\kappa / \left(\frac{A}{m} \right)^2 \simeq 10^{-1}. \quad (4.1)$$

This value is a little larger than the value which Miyachi and Nakano³⁾ evaluated by the lowest perturbation calculation, but it seems to be reasonable since the contribution of the anomalous magnetic moment interaction of the virtual baryon had not been considered in their calculation.

We have shown that even in the case of (a), it is possible to reproduce the correct mass difference of kaon by means of a non-simple form factor which is concentrated near $r \simeq 0$. However, it may be quite difficult to expect that a curious form factor adopted in the present paper can be derived from the field-theoretical picture such as those in which the charge and the magnetic moment of kaons are considered to be smeared out around the center of the kaon through the virtual pairs of baryons.

The sign of $\mu (>0$ for kaons and <0 for pions) seems to be natural according to the result of the calculation of perturbation in the lowest order. The Feynmann graph corresponding to our polarizability term is shown in Fig. 2. The



case of Fig. 2 (b) need not be considered, since this effect may be reduced to the mass renormalization of baryon by assuming that the mass of the virtual baryon is not so different from that of real one. In the graph (a), the effect caused by the electromagnetic interaction of the ordinary current-type of baryons makes μ minus for kaon and pion; namely it makes charged meson heavier than the neutral. On the other hand, considering the effect from the interaction due to the anomalous magnetic moment of the virtual baryon, we can reproduce the same result for μ as that obtained in the previous section by choosing some suitable coupling constants of $(N-A-K)$, etc., and by taking some moderate magnitude of the anomalous magnetic moment of baryons. It is remarkable that the sign of μ of kaon to be minus under the assumption of the global symmetry. Since the more detailed calculation of the magnitude of μ may be of no use at the present stage, we confine ourselves to pointing out the possibility of deriving the constant μ from the field theoretical consideration and to making a comparison of our μ with that of the nucleon. The energy shift of meson in a constant electric field E is as follows:

$$\Delta E \equiv \frac{1}{2} \alpha E^2 = \langle H_{pot}(\vec{x}) \rangle \simeq \frac{\mu e^2}{2m^3} E^2. \quad (4.2)$$

According to the results on μ in § 3, α may be of the order of 10^{-43} cm^3 for pion and kaon. The experimental value for neutron is shown to be $10^{-41} \sim 10^{-42} \text{ cm}^3$.⁷⁾ Taking into account that the polarization of the meson is caused by baryon pair clouds, this result may be regarded as reasonable.

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A Note on the Leptonic Decay of Hyperons*

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The leptonic decay modes of pions and K -mesons are studied from the viewpoint of compound model (Sakata model). By comparing these processes ($\pi \rightarrow \mu + \nu$ and $K \rightarrow \mu + \nu$), a conjecture for the leptonic decay of hyperons (*e.g.* $\Lambda \rightarrow p + \mu^- + \bar{\nu}$) is given which suggests that the (squared) bare coupling constant of this process is smaller than that of ordinary β -decay or μ -capture process of nucleons by a factor ~ 10 .

§ 1. Introduction

Recent experimental investigations on the weak interaction of elementary particles seem to suggest that all the leptonic decay processes of non-strange particles are caused by the “ V (vector) — A (axial-vector) combination of four-fermion interactions”, the scheme of which was proposed by many authors.¹⁾ As is well known, all these interactions have the coupling constants of the same magnitude (1.4×10^{-49} erg cm³) apart from the effects of renormalization due to the strong interactions. Under these circumstances, it may be quite interesting to clarify whether or not the above-mentioned universal V — A scheme can further be extended to the leptonic decay process of strange-particles such as, for example,

$$\Lambda \rightarrow p + \mu^- (e^-) + \bar{\nu} \quad (a)$$

$$\Sigma^- \rightarrow n + \mu^- (e^-) + \bar{\nu}. \quad (a')$$

At present, however, experimental evidences²⁾ concerning the process (a) or (a') are not yet sufficient to extract vital information about the structure of decay interactions of this process. Nevertheless, if we suppose that the leptonic decay of K -mesons should be induced by four-fermion interactions of the type (a) and/or (a'), it would be possible to some extent to investigate into the nature of the leptonic process of strange particles by using, as a clue, the transition probability of the decay

$$K \rightarrow \mu + \nu \quad (b)$$

which has been well established by experiments.

In this note, we shall discuss the decay process (b) from the viewpoint of the Sakata model,³⁾ and in comparing this process with the decay $\pi \rightarrow \mu + \nu$ a con-

* The essential part of this work was prepared for the Kiev Conference (1959). See “Proceedings of the 9-th Conference on High Energy Nuclear Physics” (Kiev, 1959).

jecture will be given that the (squared) *unrenormalized* coupling constants of the strangeness-non-conserving four-fermion interaction such as (a) should be smaller than those of the ordinary Fermi interactions by a factor 10, and in this sense the "universality" of the $V-A$ scheme is somewhat violated in so far as we are concerned with the leptonic process of strange particles. It is to be mentioned that a similar result has been suggested also by Oneda⁴⁾ in the phenomenological analysis of various decay processes including K -mesons and baryons.

§ 2. Relation between decays $K \rightarrow \mu + \nu$ and $\Lambda \rightarrow p + \mu^- + \bar{\nu}$ in the Sakata model

The most direct way of clarifying the relation between the processes (a) and (b) is to treat our problem on the basis of the composite model proposed by Sakata in which, for example, the K^+ -meson is considered to be a compound particle composed of the basic particles p and \bar{A} (anti- A). Let us assume that the decays $\pi \rightarrow \mu + \nu$ and $K \rightarrow \mu + \nu$ are induced by the following local (V, A) four-fermion interactions respectively:

$$H_1 = f_V (\bar{\psi}_\mu \gamma_\lambda (1 + \gamma_5) \psi_\nu) (\bar{\psi}_n \gamma_\lambda \psi_p) + f_A (\bar{\psi}_\mu i \gamma_\lambda \gamma_5 (1 + \gamma_5) \psi_\nu) (\bar{\psi}_n i \gamma_\lambda \gamma_5 \psi_p) + \text{h. c.} \quad (1)$$

$$H_2 = f_V' (\bar{\psi}_\mu \gamma_\lambda (1 + \gamma_5) \psi_\nu) (\bar{\psi}_A \gamma_\lambda \psi_p) + f_A' (\bar{\psi}_\mu i \gamma_\lambda \gamma_5 (1 + \gamma_5) \psi_\nu) (\bar{\psi}_A i \gamma_\lambda \gamma_5 \psi_p) + \text{h. c.} \quad (2)$$

with obvious notations. We have only to remark that f 's and ψ 's are *unrenormalized* quantities. We assume here K -mesons are pseudoscalar as well as pions, and the relative parity of A and nucleon is even. This parity assignment seems to be preferable according to the recent analyses.⁵⁾ In our model, pions and K -mesons are described by the Bethe-Salpeter wave functions for bound states as follows^{6), 7)}:

$$\varphi_\pi(x, y) = \langle \mathcal{Q} | T[\psi_p(x) \bar{\psi}_n(y)] | \pi \rangle \quad (3)$$

$$\varphi_K(x, y) = \langle \mathcal{Q} | T[\psi_p(x) \bar{\psi}_A(y)] | K \rangle, \quad (4)$$

where $|\mathcal{Q}\rangle$ and $|\pi\rangle$ (or $|K\rangle$) respectively denote the vacuum and the one-pion (or one-kaon) state.* Then, in terms of these amplitudes the S -matrix for the decay, e.g. $\pi \rightarrow \mu + \nu$, is expressed (in the lowest order perturbation of (1)) by

$$\langle \mu, \nu | S | \pi \rangle = i(2\pi)^4 \delta(p_\mu + p_\nu - p_\pi) (\bar{u}(p_\mu) \gamma_\lambda \gamma_5 (1 + \gamma_5) u(p_\nu)) f_A \text{Tr}[\gamma_\lambda \gamma_5 \varphi_\pi(0)], \quad (5)$$

where p 's are the momenta of respective particles and u the Dirac wave function, noticing the relation:

$$\langle \mathcal{Q} | \bar{\psi}_n \gamma_\lambda \gamma_5 \psi_p(0) | \pi \rangle = \text{Tr}[\gamma_\lambda \gamma_5 \varphi_\pi(x, y)]|_{x=y=0} \equiv \text{Tr}[\gamma_\lambda \gamma_5 \varphi_\pi(0)].$$

In the same way, for the decay $K \rightarrow \mu + \nu$, we have

* (3) and (4) are the expressions for positively charged bosons.

$$\langle \mu, \nu | S | K \rangle = i(2\pi)^4 \delta(p_\mu + p_\nu - p_K) (\bar{u}(p_\mu) \gamma_\lambda (1 + \gamma_5) u(p_\nu)) f_A' \text{Tr}[\gamma_\lambda \gamma_5 \varphi_K(0)]. \quad (6)$$

The φ_π and φ_K in these formulae are the *normalized* amplitudes which should be determined by the normalization condition:

$$i \int_0 \langle B | \bar{\psi}_p \gamma_\mu \psi_p | B \rangle d\sigma_\mu = 1, \quad (7)$$

where $|B\rangle$ denotes $|\pi\rangle$ or $|K\rangle$. For later purposes, we give here a general method to find the normalized amplitude. First, we start with the space-time integration of the expectation value for the current operator density,⁸⁾ viz.

$$\begin{aligned} \int \langle B | j_\mu(x_0) | B \rangle d^4x_0 &= \langle B | j_\mu(0) | B \rangle VT \\ &= \int d^4x_1 \cdots d^4x_4 \bar{\varphi}_B(x_1, x_2) G_\mu(x_1, x_2; x_3, x_4) \varphi_B(x_3, x_4), \end{aligned} \quad (8)$$

where VT is the space (V)-time (T) volume of the world, G_μ the suitable Green's function and $\bar{\varphi}_B$ the adjoint wave function of φ_B . Then, separating the centre-of-mass motion from φ_B :

$$\varphi_B(x, y) = \tau_B(\xi) e^{iPX} \quad (9)$$

in which $\xi = x - y$, P the total energy-momentum and X the centre of mass, (8) may be transformed into the form

$$\langle B | j_\mu(0) | B \rangle = N \int d^4\xi d^4\eta \bar{\nu}_B(\xi) J_\mu(\xi, \eta; P) \nu_B(\eta), \quad (10)$$

where $\tau_B(\xi)$ is replaced by $N^{1/2} \nu_B(\xi)$. In (10), N denotes the normalization factor and ν_B means a norm-undetermined amplitude which is directly determined from the homogeneous equation for the bound states. Defining $\mathcal{J}(P^2)$ to be

$$\int \bar{\nu}_B J_\mu \nu_B \equiv 2P_\mu \mathcal{J}(P^2), \quad (11)$$

we find the required amplitude:

$$\tau_B(\xi) = \frac{1}{\sqrt{2P_0 V}} \mathcal{J}^{-1/2}(P^2) \nu_B(\xi). \quad (12)$$

For a pseudoscalar boson, $\nu_B(0)$ generally takes the form

$$\nu_B(0) = a(P^2) \gamma_5 + b(P^2) \gamma_5 (\gamma P). \quad (13)^*$$

Accordingly, the terms $\text{Tr}(\cdots)$ in (5) and (6) can read

* If we define the wave function φ_B by $\varphi_B = \langle \mathcal{Q} | T(\psi_p, \psi_{B'}) | B \rangle$ ($\psi_{B'} = \psi_{n'}$ (or $\psi_{A'}$) for $|B\rangle = |\pi\rangle$ (or $|K\rangle$)), where $\psi_{B'} = C \psi_B$ (C : the charge conjugation matrix), we have $\nu_B = [a\gamma_5 + b\gamma_5(\gamma P)] C$ instead of (13).

$$\text{Tr}[\gamma_\lambda \gamma_5 \tau_B(0)] = \frac{4}{\sqrt{2P_0 V}} \mathcal{J}^{-1/2}(P^2) b(P^2) P_\lambda. \quad (14)$$

It may be remarked here that the factor

$$4 \mathcal{J}^{-1/2}(P^2) a(P^2) \Big|_{P^2 + \mu_B^2 = 0} \quad (\mu_B: \text{the mass of the particle } B)$$

is to be considered as a quantity which corresponds to the renormalization constant $Z_3^{1/2}$ for one-body boson propagator, although in our case it might not be interpreted as the bare particle probability.

Let us turn to the experimental situations which provide information about the decays $\pi \rightarrow \mu + \nu$ and $K \rightarrow \mu + \nu$. When we describe these processes by the phenomenological Hamiltonian:

$$H_{\text{eff}}^1 = (g_\pi / \mu_\pi) (\bar{\psi}_\mu i \gamma_\lambda (1 + \gamma_5) \psi_\nu) \frac{\partial \phi_\pi}{\partial x_\lambda} + \text{h. c.} \quad (15)$$

$$H_{\text{eff}}^2 = (g_K / \mu_K) (\bar{\psi}_\mu i \gamma_\lambda (1 + \gamma_5) \psi_\nu) \frac{\partial \phi_K}{\partial x_\lambda} + \text{h. c.}, \quad (16)$$

where ϕ_π and ϕ_K are the field operators for pions and K -mesons respectively, we find the relation

$$\left(\frac{g_\pi \mu_K}{g_K \mu_\pi} \right)^2 \approx 16 \quad (17)$$

from the magnitudes of respective transition probabilities.⁹⁾ In terms of the expression (14), we obtain instead of (17) the relation

$$\left(\frac{f_A \beta_\pi}{f_A' \beta_K} \right)^2 \approx 16 \quad (17')$$

where $\beta \equiv 4 \mathcal{J}^{-1/2}(-\mu_B^2) b(-\mu_B^2)$ (see (14)), which correlates the magnitudes of the wave functions of ϕ_π and ϕ_K at the origin to the (unrenormalized) coupling constants in the decay processes (1) and (2).*

Up to this stage, we have not introduced any specific assumptions or approximations concerning the structure of pions and K -mesons. The relation (17') is, therefore, valid even for the standard theory of pions and K -mesons which describes these particles as 'elementary'. However, the theoretical predictions for the ratio β_π / β_K may be different according to the kind of the model for pions and K -mesons adopted. In any case, it is to be remarked that, if we assume the scheme of "universal $V-A$ Fermi interaction" to hold even for the strangeness-violating leptonic decays, we should have the relation $\beta_\pi^2 \approx 16 \beta_K^2$ from experiments.

* A similar attempt has been developed also by S. Tanaka.¹⁰⁾ But since his approach was essentially restricted to the nonrelativistic two-body approximation, no general relations such as (17') were obtained.

We shall now try to estimate the ratio β_π/β_K according to the Sakata model. As was investigated previously,¹¹⁾ the pions and K -mesons are described as the bound states resulting from the effective direct interactions of the type:

- (i) $(\bar{p}On)(\bar{n}Op)$ for π^\pm
 (ii) $(\bar{p}O'A)(\bar{\Lambda}O'p)$ for K^\pm ,

where O and O' denote the Dirac matrices. The general forms of these interactions required for obtaining the pseudoscalar mesons are found to be an arbitrary linear combination of P and A , if we confine ourselves to the ladder approximation (or chain-approximation⁷⁾).

We can, however, simplify our situation one step further, assuming that the structure of the interaction of (i) and (ii) are the same and the only one type of interactions viz. P would be dominant. The first assumption is a consequence of a desired symmetry property of the Sakata model recently introduced by Ogawa and others.¹²⁾ In this proposed symmetry, the three basic particles p , n and Λ play essentially the same role in the system of strongly interacting particles. In an ideal case in which p , n and Λ are treated completely on the equal footing, we immediately obtain the result:

$$\beta_\pi = \beta_K. \quad (18)$$

If β does not critically depend on μ_B , we may conclude that $(\beta_\pi/\beta_K)^2 \sim 1$ even for the actual case in which the mass difference $\mu_K - \mu_\pi$ and κ_1 (mass of Λ) - κ (mass of the nucleon) are not equal to zero. In fact, this is the case when we introduce the second assumption (the predominance of pseudoscalar interaction). We feel that this assumption is also a conceivable one, since it provides the effective $PS(ps)$ -coupling as the main part of the pion-nucleon interactions.⁷⁾ Under these assumptions and following the general prescription of normalization developed at the first stage of this section, the β_π and β_K are easily evaluated to be*

$$\beta_\pi = -\frac{\sqrt{2}}{2\pi} \kappa \left(\ln \frac{\lambda^2}{\kappa^2} + \frac{1}{6} \frac{\mu_\pi^2}{\kappa^2} + \dots \right) \left(\ln \frac{\lambda^2}{\kappa^2} + \frac{1}{2} + \frac{1}{2} \frac{\mu_\pi^2}{\kappa^2} + \dots \right)^{-1/2} \quad (19)$$

$$\beta_K = -\frac{\sqrt{2}}{2\pi} \left(\frac{\kappa + \kappa_1}{2} \right) \left(\ln \frac{\lambda^2}{\kappa_1^2} + \frac{1}{6} \frac{\mu_K^2}{\kappa_1^2} + \dots \right) \left(\ln \frac{\lambda^2}{\kappa_1^2} + \frac{1}{2} + \frac{1}{2} \frac{\mu_K^2}{\kappa_1^2} + \dots \right)^{-1/2} \quad (20)$$

respectively, where λ is the cutoff mass in Feynman's cutoff device ($\lambda \gtrsim \kappa$). We have omitted in these expressions the terms of higher order of expansion in μ_π^2/κ^2 or μ_K^2/κ_1^2 and some terms which depend on $(\kappa_1^2 - \kappa^2)/\kappa_1^2$ since the corrections due to these terms seem to be too small to modify our discussion. From (19) and (20), one obtains

* Calculations are made along the same line as developed in reference 7).

$$(\beta_\pi/\beta_K)^2 = 0.9 \sim 0.8 \quad (21)$$

for $\ln(\lambda^2/\kappa^2)$ or $\ln(\lambda^2/\kappa_1^2) = 1 \sim 5$.

It is to be noted that whenever the magnitude of β_π and β_K are essentially determined by the binding energies required for constructing pions and K -mesons from the constituent particles, we always obtain the relation $(\beta_\pi/\beta_K)^2 \approx 1$. Thus, if we accept this relation, implying $f_A^2 \approx 16f_A'^2$, we arrive at the conjecture stated in the preceding section.

§ 3. Discussions

In connection with our arguments, it would be interesting to compare our results with that obtained by using the dispersion technique based on the current meson theories. Recently, Sakita¹³⁾ discussed this problem, applying the method developed by Goldberger and Treiman.¹⁴⁾ His results seem to be not so different from those obtained in this paper. The essential difference between both theories, however, lies in the fact that his results are concerned with the renormalized coupling constants (for the leptonic decay of hyperons) and depend on the other interaction constants such as of pion-nucleon and K -meson-baryon systems, whereas our conjecture is the one which correlates the unrenormalized constants directly with the magnitudes of wave functions of pions and K -mesons. The relation between renormalized and unrenormalized constants should be further investigated to clarify the true structure of weak interaction of elementary particles. If our tentative arguments for the smallness of the strangeness-violating interaction were found to reflect the actual case, the reason for the violation of "universality" of weak interaction would have to be explained from more profound knowledge on the internal structure of elementary particles.

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Macroscopic Causality and Analyticity of Scattering Amplitude in Quantum Field Theory

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On the basis of the macroscopic causality and the relativistic covariance it is shown that the scattering amplitudes are regular in the upper half plane as a function of the energy of the bombarding particle. This conclusion also holds for the theory in which the Hamiltonian does not exist, such as non-local field theory, so far as the theory is covariant. In order to require the macroscopic causality the scattering is investigated by means of the wave-packet-formalism. The comparison with the result of non-relativistic theory is also discussed.

§ 1. Introduction

The aim of this paper is to derive the analytic property of the scattering amplitudes as a function of the energy of the bombarding particle on the basis of the macroscopic causality and the relativistic covariance. This analyticity is usually expressed in terms of the dispersion relation and was widely discussed by many authors.¹⁾⁻⁴⁾ However, in the covariant field theory, we have so far stood always on the microscopic causality.

Recent information⁵⁾ obtained from high energy experiments on elementary particles seems to urge us to take into account some internal structure of elementary particles or to introduce the so-called non-local form factor in the interaction. It will be worth-while to make a research on the possibility of introducing the non-local interaction without violating at least the macroscopic causality or the possible type of form factors which are Lorentz-covariant and compatible with the macroscopic causality. In addition, it may be of use to investigate the relationship between the macroscopic and microscopic causality without assuming the existence of the Hamiltonian. Here it must be noted that the definition of microscopic causality is quite clear, namely, this means the commutability of field operators with space-like separation, on the other hand the "macroscopic causality" does not seem so obvious as the former. The precise definition of the macroscopic causality of the present paper will be given later.

For the purpose stated above we shall inquire into the analytic property of the scattering amplitude on the basis of the macroscopic causality, since the analytic behaviour will show us the difference between the both kinds of causalities as we have known that in the non-relativistic theory the analyticity depends on the extension of the interaction.⁶⁾⁻⁷⁾

Contrary to our expectation, we have obtained the conclusion that the macroscopic causality gives the same analytic behaviour of the scattering amplitude as that derived from the microscopic causality. This result seems to suggest us that the postulate of the relativistic covariance is so strong that the macroscopic causality is effectively reduced to the microscopic causality.

In dealing with the macroscopic causality it may be of convenience to make use of the wave-packet-formalism.⁸⁾ In section 2 we shall give a brief review of the wave-packet-formalism in quantum field theory. Section 3 will be devoted to the discussion on the transition matrix in the wave-packet-formalism. In section 4 the analyticity of the scattering amplitude will be derived. In section 5 we discuss the analyticity of the scattering amplitude for the non-relativistic theory. In section 6 some discussions are made as the conclusion of sections 4 and 5. Although our discussions can be made without regard to the Hamiltonian, the assumption of the existence of the Hamiltonian makes it very easy to derive expressions (2.1) and (2.2), the derivation of which will be given in Appendix A for the case where the Hamiltonian is present, and B for the case where the Hamiltonian is not assumed to be present.

§ 2. Wave packet

In this section we shall give a review of the wave-packet-formalism in the field theory, on which a paper has been published by the present author⁸⁾ from a somewhat general viewpoint. According to this paper, a state of two particles A and B having the average momenta \mathbf{p} and $-\mathbf{p}$ and the average positions \mathbf{R} and \mathbf{R}' respectively is denoted by $|\mathbf{p}, \mathbf{R}, A; -\mathbf{p}, \mathbf{R}', B\rangle$, which is represented in terms of outgoing or incoming waves in the following way:

$$|\mathbf{p}, \mathbf{R}, A; -\mathbf{p}, \mathbf{R}', B\rangle = \int d\mathbf{k}_1 d\mathbf{k}_2 F(\mathbf{p}, \mathbf{R}, \mathbf{k}_1) F(-\mathbf{p}, \mathbf{R}', \mathbf{k}_2) |\mathbf{k}_1, A; \mathbf{k}_2, B\rangle^+ \quad (2.1)$$

for the case where the spatial separation $\mathbf{r} = \mathbf{R}' - \mathbf{R}$ and the momentum \mathbf{p} are parallel to each other, or

$$|\mathbf{p}, \mathbf{R}, A; -\mathbf{p}, \mathbf{R}', B\rangle = \int d\mathbf{k}_1 d\mathbf{k}_2 F(\mathbf{p}, \mathbf{R}, \mathbf{k}_1) F(-\mathbf{p}, \mathbf{R}', \mathbf{k}_2) |\mathbf{k}_1, A; \mathbf{k}_2, B\rangle^- \quad (2.2)$$

for the case where \mathbf{r} and \mathbf{p} are antiparallel. Here $|\mathbf{k}_1, A; \mathbf{k}_2, B\rangle^\pm$ is the scattering state of A and B particles with the incident momenta \mathbf{k}_1 and \mathbf{k}_2 , and the subscripts \pm refer to the boundary condition of the outgoing-wave and incoming-wave respectively. $F(\mathbf{p}, \mathbf{r}, \mathbf{k})$ is the wave function in momentum representation describing the wave packet. The expressions (2.1) and (2.2) are most important in the present paper in the sense that the limitation of the validity and the applicability of the result of the present paper depends on that of (2.1) and (2.2). The derivation of (2.1) and (2.2) will be made in Appendices A and B.

§ 3. Transition matrix

Let us consider the following expression :

$$\int dk_1 dk_2 dk_3 dk_4 F^*(\mathbf{p}', \mathbf{R}', \mathbf{k}_3) F^*(-\mathbf{p}', \mathbf{R}' - \mathbf{r}', \mathbf{k}_4) F(\mathbf{p}, \mathbf{R}, \mathbf{k}_1) F(-\mathbf{p}, \mathbf{R} + \mathbf{r}, \mathbf{k}_2) \\ \times \langle \mathbf{k}_3, A; \mathbf{k}_4, B | \mathbf{k}_1, A; \mathbf{k}_2, B \rangle^+ \exp[-i\{E_A(\mathbf{k}_3) + E_B(\mathbf{k}_4)\}t], \quad (3.1)$$

where $\langle \mathbf{k}_3, A; \mathbf{k}_4, B | \mathbf{k}_1, A; \mathbf{k}_2, B \rangle^+$ is the so-called S -matrix and can be decomposed into two terms :

$$\langle \mathbf{k}_3, A; \mathbf{k}_4, B | \mathbf{k}_1, A; \mathbf{k}_2, B \rangle^+ = \delta(\mathbf{k}_1 - \mathbf{k}_2) \delta(\mathbf{k}_3 - \mathbf{k}_4) + \delta(\mathbf{k}_3 + \mathbf{k}_4 - \mathbf{k}_1 - \mathbf{k}_2) \\ \times \delta(E_A(\mathbf{k}_3) + E_B(\mathbf{k}_4) - E_A(\mathbf{k}_1) - E_B(\mathbf{k}_2)) \{E_A(\mathbf{k}_1) E_B(\mathbf{k}_2) E_A(\mathbf{k}_3) E_B(\mathbf{k}_4)\}^{-1/2} t(W, \Delta^2), \quad (3.2)$$

where W is the square of the energy in the center-of-mass system and Δ^2 is the square of the invariant momentum transfer, that is,

$$W = (\mathbf{k}_3 + \mathbf{k}_4)^2 - \{E_A(\mathbf{k}_3) + E_B(\mathbf{k}_4)\}^2$$

and

$$\Delta^2 = (\mathbf{k}_2 - \mathbf{k}_4)^2 - \{E_B(\mathbf{k}_2) - E_B(\mathbf{k}_4)\}^2$$

in the arbitrary system of coordinate, and $t(W, \Delta^2)$ is the so-called scattering amplitude whose analyticity is the main object of our investigation. If both $|\mathbf{r}|$ and $|\mathbf{r}'|$ are sufficiently larger than the extension of the wave packet $F(\mathbf{p}, \mathbf{R}, \mathbf{k})$, and the directions of \mathbf{p} and \mathbf{p}' are nearly equal to those of \mathbf{r} and \mathbf{r}' respectively, then we can see that (3.1) becomes the transition matrix

$$\langle \mathbf{p}', \mathbf{R}', A; -\mathbf{p}', \mathbf{R}' - \mathbf{r}', B | \exp(-iHt) | \mathbf{p}, \mathbf{R}, A; -\mathbf{p}, \mathbf{R} + \mathbf{r}, B \rangle \quad (3.3)$$

in the center-of-mass system by using (2.1) and (2.2). We have assumed the existence of the Hamiltonian in so far as we are concerned with (3.3). However, this assumption is superfluous, since, as we shall see in section 4, it is sufficient to consider only the case of $t=0$ for (3.1) and (3.3). A prescription for getting the transition probability from (3.3) has already been given in reference 8).

Now we shall inquire into the restriction on $\mathbf{p}, \mathbf{p}', \mathbf{R}, \mathbf{r}, \mathbf{R}', \mathbf{r}'$ for getting a non-vanishing (3.1). For this purpose we shall discuss (3.1) without recalling the restrictions on $\mathbf{r}, \mathbf{p}, \mathbf{r}', \mathbf{p}'$. In (3.1) only the second term of (3.2) is of interest, since the first term describes the transition without mutual interactions.

Let us look over the properties of the wave function describing the wave packet. In the first place, $F(\mathbf{p}, \mathbf{r}, \mathbf{k})$ is assumed to be zero except for $\mathbf{k} \approx \mathbf{p}$. We denote this uncertainty of \mathbf{k} by $\Delta \mathbf{p}$. The wave function of the ordinary space is

$$f(\mathbf{p}, \mathbf{r}, \mathbf{x}) = (2\pi)^{-3/2} \int d\mathbf{k} \exp(i\mathbf{k} \cdot \mathbf{x}) F(\mathbf{p}, \mathbf{r}, \mathbf{k}). \quad (3.4)$$

It is also assumed that the wave function $f(\mathbf{p}, \mathbf{r}, \mathbf{x})$ is zero except for $\mathbf{x} \approx \mathbf{r}$. We denote this uncertainty of \mathbf{x} by $\Delta \mathbf{r}$. We have $\Delta \mathbf{p} \cdot \Delta \mathbf{r} \approx 1$ by uncertainty principle. The reason why (3.4) vanishes for $|\mathbf{x} - \mathbf{r}| \gg \Delta \mathbf{r}$ is considered to be that a strong

cancellation occurs owing to the large variation of the phase of the integrand for the small variation of \mathbf{k} . Therefore, the postulate that the our wave packet $f(\mathbf{p}, \mathbf{r}, \mathbf{x})$ is located in the vicinity of \mathbf{r} gives to the function $F(\mathbf{p}, \mathbf{r}, \mathbf{k})$ a strong restriction that the main part of the phase of $F(\mathbf{p}, \mathbf{r}, \mathbf{k})$ depending on \mathbf{k} is $\exp[-i\mathbf{k} \cdot \mathbf{r}]$, namely

$$F(\mathbf{p}, \mathbf{r}, \mathbf{k}) = \exp[-i\mathbf{k} \cdot \mathbf{r}] F'(\mathbf{p}, \mathbf{r}, \mathbf{k})$$

where the phase of $F'(\mathbf{p}, \mathbf{r}, \mathbf{k})$ varies in a negligibly small amount as compared with $\exp[-i\mathbf{k} \cdot \mathbf{r}]$. Then, if the variation of the absolute value of $t(W, \Delta^2)$ can be neglected, the restriction on $\mathbf{p}, \mathbf{R}, \mathbf{r}, \mathbf{p}', \mathbf{R}', \mathbf{r}'$ for getting non-vanishing (3.1) will be obtained by estimating the phase variation of the integrand of (3.1) for the variation $\Delta \mathbf{k}$ of $\mathbf{k} (\approx \mathbf{p})$. However, in general the variation of the absolute value of $t(W, \Delta^2)$ cannot be neglected; accordingly we shall take into account the influence of the variation of the absolute value in the later argument.

For the variation Δk_i of k_i in (3.1), we have

$$\Delta W = -2\{\Delta k_3 \cdot \mathbf{V}_A(\mathbf{p}') + \Delta k_4 \cdot \mathbf{V}_B(-\mathbf{p}')\} \cdot \{E_A(\mathbf{p}') + E_B(-\mathbf{p}')\}$$

and
$$\Delta(\Delta^2) = 2(\Delta k_2 - \Delta k_4) \cdot (\mathbf{p}' - \mathbf{p}),$$

where $\mathbf{V}(\mathbf{p}) = \mathbf{p}/E(\mathbf{p})$ is the group velocity of the wave packet. Therefore $\Delta(\Delta^2)$ is the scalar product of $\Delta k_2 - \Delta k_4$ and the finite vector so long as the momentum transfer is finite. Then we can neglect the effect of Δ^2 , since, as we shall see later, in the present paper we shall be concerned with the case of the infinite separation \mathbf{r}' between the two scattered packets.

In order to take into account the effect of the variation of the absolute value of $t(W, \Delta^2)$ together with its phase variation, it is convenient to introduce the Fourier transform by

$$t(W, \Delta^2) = \int_{-\infty}^{\infty} d\alpha \exp[i\alpha W] T(\alpha, \Delta^2), \quad (3.5)$$

because the variation of $T(\alpha, \Delta^2)$ with respect to \mathbf{k} is negligible owing to the above stated remark. At this stage of the discussion it should be noted that $T(\alpha, \Delta^2)$ is not unique since $t(W, \Delta^2)$ is defined only in the domain of

$$\left. \begin{aligned} W \leq -M_A^2 - M_B^2 - \Delta^2 - \sqrt{\Delta^2 + 2M_A^2} \sqrt{\Delta^2 + 2M_B^2} \equiv -f(\Delta^2) \\ \text{and} \quad \Delta^2 \geq 0. \end{aligned} \right\} \quad (3.6)$$

Without varying the value of $t(W, \Delta^2)$ in the domain of (3.6), we can add $T'(\alpha, \Delta^2)$ to $T(\alpha, \Delta^2)$ provided that $T'(\alpha, \Delta^2) \cdot \exp[if(\Delta^2)\alpha]$ is regular in the lower half plane of α . Making use of this ambiguity of $T(\alpha, \Delta^2)$ we can consider the infinite kinds of scattering amplitudes any one of which agrees with others in the domain of (3.6). Well-known examples of these amplitudes in the local theory are Feynman and causal amplitudes. In the discussion on the analyticity, we must

select the amplitude corresponding to the causal one in the local theory. This problem will be investigated in the following section.

In neglecting the variation of $t(W, \Delta^2)$ due to Δ^2 the effective phase variation of (3.1) is

$$\begin{aligned} \Delta = & \Delta k_3 \cdot R' + \Delta k_4 \cdot (R' - r') - \Delta k_1 \cdot R - \Delta k_2 \cdot (R + r) - \Delta k_3 \cdot V_A(p')t - \Delta k_4 \cdot V_B(-p')t \\ & - 2\{\Delta k_3 \cdot V_A(p') + \Delta k_4 \cdot V_B(-p')\}\{E_A(p') + E_B(-p')\}\alpha. \end{aligned} \quad (3.7)$$

Now Δk_i 's are restricted by the law of conservation of the total energy momentum. These restrictions are

$$\Delta k_1 + \Delta k_2 - \Delta k_3 - \Delta k_4 = 0 \quad (3.8)$$

$$\text{and} \quad \Delta k_1 \cdot V_A(p) + \Delta k_2 \cdot V_B(-p) - \Delta k_3 \cdot V_A(p') - \Delta k_4 \cdot V_B(-p') = 0. \quad (3.9)$$

In (3.7) and (3.9) we have replaced k with p . The error due to this substitution is of the order of $\Delta k \cdot \Delta p$ and can be neglected as compared with the remainders. By using (3.8) Δk_1 can be eliminated from (3.7) and (3.9), which lead

$$\begin{aligned} \Delta = & \Delta k_3 \cdot \{R' - R - V_A(p')t - 2V_A(p')[E_A(p') + E_B(-p')]\alpha\} \\ & + \Delta k_4 \cdot \{R' - r' - R - V_B(-p')t - 2V_B(-p')[E_A(p') + E_B(-p')]\alpha\} - \Delta k_2 \cdot r \end{aligned} \quad (3.10)$$

and

$$\Delta k_3 \cdot \{V_A(p) - V_A(p')\} + \Delta k_4 \cdot \{V_A(p) - V_B(-p')\} - \Delta k_2 \cdot \{V_A(p) - V_B(-p)\} = 0. \quad (3.11)$$

In order to make (3.1) non-vanishing, Δ must be nearly equal to zero for the variation Δk_i with the restriction (3.11). Using the undeterminate multiplier T , we get

$$R' - R - V_A(p')\{t - T + 2[E_A(p') + E_B(-p')]\alpha\} - V_A(p)T \approx 0, \quad (3.12)$$

$$R' - r' - R - V_B(-p')\{t - T + 2[E_A(p') + E_B(-p')]\alpha\} - V_A(p)T \approx 0 \quad (3.13)$$

$$\text{and} \quad r - V_A(p)T + V_B(-p)T \approx 0. \quad (3.14)$$

The last relation shows that the two initial packets make a collision after a time T . In above three equations, \approx means that the equality holds provided that the extension of the wave packet is neglected which is of the order of Δp or Δr , and furthermore that the variation of $t(W, \Delta^2)$ with respect to Δ^2 is omitted. Here it may be of advantage to give a precise meaning to the term "macroscopic." If some relationship holds when the size of the wave packet is neglected, this relationship is called a "macroscopic" one.

§ 4. Macroscopic causality and analyticity of scattering amplitude

We shall investigate the conditions for the scattering amplitude in order that

the macroscopic causality holds in the covariant field theory. Here the "macroscopic causality" means that the scattered wave does not emerge before the incident wave arrives at the scatterer provided that the size of the packets is neglected. In other words it means that the incident state is orthogonal to the scattered one in the above approximation.

Consider the case where the incident energy $E_A(\mathbf{p}) + E_B(-\mathbf{p})$ is extremely large compared with the momentum transfer $|\Delta|$. In such a case we have from (3.12) and (3.13) the following relation:

$$\mathbf{r}' \approx [\mathbf{V}_A(\mathbf{p}') - \mathbf{V}_B(-\mathbf{p}')] \{t - T + 2[E_A(\mathbf{p}') + E_B(-\mathbf{p}')] \alpha\}, \quad (4.1)$$

where the terms depending on the size of packets or those including Δ have been neglected owing to the large energy as above stated. In the relation (4.1), if we take the limit $E_A(\mathbf{p}') + E_B(-\mathbf{p}') \rightarrow \infty$, \mathbf{r}' tends to infinity, and in addition if α is assumed to be positive, the direction and the sense of \mathbf{r}' approximately agree with those of \mathbf{p}' respectively, since the relative group velocity $\mathbf{V}_A(\mathbf{p}') - \mathbf{V}_B(-\mathbf{p}')$ has the same direction and sense as those of \mathbf{p}' . Accordingly, it becomes possible to observe the two scattered packets with an infinite separation from each other after a finite time interval t . This curious situation obviously violates the principle of the macroscopic causality, because in the present theory the group velocity of the packets is still finite and does not exceed the light velocity even if the energy tends to infinity.

On the other hand, if α is negative \mathbf{r}' becomes antiparallel with \mathbf{p}' . In this case, the statevector

$$\int d\mathbf{k}_1 d\mathbf{k}_2 F(\mathbf{p}', \mathbf{R}', \mathbf{k}_1) F(-\mathbf{p}', \mathbf{R}' - \mathbf{r}', \mathbf{k}_2) |\mathbf{k}_1, A; \mathbf{k}_2, B\rangle^{-}$$

does not give such a simple physical picture as the two incoming packets and does not give rise to any paradoxical result. As to the detailed explanation on this complicated state, the reader is recommended to refer to the paper 8).

Thus we reach the conclusion that for the collision of the finite momentum transfer it is sufficient for satisfying the causality condition that $T(\alpha, \Delta^2)$ vanishes for the positive value of α . It will be seen that the above conclusion is always true without regard to the magnitude of t so long as the time interval $t - T$ after the collision is finite. For example, consider the case where t is equal to zero. In this case since $t - T < 0$ the collision of incident packets does not take place but the scattered outgoing packets can be found at infinitely remote points contrary to our common knowledge. Since t is zero, we have no collision, accordingly it is inadequate to talk about the temporal change of the colliding packets. This contradiction should rather be interpreted as follows: The incident states where two packets are approaching to each other is not orthogonal to the scattered state and the former includes the latter to some extent. Consequently it violates the definition of the principle of macroscopic causality in the present paper.

Speaking in general, the causality condition can be satisfied if the contribution

from the parts of the transition matrix which violates the causality vanishes after the integration with respect to α from $-\infty$ to $+\infty$, even though the above unfavourable parts do not identically vanish for $\alpha > 0$. The above statement can be easily accepted by remembering the ambiguity of $\mathbf{T}(\alpha, \Delta^2)$ in the previous section. Therefore we are able to make $\mathbf{T}(\alpha, \Delta^2)$ zero for $\alpha > 0$ by a suitable choice of $\mathbf{T}'(\alpha, \Delta^2)$. This choice of $\mathbf{T}'(\alpha, \Delta^2)$ is equivalent to selecting the amplitude corresponding to the causal one in the local theory. Strictly speaking, it is not sure that $\mathbf{T}(\alpha, \Delta^2)$ can always be made to vanish for $\alpha > 0$ by suitably choosing the ambiguous function $\mathbf{T}'(\alpha, \Delta^2)$, but we cannot simply say that the causality is violated even if $\mathbf{T}(\alpha, \Delta^2) \neq 0$ for $\alpha > 0$ as was already pointed out above.

If we denote by ω the energy of B particle in the laboratory system we have

$$W = -M_A^2 - M_B^2 - 2\omega M_A.$$

Then we can say that, except for the particular case discussed above, the scattering amplitude is analytic in the upper half plane as a function of the energy of the bombarding particle in the laboratory system.

§ 5. Non-relativistic theory

We shall apply the foregoing discussions to the non-relativistic theory and investigate the difference between the relativistic and non-relativistic theories. For this purpose we shall consider the scattering of the non-relativistic particle by the spherical symmetric potential. In this case (2.1) and (2.2) are rewritten as:⁹⁾

$$|\mathbf{p}, \mathbf{r}\rangle = \int d\mathbf{k} F(\mathbf{p}, \mathbf{r}, \mathbf{k}) |\mathbf{k}\rangle^+ \quad (5.1)$$

$$\text{and} \quad |-\mathbf{p}, \mathbf{r}\rangle = \int d\mathbf{k} F(-\mathbf{p}, \mathbf{r}, \mathbf{k}) |\mathbf{k}\rangle^-, \quad (5.2)$$

where \mathbf{r} is the position of the particle relative to the center of the potential. The expressions (5.1) and (5.2) are valid provided that the direction of \mathbf{r} is nearly equal to that of \mathbf{p} .

The transition matrix corresponding to (3.1) is

$$\int d\mathbf{k}_1 d\mathbf{k}_2 F^*(\mathbf{p}', \mathbf{r}', \mathbf{k}_2) F(\mathbf{p}, \mathbf{r}, \mathbf{k}_1) \langle \mathbf{k}_2 | \mathbf{k}_1 \rangle^+ \exp[-iE_2 t], \quad (5.3)$$

where $E_i = \mathbf{k}_i^2/2m$. The S -matrix $\langle \mathbf{k}_2 | \mathbf{k}_1 \rangle^+$ can be written as

$$\langle \mathbf{k}_2 | \mathbf{k}_1 \rangle^+ = \delta(\mathbf{k}_1 - \mathbf{k}_2) + \delta(E_1 - E_2) \mathbf{t}_n(E_2, \cos \theta), \quad (5.4)$$

where θ is the scattering angle. Now we shall introduce the Fourier transform by

$$\mathbf{t}_n(E, \cos \theta) = \int_{-\infty}^{\infty} d\alpha \exp[-i\alpha E] \mathbf{T}_n(\alpha, \cos \theta). \quad (5.5)$$

Although $\mathbf{T}_n(\alpha, \cos \theta)$ is not uniquely determined as we have seen in section 3,

we do not repeat the discussion on this problem. In neglecting the variation of $t_n(E, \cos \theta)$ due to $\cos \theta$, the phase variation of the integrand of (5.3) with respect to k 's is

$$\Delta_n = \Delta k_2 \cdot (\mathbf{r}' - \mathbf{p}'t/m - \alpha \mathbf{p}'/m) - \Delta k_1 \cdot \mathbf{r}. \quad (5.6)$$

The restriction due to δ -function in (5.4) is

$$\Delta k_2 \cdot \mathbf{p}' - \Delta k_1 \cdot \mathbf{p} = 0.$$

Then the conditions for getting non-vanishing value of (5.3) are

$$\mathbf{r}' \approx \mathbf{p}'(t/m - T/m + \alpha/m) \quad (5.7)$$

and

$$\mathbf{r} \approx -T\mathbf{p}/m. \quad (5.8)$$

Here we have used $-T/m$ as an undeterminate multiplier.

From (5.7) and (5.8) we can conclude that $T_n(\alpha, \cos \theta)$ should be zero for $\alpha > 0$ from the macroscopic causality. Then $t_n(E, \cos \theta)$ is regular in the upper half plane of E . However, it should be noted that in the foregoing discussion it has been assumed that $t_n(E, \cos \theta)$ has the Fourier transform. This assumption is not proper since the behaviour of $t_n(E, \cos \theta)$ in $E < 0$ is arbitrary. For example,

$$t_n(E, \cos \theta) = \exp[-i\sqrt{2mE}r_0] \int_{-\infty}^0 d\alpha \exp[-i\alpha E] T''(\alpha, \cos \theta) \quad (5.9)$$

has not Fourier transform, although (5.9) does not contradict with the macroscopic causality. We can easily see that r_0 in (5.9) gives the extension of the interaction.

§ 6. Discussions

The discussions so far made on the analyticity of the scattering amplitude in the covariant field theory have always been based on the microscopic causality, that is, the commutator or anticommutator of two Heisenberg operators should vanish if the operators are taken at a couple of points with a space-like separation. In this paper we have concluded the same analyticity by requiring the macroscopic causality. However, the postulate of the relativistic covariance seems to have played an important role, explicitly or implicitly, for deriving the same analyticity of the scattering amplitude. For example, that which makes us able to investigate the behaviour of the function t is due to the possibility of considering t as a function of the two invariant quantities W and Δ^2 . The more important role played by the relativistic postulate is that the group velocity of the packets is finite and less than the light velocity which comes from the relativistic expression of the energy E as a function of the momentum \mathbf{p} .

One might think that the covariant field theory with a non-local invariant interaction may lead us to the violation of the microscopic causality even though

the macroscopic causality is satisfied. However, according to the conclusion of the present paper this is not the case in so far as the momentum transfer Δ^2 can be neglected. In other words, we can say that it seems impossible to establish a non-local field theory which satisfies the requirement of the macroscopic causality.¹⁰⁾

Before concluding this section, it should be emphasized that in the covariant theory the existence of the exponential factor like $\exp[-i\sqrt{2mE}r_0]$ in (5.9) always leads to the violation of the macroscopic causality. For example, we see that

$$t(W, \Delta^2) = \exp[-i\sqrt{-W}t_0] \int_{-\infty}^0 d\alpha \exp[i\alpha W] T''(\alpha, \Delta^2) \quad (6.1)$$

contradicts with the macroscopic causality by investigating the collision in the laboratory system even though (6.1) does not lead to the apparent violation of the macroscopic causality if we take the center-of-mass system. Namely, in the center-of-mass system, $t_0\{V_A(\mathbf{p}) - V_B(-\mathbf{p})\}$ gives the extension of the interaction. Then, by a Lorentz transformation, there appears an extension of the interaction with an order of magnitude $(t_0/\sqrt{1-\beta^2})\Delta V$ where ΔV is the relative velocity in new Lorentz frame. If $|\mathbf{p}|$ is sufficiently large this extension can be made measurable by a macroscopic observation by taking a sufficiently large β whatever small t_0 may be. Therefore, in order that $(t_0/\sqrt{1-\beta^2})\Delta V$ be always of microscopic order, t_0 must be zero.

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Appendix A

Assuming the existence of the Hamiltonian, let us derive (2.1) and (2.2). As is well known,

$$|\mathbf{k}_1, A; \mathbf{k}_2, B\rangle^+ = z^{-1/2} \lim_{t \rightarrow \infty} \exp[-iHt] \exp[iH_0 t] |\mathbf{k}_1, A; \mathbf{k}_2, B\rangle. \quad (A.1)$$

Here z is the renormalization constant and $|\mathbf{k}_1, A; \mathbf{k}_2, B\rangle$ is the normalized eigenstate of the free Hamiltonian.

Now let us consider the following statevector:

$$\begin{aligned} & \int d\mathbf{k}_1 d\mathbf{k}_2 F(\mathbf{p}, \mathbf{R}, \mathbf{k}_1) F(-\mathbf{p}, \mathbf{R}', \mathbf{k}_2) |\mathbf{k}_1, A; \mathbf{k}_2, B\rangle^+ \\ &= z^{-1/2} \lim_{t \rightarrow \infty} \exp[-iHt] \exp[iH_0 t] \int d\mathbf{k}_1 d\mathbf{k}_2 F(\mathbf{p}, \mathbf{R}, \mathbf{k}_1) F(-\mathbf{p}, \mathbf{R}', \mathbf{k}_2) |\mathbf{k}_1, A; \mathbf{k}_2, B\rangle. \end{aligned} \quad (A.2)$$

We can easily understand that the statevector (A.2) represents the state which is produced in the first place by transforming

$$\int dk_1 dk_2 F(\mathbf{p}, \mathbf{R}, \mathbf{k}_1) F(-\mathbf{p}, \mathbf{R}', \mathbf{k}_2) |\mathbf{k}_1, A; \mathbf{k}_2, B\rangle \quad (\text{A} \cdot 3)$$

backward to the infinitely remote past with the free Hamiltonian H_0 and then by adiabatically switching in the interaction and finally by transforming (A·3) forward to the present ($t=0$) under the influence of the total Hamiltonian H . Therefore, if the collision of the two particles did not occur while moving from the infinite past to the present under the effect of H , (A·2) represents the state of the packets of the physical particles whose average positions are \mathbf{R} and \mathbf{R}' and average momenta \mathbf{p} and $-\mathbf{p}$. On the contrary, if the collision of two particles occurred during the motion, then $\mathbf{R}, \mathbf{R}', \mathbf{p}$ and $-\mathbf{p}$ do not indicate the average positions and momenta, since the packets have been disturbed by the collision a great deal.

One might worry about the convergence of the limit of (A·2), as the packet (A·3) would become spread in a vast region during an infinitely long time interval. Such an unfavourable situation would occur if we consider the limit

$$\lim_{t, t' \rightarrow \infty} \exp[-iHt] \exp[iH_0 t'] \int dk_1 dk_2 F(\mathbf{p}, \mathbf{R}, \mathbf{k}_1) F(-\mathbf{p}, \mathbf{R}', \mathbf{k}_2) |\mathbf{k}_1, A; \mathbf{k}_2, B\rangle$$

in place of (A·2). However, in our case we are considering the particular limit where the deformation of the packets are always cancelled out to some extent by the two operators $\exp[-iHt]$ and $\exp[iH_0 t]$ and we are only dealing with the finite amount of deformation of the packets which is essentially caused by the interaction. Accordingly, it may be quite reasonable to assume the existence of the limit of (A·2) which is the basic assumption of the present paper.

From the above discussion we can obtain the conclusion of this appendix. If (A·3) describes two wave packets which do not overlap each other and if the direction of $\mathbf{R}' - \mathbf{R}$ is nearly equal to that of \mathbf{p} , then (A·2) describes the state of the physical packets of the particles whose average momenta and positions are $\mathbf{R}, \mathbf{R}', \mathbf{p}$ and $-\mathbf{p}$. On the contrary, if the direction of $\mathbf{R}' - \mathbf{R}$ is nearly equal to that of $-\mathbf{p}$, then (A·2) does not describe such a state but the state after the collision. For $|\mathbf{k}_1, A; \mathbf{k}_2, B\rangle^-$, the similar discussions hold and we can easily obtain (2·2).

Appendix B

Contrary to the previous appendix, without assuming the existence of the Hamiltonian, we shall derive (2·1) and (2·2). Let us consider the following quantity:

$$^+ \langle \mathbf{k}_1, A_1; \mathbf{k}_2, B | \mathbf{p}, \mathbf{R}, A; -\mathbf{p}, \mathbf{R}', B \rangle \quad (\text{B} \cdot 1)$$

for the case where $\mathbf{R}' - \mathbf{R}$ and \mathbf{p} are parallel to each other. The statevector $|\mathbf{k}_1, A_1; \mathbf{k}_2, A_2\rangle^+$ can be separated into three parts: the incident plane wave, the outgoing spherical wave and the rest which vanishes rapidly for a large distance between the colliding particles. In what follows, we call these the first, the second and

the third part of $|\mathbf{k}_1, A_1; \mathbf{k}_2, A_2\rangle^+$. The only contribution to (B.1) may be expected to come from the first part of $|\mathbf{k}_1, A_1; \mathbf{k}_2, A_2\rangle^+$, since $|\mathbf{p}, \mathbf{R}, A; -\mathbf{p}, \mathbf{R}', B\rangle$ describes the wave packets which are approaching to each other. Furthermore, we see that (B.1) is not zero only in the case that A_1 and A_2 agree with A and B , and

$$^+\langle \mathbf{k}_1, A; \mathbf{k}_2, B | \mathbf{p}, \mathbf{R}, A; -\mathbf{p}, \mathbf{R}', B \rangle = F(\mathbf{p}, \mathbf{R}, \mathbf{k}_1) F(-\mathbf{p}, \mathbf{R}', \mathbf{k}_2). \quad (\text{B} \cdot 2)$$

This fact will be understood by considering that, if $|\mathbf{R}' - \mathbf{R}|$ is sufficiently larger than the extension of the wave packet, the statevector of the system can be given in some meaning as a product of the quantities corresponding to such packet as there is no overlapping.

Next we shall show that the statevector $|\mathbf{p}, \mathbf{R}, A; -\mathbf{p}, \mathbf{R}', B\rangle$ is orthogonal to the outgoing states where more than three particles are incident. We shall try to illustrate this fact by using

$$^+\langle \mathbf{k}_1, A_1; \mathbf{k}_2, A_2; \mathbf{k}_3, A_3 | \mathbf{p}, \mathbf{R}, A; -\mathbf{p}, \mathbf{R}', B \rangle \quad (\text{B} \cdot 3)$$

as an example. The statevector $|\mathbf{k}_1, A_1; \mathbf{k}_2, A_2; \mathbf{k}_3, A_3\rangle^+$ can be decomposed into the following nine parts:

- 1) the plane wave of A_1, A_2 , and A_3 ,
- 2) the plane wave of A_1 and the third part of $|\mathbf{k}_2, A_2; \mathbf{k}_3, A_3\rangle^+$,
- 3) the plane wave of A_2 and the third part of $|\mathbf{k}_1, A_1; \mathbf{k}_3, A_3\rangle^+$,
- 4) the plane wave of A_3 and the third part of $|\mathbf{k}_1, A_1; \mathbf{k}_2, A_2\rangle^+$,
- 5) the plane wave of A_1 and the second part of $|\mathbf{k}_2, A_2; \mathbf{k}_3, A_3\rangle^+$,
- 6) the plane wave of A_2 and the second part of $|\mathbf{k}_1, A_1; \mathbf{k}_3, A_3\rangle^+$,
- 7) the plane wave of A_3 and the second part of $|\mathbf{k}_1, A_1; \mathbf{k}_2, A_2\rangle^+$,
- 8) the outgoing spherical wave of the three particles,
- 9) the rest which vanishes rapidly for a large distance between the colliding particles.

5)–9) give no contribution to (B.3) since such parts do not contain the state of approaching particles whereas $|\mathbf{p}, \mathbf{R}, A; -\mathbf{p}, \mathbf{R}', B\rangle$ is composed only of the approaching ones. If 1)–4) give some contribution to (B.3) such parts must describe two separated particles or particle-like matters which do not overlap each other. In this case, one of them is the incident particle itself, say, A_1 , and another particle-like matter is composed of the first and third parts of $|\mathbf{k}_2, A_2; \mathbf{k}_3, A_3\rangle^+$. However, the first and third parts of $|\mathbf{k}_2, A_2; \mathbf{k}_3, A_3\rangle^+$ are orthogonal to one particle state, because the former is equal to $|\mathbf{k}_2, A_2; \mathbf{k}_3, A_3\rangle^+$ minus the second part of $|\mathbf{k}_2, A_2; \mathbf{k}_3, A_3\rangle^+$ and this second part is obviously orthogonal to one particle state.

From the above discussion, we see that the statevector which is not orthogonal to $|\mathbf{p}, \mathbf{R}, A; -\mathbf{p}, \mathbf{R}', B\rangle$ is only $|\mathbf{k}_1, A; \mathbf{k}_2, B\rangle^+$ and the scalar product of one with the other is given by (B.2). Then by virtue of the completeness of $\{|\mathbf{k}_1, A_1; \mathbf{k}_2, A_2; \dots; \mathbf{k}_n, A_n\rangle^+\}$ we get (2.1). Therefore (2.1) is correct if two wave packets do not overlap each other.

Similar discussions hold for the case where $\mathbf{R}' - \mathbf{R}$ and \mathbf{p} are antiparallel to each other by using incoming state, and we can get (2.2).

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Note added in proof : From the somewhat different viewpoint we shall consider that at which point the macroscopic causality played its role in deriving our conclusion. In appendix B we required that, if any physical system can be perfectly (dynamically) separated from each other with a mutual distance of the macroscopic order, the statevector can be represented as a direct product of factors corresponding to each physically independent partial system. In our case the above separation can occur at least when the spatial separation is of the order of the size of wave packets or larger than the latter. We can interpret this requirement as a mathematical expression of the macroscopic causality, since if this requirement is not satisfied the disturbance given at some point will give instantly influences on spatially remote points. The analyticity of the scattering amplitude is mainly due to the above mentioned requirement.

Inclusion of Hole Motions in Brueckner Theory

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In order to include hole motions in the Brueckner method a systematic theory is developed based on a familiar treatment of small oscillations of some dynamical system about the stable point. It is shown that a slight modification in the Bethe-Goldstone equation enables us to take into account all effects coming from the couplings of particle-particle and hole-hole pairs with the same total momentum. Ground state energy is expressed with the sum of zero point energy shifts of all hole pair oscillators. Superfluidity condition found by Bogoliubov, Tolmachev and Shirkov and by Cooper, Mills and Sessler is also derived here as the condition for the stability of the degenerate Fermi gas state. Expression of ground state vector is given. A remark is added on the interpretation of the perturbation expansion.

§ 1. Introduction

One of the important subjects in many-body problem is the treatment of interactions with repulsive core as in the case of nuclear matter or liquid helium, where the simple Hartree method is inapplicable because of the divergence of Hartree potential. Brueckner¹⁾ overcame this difficulty by summing up those higher order perturbation terms which correspond to the movement of particular two particles while the other particles are fixed. This is accomplished by solving the Bethe-Goldstone equation²⁾:

$$(E - T_1 - T_2) \phi(r_1, r_2) = QV(r_1, r_2) \phi(r_1, r_2). \quad (1)$$

This equation differs from the usual two-body problem in two respects: The first is the appearance of projection operator Q which keeps both particles outside the Fermi sea in accordance with the Pauli principle. The second is T which does not mean simple $p^2/2m$, but contains average potential for single particle motion to be determined self-consistently.

In a previous note³⁾ we have reported an extension of this theory to include hole motions by modifying the above equation as

$$(E - T_1 - T_2) \phi = (Q - P) V \phi, \quad (2)$$

P being the projection operator inside the Fermi sea. A simple derivation of this equation together with the physical interpretation of ϕ was briefly described there. It should be remembered that this slight change in the equation enables us to include not only the movement of two particles and two holes as in the recent formalism of Chisholm and Squires⁴⁾ but also all processes arising from the coup-

lings of particle-particle and hole-hole pairs of the same total momentum. Essentially the same equation was also obtained by Galitzkii⁶⁾ in his theory with Green functions.*

In this paper we shall set up an orthodox theory concerning the above equation based on the following familiar concept: In the discussion of small oscillations of some dynamical system it is a customary way to approximate the Hamiltonian as quadratic form by expanding it about the stable point and to find normal modes by diagonalization. From this standpoint it is instructive to compare Sawada's electron gas theory⁶⁾ with the theory of superconductivity of Bardeen, Cooper and Schrieffer,⁷⁾ since our theory will be constructed in a similar way. Although particle-hole pairs are treated in Sawada's theory while BCS consider particle-particle and hole-hole pairs, still in both cases their reduced Hamiltonians have the same mathematical structure: quadratic forms of these fermion pair operators. Essential difference is the sign of interaction: The interaction of Sawada's Hamiltonian being repulsive, the degenerate Fermi gas configuration remains stable and we can safely treat fermion pairs as bosons as was remarked by Wentzel.⁸⁾ On the contrary, BCS interaction being attractive, the stable configuration changes drastically from degenerate Fermi gas and we must search for the stable point, for example, by the Bogoliubov transformation.⁹⁾

With these things in mind we shall start in § 2 with the construction of reduced Hamiltonian taking into account the couplings between particle-particle and hole-hole pairs with the same total momentum as in the case of BCS. However, we shall treat these fermion pairs as bosons assuming the degenerate Fermi gas configuration remains stable since our interaction contains repulsive core. In § 3 we shall show that our equation (2) is just the secular equation for the diagonalization of the reduced Hamiltonian. Ground state energy will be given as the sum of zero point energy shifts of all hole pair oscillators. Superfluidity condition found by Bogoliubov, Tolmachev and Shirkov¹⁰⁾ and by Cooper, Mills and Sessler¹¹⁾ will also be obtained in § 4 by demanding that all eigenvalues should be positive for the stability of the configuration from which we have started. We shall give in § 5 the expression of physical ground state vector which shows configuration mixings due to zero point oscillations. Another expression of the ground state vector will be given in the Appendix. In § 6 we shall remark on the interpretation of perturbation treatment of Eq. (2). The relation between the current Brueckner theory and ours will be discussed in § 7. To avoid complications we shall not discuss here self-consistent treatment of the average potential for single particle motion.

§ 2. Reduced Hamiltonian. Boson approximation of fermion pairs.

In this section the reduced Hamiltonian will be set up by neglecting couplings

* The author is indebted to Dr. Sawicki for informing him of this work.

between fermion pairs with different total momentum. It will be shown that these fermion pairs behave approximately as bosons when the system is not so different from the degenerate Fermi gas distribution.

Let us consider a system of fermions whose Hamiltonian is given by

$$H = \sum_k T_k c_k^* c_k + \sum_K \sum_k \sum_{k'} c_{K-k}^* c_{K+k}^* (k|V|k') c_{K+k'} c_{K-k'}, \quad (3)$$

where c_k^* and c_k are creation and annihilation operators of a fermion with momentum k :

$$\{c_k, c_{k'}^*\} = \delta_{kk'}, \quad \{c_k, c_{k'}\} = \{c_k^*, c_{k'}^*\} = 0. \quad (4)$$

\sum_k' means that k -sum should be performed over the half k -space: $k_z > 0$. Then $(k|V|k')$ contains direct and exchange terms:

$$(k|V|k') = V(|k-k'|) - V(|k+k'|), \quad (5)$$

$V(|k|)$ being the Fourier transform of the interaction potential $V(r)$. In the presence of spin, isotopic spin and possible exchange forces we can also proceed in an analogous way.

Equation of motion of a fermion pair $c_{K+k} c_{K-k}$ ($k_z > 0$) is

$$\begin{aligned} [c_{K+k} c_{K-k}, H] &= (T_{K+k} + T_{K-k}) c_{K+k} c_{K-k} \\ &+ [c_{K+k} c_{K-k}, c_{K-k'}^* c_{K+k'}^*] \sum_{k'}' (k|V|k') c_{K+k'} c_{K-k'} \\ &+ \text{coupling terms with } K' (\neq K). \end{aligned} \quad (6)$$

Now two approximations will be made: The first is to neglect K' -coupling terms.* The second is the boson approximation of fermion pairs.** Calculating the commutator appearing in (6), we find

$$[c_{K+k} c_{K-k}, c_{K-k'}^* c_{K+k'}^*] = (1 - n_{K+k} - n_{K-k}) \delta_{kk'}, \quad (7)$$

which is approximated as

$$1 - n_{K+k} - n_{K-k} = \begin{cases} +1 & |K \pm k| > k_F \\ -1 & |K \pm k| < k_F \\ 0 & \text{otherwise,} \end{cases} \quad (8)$$

k_F being Fermi momentum. This is allowed when the system is not so different from the degenerate Fermi gas distribution. Later we shall see the condition of

* In an analogous treatment of electron self-energy in quantum electrodynamics we can take into account some average effect of K' -coupling terms by determining single particle potential self-consistently. To avoid complications we shall discuss it in another chance.

** Actually fermion pair operators satisfy spin commutators rather than boson commutators as was remarked by Wada, Takano and Fukuda.¹²⁾ It is well known in spin wave theory that spin operators can be expanded by boson operators. The approximation assumed here is permitted in the case of small oscillations about stable configuration. See, for example, review articles by Van Kranendonk and Van Vleck.¹³⁾

the validity of the approximation, which depends on the nature of interaction potentials. Then, introducing particle and hole pair operators $b_k^*(K)$, $b_k(K)$ and $a_k^*(K)$, $a_k(K)$ ($k_F > 0$) by

$$c_{K+k} c_{K-k} = \begin{cases} b_k(K) & |K \pm k| > k_F \\ a_k^*(K) & |K \pm k| < k_F, \end{cases} \quad (9)$$

we shall regard them as boson operators:

$$[b_k(K), b_{k'}^*(K')] = [a_k(K), a_{k'}^*(K')] = \delta_{kk'} \delta_{KK'}, \quad (10)$$

all others are zero.

Now the reduced Hamiltonian is constructed so as to give the same equation of motion for pairs:

$$\begin{aligned} H &= \sum_k^{<k_F} T_k + \sum_K H(K), \\ H(K) &= \sum_k' \epsilon_k \epsilon_k(K) (a_k^*(K) a_k(K) + b_k^*(K) b_k(K)) \\ &\quad + \sum_k \sum_{k'}' (a_k(K) + b_k^*(K)) (k|V|k') (a_{k'}^*(K) + b_{k'}(K)), \end{aligned} \quad (11)$$

where

$$\epsilon_k = \begin{cases} +1 & |k| > k_F \\ -1 & |k| < k_F, \end{cases} \quad (12)$$

and

$$\epsilon_k(K) = T_{K+k} + T_{K-k} - 2T_{k_F}. \quad (13)$$

Thus kinetic energies of particle and hole pairs are positive:

$$\epsilon_k \epsilon_k(K) > 0. \quad (14)$$

Subtraction of $2T_{k_F}$ in (13) is allowed due to number conservation:

$$\sum_k' \epsilon_k (a_k^*(K) a_k(K) + b_k^*(K) b_k(K)) = 0. \quad (15)$$

That the constant term $\sum_k^{<k_F} T_k$ appearing in (11) is correct can be seen by comparing the expectation value of (3) with respect to degenerate Fermi gas state with that of (11).

§ 3. Normal modes. Ground state energy.

Now it is straightforward to transform the reduced Hamiltonian to the principal axis. We shall show that our equation (2) is just the eigenvalue equation for the diagonalization. In the following we shall sometimes suppress K and drop $(')$ on \sum when no confusion arises.

The reduced Hamiltonian, which we want to diagonalize, is written as

$$H(K) = - \sum_k^{< k_F} \epsilon_k \epsilon_k + (\dots a_k \dots b_k^* \dots) \begin{pmatrix} \dots \dots (k|V|k') \\ \vdots \\ \epsilon_k \epsilon_{k'} \\ \vdots \\ (k|V|k') \dots \dots \end{pmatrix} \begin{pmatrix} \vdots \\ a_{k'}^* \\ \vdots \\ b_{k'} \\ \vdots \end{pmatrix}. \quad (16)$$

Let us introduce normal modes by

$$\begin{aligned} \alpha_k &= \sum_{k'} \phi_k(k') (a_{k'} - b_{k'}^*) \\ \beta_k^* &= \sum_{k'} \phi_k(k') (-a_{k'} + b_{k'}^*). \end{aligned} \quad (17)$$

They also satisfy boson commutation relations:

$$[\alpha_k, \alpha_{k'}^*] = [\beta_k, \beta_{k'}^*] = \delta_{kk'}, \text{ all others are zero,} \quad (18)$$

if $\phi_k(k')$'s are orthonormal with metric ϵ_k :

$$\sum_k \phi_{k'}^*(k) \epsilon_k \phi_{k''}(k) = \epsilon_{k'} \delta_{k'k''}. \quad (19)$$

Then follows the relation

$$\sum_k \phi_k(k') \epsilon_k \phi_k^*(k'') = \epsilon_{k'} \delta_{k'k''}. \quad (20)$$

Solving (17) with respect to a_k^*, b_k we get

$$\begin{Bmatrix} a_k^* \\ b_k \end{Bmatrix} = \sum_{k'} \phi_{k'}(k) \begin{pmatrix} \alpha_{k'}^* + \beta_{k'} \end{pmatrix}. \quad (21)$$

Inserting (21) into (16) and requiring that $H(K)$ should be diagonal in α_k, β_k :

$$H(K) = \sum_k^{< k_F} (\epsilon_k E_k - \epsilon_k \epsilon_k) + \sum_k \epsilon_k E_k (\alpha_k^* \alpha_k + \beta_k^* \beta_k), \quad (22)$$

we find that $\phi_k(k')$ must satisfy

$$\sum_{k_1} \sum_{k_2} \phi_k^*(k_1) \{ \epsilon_{k_1} \epsilon_{k_2} \delta_{k_1 k_2} + (k_1|V|k_2) \} \phi_{k'}(k_2) = \epsilon_k E_k \delta_{k k'}. \quad (23)$$

Using (20) we have the eigenvalue equation

$$(E_k - \epsilon_{k'}) \phi_k(k') = \epsilon_{k'} \sum_{k''} (k'|V|k'') \phi_k(k''). \quad (24)$$

Defining $\phi(k)$ at $k_z < 0$ to be antisymmetric

$$\phi(-k) = -\phi(k), \quad (25)$$

and considering (5) and (13), we can immediately see that the eigenvalue equation (24) is nothing but the Fourier transform of our equation (2).

From (11) and (22) the ground state energy is expressed with the sum of zero point energy shifts of all hole pair oscillators:

$$\mathcal{E}_{\text{ground}} = \sum_k^{< k_F} T_k + \sum_K \sum_k^{< k_F} \{ \epsilon_k E_k(K) - \epsilon_k \epsilon_k(K) \}. \quad (26)$$

§ 4. Superfluidity condition

We have set up our theory by assuming that the system is not so different from the degenerate Fermi gas configuration. However, if it happens that there appear some negative $\varepsilon_k E_k$'s, our theory would be inconsistent, because (22) would be an expansion of the original Hamiltonian about a saddle point rather than about the minimum point and the degenerate Fermi gas configuration would no longer be stable. Such a case was encountered in Bardeen, Cooper and Schrieffer's theory⁷⁾ on superconductivity. Then we would have energy gap spectrum if we expand the Hamiltonian about the stable point to be found by Bogoliubov transformation.⁹⁾

Rewriting (23) for $K=0$, we have

$$\varepsilon_k E_k(0) = \int \phi_k^*(r_1 - r_2) \{ |T_1 + T_2 - 2T_{k_F}| + V(r_1 - r_2) \} \phi_k(r_1 - r_2) dr_1 dr_2 / \Omega. \quad (27)$$

That this is negative for some ϕ_k is just the superfluidity condition obtained by Bogoliubov, Tolmachev and Shirkov¹⁰⁾ and by Cooper, Mills and Sessler.¹¹⁾

In the case of nuclear matter it is especially interesting whether (27) becomes negative or not in connection with the possible occurrence of energy gap in heavy nuclei suggested by Bohr, Mottelson and Pines.¹⁴⁾ Although CMS found no gap in normal density with their trial function, there must be some critical density beyond which (27) becomes negative, because at density zero the condition is equivalent to the existence of the deuteron state. This consideration suggests that the energy gap in heavy nuclei may be caused by the cooperation of nucleons at the surface where density is low.

5. The ground state vector

Let us go over to the explicit construction of the ground state vector Ψ_0 , which contains virtual excitations from the degenerate Fermi gas state Φ_0 due to zero point oscillations. There are two ways to get Ψ_0 . Here we shall follow the first method. The second and its relation to the first will be explained in the Appendix.

Obviously, degenerate Fermi gas state Φ_0 is the ground state of the kinetic energy part of the reduced Hamiltonian:

$$\sum_K \sum_k' \varepsilon_k \epsilon_k(K) \{ a_k^*(K) a_k(K) + b_k^*(K) b_k(K) \}. \quad (28)$$

If we introduce canonical variables $p_k(K)$ and $q_k(K)$ in the usual way:

$$\begin{aligned} p_k(K) &= (i/\sqrt{2}) (a_k^*(K) - a_k(K)), & (i/\sqrt{2}) (b_k^*(K) - b_k(K)), \\ q_k(K) &= (1/\sqrt{2}) (a_k^*(K) + a_k(K)), & (1/\sqrt{2}) (b_k^*(K) + b_k(K)), \end{aligned} \quad (29)$$

then

$$\begin{aligned} [p_k(K), q_{k'}(K')] &= -i\delta_{kk'}\delta_{KK'}, \\ [p_k(K), p_{k'}(K')] &= [q_k(K), q_{k'}(K')] = 0, \end{aligned} \quad (30)$$

and (28) is expressed as

$$(1/2) \sum_K \sum_k' \epsilon_k \epsilon_k(K) (p_k^2(K) + q_k^2(K) - 1). \quad (31)$$

The ground state Φ_0 is given by

$$\Phi_0 = \prod_K \exp \left\{ - (1/2) \sum_k' q_k^2(K) \right\}. \quad (32)$$

In the same way, introducing $P_k(K)$, $Q_k(K)$ by

$$\begin{aligned} P_k(K) &= (i/\sqrt{2}) (\alpha_k^*(K) - \alpha_k(K)), & (i/\sqrt{2}) (\beta_k^*(K) - \beta_k(K)), \\ Q_k(K) &= (1/\sqrt{2}) (\alpha_k^*(K) + \alpha_k(K)), & (1/\sqrt{2}) (\beta_k^*(K) + \beta_k(K)), \end{aligned} \quad (33)$$

we can write the ground state Ψ_0 of the total Hamiltonian (22) as

$$\Psi_0 = \prod_K \exp \left\{ - (1/2) \sum_k' Q_k^2(K) \right\}, \quad (34)$$

which is rewritten as

$$\begin{aligned} \Psi_0 &= \prod_K \Lambda(K) \Phi_0, \\ \Lambda(K) &= \exp \left\{ - (1/2) \sum_k' Q_k^2(K) \right\} \cdot \exp \left\{ (1/2) \sum_k' q_k^2(K) \right\}. \end{aligned} \quad (35)$$

From (17), (29) and (33), if we assume $\psi_k(k')$ to be real, we get the expression:

$$\begin{aligned} \Lambda(K) &= \exp \left\{ (1/4) \sum_{k'}' \sum_{k''}' (a_{k'}^* + b_{k'}^* + a_{k'} + b_{k'}) \right. \\ &\quad \times (\delta_{k'k''} - \epsilon_{k'} \epsilon_{k''}) \sum_k' \psi_k(k') \psi_k(k'') (a_{k'}^* + b_{k'}^* + a_{k'} + b_{k'}) \left. \right\}. \end{aligned} \quad (36)$$

§ 6. Interpretation of perturbation expansion

In this section we shall remark on the perturbation expansion of ground state energy since it contains apparently curious terms.

From (24) and (26) perturbation expansion of the ground state energy shift may be written*

$$\begin{aligned} \Delta \mathcal{E} &= \sum_K \Delta \mathcal{E}(K), \\ \Delta \mathcal{E}(K) &= \sum_{k_1}'^{< k_F} (1|V|1) + \sum_{k_1}'^{< k_F} \sum_{k_2}' (1|V|2) \epsilon_2 (2|V|1) / (\epsilon_1 - \epsilon_2) \\ &\quad + \sum_{k_1}'^{< k_F} \sum_{k_2}' \sum_{k_3}' (1|V|3) \epsilon_3 (3|V|2) \epsilon_2 (2|V|1) / (\epsilon_1 - \epsilon_3) (\epsilon_1 - \epsilon_2) \\ &\quad + \dots \end{aligned} \quad (37)$$

* Actually, the energy denominator in this expansion is not correct even in the limit of large normalization volume. This would lead to $\tan \delta$ rather than δ . See the discussions of Fukuda and Newton¹⁵⁾ and of deWitt.¹⁶⁾

Now the first sum is the well-known Hartree-Fock term. Together with the usual second order perturbation terms the second sum contains terms with $k_2 < k_F$, which contradict the Pauli principle. However, we need not worry about it since corresponding to the term $k_1 \rightarrow k_2 (< k_F) \rightarrow k_1$ there is a counter term $k_2 \rightarrow k_1 (< k_F) \rightarrow k_2$ which has the same numerator but the denominator with different sign so that they cancel each other in the sum.

The same situation also occurs in any higher order terms: The sum of those terms is zero which correspond to transitions only inside the Fermi sea. Moreover, when the transition involves some inside k 's and some outside k 's, the sum of these terms gives just what we want as the improvement over the Brueckner method.

We shall exhibit the situation taking the fourth order terms with $k_1, k_3 < k_F$ and $k_2, k_4 > k_F$ as an example. There are two terms in (37): one corresponding to the transition $k_1 \rightarrow k_2 \rightarrow k_3 \rightarrow k_4 \rightarrow k_1$:

$$-(1|V|4)(4|V|3)(3|V|2)(2|V|1)/(\epsilon_1 - \epsilon_4)(\epsilon_1 - \epsilon_3)(\epsilon_1 - \epsilon_2), \quad (38)$$

while the other starting from k_3 :

$$-(3|V|2)(2|V|1)(1|V|4)(4|V|3)/(\epsilon_3 - \epsilon_2)(\epsilon_3 - \epsilon_1)(\epsilon_3 - \epsilon_4). \quad (39)$$

Both terms are apparently violating the Pauli principle. On the contrary, what we want is the process such that the first step is pair creation $k_1 \rightarrow k_2$, the second step pair creation $k_3 \rightarrow k_4$, the third step pair annihilation $k_2 \rightarrow k_3$, and the final step pair annihilation $k_4 \rightarrow k_1$. This would be represented by

$$(1|V|4)(3|V|2)(4|V|3)(2|V|1)/(\epsilon_1 - \epsilon_4)(\epsilon_1 + \epsilon_3 - \epsilon_2 - \epsilon_4)(\epsilon_1 - \epsilon_2). \quad (40)$$

There are three more terms of this kind corresponding to the different time sequences of pair creations and pair annihilations:

$$(3|V|2)(1|V|4)(4|V|3)(2|V|1)/(\epsilon_3 - \epsilon_2)(\epsilon_1 + \epsilon_3 - \epsilon_2 - \epsilon_4)(\epsilon_1 - \epsilon_2), \quad (41)$$

$$(1|V|4)(3|V|2)(2|V|1)(4|V|3)/(\epsilon_1 - \epsilon_4)(\epsilon_1 + \epsilon_3 - \epsilon_2 - \epsilon_4)(\epsilon_4 - \epsilon_3), \quad (42)$$

$$(3|V|2)(1|V|4)(2|V|1)(4|V|3)/(\epsilon_3 - \epsilon_2)(\epsilon_1 + \epsilon_3 - \epsilon_2 - \epsilon_4)(\epsilon_4 - \epsilon_3). \quad (43)$$

Now our remark is that the sum of (40), (41), (42) and (43) is just equal to the sum of (38) and (39); both give

$$\frac{(1|V|4)(4|V|3)(3|V|2)(2|V|1)(\epsilon_3 + \epsilon_4 - \epsilon_1 - \epsilon_3)}{(\epsilon_1 - \epsilon_2)(\epsilon_1 - \epsilon_4)(\epsilon_2 - \epsilon_3)(\epsilon_3 - \epsilon_4)}. \quad (44)$$

In the same way we can see in any desired order that the sum of all terms apparently violating the Pauli principle gives just the correct result.

The simple structure of the series (37) should be contrasted with cumbersome expressions such as (40) ~ (43). This has been the merit of the Feynman perturbation theory over the old perturbation theory.

§ 7. Discussions

We have set up an improved Brueckner theory so as to include hole motions. Here it is of some interest to compare the current Brueckner theory with ours from the following point of view.

In the current Brueckner theory we take into account only two particle jumped configurations with two fixed holes. Thus denoting two-hole state by $\phi(k, k') = c_k c_{k'} \Phi_0$; ($|k|, |k'| < k_F$), we approximate the ground state as

$$\Psi_0 = \sum_p \sum_{p'} \phi_{kk'}(p, p') c_p^* c_{p'}^* \Phi(k, k'), \quad (45)$$

and the Bethe-Goldstone equation would follow from

$$\langle c_p^* c_{p'}^* \Phi(k, k'), (H - E) \Psi_0 \rangle = 0. \quad (46)$$

This is the well-known old Tamm-Dancoff approximation.¹⁷⁾ On the contrary, our theory is based on the new Dyson-Tamm-Dancoff scheme,¹⁸⁾ for from (21) and (22) our ϕ may be defined by

$$\phi_k(k') = \begin{cases} \langle \Psi_k^{N-2}, a_{k'}^* \Psi_0 \rangle, & \langle \Psi_k^{N-2}, b_{k'} \Psi_0 \rangle, \\ \langle \Psi_k^{N+2}, a_{k'} \Psi_0 \rangle^*, & \langle \Psi_k^{N+2}, b_{k'}^* \Psi_0 \rangle^*, \end{cases} \quad (47)$$

where Ψ_0 , $\Psi_k^{N-2} = \alpha_k^* \Psi_0$ and $\Psi_k^{N+2} = \beta_k^* \Psi_0$ are the physical ground state, physical two-hole state and physical two-particle state, respectively. Although ϕ has somewhat different meaning, a satisfactory feature of our result is that Eq. (2) to be solved is not more complicated than the Bethe-Goldstone equation.

Up to now we have no estimate of the possible contributions which would come from the processes neglected in the present formalism.

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Appendix

Another expression of Ψ_0

If we find unitary operator $S(K)$ such that

$$\begin{aligned} \alpha_k &= S a_k S^{-1} = \sum_{k'} \phi_k(k') (a_{k'} - b_{k'}^*) \\ \beta_k^* &= S b_k^* S^{-1} = \sum_{k'} \phi_k(k') (-a_{k'} + b_{k'}^*), \end{aligned} \quad (A1)$$

then

$$\Psi_0 = \prod_K S(K) \Phi_0. \quad (A2)$$

Let us assume S to be the following form:

$$S(K) = \exp \left\{ i \sum_k \sum_{k'} (a_k^* + b_k) s_{kk'} (a_{k'} + b_{k'}^*) \right\} \quad (\text{A3})$$

with

$$s_{kk'}^* = s_{k'k}. \quad (\text{A4})$$

According to the general formula

$$e^A B e^{-A} = B + [A, B] + (1/2!) [A, [A, B]] + \dots, \quad (\text{A5})$$

it is easy to show

$$\left. \begin{array}{l} S a_k S^{-1} \\ S b_k^* S^{-1} \end{array} \right\} = \sum_{k'} (e^{i(\varepsilon s)})_{kk'} (a_{k'} + b_{k'}^*), \quad (\text{A6})$$

where (εs) is a matrix with element $\varepsilon_k s_{kk'}$. From (A1) and (A6) we have the relation

$$(e^{i(\varepsilon s)})_{kk'} = \varepsilon_k \varepsilon_{k'} \psi_k(k'). \quad (\text{A7})$$

However, it is difficult to express $s_{kk'}$ in an explicit way contrary to our previous expression for $\Lambda(K)$.

The situation becomes clear if we note that $S(K)$ transforms also any free state Φ_λ to the corresponding physical state Ψ_λ :

$$\Psi_\lambda = \prod_K S(K) \Phi_\lambda, \quad (\text{A8})$$

while $\Lambda(K)$ has no such general property. When we confine ourselves only to the ground state, SR also transforms Φ_0 to Ψ_0 , R being any function that satisfies $R\Phi_0 = \Phi_0$. Therefore, $\Lambda(K)$ in (35) would be written as $S(K)R(K)$ if we choose $R(K)$ in the following way.

Assuming $\psi_k(k')$ to be real, we have the relation from (35)

$$\begin{aligned} \Lambda a_k \Lambda^{-1} &= \sum_{k'} \psi_{k'}(k) (\alpha_{k'} - \beta_{k'}), \\ \Lambda b_k \Lambda^{-1} &= \sum_{k'} \psi_{k'}(k) (-\alpha_{k'} + \beta_{k'}). \end{aligned} \quad (\text{A9})$$

Then, from (A1), $R(K)$ is determined by

$$\begin{aligned} R a_k R^{-1} &= \sum_{k'} \psi_{k'}(k) (a_{k'} - b_{k'}), \\ R b_k R^{-1} &= \sum_{k'} \psi_{k'}(k) (-a_{k'} + b_{k'}). \end{aligned} \quad (\text{A10})$$

In the same way as $S(K)$, assuming

$$R(K) = \exp \left\{ \sum_k \sum_{k'} (a_k^* + b_k) r_{kk'} (a_{k'} + b_{k'}^*) \right\}, \quad (\text{A11})$$

we can derive the relation

$$(e^{-(r)})_{kk'} = \varepsilon_k \varepsilon_{k'} \psi_{k'}(k), \quad (\text{A12})$$

(r) being a matrix with elements $r_{kk'}$. Again it is difficult to get an explicit expression for $r_{kk'}$.

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Note added in proof : After submittance the author happened to know that Prof. Tomonaga had already obtained the expression of the ground state vector for the Hamiltonians of the type in meson pair theory.

(N. Fukda and Y. Wada, *Soryushiron Kenkyu* (mimeographed circular in Japanese **19** (1959),) 185.)

Classification of Composite Bosons in the Sakata Model

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Assuming the Sakata model (p, n, A are basic, all other strongly interacting particles are composite particles) and neglecting moderately strong interactions which contribute to N - A mass splitting, we find the complete symmetry between three fundamental fields (referred to as global symmetry). Under this global approximation, classification of two baryon pair states—which are supposed to represent physical mesons—is described.

§ 1. Introduction

Recently, Ogawa and the present author¹⁾ have independently developed a composite theory of strongly interacting particles, which is a special case of the so-called Sakata model.²⁾ We assume that all strongly interacting particles are composed of three basic particles (bare p, n and A), and introduce two different stages of approximations, the global and charge independent approximations (see I and II).** If we ignore the moderately strong interactions (MSI) responsible for N - A mass splitting, we find a complete symmetry among three basic particles. We refer to this symmetry property as to the global symmetry. The approximation in which MSI's are neglected shall be called the global approximation. In the charge independent approximation, MSI's are taken into account but electromagnetic and weak couplings are still neglected.

In the global approximation, basic fields, $B \equiv (p, n, A)$, have two invariant groups (the isospin rotation and the permutation $n \leftrightarrow A$). Thus (bare) p, n, A can be regarded as the three-dimensional irreducible representation of our groups. Mesons*** are supposed to be composed of baryon pairs.¹⁾⁻³⁾ Group theoretically, meson states would be equivalent to, and hence be contained in, the representations $\bar{B} \times B$, $\bar{B} \times B \times \bar{B} \times B$, etc. The simplest representation corresponding to $\bar{B} \times B$ and its irreducible decomposition have already been given:³⁾ $9 (= 3 \times 3)$ -dimensional representation can be decomposed into 1- and 8-dimensional representations, and the latter has been assigned in I and II as π (pions), K, \bar{K} (kaons), and π' (the isosinglet, non-strange, neutral meson). However, the π - π resonance state ($I=J=1$), suggested recently,⁴⁾ and the Dubna-particle,⁵⁾ if they exist, are not contained

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** See references 1) and 3).

*** Baryons have been discussed fully in I and II.

in this simplest choice of "configurations". Therefore we are going to investigate in this paper the representation equivalent to $\bar{B} \times B \times \bar{B} \times B$ (which corresponds, roughly speaking, to the two-baryon-pair states).

Perhaps it would be appropriate to state again the philosophy underlying our analysis (see I, II). First of all, we assume that the global approximation does make sense and the MSI's give rise to relatively small and smooth changes in the physical situation. The very strong interactions (which do not split N - A masses) are supposed to be chosen in such a way that only a very small number of particles corresponding to the irreducible representations be allowed to appear as stable bound states and or resonance levels: All members of the irreducible group represent the physical particles (baryons or mesons)—in the global approximation—and must have the same spin-parity as well as identical mass in the global approximation. The introduction of MSI's splits the members in the irreducible set into several isomultiplets, which correspond to physical particles found in nature.

We describe the results of irreducible decomposition of $\bar{B} \times B \times \bar{B} \times B$ in Section 2, and give some discussions in Section 3.

§ 2. Irreducible decomposition of $\bar{B} \times B \times \bar{B} \times B$

The representation corresponding to $\bar{B} \times B \times \bar{B} \times B$ has $3 \times 3 \times 3 \times 3 = 81$ dimensions, and can be decomposed into the following irreducible sets in the global approximation, (I) — (IV); $\mathbf{1} \times \mathbf{2} + \mathbf{8} \times \mathbf{4} + \mathbf{27} + \mathbf{10} \times \mathbf{2} = 81$.

(I) *Trivial representation* ($\mathbf{g}^{(0)}$), containing only one particle.

It appears twice, and is equivalent to π_0^0 in II.

(II) *8-dimensional representation* ($\mathbf{g}^{(8)}$)

It appears 4 times, and is equivalent to π , K , \bar{K} , π' in II.

(III) *27-dimensional representation* ($\mathbf{g}^{(27)}$). Its members can conveniently be expressed as $A_{+}^{I,S,Q}$, where I , S and Q are the isospin, the strangeness and the electric charge in units of protonic charge. There are A^{20} , $A^{3/2\ 1}$, $A^{3/2\ -1}$, A^{12} , A^{10} , $A^{1\ -2}$, $A^{1/2\ 1}$, $A^{1/2\ -1}$ and A^{00} .

Under the permutation $n \leftrightarrow 1$ of basic fields (called the G -transformation in II), these particles transform as follows:

$$\begin{aligned}
 Q=2 \quad & A_{++}^{20} \leftrightarrow A_{++}^{12} \\
 & A_{++}^{3/2\ 1} \leftrightarrow A_{++}^{3/2\ -1} \\
 Q=1 \quad & A_{+}^{20} \rightarrow \frac{1}{\sqrt{6}} A_{+}^{3/2\ 1} - \sqrt{\frac{5}{6}} A_{+}^{1/2\ 1} \\
 & A_{+}^{10} \rightarrow \sqrt{\frac{5}{6}} A_{+}^{3/2\ 1} + \sqrt{\frac{1}{6}} A_{+}^{1/2\ 1} \\
 & A_{+}^{3/2\ 1} \rightarrow \sqrt{\frac{1}{6}} A_{+}^{2,0} + \sqrt{\frac{5}{6}} A_{+}^{1,0}
 \end{aligned}$$

$$\begin{aligned}
& A_+^{1/2\ 1} \rightarrow -\sqrt{\frac{5}{6}} A_+^{2,0} + \sqrt{\frac{1}{6}} A_+^{1,0} \\
& A_+^{3/2\ -1} \rightarrow A_+^{1,2} \\
Q=0 \quad & A_0^{20} \rightarrow \frac{1}{6} A_0^{20} + \frac{\sqrt{5}}{2\sqrt{3}} A_0^{10} + \frac{\sqrt{5}}{3} A_0^{00} \\
& A_0^{10} \rightarrow \frac{\sqrt{5}}{2\sqrt{3}} A_0^{20} + \frac{1}{2} A_0^{10} - \frac{1}{\sqrt{3}} A_0^{00} \\
& A_0^{00} \rightarrow \frac{\sqrt{5}}{3} A_0^{20} - \frac{1}{\sqrt{3}} A_0^{10} + \frac{1}{3} A_0^{00} \\
& A_0^{3/2\ 1} \rightarrow \frac{2}{3} A_0^{3/2\ -1} + \frac{\sqrt{5}}{3} A_0^{1/2\ -1} \\
& A_0^{1/2\ 1} \rightarrow -\frac{\sqrt{5}}{3} A_0^{3/2\ -1} + \frac{2}{3} A_0^{1/2\ -1} \\
& A_0^{3/2\ -1} \rightarrow -\frac{2}{3} A_0^{3/2\ 1} - \frac{\sqrt{5}}{3} A_0^{1/2\ 1} \\
& A_0^{1/2\ -1} \rightarrow \frac{\sqrt{5}}{3} A_0^{3/2\ 1} + \frac{2}{3} A_0^{1/2\ 1} \\
Q=-1 \quad & A_-^{20} \rightarrow \frac{1}{\sqrt{6}} A_-^{3/2\ -1} + \sqrt{\frac{5}{6}} A_-^{1/2\ -1} \\
& A_-^{10} \rightarrow \sqrt{\frac{5}{6}} A_-^{3/2\ -1} - \sqrt{\frac{1}{6}} A_-^{1/2\ -1} \\
& A_-^{3/2\ -1} \rightarrow \frac{1}{\sqrt{6}} A_-^{2,0} + \sqrt{\frac{5}{6}} A_-^{1,0} \\
& A_-^{1/2\ -1} \rightarrow \sqrt{\frac{5}{6}} A_-^{2,0} - \sqrt{\frac{1}{6}} A_-^{1,0} \\
& A_-^{3/2\ 1} \rightarrow A_-^{1,-2} \\
Q=-2 \quad & A_{--}^{20} \leftrightarrow A_{--}^{1,-2} \\
& A_{--}^{3/2\ -1} \leftrightarrow A_{--}^{3/2\ -1}
\end{aligned}$$

(IV) 10-dimensional representation

There appear two 10-dimensional representations, $\mathfrak{g}^{(10)}$ and $\bar{\mathfrak{g}}^{(10)}$. However, each particle in $\bar{\mathfrak{g}}^{(10)}$ is the antiparticle of that belonging to $\mathfrak{g}^{(10)}$, respectively. $\mathfrak{g}^{(10)}$ contains the following members (B^{IS}): $B^{3/2,1}$, $B^{1/2,-1}$, $B^{1,0}$ and $B_-^{0,-2}$ ($4+2+3+1=10$). The G -transformation for B_Q^{IS} is given as follows:

$$Q=2 \quad B_{++}^{3/2\ 1} \leftrightarrow B_{++}^{3/2\ 1}$$

$$Q=1 \quad B_{+}^{3/2\ 1} \leftrightarrow B_{+}^{1,0}$$

$$Q=0 \quad B_0^{3/2\ 1} \leftrightarrow B_0^{1/2-1}$$

$$B_0^{1\ 0} \leftrightarrow -B_0^{10}$$

$$Q=-1 \quad B_-^{3/2\ 1} \leftrightarrow B_-^{0-2}$$

$$B_-^{10} \leftrightarrow B_-^{1/2-1}.$$

It should be noticed that particles, as well as their antiparticles, are contained in the same irreducible groups, except for $g^{(10)}$ and $\bar{g}^{(10)}$. Therefore if the particles corresponding to $g^{(10)}$ are supposed to exist, $\bar{g}^{(10)}$ should be taken into account simultaneously.

§ 3. Discussions

In the preceding section we have found 4 different classes of possible boson states. We must now discuss which of these groups of bosons should be identified as the mesons existing in nature and what follows from a special choice. At least we must pick up a group which contains bosons with $(I=1, S=0)$ and $(I=1/2, S=\pm 1)$. There are three possibilities $g^{(8)}$, $g^{(27)}$ and $g^{(10)}$ (plus its anti, $\bar{g}^{(10)}$).

The possibility of picking up $g^{(8)}$ has already been discussed fully (see I and II), and no further comments will be given here.

$g^{(27)}$ contains too many members and will not be welcome by anybody.*

The most interesting possibility is provided by $g^{(10)}$ and its anti $\bar{g}^{(10)}$. There are two kinds of particles, B^{10} and \bar{B}^{10} , which have $I=1, S=0$, which have equal masses in the global approximation, and are the antiparticles of each other. Nevertheless, by a suitable choice of MSI's we can assure that

$$B_1^{10} \equiv \frac{B^{10} + \bar{B}^{10}}{\sqrt{2}} \quad \text{and} \quad B_2^{10} \equiv i \frac{B^{10} - \bar{B}^{10}}{\sqrt{2}}$$

should be properly called the physical particles. Then we can establish the following property,

$$B_1^{10} \rightarrow B_1^{10}$$

$$B_2^{10} \rightarrow -B_2^{10},$$

under the charge conjugation (cf. the relationship between K^0 , \bar{K}^0 and K_1^0, K_2^0). There are no *a priori* reason why B_1^{10} and B_2^{10} should have equal masses in the charge independent approximation. There are two interesting assignments one can think of.

(a) We can try to identify B_1^{10} , $B^{1/2\ 1}$ and $\bar{B}^{1/2-1}$ as π , K and \bar{K} , respectively. Then all B^{1S} must have spin-parity 0^- even in the charge independent approximation. Notably, we can identify pseudoscalar B_-^{0-2} and \bar{B}_+^{02} as the Dubna particles⁵⁾ whose possible existence has been suggested recently. There are also remaining three:

* We have excluded the possibility of the Dubna particles being isotriplet.

B_2^{10} , $B^{3/2+1}$ and $B^{3/2-1}$. They must be either stable or quasistable (say, resonance states appearing in collision processes).

(b) Another possibility is also worth mentioning. We assume that π , K and \bar{K} belong to $g^{(8)}$, whereas B_1^{10} is nothing but the π - π resonance state ($J=I=1^-$).⁵⁾ Then all particles of resonance levels containing $g^{(10)}$ and $\bar{g}^{(10)}$ must have spin-parity 1^- . There are two meson-meson resonance levels, B_1^{10} and B_2^{10} , which might be responsible for the double humps in π^-p total cross section around the so-called second resonance. $B^{3/2,\pm 1}$ and $B^{1/2,\pm 1}$ should be regarded as π - K or π - \bar{K} resonances perfectly analogous to the π - π resonance level (unless they are stable). Finally, we can conclude that the Dubna particles, $B^{0\pm 2}$, are of vector type.

It is extremely interesting to see the spin-parity of Dubna particles experimentally, provided, of course, that such particles do exist. We can predict that the Dubna particles must be either pseudoscalar or vector.

Since rather comprehensive discussions on our composite theory have already been given in I and II, we do not think it necessary to add here any further comments on our subject.

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S-Wave Pion-Nucleon Interaction

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Pion-nucleon scattering in the limit of low energy is investigated in order to see some characteristic property of pion-nucleon interaction in the nucleon core. It is pointed out that the value of coupling constant in nucleon core ought to be reduced in appearance to $f = (\mu/2M)g$ in spite of the fact that its value in the neighborhood of pion cloud is g . Moreover, some attempt to eliminate the divergence included in the dispersion relation is made on the basis of the above result, and the experimental results for s -wave phase shifts can be explained satisfactorily.

§ 1. Introduction

Since the experimental discovery of the nucleon core in the electron-proton scattering¹⁾ the problems for the nucleon core have been studied by many authors from various points of view. But there has been no experimental evidence for validity of the current field theory within the nucleon core; in addition to this fact it seems that there exist some phenomena which may not be explained by the theory. So far as the phenomena in the neighborhood of pion cloud are concerned, the nucleon core may be treated as if it were a point. However, such a treatment as this will no longer be valid in describing the phenomena within the range of $1/M$, because the effect due to the spread of nucleon core will turn out to be so remarkable that it cannot be neglected, where M and μ are the masses of nucleon and pion respectively. The purpose of this paper is to examine how the current pion field theory should be modified in this region. For this purpose we think it appropriate to study the problems for the s -wave pion-nucleon scattering as well as those for the high energy phenomena.

Among a great number of approaches to s -wave pion-nucleon interactions, a remarkable result will be the explanation of $(\alpha_1 - \alpha_3)$ which has been given by the application of dispersion relation to this problem,²⁾⁻⁴⁾ where α_1 and α_3 are the s -wave phase shifts for the states of isotopic spin $I=1/2$ and $3/2$ respectively. But it seems to be difficult to explain the individual values of α_1 and α_3 . As will be mentioned in § 2, these may be due to the following situation. When the forward scattering amplitudes for π^+p and π^-p scattering at the energy ω are denoted by $D_+(\omega) + iA_+(\omega)$ and $D_-(\omega) + iA_-(\omega)$, respectively, in the low energy limit of pion-nucleon scattering, the effects due to the internal structure of nucleon will be almost cancelled out in the $D^{(2)}(\mu) = 1/2[D_-(\mu) - D_+(\mu)]$ which can be

expressed in terms of the phase shifts $(\alpha_1 - \alpha_3)$, while these effects will play an important role in the $D^{(1)}(\mu) = 1/2[D_-(\mu) + D_+(\mu)]$ which can be expressed in terms of $(\alpha_1 + 2\alpha_3)/3$. Thus it may be expected that there will be a clue to our study in the unsolved problem for $D^{(1)}(\omega)$.

In § 2 we concentrate our consideration to this problem for the $D^{(1)}(\omega)$. Comparing the ordinary perturbation result for $D^{(1)}(\mu)$ with the contribution from the bound state to $D^{(1)}(\mu)$ in dispersion relation, we can obtain the following conclusion. The correct dynamics should have a property such that the effective coupling constant g is reduced to $(\mu/2M)g=f$ in appearance when the dynamics is applied to the problems within the nucleon core, where g is the renormalized pseudoscalar coupling constant of symmetrical pseudoscalar theory. This conclusion is consistent with the experimental results both for high energy pion phenomena and for low energy ones.

On the basis of this result some attempt is made in § 3 in order to derive a significant part from the expression for $D^{(1)}(\omega)$ in dispersion relation in which a divergent one may be contained, and it is shown that the experimental results⁵⁾ for s -wave phase shifts $\alpha_1 = 0.16\gamma$, $\alpha_3 = -0.11\gamma$ can be explained by our method, where γ is the pion momentum in the center-of-mass system in units of μ .

§ 2. Pion-nucleon interaction in nucleon core

As is well known, the dispersion relations²⁾³⁾ are written down as follows:

$$D^{(1)}(\omega) = 1/2[D_-(\omega) + D_+(\omega)] = F_1(\omega) + G_1(\omega), \quad (1)$$

$$D^{(2)}(\omega) = 1/2[D_-(\omega) - D_+(\omega)] = F_2(\omega) + G_2(\omega), \quad (2)$$

$$F_1(\omega) = 2f^2 \frac{(\mu^2/2M)^2 - \mu^2}{\omega^2 - (\mu^2/2M)^2} \cdot \frac{1}{2M}, \quad G_1(\omega) = \frac{1}{4\pi^2} P \int_{\mu}^{\infty} \frac{\omega' k' [\sigma_-(\omega') + \sigma_+(\omega')]}{\omega'^2 - \omega^2} d\omega', \quad (3)$$

$$F_2(\omega) = -2f^2 \frac{(\mu^2/2M)^2 - \mu^2}{\omega^2 - (\mu^2/2M)^2} \cdot \frac{\omega}{\mu^2}, \quad G_2(\omega) = \frac{\omega}{4\pi^2} P \int_{\mu}^{\infty} \frac{k' [\sigma_-(\omega') - \sigma_+(\omega')]}{\omega'^2 - \omega^2} d\omega', \quad (4)$$

where k is the wave number of pion. If the π -interaction Lagrangian contains, for example, $\phi_{\pi}^4(x)$ -term explicitly, an additional term C must be involved on the right-hand side of Eq. (1). But we assume for a moment that $\phi_{\pi}^4(x)$ -interaction term is not explicitly contained in the Lagrangian, but may be derived from the interaction of Yukawa type. Some discussion about this problem will be done in § 4.

According to the experimental results,⁶⁾⁷⁾ in the energy region ≥ 1.9 BeV it seems that $\sigma_-(\omega') \cong \sigma_+(\omega') \cong \text{constant}$ although it will be difficult to confirm this result up to $\omega' \rightarrow \infty$. Then the value of $D^{(2)}(\omega)$ turns out to be finite, while Eq. (1) loses its meaning because $G_1(\omega)$ turns out to be a divergent quantity. It

may be said that the cause that gives rise to this circumstance is attributed to the divergence which appears in the integral along the contour of semicircle with radius $R \rightarrow \infty$ on the upper half plane of ω , when the dispersion relation is derived.* If a significant physical quantity $G_{1r}(\omega)$ can be derived from $G_1(\omega)$ by a suitable method, Eq. (1) should be rewritten as follows,

$$D^{(1)}(\omega) = 1/2[D_-(\omega) + D_+(\omega)] = F_1(\omega) + G_{1r}(\omega). \quad (1')$$

Some attempt to obtain this $G_{1r}(\omega)$ is made in § 3.

In this section we pay our attention to the $F_1(\mu)$ and $F_2(\mu)$ which represent the contributions from the bound state to pion-nucleon scattering in the limiting case as $k=0$. We may now regard these $F_1(\mu)$ and $F_2(\mu)$ as the perturbation results in the true pion field theory, since $F_1(\mu)$ and $F_2(\mu)$ correspond to the contributions from the zero-pion state in the Chew-Low theory⁸⁾ based only on the general arguments for the transformation property of matrix elements. Our point of view may be supported by the fact that the qualitative character of experimental results for s -wave phase shifts can be explained fairly well by employing the approximate relations $D^{(1)}(\mu) = F_1(\mu)$, $D^{(2)}(\mu) = F_2(\mu)$.

From the expressions of $D^{(2)}(\omega)$ in Eq. (2) and $D^{(1)}(\omega)$ in Eq. (1'), it may be supposed that the effects due to the internal structure of nucleon core are almost cancelled out in the former, while the effects play an important role in the latter. Then $F_1(\mu)$ and $F_2(\mu)$ will mainly represent the contributions from the nucleon core (inner region $\sim 1/M$) and those from the pion cloud (outer region $\sim 1/\mu$) respectively. This can also be seen through the following expressions,

$$F_1(\mu) = -f^2/M, \quad F_2(\mu) = 2f^2/\mu. \quad (5)$$

On the other hand, the practical straightforward calculation^{9)**} based on the Lorentz-covariant perturbation (the second order) gives the following results,

$$D_{\pm}(\mu) = \frac{-g^2}{M(1+\mu/M)(1 \mp \mu/2M)}. \quad (6)$$

Therefore

$$D^{(1)}(\mu) \cong -g^2/\{M(1+\mu/M)\} \cong -g^2/M, \quad (7)$$

$$D^{(2)}(\mu) \cong g^2(\mu/2M)/\{M(1+\mu/M)\} \cong 2f^2/\mu. \quad (8)$$

$D^{(2)}(\mu)$ in Eq. (8) is almost identical with $F_2(\mu)$, while $D^{(1)}(\mu)$ in Eq. (7) differs from $F_1(\mu)$. This discrepancy may be due to an incorrect application of the theory to the region of nucleon core in spite of the fact that the theory has not been established except in the region of pion cloud, and may correspond to the fact that the usual renormalized coupling constant g differs from the g_s which has been defined

* This is the viewpoint that has been emphasized by H. Miyazawa.

** Note that in Marshak's book g refers to the coupling constant between charged pion and nucleon which is $\sqrt{2}g$ in our notation.

by Deser, Thirring and Goldberger.¹⁰⁾¹¹⁾ Matthews and Salam¹²⁾ have said in their study for K - N scattering that, since the similar relation to (1) does not check in the lowest order perturbation calculation, one may take the attitude that either a perturbation calculation is not valid or the failure of (1) may be ascribed to the lack of convergence of the integrals involved on the right-hand side. Our viewpoint may rather belong to the former. Moreover, this discrepancy is here regarded as an important clue to the true theory by which the phenomena in the nucleon core can be described correctly. Comparing Eq. (5) with Eq. (7), we can see that the phenomena in the neighborhood of pion cloud can be explained by the current field theory, while those in the neighborhood of the nucleon core cannot be explained. This defect, however, can be amended if only g^2 in the expression of $D^{(1)}(\mu)$ in Eq. (7) is replaced by f^2 . From this result* we may conclude as follows. The correct dynamics should have such a property as the effective coupling constant g is reduced to $(\mu/2M)g=f^{**}$ in appearance when the dynamics is applied to the problems within the nucleon core. And we may regard the value of effective coupling constant as a good parameter in the description of the phenomena in the nucleon core. When pv -coupling theory is adopted, we obtain a similar one with the expression in Eq. (6) except the difference in coupling constant between f^2 and g^2 . Thus it may be said that pv -coupling theory is promising to describe the behavior of pion-nucleon interaction in the nucleon core.

Deser, Thirring and Goldberger¹⁰⁾ have investigated pion-nucleon scattering in the limit of low energy and have shown the following result,

$$D_+(0) = D_-(0) = -g_s^2/M, \quad (9)$$

where g_s^2 differs from the usual renormalized one and turns out to be very small, but it seems to be difficult to obtain the correct value of g_s^2 . We think their result very natural, because it will be difficult to estimate correctly the contribution from the nucleon core which may play the most important role in the scattering in the limiting case of $\mu=0$ and $\omega \rightarrow 0$. But, if the practical effect brought by the application of the correct dynamics to the phenomena in the region of the nucleon core is expressed in terms of the change of the value of coupling constant as was mentioned above, the unknown value of g_s^2 should be taken to be equal to f^2 . Then

$$D_+(0) = D_-(0) = -f^2/M. \quad (9')$$

§ 3. Elimination of divergence and s -wave phase shifts

Although there is no clue to obtain the correct $G_{1r}(\omega)$, in this section some attempts to evaluate the value of $(\alpha_1 + 2\alpha_3)/3$ are made on the basis of the result

* This result will also make it possible to obtain some knowledge about the distribution of mesonic charge in the nucleon core.

** We think it natural to interpret this f as the value of coupling constant averaged over the spread of nucleon core. Its value in the center of nucleon will probably tend to zero. This may be closely connected with elimination of the divergence inherent in the current field theory.

obtained in § 2.

Method (I)

First of all, let us try to derive $G_{1r}(\omega)$ from $G_1(\omega)$ under the simple assumption that the correct $D^{(1)}(\mu)$ in the limiting case of $\omega=\mu$ can be expressed by the perturbation result in $p\bar{\nu}$ -coupling theory. This assumption may be supported by the fact that, as was mentioned above, the $p\bar{\nu}$ -coupling theory is promising to describe the behavior of pion-nucleon interaction in the nucleon core. Since the perturbation calculation in $p\bar{\nu}$ -coupling theory predicts $D^{(1)}(\mu) \cong F_1(\mu)$ as was mentioned in § 2, $G_{1r}(\mu)$ must be equal to zero. The form of $G_{1r}(\omega)$ is now chosen so that $D^{(1)}(\omega)$ may satisfy the following relation which has been applied to the problems of pion-nucleon interaction by many authors,²⁾¹³⁾

$$D^{(1)}(\omega) - D^{(1)}(\mu) = 2f^2 \frac{k^2}{\omega^2 - (\mu^2/2M)^2} \cdot \frac{1}{2M} + \frac{k^2}{4\pi^2} \int_{\mu}^{\infty} \frac{\omega'}{k'} \frac{[\sigma_-(\omega') + \sigma_+(\omega')]}{\omega'^2 - \omega^2} d\omega'. \quad (1'')$$

Then $G_{1r}(\omega)$ in Eq.(1') can be expressed as follows:

$$G_{1r}(\omega) = \frac{1}{4\pi^2} P \int_{\mu}^{\infty} \left(\frac{1}{\omega'^2 - \omega^2} - \frac{1}{\omega'^2 - \mu^2} \right) \omega' k' [\sigma_-(\omega') + \sigma_+(\omega')] d\omega'. \quad (10)$$

In the limit as $k \rightarrow 0$,

$$D_+(\mu) = (1/\mu) [1 + (\mu/M)] a_3, \quad (11)$$

$$D_-(\mu) = (1/\mu) [1 + (\mu/M)] [(2/3) a_1 + (1/3) a_3], \quad (12)$$

where $\alpha_1 = a_1 \eta$, $\alpha_3 = a_3 \eta$.

Under our assumption,

$$D^{(1)}(\mu) = 1/2 [D_-(\mu) + D_+(\mu)] = F_1(\mu) = -f^2/M, \quad (13)$$

therefore

$$[1 + (\mu/M)] (a_1 + 2a_3) / 3 = -f^2(\mu/M). \quad (14)$$

Putting the values of $\mu/M=0.15$ and $f^2=0.08$ into Eq. (14),

$$a_1 + 2a_3 = -0.03. \quad (15)$$

By using this together with the result²⁾

$$a_1 - a_3 = 0.27 \quad (16)$$

which has been derived from Eq. (2),

$$a_1 = 0.17, \quad a_3 = -0.10. \quad (17)$$

These values agree very well with those which have been obtained from an analysis of low energy pion scattering data such as has been made by Orear.⁵⁾ This method may probably be regarded as one of the simplest ones. But there is no theoretical enough ground to be persisted as the most suitable one. Let us now try to perform another method.

Method (II)

Here $G_{1r}(\omega)$ in Eq. (1') is evaluated by employing the relation of Eq. (9'). Since both $G_1(\omega)$ and $G_{1r}(\omega)$ are some functions of variables ω, μ, M and f^2 or of variables k, μ, M and f^2 , they can be rewritten as follows:

$$\begin{aligned} G_1(\omega) &= G(k, \mu, M, f^2), \\ G_{1r}(\omega) &= G_r(k, \mu, M, f^2). \end{aligned} \quad (18)$$

In order to derive $G_r(k, \mu, M, f^2)$ from the $G(k, \mu, M, f^2)$ which may be a divergent quantity, the following assumption is introduced. In the limit as $k \rightarrow 0$, $G_r(k, \mu, M, f^2)$ is reduced to $G_r(0, \mu, M, f^2)$, that is,

$$G_r(0, \mu, M, f^2) = G_r(0, 0, M, f^2). \quad (19)$$

This method can be interpreted as meaning that $G_r(0, \mu, M, f^2)$ is renormalized to $G_r(0, 0, M, f^2)$.

The $G_r(k, \mu, M, f^2)$ can generally be expressed by the following form,

$$\begin{aligned} G_r(k, \mu, M, f^2) &= \frac{1}{4\pi^2} P \int_{\mu}^{\infty} \left(\frac{1}{\omega'^2 - \omega^2} - \frac{1}{\omega'^2 - \mu^2} \right) \omega' k' [\sigma_-(\omega') + \sigma_+(\omega')] d\omega' + \alpha(\mu, M, f^2) \\ &= \frac{1}{4\pi^2} P \int_0^{\infty} \left(\frac{1}{k'^2 - k^2} - \frac{1}{k'^2} \right) k'^2 [\sigma_-(k') + \sigma_+(k')] dk' + \alpha(\mu, M, f^2). \end{aligned} \quad (20)^*$$

But to determine this unknown $\alpha(\mu, M, f^2)$ is a matter of importance. On the other hand, in the case of $\mu=0$,

$$\begin{aligned} G_r(k, 0, M, f^2) &= \frac{1}{4\pi^2} P \int_0^{\infty} \left(\frac{1}{k'^2 - k^2} - \frac{1}{k'^2} \right) k'^2 [\sigma_-(k') + \sigma_+(k')] dk' + \alpha(0, M, f^2). \end{aligned} \quad (21)$$

Since $G_r(0, \mu, M, f^2) = \alpha(\mu, M, f^2)$ and $G_r(0, 0, M, f^2) = \alpha(0, M, f^2)$, under the assumption of (19) $\alpha(\mu, M, f^2)$ does not depend on μ in spite of the μ -dependence of both $\sigma_-(k')$ and $\sigma_+(k')$ in Eq. (20). Therefore $\alpha(\mu, M, f^2)$ can be rewritten as follows:

$$\alpha(\mu, M, f^2) = \beta(M, f^2). \quad (22)$$

Then

$$D^{(1)}(\omega) = \frac{1}{2} [D_-(\omega) + D_+(\omega)] = F_1(\omega) + \frac{1}{4\pi^2} P \int_{\mu}^{\infty} \left(\frac{1}{\omega'^2 - \omega^2} - \frac{1}{\omega'^2 - \mu^2} \right)$$

* It must be noted that

$$\frac{1}{4\pi^2} P \int_{\mu}^{\infty} \frac{\omega' k' [\sigma_-(\omega') + \sigma_+(\omega')]}{\omega'^2 - \mu^2} d\omega'$$

does not depend on ω , and that the $D^{(1)}(\omega)$ derived from this $G_r(k, \mu, M, f^2)$ satisfies the relation of (1'').

$$\times \omega' k' [\sigma_-(\omega') + \sigma_+(\omega')] d\omega' + \beta(M, f^2). \quad (23)$$

This unknown function $\beta(M, f^2)$ is determined on the basis of some consideration about the limiting case $\mu=0$ and $\omega \rightarrow 0$. In this limiting case, $F_1(\omega)=0^*$ and

$$D^{(1)}(0) = \beta(M, f^2). \quad (24)$$

Making use of the relation of (9'), we get

$$\beta(M, f^2) = -f^2/M. \quad (25)$$

Thus $D^{(1)}(\mu)$ in Eq. (23) can be expressed as follows:

$$\begin{aligned} D^{(1)}(\mu) &= 1/2 [D_-(\mu) + D_+(\mu)] = F_1(\mu) + \beta(M, f^2) \\ &= -2f^2/M. \end{aligned} \quad (26)$$

Since the value of $D^{(1)}(\mu)$ in Eq. (26) is twice as large as that in Eq. (13),

$$a_1 + 2a_3 = -0.06. \quad (27)$$

By using this together with the relation of (16),

$$a_1 = 0.16, \quad a_3 = -0.11. \quad (28)$$

These values also agree very well with those which have been obtained by Orear.⁵⁾

§ 4. Discussions

In § 3 we have tried two kinds of subtraction methods for the calculation of $D^{(1)}(\mu)$ and have shown that each of them leads to the satisfactory explanation of s -wave phase shifts. An essential point in our method is that the coupling constant in the nucleon core is reduced to $f = (\mu/2M)g$, and this may be the reason why the experimental results for s -wave can be reproduced. This reduction of coupling constant is also consistent with the experimental results for other pion phenomena. At low energies it may guarantee the availability of static approximation and may predict that p -wave interaction plays an important role in pion-nucleon scattering at energies 100~200 Mev. So long as the $G_{1\pi}(\omega)$ mentioned in § 3 is adopted, the characteristic properties not only of s -wave but also of p -wave interaction can, of course, be explained by relations (1') and (2). This statement is based on the following result shown by Anderson et al.¹³⁾ The pion-nucleon scattering in low energy region, in particular ($I=3/2, J=3/2$)-resonance scattering, can be described fairly well by the dispersion relation if only the correct values of a_1 and a_3 are given. In high energy pion phenomena, such as pion-nucleon collisions at Bev, our model may also predict that during a collision the nucleon does not change its direction remarkably because pion-nucleon interaction takes

* If the limiting procedure is performed in such a way as $\mu=\omega \rightarrow 0$, we have $F_1(\omega)_{\omega=0} = -f^2/M$. Then the relation corresponding to Eq. (25) is $\beta(M, f^2)=0$. In this case the expression for $D^{(1)}(\omega)$ is reduced to the same form as in the case of Method (I).

place mostly in the outer region than $1/M$. It is interesting that the characteristic feature^{14,15)} in pion-nucleon collision at high energy may be explained in terms of the reduction of coupling constant in the nucleon core.

Some efforts have been made in order to interpret the behavior of s -wave in terms of the “ s -wave damping” or “pair suppression” which means a suppression of contributions from virtual nucleon-antinucleon pairs.¹⁶⁾ Such approaches to s -wave interaction will be different from ours because we suppose that the treatment based on the current field theory may no longer be valid in describing the phenomena within the range of $1/M$.

Recently several approaches to the π - π interaction¹⁷⁾ have been made with the object of reproducing the experimental results for high energy pion phenomena or those for s -wave pion-nucleon interaction. The π - π interaction, for example $\phi_\pi^4(x)$, will make it possible to strengthen the interaction in pion cloud. When the π -interaction Lagrangian explicitly contains $\phi_\pi^4(x)$ -term, an additional term C must be involved on the right-hand side of Eq. (1). If we are forced to reinterpret our subtraction method under the assumption that interaction Lagrangian contains $\phi_\pi^4(x)$ -term, it may be said that the additional C term ought to be included in our expression $G_{1r}(\omega) - G_1(\omega)$. Finally we want to add a remark about the additional C term in the expression of $D^{(2)}(\omega)$, although some discussion has been made by Matthews and Salam.¹²⁾ So far as the dispersion relation (2) is concerned, it is needless to perform any subtraction procedure by the following reason. Even if a term $C'(\mu, M, f^2)$ is added to the right-hand side of Eq. (2), this $C'(\mu, M, f^2)$ must be equal to zero on account of the relation $D_-(0) = D_+(0)$ based on the crossing symmetry.

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On Stellar Models with Double Energy-Sources*

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To investigate the characteristics of stellar models having double energy-sources, three sample models ($M=1.2M_{\odot}$) consisting of the following regions were constructed using the newest rate of the CN-cycle: (1) hydrogen-rich envelope, (2) radiative helium region and (3) convective helium core. A model for which the mass fraction of the helium regions is 0.6 shifts towards the left of the RR Lyrae gap in the HR-diagram from the red giant region, in which the corresponding model of Hoyle and Schwarzschild lies. This result shows that the properties of such models are very sensitive to the rates of both the hydrogen- and helium-burning.

§ 1. Introduction

Calculations have been made by Morton¹⁾ on the effect on the main sequence of transfer of control of the CN-cycle from $N^{14}(p, \gamma)O^{15}$ to $C^{12}(p, \gamma)N^{13}$. His results indicate a sideward shift of the main sequence on the mass-luminosity diagram. This shift would be of the order of the observational spread in the width of the main sequence, and he has concluded that the absence of such a shift is an evidence for the absence of a level in the critical region of O^{15} just above 7.30 Mev. Recently, Hebbard and Povh^{2)***} have found a new level of O^{15} with an excitation energy of 7.17 Mev, hence it is established that the $N^{14}(p, \gamma)O^{15}$ reaction is off-resonant at stellar energies.

A few years ago, several authors³⁻⁴⁾ investigated the structure of population II stars in the helium-burning phase. Their results showed that the horizontal branch in the HR-diagram of the globular clusters corresponds to the stars in this phase. In their calculation, however, they used the old rate of the CN-cycle which was the most reasonable at that time and which was one hundred times larger than the newest correct value. On the other hand, it has been shown by Hayashi et al.⁵⁾ that in the case of massive stars the properties of the stellar models with double energy-sources are very sensitive to the rates of both the hydrogen- and helium-burning. One may expect that such a tendency will appear also in the case of population II stars at the helium-burning stage.

The purpose of this paper is to reconstruct the models in the helium-burning phase by using the newest rate of the CN-cycle.

* An outline of this paper was reported at the Symposium on Nucleogenesis and Stellar Evolution, held at the Research Institute for Fundamental Physics, Kyoto University, in October 1959.

** The author is indebted to Dr. Povh for the information in advance of publication.

§ 2. Definitions and assumptions

Closely following the evolutionary scheme as discussed by Hoyle and Schwarzschild,⁸⁾ we have adopted the following physical parameters and assumptions in our computations.

- (a) The mass is taken as $1.2 M_{\odot}$.
- (b) The models consist of the following three zones: (i) a hydrogen-rich envelope, (ii) a radiative pure helium zone and (iii) a convective helium core.
- (c) Radiation pressure is neglected.
- (d) No mixing occurs in the envelope, or between the envelope and the helium regions, so that inside the convective helium core complete mixing keeps the constant composition of essentially pure helium.
- (e) The abundance of heavy elements (*i. e.* elements other than hydrogen and helium) is assumed to be so low that they do not contribute to the opacity.
- (f) The envelope is divided into two parts, the outer part, in which the opacity is taken as arising from the free-free transitions of hydrogen and helium, and the inner zone, in which the free electron scattering determines the opacity. No intermediate region of mixed opacity has been introduced—the opacity formula is switched abruptly from free-free transitions to electron scattering at an appropriate interface, the opacity being kept continuous.
- (g) In our models energy generation takes place by the 3α -process in the helium core and by the CN-cycle in a shell just outside the radiative helium region. The rate for the former process is given by

$$\epsilon_{3\alpha} = 7.12 \times 10^{-4} \rho^2 Y^3 (T/1.35 \times 10^8)^{29.0} \text{ ergs/g} \cdot \text{sec}, \quad (1)$$

ρ being the density. The numerical values in this expression are adopted from Salpeter's work.⁹⁾ The rate for the CN-cycle is

$$\epsilon_{\text{CN}} = 3.34 \times 10^5 \rho X X_N (T/3 \times 10^7)^{16.3} \text{ ergs/g} \cdot \text{sec}, \quad (2)$$

where the newest result of the off-resonant measurements for $\text{N}^{14}(p, \gamma)\text{O}^{15}$ is used,⁷⁾ and X_N is the concentration of nitrogen.

- (h) The thickness of the shell in which the energy generation by the CN-cycle occurs is neglected. Hence the energy flux jumps discontinuously at the shell and is constant outside the shell.
- (i) The envelope retains the initial composition

$$X_e = 0.90, \quad Y_e = 0.10, \quad X_N = 0.0005 \quad (3)$$

for which the mean molecular weight in the envelope becomes $\mu_e = 0.533$. This concentration of nitrogen is just the same as adopted by Hoyle and Schwarzschild, and it is a half of the one in the revised table of the cosmical chemical compositions compiled by Cameron.⁸⁾ Recently, Greenstein and Keenan⁹⁾ have found a giant star having a logarithmic deficiency of -1.55 compared with the CN abundance in the population I stars. The models, having X_N one order of magni-

tude smaller than the one used by Hoyle and Schwarzschild, will also be discussed in § 4. The mean molecular weight in the helium regions is taken to be $4/3$.

(j) The boundary condition that the temperature and density shall tend to zero at the surface of the star is used.

(k) As for the outer convective zone, our models have higher effective temperature than 4700°K , hence this zone is not expected to play as important a role as in the low-temperature stars.

(l) Subscripts will be used with the following meaning: e for quantities in the envelope, i for quantities in the radiative helium region, c for central values, 1 for values at the interface between the envelope and the radiative helium zone, d for values at the interface between the radiative helium region and the convective core, and s for the values at the interface where the opacity formula switches.

§ 3. Basic equations and construction of models

In terms of the dimensionless variables,¹⁰⁾

$$P = p \frac{GM^2}{4\pi R^4}, \quad T = t \frac{\mu H}{k} \frac{GM}{R}, \quad M(r) = qM, \quad r = xR, \quad (4)$$

the basic equations for the conditions of mechanical and thermal equilibria take the following forms:

$$0 < x < 1; \quad \frac{dp}{dx} = -\frac{pq}{tx^2}, \quad \frac{dq}{dx} = \frac{px^2}{t}, \quad (5)$$

$$0 < x < x_d; \quad \frac{t}{p} \frac{dp}{dt} = 2.5, \quad (6)$$

$$x_d < x < x_1; \quad \frac{dt}{dx} = -C_{El}^* \frac{p}{t^4 x^2}, \quad (7)$$

$$x_1 < x < x_s; \quad \frac{dt}{dx} = -C_{El} \frac{p}{t^4 x^2}, \quad (8)$$

$$x_s < x < 1; \quad \frac{dt}{dx} = -C_{Kr} \frac{p^2}{t^{8.5} x^2}, \quad (9)$$

where

$$C_{Kr} = \frac{3}{4ac} \frac{3.68 \times 10^{20} (1 + X_e)}{(4\pi)^3} \left(\frac{k}{\mu_e HG} \right)^{7.5} \frac{LR^{0.5}}{M^{5.5}}, \quad (10)$$

$$C_{El} = \frac{3}{4ac} \frac{0.19 (1 + X_e)}{(4\pi)^2} \left(\frac{k}{\mu_e HG} \right)^4 \frac{L}{M^3}, \quad (11)$$

$$C_{El}^* = \frac{3}{4ac} \frac{0.19}{(4\pi)^2} \left(\frac{k}{\mu_e HG} \right)^4 \frac{L_{\text{core}}}{M^3}, \quad (12)$$

and

$$L = L_{\text{shell}} + L_{\text{core}}. \quad (13)$$

The homology variables are

$$U = \frac{px^3}{qt}, \quad V = \frac{q}{xt}, \quad (14)$$

$$n+1 = \frac{q}{C_{Kr}} \frac{t^{8.5}}{p^3} \quad \text{for } 1 > x > x_s, \quad (15)$$

$$n+1 = \frac{q}{C_{El}} \frac{t^4}{p} \quad \text{for } x_s > x > x_1, \quad (16)$$

$$n+1 = \frac{q}{C_{El}^*} \frac{t^4}{p} \quad \text{for } x_1 > x > x_d. \quad (17)$$

The boundary conditions at the center are

$$q=0 \quad \text{and} \quad \frac{dp}{dx}=0 \quad \text{at } x=0. \quad (18)$$

There are two disposable constants for Eqs. (5) and (6), *i. e.* t_c and p_c . Regarding x_d as a disposable constant, we can obtain a solution for $0 < x < x_1$ from Eqs. (5) and (7) with p_c , t_c and x_d all specified, where x_1 is determined by the requirement that q will take some assigned value q_1 . The transition from Eq. (6) to Eq. (7) for specified x_d is made with the following fitting conditions.

$$U_{dt} = U_{de}, \quad V_{dt} = V_{de}, \quad (n+1)_{dt} = 2.5. \quad (19)$$

The value of C_{El}^* is given by the continuity of dt/dx .

Provided that values of C_{Kr} and x_s are given, Eqs. (5), (8) and (9) can be integrated inward under the boundary condition,

$$p=0, \quad q=1, \quad t=0 \quad \text{at } x=1. \quad (20)$$

The fitting conditions at x_s are such that dimensionless variables and their derivatives are continuous. The value of C_{El} is determined by the continuity of dt/dx , that is,

$$C_{El} = C_{Kr} \frac{p_s}{t_s^{4.5}}. \quad (21)$$

If we regard q_1 as given, then p_c , t_c , x_d , C_{Kr} , x_s , and R , which is required for transforming the dimensionless variables to physical ones, remain to be determined. The continuity of U/μ and V/μ at x_1 gives two conditions. The third one is that the envelope solution has a specified q_1 at x_1 . Three other conditions are supplied by the following physical requirements.

(i) The energy generation in the core due to the helium-burning is given by⁽¹¹⁾

$$L_{3\alpha} = 7.12 \times 10^{-4} Y^3 \left(\frac{7.5}{33.5} \right)^{3/2} \left(\frac{k}{2G\mu_c H} \right)^{3/2} \rho_c^{3/2} T_c^{3/2} \left(\frac{T_c}{1.35 \times 10^8} \right)^{29.0} \quad (22)$$

and must equal the outward flux L_{core} as determined by Eq. (12).

(ii) The energy production in the hydrogen-burning shell, L_{shell} , must equal $L - L_{\text{core}}$, where

$$L_{\text{shell}} = 3.34 \times 10^5 X X_N \rho_{1e}^2 \left(\frac{T_1}{3 \times 10^7} \right)^{16.3} \frac{4\pi r_1^3}{V_{1e} [2 + (14.3)/(n+1)_{1e}] - 3} \quad (23)$$

and L is determined by (11).

(iii) The opacity from electron scattering must be equal to the opacity from free-free transitions at $x = x_s$.

§ 4. Results and discussions

We have constructed a model for $q_1 = 0.60$, which is designated as Model I. In Table I, the mathematical and physical characteristics of this model are summarized and are compared with those of the corresponding model of Hoyle and Schwarzschild, which is referred to as H-S. The results of these two models are plotted in a color-magnitude diagram in Fig. 1, where the shaded area is that occupied by the stars of the globular clusters M3¹²⁾ and M92.¹³⁾ Color index has been derived on the same system as used by Hoyle and Schwarzschild. As seen in Fig. 1, Model I shifts towards the left of the RR Lyrae gap in the Hertzsprung-Russell diagram from the red giant region, in which the H-S model lies. Since the re-

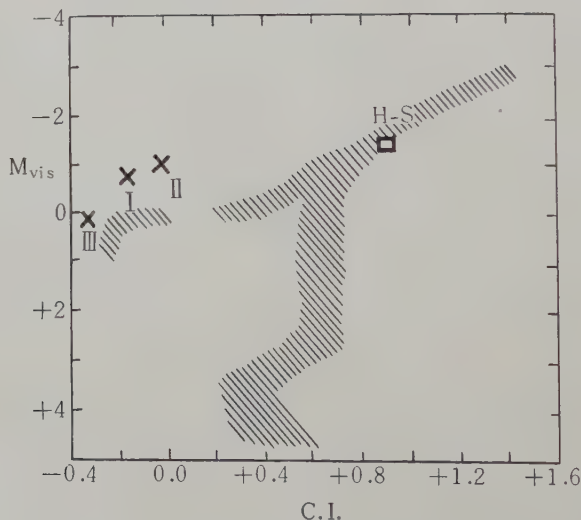


Fig. 1. Hertzsprung-Russell diagram for the star with $M = 1.2M_{\odot}$ at the helium-burning stage. The shaded area indicates the observed sequences in globular clusters.

Table I. Mathematical and physical characteristics of inhomogeneous models with helium-burning in convective core (*c* to *d*), radiative zone in helium (*d* to 1), hydrogen-burning shell (at 1), and radiative envelope (1 to surface) ($M=1.2M_{\odot}$)

	H-S	Model I	Model II	Model III
θ_d	0.658	0.658	0.658	0.658
ξ_d	1.565	1.564	1.564	1.564
$\log x_d$	-2.70	-2.18	-2.32	-1.97
$\log q_d$	-0.73	-0.72	-0.72	-0.71
$\log t_d$	+2.04	+1.13	+1.26	+0.93
$\log p_d$	+9.38	+7.30	+7.84	+6.46
$(n+1)_{1\epsilon}$	14.6	10.1	10.8	6.86
$U_{1\epsilon}$	0.076	0.13	0.11	0.16
$V_{1\epsilon}$	3.53	3.35	3.38	3.30
$(n+1)_{1\epsilon}$	3.91	3.80	3.82	3.76
$\log x_1$	-2.29	-1.82	-1.93	-1.63
$\log q_1$	-0.22	-0.22	-0.22	-0.22
$\log t_1$	+1.52	+1.06	+1.18	+0.87
$\log p_1$	+7.06	+5.29	+5.81	+4.59
$\log \bar{C}_{Hl}$	-1.76	-1.91	-1.89	-1.99
$\log x_s$	-0.39	-0.27	-0.29	-0.23
$\log p_s$	-1.20	-1.68	-1.79	-1.97
$\log t_s$	-0.55	-0.78	-0.74	-0.85
$\log R/R_{\odot}$	+1.39	+0.75	+0.86	+0.54
$\log L/L_{\odot}$	+2.44	+2.31	+2.33	+2.23
$\log T_c$	+8.00	+8.12	+8.15	+8.12
$\log \rho_c$	+3.58	+3.95	+4.03	+3.94
$\log T_1$	+7.30	+7.48	+7.50	+7.50
$\log \rho_{1\epsilon}$	+1.11	+1.86	+1.80	+1.96
$L_{\text{shell}}/L_{\text{core}}$	0.96	0.40	0.49	0.094
$\log T_{\text{eff}}$	3.67	3.96	3.91	4.10
B.C.	-0.14	-0.41	-0.20	-1.07
M_{vis}	-1.33	-0.74	-1.00	+0.14
C.I.	+0.88	-0.14	+0.00	-0.33

markable differences between Model I and H-S result from using the newest rate of the CN-cycle, it must be emphasized that the characters of the model having a double energy-source strongly depend on the ratio of the hydrogen-burning rate to the helium-burning one.

In our computations we did not take account of the effect of the conversion of helium to carbon in the core. A rough estimate indicates that the degree of depletion of the helium content in the core amounts to 0.2. A sequence of models having the helium-depleted cores will be studied in a forthcoming paper.

Recently, Eccles and Bodansky¹⁴⁾ have attempted to determine the probability that the 7.65 Mev state of C^{12} decays to its lower states and found that it is less than 0.1%. This limit is an order of magnitude smaller than the best previous experimental value¹⁵⁾ from which Salpeter estimated the rate of the helium-burning. Hence it may be of interest to construct a model by using the value of one tenth of Salpeter's reaction rate. In this case, Model I should be replaced by Model II in Table I and Fig. 1. The following feature can be pointed out by comparison of these two models. The reduced radius of the hydrogen-burning shell, x_1 , now

become smaller by a factor 0.78 for Model II than Model I. This gives rise to an increase of radius R by a factor 1.29 and consequently to a decrease of the effective temperature.

Finally, we consider the case where a smaller value of nitrogen concentration is adopted according to the argument given in § 2. The results are given in Table I as Model III. In this case, it should be noted that x_1 becomes larger than that of Model I while $L_{\text{shell}}/L_{\text{core}}$ decreases, due mainly to the smaller X_N .

The author wishes to thank Prof. C. Hayashi, Dr. J. Jugaku and Mr. S. Sakashita for their valuable discussions.

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Surface Diffuseness and Phenomenological Treatment of O^{17} Nucleus*

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The effects of surface diffuseness on the single particle level structure and wave functions are investigated in a pure single particle model. It is shown that the ground state and the first excited state of O^{17} , which are interpreted as giving $1d_{5/2}$ and $2s_{1/2}$ shell levels respectively, and low energy elastic scattering of neutrons by O^{16} can be well reproduced by a diffuse potential well. In particular, it is pointed out that the relative location of $2s_{1/2}$ and $1d_{5/2}$ levels depends sensitively on finer details of the surface diffuseness. The modifications of the single particle wave functions from pure harmonic oscillator functions are further investigated, and they are found to be small up to $1p$ states, but for higher states they may be so large that with pure oscillator functions one cannot hope to make quantitative discussions when treating near $A=16$ nuclei.

§ 1. Introduction

The nuclear shell model has had many successes in explaining the properties of the ground and low excited states of atomic nuclei.¹⁾ On the other hand, scattering of nucleon is well described by the optical model also in low energy region.²⁾ These facts suggest that we can imagine that nucleons move independently in an averaged potential under the influence of strong spin-orbit interaction. In treating a nuclear problem in such an independent particle model, we encounter with the essential problem, what potential to adopt in the first place. In the early stage of the shell model, this average potential was taken to be a square well or a harmonic oscillator potential and one guessed that true affairs might lie between these two extremes.¹⁾ Until quite recently much efforts for improving the shell model, especially in taking account of the effects of residual interactions, were also based mainly on a harmonic oscillator potential. More realistic single particle potential, however, should be negative inside a nucleus and should tend to zero outwards, which property is possessed neither by an infinite square well nor by a harmonic oscillator potential. No doubt, it is quite desirable that any future treatments aiming at quantitative agreement with observations should adopt some more realistic single particle potential as the starting point.

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Recently, Green proposed a potential,⁸⁾

$$V(r) = \begin{cases} -V_0, & r < R_0, \\ -V_0 \exp[-a(r-R_0)], & r > R_0, \end{cases} \quad (1)$$

while, Woods and Saxon invented another one,⁴⁾

$$V(r) = -V_{WS}/[1 + \exp \alpha(r-b)], \quad (2)$$

taking account of diffuse potential boundary. (2) is presented as (a real part of) the optical potential, and applied to the problem of energy levels of bound states by Ross, Mark and Lawson.⁵⁾ These authors showed that heavy nuclei are well described by these diffuse boundary potentials. The potential (2) is also supported by electron scattering experiment performed by Stanford group.⁶⁾

The lighter the nuclei, the more prominent is the diffuseness of the potential on the nuclear surface, so for the study of surface diffuseness, lighter nuclei are adequate. In fact, in lightest nuclei, surface region covers the whole nucleus. Therefore, for lighter nuclei, the central region may be reasonably represented by some harmonic oscillator potential. This point of view is supported in fact by the electron scattering experiment.⁷⁾ But in the harmonic oscillator potential, the fact that potential becomes zero outside the nucleus is not taken into account, and, of course, the potential cannot describe elastic scatterings.

Thus we propose to modify the potential as*

$$V(r) = \begin{cases} -V_0 + \frac{1}{2} \hbar \omega_0 (\alpha_0 r)^2, & r < r_0, \\ -\frac{1}{2} \hbar \omega_1 [\alpha_1 (r_1 - r)]^2, & r_0 < r < r_1, \\ 0, & r_1 < r, \end{cases} \quad (3)$$

where,

$$\begin{aligned} \alpha_0 &= (M\omega_0/\hbar)^{1/2}, & \alpha_1 &= (M\omega_1/\hbar)^{1/2}, \\ \omega_1^2 &= \omega_0^2 r_0 / (r_1 - r_0), & r_1 &= 2V_0 / \hbar \omega_0 \alpha_0^2 r_0. \end{aligned} \quad (4)$$

We proceed to compare those brought about by this potential with those brought by the harmonic oscillator potential using some particular examples. Our purpose is to get an idea of how much the so much used harmonic oscillator wave functions represent the real situation. Our choice of the potential (3) in the region $r_0 < r < r_1^{**}$ is for simplicity, allowing for the potential to become zero from region I to III smoothly. The last two equations in (4) guarantee the smoothness at $r = r_0$. Of course, spin-orbit interaction,

* For a reasonable choice of parameters in Eq. (2), the potential (2) can be made to resemble our potential (3) fairly well. Thus, there is no essential difference between the potentials (2) and (3). We will later discuss the difference between the potential (1) and ours (3) in detail.

** This region shall be called region II and the regions $r < r_0$ and $r_1 < r$ region I and III, respectively.

$$V_{ls}(r) = -\frac{\gamma \hbar^2}{2m^2 c^2} \frac{1}{r} \frac{dV(r)}{dr} \frac{(\mathbf{l} \cdot \mathbf{s})}{\hbar^2} \quad (5)$$

(where m is the nucleon mass),

should be added to (3) as usual. The number of our independent parameters is four, i.e. V_0 , ω_0 , r_0 and γ .

Let us take up, as our first example, O^{17} , which is an important nucleus, because it offers some fundamental data (e.g. spin-orbit splitting, $2s$ - $1d$ level separation) to be further used in more complicated cases. The first aim of this work is to get such fundamental data. This is possible, because we can expect that O^{17} nucleus may be regarded as is consisted of an inert closed shell core plus one extra neutron outside the

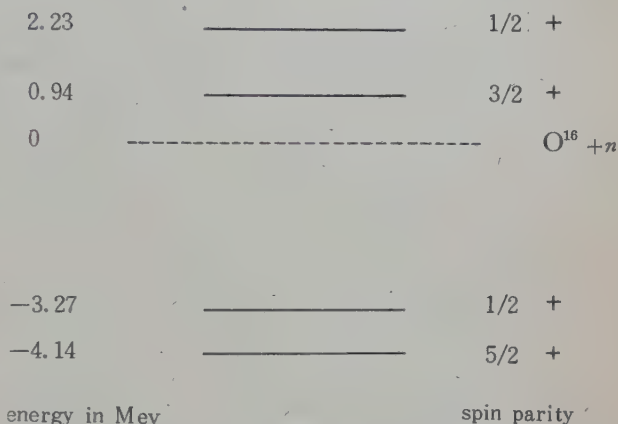


Fig. 1. Energy levels of O^{17} which are interpreted as caused by jumps of the extra neutron into higher states

core. We will neglect the effect of antisymmetry of this extra neutron with the neutrons in the core. Levels which are interpreted as formed by jumps of this neutron into higher states are presented in Fig. 1.⁸⁾ Resonances of 0.94 Mev and 2.23 Mev levels are observed in elastic scattering of neutrons by O^{16} , and these levels are established to be $d_{3/2}$ and $s_{1/2}$ respectively. -4.14 Mev (ground) state and -3.27 Mev (first excited) state respectively represents $1d_{5/2}$ and $2s_{1/2}$ single particle shell levels. An interesting feature appearing in this data is that $2s_{1/2}$ and $1d_{5/2}$ levels lie very close together. Splitting of $1d_{5/2}$ and $1d_{3/2}$ is due to spin-orbit interaction. The point in question is the relative positions of $2s$ and $1d$ levels when no spin-orbit interaction exists. The fact that $2s_{1/2}$ and $1d_{5/2}$ lie close together indicates that, without spin-orbit interaction, $2s$ level would be lower than $1d$ level. In any square well potential, on the contrary, $2s$ level will come out higher than $1d$ level, and in the harmonic oscillator case the two levels come out degenerated. Why is $2s$ level lower than $1d$ level? Will our potential (3) with (5) succeed to reproduce this level structure? To answer to this question is the second aim of this work.

At first sight, our potential (3) with (5) contains four parameters and Fig. 1, four data, so the question is simply to determine the parameters. But the significance of such study should not be underestimated. Fowler and Cohn have already studied this problem, taking Green's functional form (1) as the potential.⁹⁾ Their result shows that for a reasonable choice of parameter a in (1), which is deduced

from $d_{3/2}$ resonance width, $1d$ level is found to lie considerably lower than $2s$ level, although their result is much better than the square well case. In the present work, we will show that the data are well reproduced by our potential (3) for a reasonable set of parameters. Comparing these results, we shall have more detailed knowledge about the diffuseness of the potential. We will later on discuss this point more in detail.

In § 2, we will present our method to calculate energy level, wave function and scattering phase-shifts, and in § 3, the results of the calculations.

§ 2. Calculations

Our first task is to determine the parameters appearing in (3) and (5), so that the data shown in Fig. 1 are well reproduced. Once the parameters are fixed, energy levels, scattering phase shifts and wave functions may be readily calculated. Practically, first, we give ω_0 an arbitrary value and determine V_0 and r_0 , so that $2s$ bound state and s state scattering resonance appear in the observed energies. In s state, spin-orbit interaction (5) gives no contribution. Then, using these ω_0 , V_0 and r_0 , we decide the value of γ , which gives the resonance in $d_{3/2}$ state in the observed energy value. Finally, using these values of parameters, $1d_{5/2}$ energy level is calculated. Changing the starting value of ω_0 , these procedures are repeated, until we arrive at the observed $1d_{5/2}$ energy value.

To perform the above-mentioned calculations, we have to solve Schrödinger equations in regions I, II and III, and match the logarithmic derivatives of the solutions at $r=r_0$ and $r=r_1$. In region III a solution to Schrödinger equation is well known and need not be reproduced here. Schrödinger equations to be solved in regions I and II read as follows,

$$-\frac{\hbar^2}{2M} \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\phi}{dr} \right) + \left[-V_0 + \frac{1}{2} \hbar \omega_0 (\alpha_0 r)^2 + \frac{\hbar^2}{2M} \frac{l(l+1)}{r^2} - \frac{\gamma \hbar^2}{2m^2 c^2} \hbar \omega_0 \alpha_0^2 \frac{(\mathbf{l} \cdot \mathbf{s})}{\hbar^2} \right] \phi = E \phi, \quad (6.1)$$

and

$$-\frac{\hbar^2}{2M} \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\phi}{dr} \right) + \left[-\frac{1}{2} \hbar \omega_1 \alpha_1^2 (r_1 - r)^2 + \frac{\hbar^2}{2M} \frac{l(l+1)}{r^2} - \frac{\gamma \hbar^2}{2m^2 c^2} \hbar \omega_1 \alpha_1^2 \frac{(\mathbf{l} \cdot \mathbf{s})}{\hbar^2} \frac{r_1 - r}{r} \right] \phi = E \phi, \quad (6.2)$$

respectively, where M is a reduced mass in the system of O^{16} core and one neutron.

We will solve Eqs. (6.1) and (6.2) in power series expansions. In fact, our main interest lies in examining finer details of the shape of the potential, so that approximate methods (e.g. WKB approximation) will be of no use. The solution to Eq. (6.1) is given in a power series expansion as

$$\psi(r) = N \exp\left[-\frac{1}{2}(\alpha_0 r)^2\right] \sum_n \frac{1}{n!} \frac{\Gamma(n + \frac{1}{2}l - (\varepsilon + 1)/4 + 1)}{\Gamma(n + l + 3/2)} (\alpha_0 r)^{2n+l}, \quad (7.1)$$

where, $\varepsilon = (E + V_0 + (\gamma \hbar^2 / 2m^2 c^2) \hbar \omega_0 \alpha_0^2 (\mathbf{l} \cdot \mathbf{s}) / \hbar^2) / (1/2) \hbar \omega_0$, and N is to be determined from normalization. The solution to Eq. (6.2) is a little more complicated. Putting

$$\psi(r) = \frac{1}{r} \sum_n c_n [\alpha_1 (r_1 - r)]^n, \quad (7.2)$$

recurrence formulae are obtained from (6.2) for the coefficients c_n to be,

$$\begin{aligned} 2 \cdot 1 \ c_2 + \left(\lambda - \frac{l(l+1)}{(\alpha_1 r_1)^2} \right) c_0 &= 0, \\ 3 \cdot 2 \ c_3 + \left(\lambda - \frac{l(l+1)}{(\alpha_1 r_1)^2} \right) c_1 \\ &+ \left(v - \frac{2l(l+1)}{(\alpha_1 r_1)^2} \right) \frac{1}{(\alpha_1 r_1)} c_0 = 0, \\ 4 \cdot 3 \ c_4 + \left(\lambda - \frac{l(l+1)}{(\alpha_1 r_1)^2} \right) c_2 \\ &+ \left(v - \frac{2l(l+1)}{(\alpha_1 r_1)^2} \right) \frac{1}{(\alpha_1 r_1)} c_1 \\ &+ \left(1 + \left(v - \frac{3l(l+1)}{(\alpha_1 r_1)^2} \right) \frac{1}{(\alpha_1 r_1)^2} \right) c_0 = 0, \\ &\dots\dots\dots, \end{aligned} \quad (8)$$

where, $\lambda = 2E / \hbar \omega_1$ and $v = (\gamma \hbar^2 / m^2 c^2) \alpha_1^2 (\mathbf{l} \cdot \mathbf{s}) / \hbar^2$. In (8) the ratio c_1 / c_0 is to be determined by equating the logarithmic derivative of (7.2) at $r = r_1$ with that of the solution in region III. c_1 itself is to be decided from normalization.

Having thus obtained the solutions of Schrödinger equations for arbitrary energy, we can bring out the calculations along the lines indicated at the beginning of this paragraph. The results are shown in Table I.

Table I. Values of parameters obtained from the data

$\hbar \omega_0$	V_0	r_0	r_1	α_0^{-1}	r
15.95 Mev	56.23 Mev	3.726×10^{-13} cm	5.271×10^{-13} cm	1.669×10^{-13} cm	20.3

§ 3. Results and discussion

Now that we have determined the potential, we will investigate here the consequences derived from this potential and discuss them keeping our special interest on surface diffuseness.

3.1. Comparison with phase shift analysis and energy levels

The phase shifts of elastic scattering of a neutron by O^{16} nucleus are calculated and shown in Fig. 2. In calculating phase shifts at arbitrary energy, a slightly different method from that shown in § 2 was employed, because the method indicated in § 2 was suitable only when the solution in region III was known in

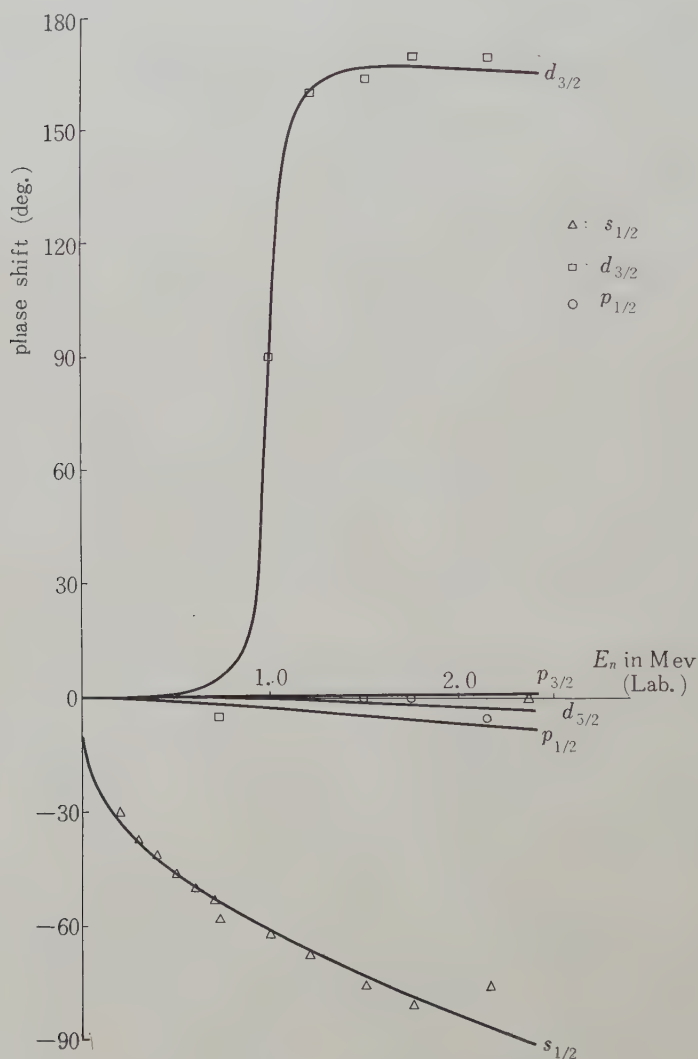


Fig. 2. Comparison of phase shifts with experiment

Experimental points above $E_n=0.73$ Mev are from reference 9), below, from Okazaki, Phys. Rev. **99** (1955), 55, and at $E_n=2.37$ Mev, from reference 10). As to the comparison of $p_{3/2}$ and $d_{5/2}$ phase shifts with observation, see the text.

the first place.* The revised method adopted in phase shift calculations is shown in the Appendix.

From Fig. 2 we see that the overall agreement of calculated phase shifts with those determined from experiments is satisfactory. To compare with experiments we must naturally take account of the compound elastic scattering effect. The result of our calculation represents only potential scattering, so that effect of compound elastic scattering should be super-imposed upon. This applies to $p_{3/2}$ wave (at two resonance energies 0.43 and 1.31 Mev) and to $s_{1/2}$ wave (at 2.37 Mev). As regards $s_{1/2}$ wave, the steep rise of the observed phase shift near 2.37 Mev is explained in this way.¹⁰⁾ As for $p_{3/2}$ wave, the phase shift in this energy region is known to be well reproduced by solely considering the contributions of the two compound resonances¹¹⁾ and our result is an excellent confirmation on the statement that potential phase shift is zero. Calculated $d_{5/2}$ phase shift is also sufficiently small, thus neglect of $d_{5/2}$ phase shift in the phase shift analysis is justified.⁹⁾ The $d_{3/2}$ phase shift at $E_n=0.73$ Mev is -5° experimentally, contradicting with our result (plus sign). But the possibility of minus phase shift is already excluded by a polarization experiment.¹¹⁾

The resonance width at $E_n=1.00$ Mev (due to $d_{3/2}$ wave) is 90 kev from Fig. 2, agreeing fairly well with the experimental value of 96 ± 5 kev. This resonance width was exploited to determine the diffuseness parameter appearing in (1) in Fowler and Cohn's work.⁹⁾ At any rate, comparison of phase shifts with experiments is equally good for both theirs and ours in so far as $s_{1/2}$ and $d_{3/2}$ waves are concerned. Calculations for other waves have not been reported, unfortunately; so let us proceed to consider the energy level of $1d_{5/2}$.

Fowler and Cohn obtained too low an energy for the $1d_{5/2}$ state, while it is correctly reproduced in our potential. Allowing for the results mentioned above, we see that our potential describes the observed facts more adequately than theirs.** As mentioned in the introduction, a square well potential gives too low $1d$ level relative to $2s$ level. Green's potential (1) somewhat resembles a square well potential, and, therefore, Fowler and Cohn's results remain to bear some resemblance to the case of the square well, although their results are certainly very much improved than the pure square well case. Thus, the situations of $1d$ and $2s$ levels are intimately connected with the diffuseness of the potential at the surface. In a harmonic oscillator potential the diffuseness is to some extent contained in itself, as a result of which $1d$ and $2s$ levels are degenerate. Our consideration may be further confirmed by noting that in a Coulomb potential, an extreme case of long tailed potential, $2s$ level is contained in a lower shell than that in which $1d$ level*** is contained.

* In § 2 we exploited the observed resonance energies of potential scattering, so we knew the wave function in region III at the energy aforesaid.

** Note that the number of independent parameters is same both for theirs and ours.

*** The $1d$ level in our notation is designated $3d$ level in the notation of atomic spectroscopy.

Thus we have shown that the situation of $1d$ and $2s$ levels and the resonance width at $E_n=1.00$ Mev can be adequately accounted for by properly treating the diffuseness of the potential at the nuclear surface. Next we will investigate the effect of the diffuseness on the wave functions.

3.2. Wave Functions

Once energy levels are determined, corresponding wave functions can be readily obtained from (7.1) and (7.2) with (8). The normalization constant is calculated by numerical integrations. Energy levels are determined by means of the method explained in § 2. The results are tabulated in Table II. Our main

Table II. Bound state in our potential

state	energy eigenvalue (Mev)	$\langle r^2 \rangle$ (α_0^{-2})	overlap integral
$1s_{1/2}$	-32.36	1.53 (1.5)	0.9998
$1p_{3/2}$	-17.95	2.65 (2.5)	0.997
$1p_{1/2}$	-14.24	2.73 (2.5)	0.997
$1d_{5/2}$	- 4.14	4.31 (3.5)	0.983
$2s_{1/2}$	- 3.27	6.07 (3.5)	0.929

Expectation values shown in the parenthesis are those for the corresponding harmonic oscillator case.

interest consists in examining the effect of potential behaviour at the nuclear surface on single particle wave functions. As our potential (3) is a modification of a harmonic oscillator potential, the comparison with the latter case is of particular interest. In this connection, in the fourth column of the table, are shown overlap integrals of our wave functions with the corresponding harmonic oscillator wave functions, i.e. wave functions which are eigen-functions in the harmonic oscillator potential with angular frequency equal to ω_0 .

For lower energy states overlap is better, as can be anticipated from an argument based on the perturbation theory. For lower states, wave functions damp out more rapidly in the external region, so that the perturbation operating only in the external region has less effect for these states. On the other hand, for the states approaching to the zero energy, the overlaps become worse and worse. This is because, for the higher states, wave functions penetrate out more and more on account of the effect of the potential behavior in regions II and III, as can be seen from Fig. 3. The statement is further confirmed by the calculations on expectation values of r^2 (the third column). Qualitatively, the argument stated here may be easily accepted. Our calculations will serve to estimate the errors caused by the ambiguity that surface and external regions are not correctly treated when use is made of harmonic oscillator wave functions. Thus, roughly speaking, use of the harmonic oscillator wave functions may be a good approximation when only $1s$

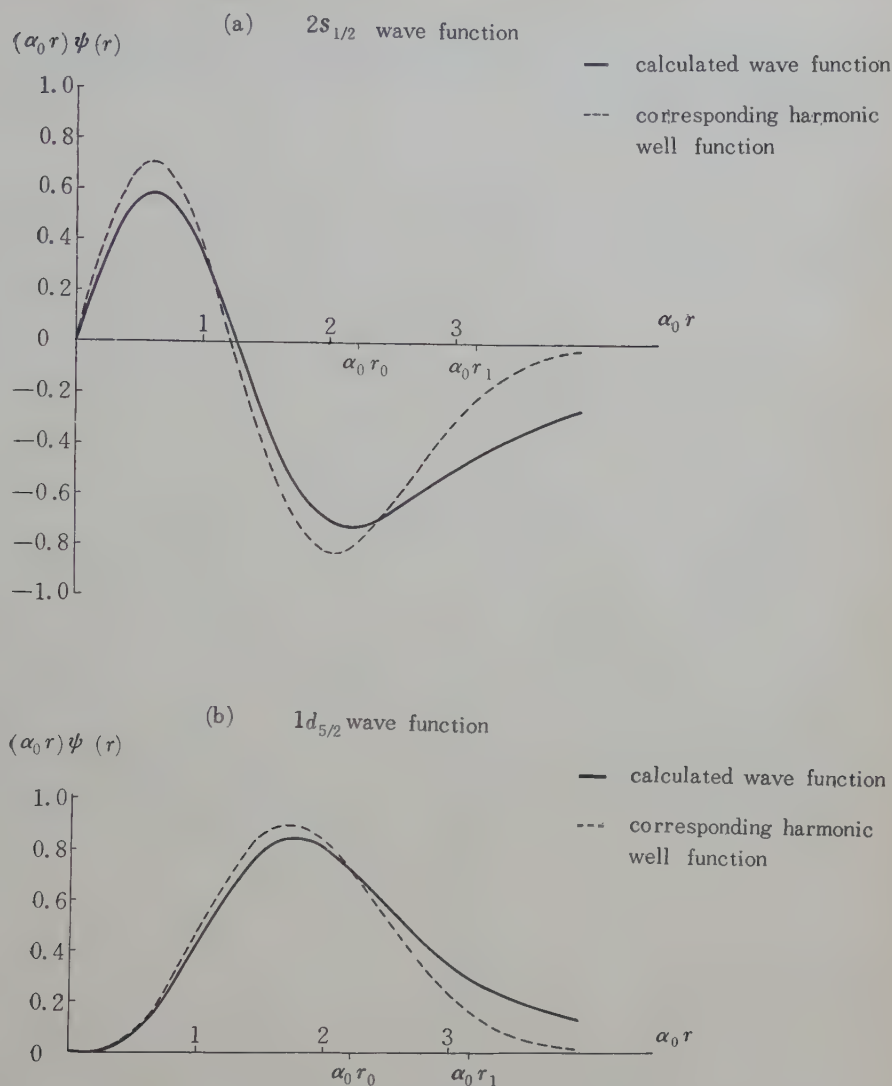


Fig. 3. Radial wave functions compared with the corresponding harmonic well case. For $1s_{1/2}$, $1p_{3/2}$ and $1p_{1/2}$ states mutual deviations are small, so they are not reproduced here. Normalization is such that

$$\int |(\alpha_0 r) \psi(r)|^2 d(\alpha_0 r) = 1.$$

and $1p$ states are involved, but in calculations involving higher states, it may cause some error because of the deviations of single particle wave functions from those of harmonic oscillator discussed here. For example, the fact that high energy elastic scattering of electrons by O^{16} nucleus is well described by the harmonic oscillator model endorses our conjecture, because the electron scattering is essentially determined by the charge distribution and this, in turn, involves $1s$ and $1p$ states

only.* On the other hand, $2s_{1/2}$ — $1d_{5/2}$ single particle transition probability is estimated to be larger by about factor 2 than the corresponding harmonic oscillator case (assuming $E2$ transition).

3.3 Nuclear Radius

As the final check of our potential we calculate here the nuclear radius, by means of the wave functions just derived before. For comparison we here calculate the equivalent radius:

$$R_{eq} = \sqrt{(5/3) \langle r^2 \rangle_{av}}.$$

Expectation values of r^2 are already calculated and shown in Table II. Thus, we only need to average these values over the nucleus.** The value is

$$R_{eq} = 3.328 \times 10^{-13} \text{ cm}$$

for the O^{16} nucleus.*** Strictly speaking, this value shows the distribution of the neutrons in the O^{16} nucleus. On the other hand, electron scattering experiment measures the distribution of the protons. The equivalent radius derived from published data is $3.41 \times 10^{-13} \text{ cm}$ for the charge distribution, slightly larger than our neutron value.⁷⁾ Qualitatively, Coulomb forces reduce $\hbar\omega_0$ in region I, while in regions II and III Coulomb barrier must be added. The former effect tends to push out the wave function and the latter to pull in. However, the latter effect would not be so large as to cancel out the former effect. Thus, it is not surprising that equivalent radius for protons comes out larger than that for neutrons. Carlson and Talmi calculated the nuclear radius from the Coulomb energy difference between mirror nuclei.¹²⁾ From their calculation, the equivalent radius for $A=17$ nuclei (F^{17} , O^{17}) becomes $3.245 \times 10^{-13} \text{ cm}$, which is smaller than ours. The Coulomb energy method, however, does not directly measure $\langle r^2 \rangle_{av}$, and moreover we should pay attention to the fact that their calculation is based on pure harmonic well wave functions. In this connection, we should like to note here that, in our calculation of the equivalent radius, if use were made of our corresponding harmonic oscillator functions, the radius would have been reduced by about 3%. Thus, we can conclude that our potential gives a reasonable radius.

§ 4. Conclusion

It is shown that $2s_{1/2}$ and $1d_{5/2}$ shell levels, and low energy elastic scatterings of neutrons by O^{16} can be well reproduced by a diffuse potential well. In particular, it is pointed out that the relative location of $2s_{1/2}$ and $1d_{5/2}$ levels depends sensitively

* What we have calculated are for neutrons; so for protons numerical conclusions cannot be drawn, but qualitative arguments will still hold.

** The effect of the center-of-mass motion should be taken into account for completeness, which amounts to a factor 15/16. This remark also applies to the values cited below.

*** We assume that our potential and wave functions apply also for O^{16} .

on finer details of the surface diffuseness. The modifications of the single particle wave functions from a pure harmonic oscillator case are small up to $1p$ states, but for higher states, they may be so large that with pure oscillator functions one cannot hope to make quantitative discussions when treating near $A=16$ nuclei.

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Appendix

The solution of the Schrödinger equation in region II shown in § 2 is suitable when the solution in region III is known beforehand. The equation can be solved also in a power series of $(r-r_0)$, which is useful when the solution in region I is already obtained. But it should be noted that because of the terms proportional to $1/r$ (which comes from the spin-orbit force) and $1/r^2$ (which comes from the centrifugal force) contained in the equation, the series diverges at $r-r_0=r_0$, so that when $r_1-r_0>r_0$ the analytic continuation is necessary. Thus, for the determination of the potential the solution in § 2 is more adequate. The potential we determined in § 2, however, admits also power series expansion in powers of $(r-r_0)$ in the whole region II; so, for the phase shift calculation the expansion in powers of $(r-r_0)$ is more profitable.

By means of the well-known expansions,

$$1/r = 1/r_0 \sum_n (-)^n [(r-r_0)/r_0]^n,$$

and

$$1/r^2 = 1/r_0^2 \sum_n (-)^n (n+1) [(r-r_0)/r_0]^n,$$

Eq. (6.2) can be solved as before. Putting

$$\phi(r) = \frac{1}{r} \sum_n d_n [\alpha_1 (r-r_0)]^n,$$

recurrence formulae for the d_n 's read as follows:

$$\begin{aligned} 2 \cdot 1 \, d_2 + \left(\lambda + [\alpha_1 (r_1 - r_0)]^2 - \frac{l(l+1)}{(\alpha_1 r_0)^2} + v \left(\frac{r_1}{r_0} - 1 \right) \right) d_0 &= 0, \\ 3 \cdot 2 \, d_3 + \left(\lambda + [\alpha_1 (r_1 - r_0)]^2 - \frac{l(l+1)}{(\alpha_1 r_0)^2} + v \left(\frac{r_1}{r_0} - 1 \right) \right) d_1 \\ + \left(-2\alpha_1 (r_1 - r_0) + \frac{2l(l+1)}{(\alpha_1 r_0)^3} - \frac{vr_1}{r_0} \frac{1}{(\alpha_1 r_0)} \right) d_0 &= 0, \end{aligned}$$

$$\begin{aligned}
& 4 \cdot 3 d_4 + \left(\lambda + [\alpha_1(r_1 - r_0)]^2 - \frac{l(l+1)}{(\alpha_1 r_0)^2} + v \left(\frac{r_1}{r_0} - 1 \right) \right) d_2 \\
& + \left(-2\alpha_1(r_1 - r_0) + \frac{2l(l+1)}{(\alpha_1 r_0)^3} - \frac{vr_1}{r_0} \frac{1}{(\alpha_1 r_0)} \right) d_1 \\
& + \left(1 - \frac{3l(l+1)}{(\alpha_1 r_0)^4} + \frac{vr_1}{r_0} \frac{1}{(\alpha_1 r_0)^2} \right) d_0 = 0, \\
& \dots\dots\dots,
\end{aligned}$$

where λ and v are already defined at just below Eq. (8).

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Theory of Relativistic Rotators and Elementary Particles. I

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In view of supplying a unified model of elementary particles the theory of relativistic rotators is developed. Clarifying the physical concept of relativistic rotators, general theory is constructed on the basis of kinematical variables. Consideration of the physical properties of the internal rotational space leads to the definition of isospin and internal chirality. Other internal constants of motion are also taken out to be identified as mass and ordinary spin. Possible rotator models are classified according to the structure of the rotational part of the Lagrangian. Various hitherto known models are automatically reproduced in this way, with their internal properties revealed. In addition, entirely new models also follow from this scheme.

Introduction

It has been conceived more or less widely that the present theory based on the assumption of representing elementary particles like geometrical points would be a bad idealization. On the other hand, attempts to unify elementary particles by attributing to them certain quantities associated with an abstract manifold independent of the Minkowski space remain as yet formal and phenomenological. Thus there are good reasons for taking the point of view to unify elementary particles by giving them a certain realistic structure in the ordinary space. In this general point of view one may start from field theory from the outset, then one would reach, for example, the non-local theory of Yukawa.¹⁾ This theory, however, does not yet succeed in explaining isospin properties of elementary particles. The approach which we would like to take is to start directly from extending the classical point model in order to maintain clear particle image.

Even though we have no definite way of such a generalization of the model in advance, it will generally imply to replace the simple point model by a certain body with a small spatial extension, and its most sensible consequence will kinematically appear as rotational degrees of freedom to be attained by the "particle". We do not know about the inside of elementary particles: the structures, forces, physical laws, etc. Moreover, even if we assume a certain kind of extended model and the validity of the physical principles known in the external world, the relativistic theory of such a model will be complicated and more or less ambiguous in general. We shall take a simpler method which starts from the assumption that the rotational degrees of freedom of the body can be clearly abstracted from

the total internal motions in some way and that to those rotational degrees relativity and quantum mechanics be applicable. It is then possible to formulate a consistent theory of such a rotating body in a quite general form without introducing the spatial extension of the body *explicitly*. We shall consider in the framework of special relativity, although it is naturally desirable to extend the treatment to general relativity.

Our theory has a likeness to that of non-relativistic rigid body in the sense that kinematically our model has rotational degrees of freedom alone besides the orbital motion of the center, i.e., that it has *directional rigidity*.* However, it must be remarked that we have *a priori* no necessity to presuppose that the rotations themselves in our model should arise like the straightforward relativistic transcription of the conventional non-relativistic rigid body motion, because we are seeking a purely relativistic theory of rotators for the model of microscopic particles. Indeed, starting from the kinematics of relativistic rotators, we proceed with the general principles, introducing step by step the conditions to characterize the model, e.g., the internal isotropy, internal reflection symmetry, "rigidity", etc. In this way we classify various possible models which have in themselves clear physical meanings but *dynamically* have more degrees of freedom than that of the customary rigid body and can take wider mode of motions.

Evidently such a rotator model possesses the ordinary spin property represented by its internal angular momentum (i.e. the spatial components of the internal angular momentum six-vector $S_{\mu\nu}$ on a fixed rest frame of external motion), including half-integral values.²⁾ The important point is that it carries isospin at the same time. In the usual theory iso-space is an abstract manifold completely decoupled from the Minkowski space, whereas in our theory it means the *ray space* consisting of the directions fixed to the body, and *as such it is fitted into the ordinary space*.

Abstractly speaking, the isospin means the existence of triple quantities (I_1, I_2, I_3), which satisfy the commutation relations between the generators of a 3-dimensional rotation:

$$[I_i, I_j] = i\epsilon_{ijk} I_k,$$

and *each of which is scalar invariant under Lorentz transformations*. Naturally, isospin is further required to commute with the ordinary spin components and other quantities, and that in electromagnetic interaction its conservation is reduced to that of I_3 only, which is related to charge.

Now a relativistic rotator means an entity to carry a Lorentz transformation and its instantaneous state of motion just defines a Lorentz frame to be called "body frame". The three spatial components of the internal angular momentum six-vector $S_{\mu\nu}$ on the body frame are identified as isospin components since they possess

* This means that the angle between arbitrary two directions fixed to the body remains unchanged by the motion (always viewed in the rest frame of the translational motion).

all of the properties mentioned above.*

Since the rotator is essentially an extended entity, it ought to have various internal geometrical properties: It can be spherical or not, reflection symmetric or not, "rigid" or not, etc. In our theory, however, we employ the variables representing the rotational orientations of the body but do not employ the variables to describe the radial extension explicitly, and accordingly the above mentioned internal geometrical properties are not expressed as such but are represented through the dynamics of the rotator, namely through the structure of the Lagrangian of the system. If the Lagrangian is spherically symmetric in the internal body space, each isospin component is necessarily conserved. On the other hand, a spherical rotator is not necessarily symmetric with respect to the internal reflection; it can also be anti-symmetric. In general, we have motions with a non-vanishing internal chirality which is properly represented by the pseudoscalar $S_{\mu\nu} \tilde{S}_{\mu\nu}$ formed from the internal angular momentum tensor $S_{\mu\nu}$ (and its dual $\tilde{S}_{\mu\nu}$). There exists always an internal constant of motion** related to $S_{\mu\nu} \tilde{S}_{\mu\nu}$ and the corresponding quantum number is interpreted as strangeness.

Generally, a relativistic rotator has six internal degrees of freedom because it has the structure corresponding to a Lorentz frame. Thus it must have one more internal degree of freedom besides spin, isospin (their magnitudes and third components) and internal chirality, which commutes with spin and isospin components. This is of course the rest mass, which signifies the absolute length of the momentum-energy 4-vector in the Minkowski space and depends only on the internal state of motion of the rotator. (On the other hand the direction of this vector in the Minkowski space represents the *external* degrees of freedom.) All of those constants of motion*** take discrete eigenvalues in quantum theory. Thus our model involves just all intrinsic properties assigned to elementary particles. It is not to be expected at once that our theory will precisely reproduce the empirical spectra of those intrinsic properties by a certain appropriate choice of the rotator model, as there are problems related to interactions, but we may say that our theory gives a unified model elementary particles at least in a qualitative way.

In Part I of this series of papers we develop the general theory of relativistic rotators in terms of the kinematical variables. Emphasis is placed on clarifying the physical foundation of theory. Simple possible models are derived and classified according to a coherent Lagrangian formalism. They include various existing models as special examples,—Weyssenhoff's particle,^{3),4)} Hönl-Papapetrou's mass-dipole,⁵⁾ Nakano's rigid body,⁶⁾ and so on, and in addition allows various

* This identification of isospin is not necessarily the unique possibility but evidently it is the simplest one.

** For the spherical rigid rotator model (6b), e.g. this quantity means the "precessional areal velocity" given by Eq. (88b).

*** Besides them it also has discontinuous conserved quantities like internal parity which is represented by Eq. (14) in classical theory.

quite new models.

In Part II we classify the internal constants of motion for each model and obtain the classical solutions explicitly by using elliptic functions. This reveals the relations between various internal properties according to the structure of a model. Finally in Part III we establish the quantum theory to obtain eigen-solutions, which correspond to each elementary particle state, by the joint use of matrix mechanics and wave equations, and also show how the internal symmetry properties are affected by interactions.

§ 1. Kinematical variables

1a. Tetrapod

A relativistic rotator is by definition an entity of which configuration be represented by the coordinates of its center x_μ and the variables describing its internal rotational orientation around the center. In this paper we deal with cases where the translational velocity V is smaller than the light velocity*

$$|V| < 1; \quad (1)$$

the case with light velocity needs special consideration. Thus there exists always a rest frame Σ , and also one can define the proper time τ and the unitary 4-velocity v_μ satisfying

$$v_\mu^2 = -1, \quad (2)$$

by the relation

$$v_\mu = \frac{dx_\mu}{d\tau} \equiv \dot{x}_\mu.$$

Σ means an inertial frame in which the orbital motion vanishes momentarily:

$$v_\mu^{(\Sigma)} = i\delta_{\mu 4}. \quad (3)$$

Now the internal rotational orientation is specified, *in the rest frame Σ* , with a set of three orthonormal 3-dimensional vectors $a_k^{r(\Sigma)}$ ($r=1, 2, 3$) attached to the body, satisfying

$$a_k^{r(\Sigma)} a_k^{s(\Sigma)} = \delta_{rs}. \quad (4)$$

We define $a_\mu^{\xi(\Sigma)}$ ($\xi=1, 2, 3, 4$) by

$$a_\mu^{r(\Sigma)} = \{a_k^{r(\Sigma)}, 0\}, \quad a_\mu^{4(\Sigma)} = -iv_\mu^{(\Sigma)}, \quad (5)$$

then (3), (4), and (5) mean**

$$a_\mu^{\xi(\Sigma)} a_\mu^{\eta(\Sigma)} = \delta_{\xi\eta}.$$

We now define three space-like vectors a_μ^r by the property that they become the above $a_\mu^{r(\Sigma)}$ in the rest frame Σ , then these a_μ^r and

* We employ the unit system in which $c=1$, and use the imaginary time component, thus $x_4=it$ and the energy of particle is p_4/i .

** In this paper lower Latin indices i, j, k, \dots range over 1, 2, 3 while Greek indices μ, ν, κ, \dots over 1 to 4. Similarly upper Latin indices r, s, t, \dots run over 1, 2, 3 while Greek ones ξ, η, ζ, \dots over 1 to 4. Summation convention is understood for any repeated indices.

$$a_\mu^4 = -iv_\mu \quad (6)$$

remain to satisfy

$$a_\mu^\xi a_\mu^\eta = \delta_{\xi\eta}, \quad a_\mu^\xi a_\nu^\xi = \delta_{\mu\nu}. \quad (7)$$

This indicates that⁷⁾ a kinematical description of a relativistic rotator is just given by a set of four orthonormal 4-vectors $a_\mu^\xi(\tau)$ called tetrapod, in which the fourth axis $ia_\mu^4 = v_\mu$ is time-like and plays physically distinguished rôle, as it signifies the orbital 4-velocity while a_μ^r represents the rotational orientation in the 3-dimensional hyperplane normal to the world-line.*

If we consider the *inertial* frame Σ^0 which momentarily coincides with the tetrapod $\{a_\mu^\xi\}$, then Σ^0 means a particular one among arbitrary rest frames Σ , and accordingly we call Σ^0 "the body rest frame". Then a_μ^ξ are nothing but the coefficients of the Lorentz transformation from the laboratory frame to the body frame.

We call a_μ^ξ the kinematical variables, but they are dynamical quantities themselves and not simple mathematical parameters. Our supposition that a_μ^r signify the radius vectors fixed to the body implies that even for spherically symmetric rotators the directions rotating with the body have physical sense. Such an assumption is characteristic for our theory and allows to define the body frame and so the isospin components in a simple way.

1b. Various representations

A relativistic rotator was described by a Lorentz frame $\{a_\mu^\xi\}$ at each instant. On the other hand, it is well-known that the homogeneous proper Lorentz group L is homomorph with the unimodular group C_2 :

$$L \sim C_2.$$

This immediately tells us that a relativistic rotator is kinematically describable with a unimodular matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad (\alpha\delta - \beta\gamma = 1)$$

equivalently.**

Again, the Lorentz group is isomorph with the complex orthogonal group O_3 :

$$L \sim O_3,$$

and hence the kinematical state of a relativistic rotator can also be represented with a set of three 3-dimensional orthonormal complex vectors (components of self-dual tensors) A_k^\dagger :

* The tetrapod variables in the present sense were originally introduced to represent the Dirac field by a continuous distribution of relativistic rotators.^{8),9)} Our a_μ^ξ variables are quite different, e.g. from Nakano's $a_{\alpha\mu}$ which are arbitrary Euclidian parameters without the physical identification (6).

** Compare with F. Gürsey, *Nuovo Cimento* 5 (1957) 784.

$$A_k^r A_k^s = \delta_{rs}, \quad A_i^r A_j^r = \delta_{ij},$$

namely a "complex tripod". This is connected with the original tetrapod $\{a_\mu^\xi\}$ through the relation*

$$A_k^r = (a_k^r a_4^4 - a_4^r a_k^4) \pm \varepsilon_{kij} a_i^r a_j^4.$$

Still another method** is to formulate the theory in terms of a 4-component spinor ζ under the constraint

$$\bar{\zeta}\zeta = 1, \quad \bar{\zeta}_{i5}\zeta = 0,$$

because such ζ is mathematically equivalent with a tetrapod $\{a_\mu^\xi\}$.

Finally we can replace $\{a_\mu^\xi\}$ by six independent parameters—three Euler angles θ, φ, ψ of spatial rotation, and two polar angles α and β and one hyperbolic angle γ corresponding to $a_\mu^4 = -iv_\mu$. This method is employed in wave mechanics (Part III).

In Part I we shall mainly employ the method of tetrapod variables as it is most suitable for general theory.

§ 2. Internal space and internal transformations

2a. Body components and internal rotations

Since $\{a_\mu^\xi\}$ was nothing but the coefficients of Lorentz transformation from the laboratory frame to the body frame Σ^0 , an arbitrary tensor—for instance, a vector V_μ or a second rank tensor $T_{\mu\nu}$ —has components in the body frame:

$$V^\xi = a_\mu^\xi V_\mu, \quad T^{\xi\eta} = a_\mu^\xi a_\nu^\eta T_{\mu\nu}, \quad (T_{\mu\nu} = a_\mu^\xi a_\nu^\eta T^{\xi\eta}) \quad (8)$$

respectively. However, noting that each a_μ^ξ (for a fixed ξ) is itself an ordinary 4-vector, each of the components V^ξ or $T^{\xi\eta}$ represents a scalar invariant with respect to Lorentz transformation of the laboratory frame. Nevertheless, $\{a_\mu^\xi\}$ represents a particular Lorentz transformation at each instant; therefore if we have an arbitrary tensor relation in the laboratory frame *which does not explicitly involve proper-time derivations*, it can be transferred to the body frame components with the invariant form. A trivial example is

$$T_{\mu\nu} T_{\mu\nu} = T^{\xi\eta} T^{\xi\eta}.$$

We now consider a 3-dimensional rotation of the spatial axes in the internal space of the rotator, namely

$$a_\mu^r \rightarrow a_\mu^{r'} = C^{rs} a_\mu^s; \quad a_\mu^4 = \text{invariant}, \quad (9a)$$

Where (C^{rs}) is a constant matrix of rotation:

* T. Takabayasi, *mémoire mimeographed* (1958); see also ref. 10).

** This method was employed in ref. 7).

$$C^{rs} C^{rt} = \delta_{st}. \quad (9b)$$

This transformation expresses the fact that we can always go over from one set of spatial axes fixed to the body to another set of axes again fixed to the body for the description of the rotator's motion. The *spherical* property of the rotator is then represented by the fact that the theory should be invariant under the transformation (9).

If we generalize (9) to full rotations in the internal 4-dimensional Lorentz space such that

$$a_{\mu}^{\xi} \rightarrow a_{\mu}^{\xi'} = C^{\xi\eta} a_{\mu}^{\eta}, \quad C^{\xi\eta} C^{\xi\xi} = \delta_{\eta\xi}, \quad (10)$$

the orthonormality condition of $\{a_{\mu}^{\xi}\}$ and the time-like character of ia_{μ}^4 remain conserved, and $T^{\xi\eta}$ transforms then as tensor components in the internal Lorentz space:

$$T^{\xi\eta} \rightarrow C^{\xi\xi} C^{\eta\eta} T^{\xi\eta}.$$

However, as the result of (10), the new fourth axis $ia_{\mu}^{4'}$ no longer signifies the orbital velocity of the rotator. Thus we have no simple physical reason to require that the theory be invariant under the 4-dimensional full rotations (10), nor there exists any theory for a single rotator which fulfills such requirement perfectly. (The situation is different in field theory. When we represent the 4-component neutrino field by a continuous distribution of rotators,* we see that this field fulfills the perfect 4-dimensional internal symmetry. Indeed, the internal Lorentz space in the present sense was originally introduced to manifest this neutrino group.⁹⁾ In Heisenberg's non-linear field¹²⁾ this symmetry reduces to the spherical symmetry, where Pauli transformation¹³⁾ is nothing but the internal rotation (9).)

2b. Internal reflection

It also belongs to our liberty whether we take a right-handed frame or a left-handed one for the frame fixed to the body. Corresponding to this fact we can consider reflections of the body frame. Since this transformation is given by $a_k^{r(2)} \rightarrow -a_k^{r(2)}$ in a rest frame, it is represented, in a laboratory frame, by

$$a_{\mu}^r \rightarrow -a_{\mu}^r, \quad a_{\mu}^4 = -iv_{\mu} = \text{invariant}. \quad (11)$$

Evidently, this transformation is independent of the usual inversion of the laboratory frame:

$$x_k \rightarrow -x_k, \quad t \rightarrow t, \quad (12)$$

where a_{μ}^{ξ} 's transform by

$$a_k^{\xi} \rightarrow -a_k^{\xi}, \quad a_4^{\xi} \rightarrow a_4^{\xi}, \quad (13)$$

as we regard a_{μ}^{ξ} 's as regular vectors. (11) is clearly different from (13). We

* Cf. footnote, p. 919.

shall call (11) the internal inversion to distinguish it from the usual external inversion (12). If the model is right-left symmetric the theory must be invariant under (11).

For this internal inversion, the usual body components of a tensor, (8), transform like those of a regular tensor:

$$\begin{aligned} V^r &\rightarrow -V^r, & V^4 &\rightarrow V^4; \\ \left\{ \begin{array}{ll} T^{rs} &\rightarrow T^{rs}, & T^{44} &\rightarrow T^{44}, \\ T^{r4} &\rightarrow -T^{r4}, & T^{4r} &\rightarrow -T^{4r}, \end{array} \right. \end{aligned}$$

irrespective of whether V_μ or $T_{\mu\nu}$ is a regular tensor or a pseudotensor with respect to external reflection.

We have a particular unit pseudoscalar for internal transformations,

$$\eta = \det\{a_\mu^\xi\}, \quad (14)$$

which is also a pseudoscalar for external (i.e. usual) transformations. η is $+1$ if the body frame and the laboratory frame are of the same chirality, and -1 if they are of different chiralities. (Here we are suppressing time reversal.)

2c. Antisymmetric tensor and duals

The distinction between the two kinds of inversions is important in the definition of the dual of an antisymmetric tensor $A_{\mu\nu}$. The usual dual defined by

$$\widetilde{A}_{\mu\nu} = -\frac{i}{2} \varepsilon_{\mu\nu\kappa\lambda} A_{\kappa\lambda} \quad (15)$$

is a pseudotensor if $A_{\mu\nu}$ is a regular tensor. The six body components of \widetilde{A} , given by

$$\widetilde{A}^{\xi\eta} = a_\mu^\xi a_\nu^\eta \widetilde{A}_{\mu\nu} = -\frac{i}{2} \varepsilon_{\mu\nu\kappa\lambda} a_\mu^\xi a_\nu^\eta A_{\kappa\lambda}, \quad (16)$$

are external pseudoscalars and form an antisymmetric tensor in the internal space.

On the other hand, we can define the dual of A regarding body components by

$$\check{A}^{\xi\eta} = -\frac{i}{2} \varepsilon_{\xi\eta\zeta\iota} A^{\zeta\iota}, \quad (17)$$

analogously to (15). Then we have the relation

$$\check{A}^{\xi\eta} = -\frac{i}{2} \varepsilon_{\xi\eta\zeta\iota} a_\mu^\zeta a_\nu^\iota A_{\mu\nu} = -\frac{i}{2} \eta \varepsilon_{\kappa\lambda\mu\nu} a_\kappa^\xi a_\lambda^\eta A_{\mu\nu} = \eta \widetilde{A}^{\xi\eta}, \quad (18)$$

with the use of (14). This indicates that $\check{A}^{\xi\eta}$ are external scalars each and constitute an internal pseudotensor.

Evidently, we have the relation

$$A_{\mu\nu} \widetilde{A}^{\mu\nu} = A^{\xi\eta} \widetilde{A}_{\xi\eta} = \eta A^{\xi\eta} \check{A}_{\xi\eta}. \quad (19)$$

Table I. Tensor character of duals

	$\tilde{A}_{\mu\nu}$	$\tilde{A}_{\xi\eta}$	$\tilde{A}_{\xi\eta}$
external	PT	PS	S
internal	S	T	PT

Another useful relation between the body components and the laboratory components of a skew-tensor is

$$\sum_{[rs]} (A^{rs})^2 = (\tilde{A}_{\mu\nu} v_\nu)^2. \quad (20)^*$$

§ 3. Angular velocity

3a. Angular velocity tensor

Each a_μ^ξ vector transforms, under an infinitesimal Lorentz transformation

$$l_{\mu\nu} = \delta_{\mu\nu} + \varepsilon_{\mu\nu}, \quad (\varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu}) \quad (21)$$

by

$$a_\mu^\xi \rightarrow a_\mu^\xi + \varepsilon_{\mu\nu} a_\nu^\xi.$$

If we suppose instead that the laboratory frame is unchanged and the Minkowski space fixed to the tetrapod $\{a_\mu^\xi\}$ makes an infinitesimal 4-dimensional rotation represented by the same (21), a_μ^ξ changes by

$$a_\mu^\xi \rightarrow a_\mu^\xi - \varepsilon_{\mu\nu} a_\nu^\xi. \quad (22)$$

Dividing $\varepsilon_{\mu\nu}$ by the proper-time interval $d\tau$, during which the infinitesimal rotation took place, we get the 4-dimensional angular velocity tensor, $\omega_{\mu\nu} = \varepsilon_{\mu\nu}/d\tau$, so that (22) is rewritten as

$$\frac{da_\mu^\xi}{d\tau} = -\omega_{\mu\nu} a_\nu^\xi. \quad (23)$$

This gives

$$\omega_{\mu\nu} = -\omega_{\nu\mu} = a_\mu^\xi \dot{a}_\nu^\xi \quad (24)$$

by the aid of (7). This exhibits that $\omega_{\mu\nu}$ is an *internal scalar* regarding full internal rotations (10) including internal reflection (11).

On the other hand, the body components of $\omega_{\mu\nu}$ are

$$\omega^{\xi\eta} = a_\mu^\xi a_\nu^\eta \omega_{\mu\nu} = \dot{a}_\mu^\xi a_\mu^\eta = -\omega^{\eta\xi}, \quad (25)$$

* $\sum_{[rs]}$ signifies the summation over $(rs) = 23, 31, 12$. To prove (20) we first note that its left side is the length of the axial vector representing the spatial part of the body components of the skew-tensor $A_{\xi\eta}$, while the right side of (20) is the length of the axial vector corresponding to the spatial part of $A_{\mu\nu}$ in an arbitrary rest frame Σ (i.e. $A_{23}^{(\Sigma)2} + A_{31}^{(\Sigma)2} + A_{12}^{(\Sigma)2}$), and second that Σ and Σ^0 transform to each other by a pure space rotation.

which are external scalars and form an internal skew-tensor. Its dual is

$$\tilde{\omega}^{\xi\eta} = -\frac{i}{2} \varepsilon_{\xi\eta\zeta} \dot{a}_\mu{}^\zeta a_\mu{}^\xi = \eta \tilde{\omega}^{\xi\eta}. \quad (26)$$

3b. Proper angular velocity

Now, $\omega_{\mu\nu}$ signified the proper-time rate with which the rotator is realizing a Lorentz transformation at each instant. The real angular velocity of spatial rotation is more properly represented in a covariant way by the "proper space part" of $\omega_{\mu\nu}$, which we denote as $\omega_{\mu\nu}^*$, rather than by $\omega_{\mu\nu}$ itself. Namely, following the general procedure (stated in the Appendix) of decomposing an arbitrary skew-tensor, $\omega_{\mu\nu}$ is uniquely resolved such that

$$\omega_{\mu\nu} = \omega_{\mu\nu}^* + \omega_{\mu\nu}^\dagger, \quad (27a)$$

with the conditions

$$\omega_{\mu\nu}^* v_\nu = 0, \quad (\omega_{\mu\nu}^\dagger)^\sim v_\nu = 0. \quad (27b)$$

This is also expressed by

$$\omega^{rs} = (\omega^*)^{rs}, \quad \omega^{r4} = (\omega^\dagger)^{r4}.$$

Explicitly,

$$\omega_{\mu\nu}^* = \frac{1}{2} (a_{[\mu}^r \dot{a}_{\nu]}^r - a_{[\mu}^4 \dot{a}_{\nu]}^4), \quad (28a)$$

$$\omega_{\mu\nu}^\dagger = a_{[\mu}^4 \dot{a}_{\nu]}^4 = -v_{[\mu} \dot{v}_{\nu]}, \quad (28b)$$

or, with the aid of a pseudovector,*

$$\omega_\mu^* \equiv \bar{\omega}_{\mu\nu} v_\nu, \quad (\omega_\mu^* v_\mu = 0), \quad (29)$$

the former is expressed in the form

$$\omega_{\mu\nu}^* = -i \varepsilon_{\mu\nu\kappa\lambda} \omega_\kappa^* v_\lambda. \quad (30)$$

That $\omega_{\mu\nu}^*$ (or equivalently ω_μ^*) represents the covariant angular velocity of the spatial rotation is justified from the fact⁸⁾ that in the rest frame Σ' its space components are

$$\omega_{ij}^{*(\Sigma)} = \omega_k^{*(\Sigma)} = \omega_{ij}^{(\Sigma)} = \left(a_i^r \frac{da_j^r}{dt} \right)^{(\Sigma)}, \quad (31a)$$

($ijk \sim 123$)

the last expression being the usual 3-dimensional angular velocity, while the time components vanish:

$$\omega_{k4}^{*(\Sigma)} = \omega_k^{*(\Sigma)} = 0. \quad (31b)$$

On the other hand, $\omega_{\mu\nu}$ comprises the angular velocity of rotation and the

* Note that $\omega_\mu^* \equiv \bar{\omega}_{\mu\nu} v_\nu = -\dot{v}_\mu$.

acceleration of the translational motion as is observed in (28b).

The internal hyper-spherical property of $\omega_{\mu\nu}$ is reduced to the simple spherical symmetry in $\omega_{\mu\nu}^*$. Naturally this applies also to their magnitudes:

$$\omega_{\mu\nu}^2 = \dot{a}_\mu^\xi \dot{a}_\nu^\xi = (\dot{a}_\mu^r)^2 - \dot{v}_\mu^2, \quad (32)$$

$$(\omega_{\mu\nu}^*)^2 = 2(\omega_\mu^*)^2 = (\dot{a}_\mu^r)^2 - (\dot{a}_\mu^4)^2 = (\dot{a}_\mu^r)^2 + \dot{v}_\mu^2. \quad (33)$$

The former is an internal 4-scalar, whereas the latter is an internal 3-scalar. On the other hand,

$$\omega_{\mu\nu} \tilde{\omega}_{\mu\nu} = 2i\varepsilon_{\mu\nu\kappa\lambda} a_\kappa^r \dot{a}_\lambda^r v_\mu \dot{v}_\nu \quad (34)$$

is an internal 4-scalar and external pseudoscalar, while

$$\omega^{\xi\eta} \tilde{\omega}^{\xi\eta} = -2\varepsilon_{\tau\delta t} a_\mu^r \dot{a}_\mu^s a_\nu^t \dot{v}_\nu = \eta \omega_{\mu\nu} \tilde{\omega}_{\mu\nu} \quad (35)$$

is an internal pseudoscalar and external scalar.

§ 4. General dynamics

4a. Lagrangian

Now the equations of motion for the kinematical variables of the rotator are derived from the Lagrangian. Its main part corresponds to a "4-dimensional rotation" denoted as \mathcal{L} , and we assume that \mathcal{L} is a function of a_μ^ξ and \dot{a}_μ^ξ only, without involving higher derivatives. The total Lagrangian for the case of free rotator* is given by adding to \mathcal{L} the terms for assuring the constraint (7) and the relation (6) which defines the physical meaning of a_μ^4 , i.e.

$$\dot{x}_\mu = v_\mu = i a_\mu^4. \quad (36)$$

Thus the Lagrangian must be of the general form:

$$L[a_\mu^\xi, x_\mu, p_\mu, \lambda^{\xi\eta}] = \mathcal{L}(a_\mu^\xi, \dot{a}_\mu^\xi) + \frac{1}{2} \lambda^{\xi\eta} (a_\mu^\xi a_\mu^\eta - \delta_{\xi\eta}) + p_\mu (\dot{x}_\mu - i a_\mu^4), \quad (37)$$

where $\lambda^{\xi\eta} = \lambda^{\eta\xi}$ are Lagrangian multipliers. The p_μ 's are also other Lagrangian multipliers, but from (37) it follows immediately that

$$\frac{\partial L}{\partial \dot{x}_\mu} = p_\mu, \quad \dot{p}_\mu = 0, \quad p_\mu = \text{const}, \quad (38)$$

indicating that p_μ are canonical momenta conjugate to x_μ and are constants of motion. Thus p_μ means the momentum-energy 4-vector of the system, required to be time-like (or null). Putting

$$p_\mu^2 = \text{const} = -m'^2, \quad (m' \geq 0) \quad (39)$$

* Throughout Parts I and II we limit the considerations to free rotator. The case with interaction is treated in Part III.

m' signifies the observable rest mass of the system, since empirically we define the rest mass through this relation.⁷⁾ It is one of the constants of motion depending solely on the internal state of motion as we shall see later.

Here we give some further remarks about the form (37). As already stated the identification (36) is essential for a relativistic rotator, meaning a kinematical coupling between orbital motion and internal rotation of the rotator. The covariant theory of rotator is formally represented as a 4-dimensional rotation, but the properly internal rotation must take place always in the 3-dimensional space-like plane normal to the orbital velocity which is taken as the fourth axis of the 4-dimensional rotation. For the rotator motion we are not introducing *kinematically* any new mysterious degrees of freedom non-existing in a simple non-relativistic rotator. (If one discard the condition (36), this amounts to introducing the second velocity ia_μ^4 along with the orbital velocity v_μ , and the theory no longer represents a pure rotator but corresponds to a more complex motion including "time-like rotation". Simple omission of (36) means the perfect decoupling between the internal and external degrees of freedom but it does not lead to a consistent theory with a clear physical meaning; one needs to assume certain alternative relation between \dot{x}_μ and a_μ^4 in the place of (36). In this kind of theory there will appear a divergence difficulty or infinite degeneracy of levels in the quantization of internal motion in so far as the internal space is an *independent* Lorentz space. If one replaces this by a 4-dimensional Euclidian internal space as was assumed by Nakano,⁶⁾ one is further away from a real rotator model.)

Now, on the other hand, the kinematical coupling between the internal rotation and orbital motion implied in the conditions (36) and (7) allows a relativistic rotator an accelerated orbital motion even for a free system, so that $v_\mu = \dot{x}_\mu$ is not colinear with p_μ which is the canonical conjugate to x_μ and represents the mean constant rectilinear translation of the system. Thus v_μ behaves as variables independent of p_μ , so that the system possesses, besides p_μ , six independent degrees of freedom $\{v_\mu, a_\mu^r\}$, i.e. the original $\{a_\mu^\xi\}$ variables. (The situation will become clearer presently.) Although v_μ is the orbital velocity it is convenient to count it among the internal variables because it represents essentially the precessional motion (Zitterbewegung) around the constant p_μ , and thus we have six internal degrees of freedom effectively.

The above situation is similar to the case of Dirac particle in which the instantaneous velocity operator α_k representing Zitterbewegung could be regarded as internal variables. The theory in which the orbital velocity works as internal variables was formerly proposed by Hönl, Papapetrou⁵⁾ and Bopp,¹⁴⁾ but because these authors started from the pole-dipole model, they introduced as internal variables only v_μ and not the spatial rotational variables a_μ^r normal to v_μ . Such model involves spin but not isospin. As we shall see, this theory is a special degenerate case of our rotator theory.

The second remark is that we need not supplement the Lagrangian (37) with

a usual term like

$$\frac{m^*}{2} (\dot{x}_\mu^2 + 1), \quad (40)$$

corresponding to the translational motion, since such term is already implied in the second and third terms of (37), and the only effect of the addition of (40) to the original Lagrangian is the renormalization of the "true rest mass"

$$m = -v_\mu p_\mu \quad (41)$$

by an amount m^* . In our theory the rest mass (m' of (39) rather than (41)) is a physical quantity which is shown to be a constant of motion related to the internal motion and takes a discrete spectrum of eigenvalues after quantization. Naturally this is possible as \mathcal{L} must contain at least one structure constant involving the dimension of mass so that \mathcal{L} has the dimension of energy.

In the above we called the expression (41) "true rest mass" because it is equal to the energy of the system in the rest frame Σ :

$$-v_\mu p_\mu = p_4^{(\Sigma)}/i.$$

On the other hand, the observable rest mass m' means the energy of the system in the "mean rest frame" denoted as Π , i.e. the frame in which the momentum vanishes:

$$p_\mu^{(\Pi)} = im' \delta_{\mu 4}.$$

Finally, the characteristics for a given rotator model are condensed in the functional structure of $\mathcal{L}(a_\mu^\xi, \dot{a}_\mu^\xi)$. For general arguments we do not need its explicit form but in any case it must be invariant under usual Lorentz transformations. This is assured by the condition

$$\frac{\partial \mathcal{L}}{\partial a_\mu^\xi} a_\nu^\xi + \frac{\partial \mathcal{L}}{\partial \dot{a}_\mu^\xi} \dot{a}_\nu^\xi = 0. \quad (42)$$

4b. Equations of motion, internal angular momentum

Now the variations of the Lagrangian (37) with respect to its arguments yield, besides Eqs. (7), (36), and (38),

$$-\dot{\alpha}_\mu^\xi + \frac{\partial \mathcal{L}}{\partial a_\mu^\xi} + \lambda^{\xi\eta} a_\mu^\eta = i \delta_{\xi 4} p_\mu, \quad (43)$$

where

$$\alpha_\mu^\xi = \frac{\partial \mathcal{L}}{\partial \dot{a}_\mu^\xi} \quad (44)$$

are canonical momenta conjugate to a_μ^ξ .

We can eliminate $\lambda^{\xi\eta}$'s from (43) by multiplying it with a_ν^ξ (summation over ξ) and noting the symmetry of $\lambda^{\xi\eta} a_\mu^\eta a_\nu^\xi$ in μ and ν . Thus we get

$$a_{[\mu}^{\xi} \dot{\alpha}_{\nu]}^{\xi} + a_{[\nu}^{\xi} \cdot \frac{\partial \mathcal{L}}{\partial a_{\mu]}^{\xi}} = i p_{[\mu} a_{\nu]}^4.$$

By the aid of (42) this is rewritten as

$$\frac{d}{d\tau} (a_{\mu}^{\xi} \alpha_{\nu}^{\xi} - a_{\nu}^{\xi} \alpha_{\mu}^{\xi}) = i (p_{\mu} a_{\nu}^4 - p_{\nu} a_{\mu}^4). \quad (45)$$

This exhibits that

$$\dot{S}_{\mu\nu} = a_{\mu}^{\xi} \alpha_{\nu}^{\xi} - a_{\nu}^{\xi} \alpha_{\mu}^{\xi} \quad (46)$$

means the internal angular momentum tensor of the system,¹⁵⁾ because (45) then takes the standard form

$$\dot{S}_{\mu\nu} = p_{\mu} v_{\nu} - p_{\nu} v_{\mu}. \quad (47)$$

Its integrated form

$$S_{\mu\nu} + (x_{\mu} p_{\nu} - x_{\nu} p_{\mu}) = J_{\mu\nu} = \text{const} \quad (48)$$

represents the covariant conservation law of the angular momentum, namely the orbital part of the angular momentum $x_{[i} p_{j]}$ plus the internal part S_{ij} is conserved.

Eq. (45) represents, at the same time, the basic equations of the rotator which are differential equations of the second order (involving \ddot{a}_{μ}^{ξ}) excepting the case in which \mathcal{L} is linear in \dot{a}_{μ}^{ξ} . Thus by integrating the set of six equations contained in (45), with respect to the six independent internal variables implied by $\{a_{\mu}^{\xi}\}$, the motion is to be determined in principle.

The structure of the internal angular momentum tensor $S_{\mu\nu}$ as functions of kinematical variables is not unique but just characterizes the model. We have a general relation

$$S_{\mu\nu} \omega_{\mu\nu} = 2 \frac{\partial \mathcal{L}}{\partial \dot{a}_{\mu}^{\xi}} \dot{a}_{\mu}^{\xi},$$

which becomes

$$\mathcal{L} = \frac{1}{2n} S_{\mu\nu} \omega_{\mu\nu} \quad (49)$$

if \mathcal{L} is a homogeneous form of the n th order in \dot{a}_{μ}^{ξ} .

The body components of the internal angular momentum tensor are evidently

$$S^{\xi\eta} = \alpha_{\mu}^{\xi} a_{\mu}^{\eta} - \alpha_{\mu}^{\eta} a_{\mu}^{\xi}. \quad (50)$$

Just as we have done for the angular velocity tensor $\omega_{\mu\nu}$, it is useful to decompose $S_{\mu\nu}$ into its proper space part $S_{\mu\nu}^*$ and proper space-time part $S_{\mu\nu}^{\dagger}$ by the aid of two vectors¹⁶⁾

$$\begin{aligned} S_{\mu}^* &= \tilde{S}_{\mu\nu} v_{\nu}, & S_{\mu}^{\dagger} &= S_{\mu\nu} v_{\nu}, \\ (S_{\mu}^* v_{\mu} &= S_{\mu}^{\dagger} v_{\mu} = 0) \end{aligned} \quad (51)$$

such that

$$\left. \begin{aligned} S_{\mu\nu} &= S_{\mu\nu}^* + S_{\mu\nu}^\dagger, \\ S_{\mu\nu}^* &= -i\varepsilon_{\mu\nu\kappa\lambda} S_\kappa^* v_\lambda, \quad S_{\mu\nu}^\dagger = v_{[\mu} S_{\nu]}^\dagger. \end{aligned} \right\} \quad (52)$$

Equally important is the decomposition⁷⁾ using p_μ in the place of v_μ ; namely, $S_{\mu\nu}$ is equivalent to the pair of pseudovector and vector,

$$\begin{aligned} \mathcal{M}_\mu &= \widetilde{S}_{\mu\nu} p_\nu / m', \quad r_\mu = S_{\mu\nu} p_\nu / m'^2, \\ (\mathcal{M}_\mu p_\mu &= 0, \quad r_\mu p_\mu = 0) \end{aligned} \quad (53)$$

since $S_{\mu\nu}$ is then expressed in the form

$$S_{\mu\nu} = -\frac{i}{m'} \varepsilon_{\mu\nu\kappa\lambda} \mathcal{M}_\kappa p_\lambda - r_{[\mu} p_{\nu]}. \quad (54)$$

The above \mathcal{M}_μ is the quantity which coincides with the internal angular momentum S_{ij} in the Π -frame:

$$\mathcal{M}_k^{(\Pi)} = S_{ij}^{(\Pi)}, \quad (ijk \sim 123); \quad \mathcal{M}_4^{(\Pi)} = 0,$$

and is exactly the covariant definition of particle *spin* given by Pauli.¹⁷⁾ On the other hand, r_μ signifies the radius vector of Zitterbewegung as we shall see presently.

In Eq. (48) we have already obtained six constants of motion $J_{\mu\nu}$, the components of the total angular momentum tensor. This is already the right number of intermediate integrals for the equation which involve six independent variables a_μ^ξ . However, $J_{\mu\nu}$'s are not the appropriate quantities representing the internal properties of the rotator, because it depends on the external motion also. It is one of the main problem to find out a right number of the constants of motion depending solely on the internal state of motion in order to identify them with various intrinsic properties of elementary particles.

4c. Zitterbewegung, internal chirality

In this subsection we shall give some general relations which follow directly from the conservation laws, (38) and (47), without going into the characteristics of the model, namely the functional form of α_μ^ξ in a_μ^ξ and \dot{a}_μ^ξ .

Now, corresponding to the separation (54), the basic equation (47) is equivalent to the pair of equations:

$$\begin{cases} \dot{\mathcal{M}}_\mu = 0, \quad \mathcal{M}_\mu = \text{const.}, \\ \dot{r}_\mu = v_\mu - \frac{m}{m'^2} p_\mu, \end{cases} \quad (55)$$

where m signifies the true rest mass defined by (41) and is not necessarily a constant of motion. Eq. (55) indicates that the spin pseudovector is generally conserved, supplying three constants of motion. Especially, the magnitude of spin

$$\mathcal{M}_\mu^2 = \mathcal{M}_k^{(II)2} \equiv \sigma^2 \quad (57)$$

is the constant of motion determined by the internal state of motion.

On the other hand, integrating (56) we obtain

$$x_\mu = \frac{p_\mu}{m'^2} \int m d\tau + r_\mu + \text{const}, \quad (58)$$

which exhibits that the orbital motion consists of two parts, the mean rectilinear motion in the direction of p_μ and a certain Zitterbewegung around it represented by r_μ . Written in the II -frame, (58) becomes

$$x_k^{(II)} = r_k^{(II)} + \text{const}, \quad t^{(II)} = \frac{1}{m'} \int m d\tau, \quad (59)$$

which gives a parametric representation of the orbit in this frame, $x_k^{(II)}(\tau(t^{(II)}))$.

Now the meaning of (54) is clear: The first term in the right side is constant while the second term just represents the part to balance the fluctuation of the orbital angular momentum due to Zitterbewegung.

Other important quantities derivable from $S_{\mu\nu}$ are the scalar $S_{\mu\nu}^2$ and the pseudoscalar $S_{\mu\nu} \tilde{S}_{\mu\nu}$. They are written as

$$\begin{aligned} S_{\mu\nu}^2 &= S^{\xi\eta} S^{\xi\eta} = 2(S_{\mu}^{*2} - S_{\mu}^{\dagger 2}) \\ &= 2(\mathcal{M}_\mu^2 - m'^2 r_\mu^2), \end{aligned} \quad (60)$$

$$S_{\mu\nu} \tilde{S}_{\mu\nu} = S^{\xi\eta} \tilde{S}^{\xi\eta} = -4m' \mathcal{M}_\mu r_\mu = -4S_\mu^* S_\mu^\dagger. \quad (61)$$

Instead of (61) we may employ the externally scalar and internally pseudoscalar quantity:

$$\chi = -\frac{1}{4} S^{\xi\eta} \tilde{S}^{\xi\eta} = \eta m' \mathcal{M}_\mu r_\mu = \eta m' \mathcal{M}_k^{(II)} r_k^{(II)}. \quad (62)$$

While (60) coincides with σ^2 in the non-relativistic limit, (62) represents a typically relativistic quantity. If $\chi=0$, $\mathcal{M}_k^{(II)}$ is orthogonal to $r_k^{(II)}$, considered in the II -frame; namely the rotator is spinning *in the same plane* with the orbital motion. But if $\chi \neq 0$ the Zitterbewegung has a component normal to the plane of spin and defines a right-handed or left-handed screw according to the sign of χ . Thus χ represents the chirality of the internal motion and the appearance of η in (62) indicates that the definition is made in the manner invariant against external reflection.¹⁸⁾

A general formula for a skew-tensor gives

$$(\frac{1}{2} S_{\mu\nu} \tilde{S}_{\mu\nu})^2 = -\det(S_{\mu\nu}).$$

Hence, if $S_{\mu\nu}$ satisfies a relation of the type $S_{\mu\nu} k_\nu = 0$, where k_ν is a certain 4-component quantity, χ vanishes identically, since the matrix $(S_{\mu\nu})$ is then of rank three and so $\det(S_{\mu\nu}) = 0$. Consequently the internal chirality vanishes almost always

for hitherto known models (e.g. for Weyssenhoff's particle $S_{\mu\nu}v_\nu=0$, for Nakano's rigid body $S_{\mu\nu}p_\nu=0$, and for Hönl-Papapetrou's particle $\tilde{S}_{\mu\nu}v_\nu=0$), but it is not the case with more general models.

Finally, we note some other relations easily following from (47) for future references.

$$p_\mu = mv_\mu - \dot{S}_{\mu\nu} v_\nu, \quad (63)$$

$$\dot{m} = \dot{S}_{\mu\nu}^+ \dot{v}_\mu, \quad (64)$$

$$\dot{r}_\mu^2 = \frac{m^2}{m'^2} - 1. \quad (65)$$

Though m is not necessarily a constant of motion, it must be greater than m' :

$$m \geq m' > 0, \quad (66)$$

taking into account the assumption (1). (66) is derived by using the definitions (2), (39) and (41), and the Schwartz inequality.*

§ 5. Internal symmetries, isospin, rigidity

5a. Spherical symmetry and isospin

We now reexpress the equation of motion (45) in terms of the body components of the internal angular momentum tensor (50) to obtain

$$\dot{S}^{\xi\eta} = \left(-\frac{\partial \mathcal{L}^0}{\partial a_\mu^{[\xi}} \dot{a}_\mu^{\eta]} + \frac{\partial \mathcal{L}^0}{\partial a_\mu^{[\xi}} a_\mu^{\eta]} \right) + i p_\mu \delta_{[\xi}^4 \cdot a_\mu^{\eta]}. \quad (67)$$

Here we shall introduce the assumption that the rotator be *spherical*. This is ensured if \mathcal{L}^0 be invariant under 3-dimensional infinitesimal rotations in the internal space belonging to (9):

$$\begin{cases} a_\mu^r \rightarrow a_\mu^r + \varepsilon^{rs} a_\mu^s, & (\varepsilon^{rs} = -\varepsilon^{sr}) \\ a_\mu^4 = \text{invariant}, \end{cases} \quad (68)$$

but this condition is expressed by

$$\left(\frac{\partial \mathcal{L}^0}{\partial a_\mu^r} a_\mu^s + \frac{\partial \mathcal{L}^0}{\partial \dot{a}_\mu^r} \dot{a}_\mu^s \right) - \left(\frac{\partial \mathcal{L}^0}{\partial a_\mu^s} a_\mu^r + \frac{\partial \mathcal{L}^0}{\partial \dot{a}_\mu^s} \dot{a}_\mu^r \right) = 0. \quad (69)$$

We now employ (69) to (67) and immediately obtain

$$\dot{S}^{rs} = 0, \quad \text{i. e.,} \quad S^{rs} = \text{const.} \quad (70)$$

Thus we get three new constants of motion, S^{23} , S^{31} , and S^{12} , depending solely on the internal state of motion of the rotator. It is appropriate to identify them as

* In fact we have one more possibility: $m > 0$, $m' = 0$ but we disregard this case in our discussions.

the components of isospin I_1 , I_2 , and I_3 (apart from a constant factor such that $S^{rs} = -\hbar \epsilon_{rst} I_t$). This identification will further be justified afterwards.* In particular the magnitude of isospin can be expressed in terms of laboratory components:

$$\tau^2 \equiv \sum_{[rs]} (S^{rs})^2 = (\tilde{S}_{\mu\nu} v_\nu)^2 \equiv S_\mu^{*2}, \quad (71)$$

recalling the formula (20).

We have thus obtained six independent constants of motion, \mathcal{M}_μ and S^{rs} , for spherical rotators in general. This is the right number of intermediate integrals but they are not yet sufficient for our purpose, because they furnish only a set of four quantities (\mathcal{M}_μ^2 , $\mathcal{M}_3^{(II)}$, S_μ^{*2} , S^{12}) with mutually vanishing Poisson brackets.

We shall add two remarks. Higher symmetry than spherical is the internal 4-dimensional symmetry, i.e. the invariance under full rotations in the internal Lorentz space (10). When \mathcal{L} has this symmetry we shall call the case "hyper-spherical". Even in this case, however, the fourth axis plays a distinguished rôle in the total Lagrangian (37) and we do not have $\dot{S}^{r4}=0$ other than (70) either.

Secondly, though we have shown that the spherical symmetry leads to (70), there are cases where (70) holds without spherical symmetry as in the example of 6a-(i). However, in such a case we cannot identify S^{rs} as isospin components, since for this purpose S^{rs} must satisfy among themselves the commutation relations corresponding to the generators of a three-dimensional rotation, and this requires that the conservation of S^{rs} must result from the invariance under a certain kind of 3-dimensional rotation.

5b. Reflection symmetry

This means the invariance under (11). If a rotator is spherical and at the same time reflection symmetric, the proper space parts of angular momentum and angular velocity tensors are usually parallel:

$$S_\mu^* = \mathcal{G} \omega_\mu^*, \quad \text{i. e.} \quad S^{rs} = \mathcal{G} \omega^{rs}, \quad (72)$$

whereas if a rotator is spherical and reflection anti-symmetric, the relation will be

$$S^{rs} = \mathcal{G}' \tilde{\omega}^{rs}, \quad \text{i. e.} \quad S_\mu^* = \eta \mathcal{G}' \tilde{\omega}_\mu^*. \quad (73)$$

5c. Rigidity

Our rotators have *directionally* rigid structures, but we now define full "rigidity" by the following condition.

We know that for a non-relativistic spherical rigid body the angular momentum \mathbf{M} and angular velocity $\boldsymbol{\omega}$ are related through a constant moment of inertia I like $\mathbf{M} = I\boldsymbol{\omega}$. Restricting ourselves to the cases satisfying (72), we shall call a relativistic rotator "rigid" if \mathcal{G} be a constant,⁷⁾ namely

* The identification of the iso-spin in our theory is essentially identical as in the Heisenberg's theory,¹²⁾ because the rotation (9) is represented as the Pauli transformation if it is expressed in spinor formalism.⁹⁾

$$S_{\mu\nu}^* = I\omega_{\mu\nu}^* \quad (I = \text{const}) \quad (74)$$

which is equivalent to

$$S_{\mu}^* = I\omega_{\mu}^*, \quad \text{or} \quad S^{rs} = I\omega^{rs}. \quad (74')$$

Still it should be remarked that our "rigidity" must not be taken too literally, since we have defined it through the dynamical relation (74), without going into the internal geometrical structures. Our rigid rotator includes the simple relativistic transcription of the non-relativistic rigid body as a special case, but it has *dynamically* more degrees of freedom, which allow the rotator to perform wider "non-classical" mode of motions.

Finally, we remark that the full proportionality

$$S_{\mu\nu} = I\omega_{\mu\nu} \quad (75)$$

is not required for a rotator to be rigid. (75) is satisfied only for the rigid hyperspherical case (6b-(ii)). However, in usual cases where (74') holds, the proper time parts of $S_{\mu\nu}$ and $\omega_{\mu\nu}$ are also parallel, but with a different proportionality factor, like $S_{\mu}^+ = k\omega_{\mu}^+$ (see Eq. (87)).

When a rotator satisfies (72) but with \mathfrak{Q} depending on a_{μ}^{ξ} and \dot{a}_{μ}^{ξ} , it is not "rigid" but corresponds to a radially deformable structure.

§ 6. Classification of relativistic rotators

On the basis of the general theory thus far developed we shall classify in this section simple possible relativistic rotators according to the forms of the rotational part of the Lagrangian, \mathcal{L} .

6a. Linear Lagrangian

If \mathcal{L} is linear in \dot{a}_{μ}^{ξ} it must have the form

$$\mathcal{L} = -\frac{1}{2} A^{\xi\eta} a_{\mu}^{\xi} \dot{a}_{\mu}^{\eta} = \frac{1}{2} A^{\xi\eta} \omega^{\xi\eta}, \quad (76)$$

where $A^{\xi\eta} = -A^{\eta\xi}$ are constant coefficients. The internal angular momentum tensor takes the form

$$S_{\mu\nu} = A^{\xi\eta} a_{\mu}^{\xi} a_{\nu}^{\eta}, \quad S^{\xi\eta} = A^{\xi\eta} = \text{const}. \quad (77)$$

and so the equation of motion is of the first order in proper-time derivation. We have $m = \text{const.}$, using (64) and (77). We shall consider the three special cases.

(i) If $A^{rs} = 0$, (76) and (77) become

$$\mathcal{L} = -\frac{1}{2} A^{rs} a_{\mu}^r \dot{a}_{\mu}^s, \quad (78)$$

and

$$S_{\mu\nu} = A^{rs} a_{\mu}^r a_{\nu}^s. \quad (79)$$

The latter evidently satisfies Weyssenhoff's condition⁴⁾

$$S_{\mu}^+ \equiv S_{\mu\nu} v_{\nu} = 0, \quad (80)$$

which is also expressed as

$$S^{r4} = 0. \quad (80')$$

Although (78) does not fulfill the condition of spherical symmetry, S^{rs} are constant :

$$S^{rs} = A^{rs}. \quad (81)$$

This indicates, however, that S^{rs} become structure constants rather than constants of motion and cannot be identified as isospin, since they remain c -numbers in quantum theory and cannot fulfill the commutation relations for isospin components.

Further, it is proved that the condition (72) is also satisfied as the result of equations of motion.

It can really be proved* that the above theory is equivalent to Matthison-Weysenhoff theory.^{3),4)} In the usual form of this theory the internal rotation is hidden and the theory is represented directly by (47) and (80), while in our method the theory is automatically derived only if we take the form (78) for the rotational part of the Lagrangian. Moreover, we shall see that more significant form of \mathcal{L} corresponding to Weyssenhoff's theory is given by a particular second order form.

(ii) The next case is $A^{rs} = 0$, then we have

$$\begin{aligned} \mathcal{L} &= -A^r a_{\mu}^r \dot{v}_{\mu} = A^r \dot{a}_{\mu}^r v_{\mu}, \\ S_{\mu\nu} &= A^r (a_{\mu}^r v_{\nu} - a_{\nu}^r v_{\mu}), \end{aligned}$$

so that

$$\begin{cases} S_{\mu}^* = 0, & \text{i.e. } S^{rs} = 0, \\ S^{r4} = iA^r = \text{const.} \end{cases}$$

This model is similar to the case 6*b*-(iii) of the quadratic Lagrangian.

(iii) If A^{r4} is proportional to A^{st} , such that

$$A^{r4} = \frac{\Gamma}{2} \varepsilon_{rst} A^{st}, \quad \text{so} \quad S^{r4} = \frac{\Gamma}{2} \varepsilon_{rst} S^{st},$$

this gives the model proposed in a previous paper.⁷⁾ This model is distinguished from the preceding ones by the feature that the internal chirality $\chi = -(\eta/4) S_{\mu\nu} \tilde{S}_{\mu\nu}$ is a non-vanishing constant of motion.

Generally speaking, linear Lagrangian models are exceptional in that the equations of motion are first order differential equations, and as a result they have not sufficient number of internal constants of motion, lacking isospin freedoms. In

* To be published separately.

fact they are generally regarded as representing particular modes of motions implied by the models belonging to quadratic \mathcal{L} as to be stated below. On the other hand, it has been shown that the Dirac field is interpreted as a continuous distribution of relativistic rotators of the linear Lagrangian type.⁸⁾

6b. Quadratic Lagrangian

We shall now consider the rotators whose \mathcal{L} is quadratic in \dot{a}_μ^t , restricting ourselves to *spherical* rotators. We have then two possibilities, the first is internal 3-scalar,

$$\mathcal{L} = \frac{I}{4} \dot{a}_\mu^r \dot{a}_\mu^r + \frac{K}{4} \dot{a}_\mu^4 \dot{a}_\mu^4, \quad (82)$$

and the second is internal pseudoscalar (cf. Eqs. (34) and (35)),

$$\begin{aligned} \mathcal{L} &= \frac{\Gamma}{4} \omega^{\epsilon\eta} \tilde{\omega}^{\epsilon\eta} = -\frac{\Gamma}{2} \epsilon_{rst} a_\mu^r \dot{a}_\mu^s a_\nu^t \dot{v}_\nu \\ &= \frac{\gamma\Gamma}{4} \omega_{\mu\nu} \tilde{\omega}_{\mu\nu} = -\gamma\Gamma \omega_\mu^* \omega_\mu^\dagger = \frac{i\gamma\Gamma}{2} \epsilon_{\mu\nu\kappa\lambda} a_\kappa^r \dot{a}_\lambda^r v_\mu \dot{v}_\nu. \end{aligned} \quad (83)$$

(We are using the method¹⁸⁾ in which the Lagrangian is always an external scalar, and accordingly the equations of motion is always covariant under external reflection. In this method the right-left asymmetrical object is distinguished by the appearance of the pseudoscalar η in the equation of motion.)

The former, (82), contains two structure constants I and K , while the latter, (83), involves only one structure constant Γ and is necessarily hyper-spherical.

We shall first consider the form (82). The internal angular momentum tensor is

$$\begin{aligned} S_{\mu\nu} &= \frac{I}{2} a_{[\mu}^r \dot{a}_{\nu]}^r + \frac{K}{2} a_{[\mu}^4 \dot{a}_{\nu]}^4 \\ &= I \omega_{\mu\nu} + \frac{K-I}{2} a_{[\mu}^4 \dot{a}_{\nu]}^4 \\ &= I \omega_{\mu\nu}^* - \frac{J}{2} v_{[\mu} \dot{v}_{\nu]}, \quad (J \equiv I+K) \end{aligned} \quad (84)$$

which gives

$$S_\mu^* \equiv \tilde{S}_{\mu\nu} v_\nu = I \omega_\mu^*, \quad (85)$$

indicating that the rotator is "rigid". The constant I corresponds to the moment of inertia for the non-relativistic rigid body, and so we have to assume $I \geq 0$, while K is allowed to be any constant value $K \leq 0$. According to the general theory of 5a we have

$$S^{rs} = I \omega^{rs} = I \dot{a}_\mu^r a_\mu^s = \text{const.}, \quad (86)$$

signifying the isospin components.

On the other hand, (84) gives

$$S_{\mu}^{\dagger} \equiv S_{\mu\nu} v_{\nu} = -\frac{J}{2} \dot{v}_{\mu} = -\frac{J}{2} \omega_{\mu}^{\dagger}, \quad (87)$$

or equivalently,

$$S^{r4} = -\frac{J}{2} \dot{a}_{\mu}^r a_{\mu}^4 = -\frac{J}{2} \omega^{r4}. \quad (87')$$

(87) is the characteristic relation for the spherical rigid rotator.

Internal constants of motion for this model other than (86), (39) and (55) are the following two remarkable quantities (see Part II) :

$$m + \frac{1}{4} J \dot{v}_{\mu}^2, \quad (88a)$$

$$\chi^2 + J \mathcal{M}_{\mu} S_{\mu}^*. \quad (88b)$$

This model is especially important, including the following special cases :

$$(i) \quad K = -I \quad \text{i.e.} \quad J = 0.$$

Then (cf. Eq. (33)),

$$\mathcal{L} = \frac{I}{4} (\dot{a}_{\mu}^r \dot{a}_{\mu}^r + \dot{v}_{\mu}^2) = \frac{I}{4} (\omega_{\mu\nu}^*)^2 = \frac{I}{2} (\omega_{\mu}^*)^2, \quad (89)$$

$$S_{\mu\nu} = I \omega_{\mu\nu}^*.$$

This case may be regarded as the natural covariant generalization of the usual non-relativistic spherical rigid body with the moment of inertia I , since ω_{μ}^* in (89) signifies the covariant angular velocity of spatial rotation. Indeed, this case has particular simplicity because (87) simplifies to

$$S_{\mu\nu} v_{\nu} = 0, \quad \text{i.e.} \quad S^{r4} = 0, \quad (90)$$

i.e. the Weyssenhoff's condition. By virtue of this the motion necessarily satisfies

$$m = \text{const.}, \quad S_{\mu\nu}^2 = \text{const.}, \quad \chi = 0.$$

Still, p_{μ} is not colinear to v_{μ} in general.

The present model contains the model given by (78) and is regarded as the richer representation of Mathison-Weyssenhoff's theory in terms of the internal space. To see this we first note that

$$S_{\mu\nu} = a_{\mu}^{\xi} a_{\nu}^{\eta} S^{\xi\eta} = S^{rs} a_{\mu}^r a_{\nu}^s \quad (91)$$

because of (90). But, since we know that S^{rs} here are constants owing to (86), $S_{\mu\nu}$ of (91) is equivalent to (79), and especially the equations of motion (47) become of the same structure in both cases. Thus, if one fixes the constants of motion S^{rs} in the present case to certain specified values, the present model reduces to the former model defined by the linear Lagrangian (78). The advantage of the present theory over the former one is clear, since here the arbitrary structure constants A^{rs} are replaced by the constants of motion meaning isospin components.

In this model magnitudes of spin and isospin stand in a simple relation,

$$\tau/\sigma = m'/m,$$

and we have *five* internal degrees of freedom only (see Part II).

(ii) $K=I$.

The rotator is hyper-spherical, with (cf. Eq. (32))

$$\mathcal{L} = \frac{I}{4} \dot{a}_\mu^{\xi} \dot{a}_\mu^{\xi} = \frac{I}{4} (\dot{a}_\mu^r \dot{a}_\mu^r - \dot{v}_\mu^2) = \frac{I}{4} \omega_{\mu\nu}^2, \quad (92)$$

$$S_{\mu\nu} = I \omega_{\mu\nu}. \quad (93)$$

This case was adopted by Vigier et al.^{(10),(11)*} Important property is that *only in this case and in the preceding case* (i) the relation

$$\frac{1}{4I} S_{\mu\nu}^2 = \mathcal{L} \quad (94)$$

holds. (Compare with the general relation (49).) On the other hand, there is no *a priori* physical reason to pick out this hyper-spherical case, since the fourth axis has physically distinguished meaning, as was stated before. Nor this hyper-spherical property changes the general character of the motion (cf. 5a).

(iii) If one imposes the subsidiary condition

$$S_{\mu\nu} p_\nu \equiv m'^2 r_\mu = 0, \quad (95)$$

on our rigid rotator defined by the general Lagrangian (82), realizable motions are restricted to special ones in which p_μ is colinear to v_μ with complete decoupling between internal rotation and orbital motion. This is just the case considered by Nakano⁽⁶⁾ as the relativistic rigid body by his definition. (Such "classical mode" of motions is always allowed as special solutions *irrespective of the value of K* for our general rigid rotator (82). In order to confine the motion to "classical" mode a simple procedure is to replace the constant K in the Lagrangian (82) by a Lagrangian multiplier λ . Nakano⁽⁶⁾ starts from a complicated Lagrangian in terms of his Euclidian parameters, but the resulting motion is the same as far as classical theory is concerned.)

It is to be noted, however, that, as we have stated, in order to have the theory of a relativistic rotator with clear physical meaning, the condition (36) was essential, while the simple analogy with the conventional non-relativistic rigid body by (95) is by no means necessary. Indeed r_μ , the time-like part of the internal angular momentum tensor $S_{\mu\nu}$, was physically well understood as the radius vector of Zitterbewegung. If one suppresses *three* internal degrees of freedom by the condition (95) in order to restrict the motion to "classical" mode, the disparities between

* They imposed the subsidiary condition $m=\text{const.}$ which suppresses two degrees of freedom for the motion (since then χ must necessarily vanish), while we deal with the general motion without such condition.

two rest masses m and m' and between the magnitudes of spin and isospin, σ and τ , necessarily disappear (see Part II).

(iv) $I=0$, ($K \neq 0$)

This is nothing but the Hönl-Papapetrou particle,⁵⁾ which corresponds to a most primitive non-local particle model. The internal rotation degenerates completely and the internal angular momentum tensor takes the form

$$S_{\mu\nu} = -\frac{K}{2} (\dot{v}_\mu \dot{v}_\nu - \dot{v}_\nu \dot{v}_\mu), \quad (96)$$

originating exclusively from acceleration. Thus this model can supply the ordinary spin but not the isospin:

$$S_\mu^* = 0, \quad \text{i.e.} \quad S^{rs} = 0. \quad (97)$$

The equations of motion are reduced to

$$-\dot{S}_\mu^* = \frac{K}{2} \dot{v}_\mu = \left(\frac{K}{2} \dot{v}_\rho^2 + v_\rho p_\rho \right) v_\mu + p_\mu. \quad (98)$$

Since in this case the variables a_μ^r are entirely redundant, the Lagrangian can be simplified to the form

$$\mathcal{L} = -\frac{K}{4} \dot{v}_\mu^2 - \frac{\dot{\lambda}}{2} (\dot{v}_\mu^2 + 1) + p_\mu (\dot{x}_\mu - v_\mu),$$

which directly leads to (98). Taking into account that $\dot{p}_\mu = 0$, it is readily proved that (98) is equivalent to the Hönl-Papapetrou's equation of motion.

Our considerations need not be limited to the special cases (i)–(iv), above stated. In later sections we shall investigate the spherical rigid rotators without taking those special choices of I and K ratio.

Next we consider the model given by (83) which is quite new one and represents a typical example with internal chirality. In this case

$$S_{\mu\nu} = \eta \Gamma \tilde{\omega}_{\mu\nu}, \quad \text{i.e.} \quad S^{\xi\eta} = \Gamma \tilde{\omega}^{\xi\eta}, \quad (99)$$

hence

$$S_\mu^* = \eta \Gamma \dot{v}_\mu. \quad (100)$$

The isospin components are

$$S^{rs} = \Gamma \varepsilon_{rst} a_\mu^t \dot{v}_\mu = \text{const.}, \quad (101)$$

and accordingly we have

$$\dot{v}_\mu^2 = \text{const.} \quad (102)$$

The appearance of η in (100) indicates that the motion defines a right- or left-handed screw (depending on the sign of Γ). The reason why the internal chirality can have relativistically invariant sense for non-zero mass particle is due to the acceleration \dot{v}_μ which does not vanish even for free particle, defining a screw with

the pseudovector S_μ^* .

Finally, the mixture of the linear Lagrangian (78) and the quadratic Lagrangian (82) defines a more general model,* which has interesting properties.

In this section we have enumerated and classified various possible relativistic rotators starting from the Lagrangian and outlined their features. In later sections we shall especially work out the properties of the quadratic Lagrangian model more deeply and show that it leads to a unified model of elementary particles.

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Appendix

Relativistic decomposition of anti-symmetric tensors

An arbitrary skew-tensor $A_{\mu\nu}$ can be decomposed uniquely into its "proper space part" $A_{\mu\nu}^*$ and "proper time part" $A_{\mu\nu}^\dagger$ such that

$$\begin{cases} A_{\mu\nu} = A_{\mu\nu}^* + A_{\mu\nu}^\dagger, & (A \cdot 1) \end{cases}$$

$$\begin{cases} A_{\mu\nu}^* v_\nu = 0, \quad \text{i.e.} \quad A_{\mu 4}^{*(X)} = 0, & (A \cdot 2) \end{cases}$$

$$\begin{cases} (A_{\mu\nu}^\dagger) \sim v_\nu = 0, \quad \text{i.e.} \quad A_{ij}^{\dagger(X)} = 0. & (A \cdot 3) \end{cases}$$

Using the projection tensor

$$\begin{aligned} \eta_{\mu\nu} &\equiv \delta_{\mu\nu} + v_\mu v_\nu = a_\mu^r a_\nu^r, \\ (\eta_{\mu\nu} &= \eta_{\nu\mu}, \quad \eta_{\mu\nu} v_\nu = 0, \quad \eta_{\mu\nu} \eta_{\nu\lambda} = \eta_{\mu\lambda}), \end{aligned} \quad (A \cdot 4)$$

$A_{\mu\nu}^*$ is derived by

$$A_{\mu\nu}^* = \eta_{\mu\rho} \eta_{\nu\sigma} A_{\rho\sigma}. \quad (A \cdot 5)$$

$A_{\mu\nu}^*$ and $A_{\mu\nu}^\dagger$ are mathematically equivalent to the pseudovector

$$A_\mu^* = \widetilde{A}_{\mu\nu} v_\nu, \quad (A_\mu^* v_\mu = 0) \quad (A \cdot 6)$$

and the vector

$$A_\mu^\dagger = A_{\mu\nu} v_\nu, \quad (A_\mu^\dagger v_\mu = 0) \quad (A \cdot 7)$$

respectively, since we have

$$\begin{cases} A_{\mu\nu}^* = -i\varepsilon_{\mu\nu\kappa\lambda} A_\kappa^* v_\lambda, & (A \cdot 8) \end{cases}$$

$$\begin{cases} A_{\mu\nu}^\dagger = v_{[\mu} A_{\nu]}^\dagger. & (A \cdot 9) \end{cases}$$

In a rest frame we have

* This is an important example of rotators with a composite internal structure.

$$\begin{cases} A_{ij}^{(\Sigma)} = A_{ij}^{*(\Sigma)} = A_k^{*(\Sigma)}, & (ijk \sim 123) \\ A_{i4}^{(\Sigma)} = A_{i4}^{\dagger(\Sigma)} = -iA_i^{\dagger(\Sigma)}. \end{cases} \quad (\text{A} \cdot 10)$$

The dual operation and the $*$ or $+$ projection is not commutative, but we have

$$(\tilde{A}_{\mu\nu})^* = (A_{\mu\nu}^\dagger)^\sim = -i\varepsilon_{\mu\nu\kappa\lambda} v_\kappa A_\lambda^\dagger, \quad (\text{A} \cdot 11)$$

$$(\tilde{A}_{\mu\nu})^\dagger = (A_{\mu\nu}^*)^\sim = \tilde{A}_{\mu\nu} + i\varepsilon_{\mu\nu\kappa\lambda} v_\kappa A_\lambda^\dagger = v_{[\mu} A_{\nu]}^*, \quad (\text{A} \cdot 12)$$

which involve, e.g., the relation

$$(\tilde{A}_{\mu\nu})^* = \tilde{A}_{\mu\nu} - (A_{\mu\nu}^*)^\sim. \quad (\text{A} \cdot 13)$$

For arbitrary two skew-tensors, $A_{\mu\nu}$ and $B_{\mu\nu}$, we have the formulas:

$$\begin{cases} A_{\mu\nu}^* B_{\mu\nu}^* = 2A_\mu^* B_\mu^*, & A_{\mu\nu}^\dagger B_{\mu\nu}^\dagger = -2A_\mu^\dagger B_\mu^\dagger, \\ A_{\mu\nu}^* B_{\mu\nu}^\dagger = 0, \end{cases} \quad (\text{A} \cdot 14)$$

and accordingly

$$A_{\mu\nu} B_{\mu\nu} = 2(A_\mu^* B_\mu^* - A_\mu^\dagger B_\mu^\dagger). \quad (\text{A} \cdot 15)$$

Taking into account that $(\tilde{B}_{\mu\nu})^\sim = -B_{\mu\nu}$, (A·15) gives

$$A_{\mu\nu} \tilde{B}_{\mu\nu} = -2(A_\mu^* B_\mu^\dagger + A_\mu^\dagger B_\mu^*), \quad (\text{A} \cdot 16)$$

and, in particular,

$$A_{\mu\nu} \tilde{A}_{\mu\nu} = -4A_\mu^* A_\mu^\dagger. \quad (\text{A} \cdot 17)$$

Finally a similar decomposition of a skew-tensor can be made by using any (non-null) vector, say z_μ , in the place of v_μ , thus

$$z_\rho^2 A_{\mu\nu} = i\varepsilon_{\mu\nu\kappa\lambda} (\tilde{A}_{\kappa\rho} z_\rho) z_\lambda + A_{[\mu\rho} z_\rho \cdot z_{\nu]}. \quad (\text{A} \cdot 18)$$

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Note added in proof: 1) To make the argument clear we have restricted ourselves in this Part to the simplest types of rotators with directional rigidity. However, we do not mean to exclude the possibility of more flexible models. Indeed we can introduce the degrees of freedom corresponding to dilatations and deformations by slightly generalizing our formulation. It is also possible and useful to investigate the rotators containing higher derivatives, rotators with composite internal structures,* partially "time-like rotations", rotations on the light-cone,¹⁹⁾ etc. [*An example is given by the Lagrangian $L = A \dot{a}_k \cdot \dot{b}_k$, where $\{a_k\}$ and $\{b_k\}$ are independent sets of 3-dimensional orthonormal vectors (taking the non-relativistic case for simplicity). Another example was recently proposed by H. Fukutome (Soryusiron-kenkyu (mimeographed circular in Japanese) 20 (1959), 331).]

2) Recently, a quantization of relativistic rotator was investigated by Vigier et al. (to be published). Our theory is different from theirs in the following main respects: (1) Identification of intrinsic properties of elementary particles are quite different. (2) Their treatment discards the kinematical coupling between orbital motion and internal rotation. This implies to admit unrestricted "time-like rotations". (3) They quantize the motion in terms of three complex Euler angles, which involve three hyperbolic parameters varying from zero to ∞ , by introducing an indefinite metric, while we quantize, in a normal method, by five angular variables and only one hyperbolic parameter γ ($\cosh \gamma = v_0 \equiv v_4/i$), which works just like a radius variable, and the corresponding principal quantum number determines the discrete eigenvalues of rest mass that depends also on other quantum numbers (spin, isospin and hypercharge).

3) Our quantum theory for simple spherical rotator deduces main characteristics of elementary particles, while its most serious defect is the fact that the spin and isospin for each particle have both integer or both half-integer values, as was remarked especially by Yukawa. This is due to the fact that in our theory the ordinary spin is, roughly speaking, the sum of the angular momentum of internal rotation (which is related to isospin) and that of the orbital precession which takes only integral eigenvalues. There are various ways to revise this defect.

Dispersion Relations in Nucleon-Nucleon Scattering

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Two-pion contribution to the absorptive part of nucleon-nucleon scattering amplitudes in the unphysical region is calculated using the dispersion relations for pion-nucleon scattering. The dispersion relations with this absorptive part are used for analyzing nucleon-nucleon scattering data at low energy and at moderate energy, and we find good agreement if we choose the coupling constant as

$$\frac{f^2}{4\pi} = 0.08 \pm 0.01.$$

§ 1. Introduction

Although the dispersion relations have proved so successful in the problem of pion-nucleon phenomena, the corresponding relationship for nucleon-nucleon case has not been exploited in detail.¹⁾ This is due to the appearance of large unphysical regions which do not correspond to directly observable quantities. However, it is possible to calculate this unphysical contribution in terms of the meson theory with sufficient accuracy to compare with experimental information. The application of this relation to experimental data will be an important test for our understanding of local field theories and this is the purpose of this paper.

The validity of dispersion relations for nucleon-nucleon scattering has not been proved with the same rigor as the pion-nucleon case, but for forward scattering the relation has been proved in every order in perturbation expansion.²⁾ We expand the S -matrix in the number of exchanged pions and calculate one- and two-pion exchange terms of its absorptive part. In doing so, use is made of the dispersion relations for pion-nucleon scattering. Three- and more-pion exchange terms, after one subtraction, turned out to be unimportant for moderate nucleon energies.

The dispersion relation with this absorptive part in the unphysical region is used for analyzing proton-proton scattering below 40 Mev where phase shift analysis was carried out.³⁾ It is also applied to low energy neutron-proton scattering. We find good agreement if we choose the coupling constant as

$$\frac{f^2}{4\pi} = 0.08 \pm 0.01.$$

In § 2 we calculate the two-pion contribution to the absorptive part in the unphysical region. This dispersion relation is applied to proton-proton scattering in § 3, and to low energy nucleon-nucleon scattering in § 4. Conclusion and dis-

cussions are given in § 5.

§ 2. One- and two-pion contribution to the absorptive part

In this section we calculate the absorptive part of the forward scattering amplitudes in the unphysical region. Here the momenta of the incoming particles are in general complex and the scattering matrices are not defined in the usual sense. However, there are various ways of extending into complex variables. For example, in Feynman's perturbation expansion the scattering matrices are written as rational functions in the momentum variables and they are well defined even for complex momenta.

The S -matrix elements are divided into two parts, i.e. dispersive part D and absorptive part A :

$$\begin{aligned} S(E) &= (2\pi)^4 \delta^4(\Sigma p) i(D(E) + iA(E)) \quad E \geq m, \\ &= (2\pi)^4 \delta^4(\Sigma p) i(D(E) - iA(E)) \quad E < m, \end{aligned} \quad (2.1)$$

where E is the laboratory energy of the incoming nucleon and m is the nucleon mass. D and A are written in the form

$$D(E) = \bar{u}(p) \bar{u}(q) d(E) u(p) u(q)$$

$$A(E) = \bar{u}(p) \bar{u}(q) a(E) u(p) u(q)$$

and $d(E)$ and $a(E)$ satisfy dispersion relations.¹⁾ In this paper we are interested only in low energy phenomena in which relativistic effects are unimportant. We often make non-relativistic approximations whenever it is convenient and we do not distinguish between d and D or a and A since energy dependence of the Dirac spinor u , \bar{u} , is small. For the case of forward scattering in definite spin state, D and A are both real and

$$A(E) = -\frac{1}{(2\pi)^4 \delta^4(0)} \text{Re } S(E), \quad E < m. \quad (2.2)$$

The problem is thus the calculation of the real part of the S -matrix for $E < m$.

The corresponding Feynman diagrams are drawn in Fig. 1 and Fig. 2. α, α', β or β' stand for spin and isotopic spin state of each particle. As is easily seen,

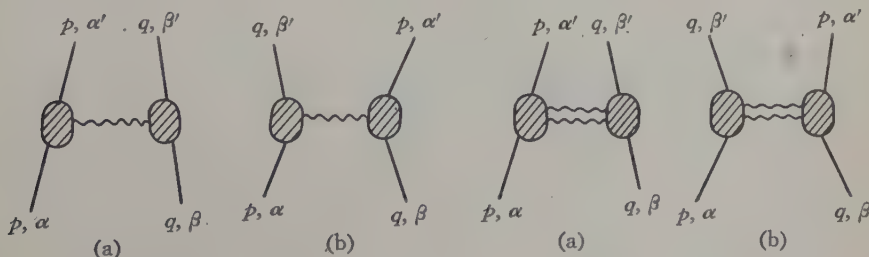


Fig. 1.

Fig. 2.

Fig. 1(a) and Fig. 2(a) give no contribution to the absorptive part in the unphysical region¹⁾ in which we are interested (Fig. 1(a) is actually zero), so we concern ourselves with exchange type diagrams, i.e. Fig. 1(b) and Fig. 2(b). The momentum p is held fixed and q is varied in such a way that

$$\begin{aligned} p_\mu &= (m; 0), \\ q_\mu &= (E; q\mathbf{e}), \quad q^2 = E^2 - m^2. \end{aligned}$$

The absorptive part for one-pion exchange, $A_{1\pi}$, is rather trivial and*

$$A_{1\pi}(E) = \frac{f^2}{4\pi} \frac{2\pi^2}{m} (\alpha' | \sigma \mathbf{e} \tau_i | \beta) (\beta' | \sigma \mathbf{e} \tau_i | \alpha) \delta \left(E - m + \frac{1}{2m} \right) \quad (2.3)$$

where f is the renormalized coupling constant.**

For the calculation of the two-pion part, $A_{2\pi}$, it is convenient to choose the center-of-mass system. In this case

$$\begin{aligned} p'_\mu &= \left(E'; -\frac{q}{2} \mathbf{e} \right), \quad q'_\mu = \left(E'; \frac{q}{2} \mathbf{e} \right). \\ E'^2 &= \frac{q^2}{4} + m^2. \end{aligned}$$

The S -matrix is expressed as⁴⁾

$$S_{2\pi} = -\frac{1}{2} \frac{1}{(2\pi)^8} \iint \frac{\langle j, l | S^1 | i, k \rangle \langle j, -l | S^2 | i, -k \rangle}{(k_0^2 - \omega_k^2)(l_0^2 - \omega_l^2)} d^4k d^4l, \quad (2.4)$$

where $\langle j, l | S^1 | i, k \rangle$ is the scattering amplitude of an i -th pion with momentum k into a j -th pion with momentum l by the first nucleon. Similarly, $\langle j, -l | S^2 | i, -k \rangle$ describes the scattering by the second nucleon. The momentum transfer is given by

$$\begin{aligned} l - k &= p' - q' = t, \\ t_\mu &= (0; -q\mathbf{e}). \end{aligned}$$

$A_{2\pi}$ has non-vanishing value for $q^2 < -4$.¹⁾ In this unphysical region the momentum transfer is purely imaginary and the scattering matrices $\langle | S^i | \rangle$ must be defined in some way. Here we proceed as follows. The scattering matrix with constant momentum transfer can be expressed in a dispersion formula in the pion energy. Since we are interested only in the region of fairly small t_μ^2 , it can be sufficiently well approximated by the following expression⁴⁾:

$$\begin{aligned} \langle j, l | S | i, k \rangle &= (2\pi)^4 i \delta^4(l - k - t) [A(k_0) \tau_i \tau_j \sigma \mathbf{k} \sigma \mathbf{l} + B(k_0) \\ &\quad \times (\tau_i \tau_j \sigma \mathbf{l} \sigma \mathbf{k} + \tau_j \tau_i \sigma \mathbf{k} \sigma \mathbf{l}) + C(k_0) \tau_j \tau_i \sigma \mathbf{l} \sigma \mathbf{k}] \end{aligned} \quad (2.5)$$

* We take $\hbar = c = \text{pion mass} = 1$.

** $2mf = g$ is the pseudoscalar coupling constant.

$$\begin{aligned}
A(k_0) &= f^2 \frac{1}{k_0 - \varepsilon} + \frac{1}{4\pi} \int_0^\infty \frac{dp}{\omega_p} \frac{\sigma_{33}(p)}{\omega_p - k_0} + \frac{1}{36\pi} \int_0^\infty \frac{dp}{\omega_p} \frac{4\sigma_{11} + 4\sigma_{13} + \sigma_{33}}{\omega_p + k_0}, \\
B(k_0) &= \frac{1}{12\pi} \int_0^\infty \frac{dp}{\omega_p} \frac{\sigma_{33} + 2\sigma_{13}}{\omega_p - k_0} + \frac{1}{12\pi} \int_0^\infty \frac{dp}{\omega_p} \frac{\sigma_{33} + 2\sigma_{13}}{\omega_p + k_0}, \\
C(k_0) &= -f^2 \frac{1}{k_0 + \varepsilon} + \frac{1}{36\pi} \int_0^\infty \frac{dp}{\omega_p} \frac{4\sigma_{11} + 4\sigma_{13} + \sigma_{33}}{\omega_p - k_0} + \frac{1}{4\pi} \int_0^\infty \frac{dp}{\omega_p} \frac{\sigma_{33}}{\omega_p + k_0}, \quad (2.6)^* \\
\varepsilon &= \frac{\mathcal{A}^2 - 2}{4m}, \quad \mathcal{A}^2 = -q^2 > 0.
\end{aligned}$$

Except for the first terms of A and C, (2.6) is equal to the static formula. However, the ε in the denominator cannot be neglected since $A_{2\pi}$ (Eq. (2.8)) is divergent without this term. In this sense $A_{2\pi}$ does not have a static limit as $1/m \rightarrow 0$. This scattering matrix is defined also for complex momenta.

The momentum transfer t is time-like in the sense that $t_\mu^2 = t_0^2 - t^2 = \mathcal{A}^2 > 0$, and it is possible to choose a Lorentz frame in which $t_\mu' = (\mathcal{A}, 0)$. Notice that the S -matrix as defined by the Feynman method is invariant under (complex) Lorentz transformation. In this reference frame the calculation of the absorptive part is straightforward since it is identical with the calculation of total cross section for the production of two (physical) pions.** The momenta of the two pions are

$$\begin{aligned}
l_\mu' &= \left(\frac{\mathcal{A}}{2}; k_1, k_2, k_3 \right), \quad -k_\mu' = \left(\frac{\mathcal{A}}{2}; -k_1, -k_2, -k_3 \right), \\
k_1^2 + k_2^2 + k_3^2 &= \frac{\mathcal{A}^2}{4} - 1,
\end{aligned}$$

or in the original reference frame,

$$k_\mu = \left(ik_3; k_1, k_2, i\frac{\mathcal{A}}{2} \right), \quad l_\mu = \left(ik_3; k_1, k_2, -i\frac{\mathcal{A}}{2} \right),$$

if e is chosen as the third axis. In this way, if

$$S_{2\pi} = \delta^4(0) \int \frac{d^4k d^4l}{(k_0^2 - \omega_k^2)(l_0^2 - \omega_l^2)} \delta^4(l - k - t) f(k, l),$$

its absorptive part in the unphysical region is given by

$$\begin{aligned}
A_{2\pi} &= -\frac{1}{16\pi^2 \mathcal{A}} \int d^3k \delta \left(k^2 - \left(\frac{\mathcal{A}^2}{4} - 1 \right) \right) f \left(ik_3; k_1, k_2, i\frac{\mathcal{A}}{2}, ik_3; k_1, k_2, -i\frac{\mathcal{A}}{2} \right), \\
\mathcal{A}^2 &\geq 4. \quad (2.7)
\end{aligned}$$

(2.4), (2.5), (2.6) and (2.7) yield

* In the following, we neglect σ_{11} and σ_{13} .

** This procedure is not always justified when unphysical threshold appears.

$$\begin{aligned}
A_{2\pi}(E) = & -\frac{1}{16\pi A} \int_{-\sqrt{\Delta^2/4-1}}^{\sqrt{\Delta^2/4-1}} dk_3 \left\{ (\boldsymbol{\tau}^1 \boldsymbol{\tau}^2)^2 \left[\left(\frac{\mathcal{L}^2}{2} - 1 - k_3^2 \right)^2 + \frac{1}{2} \mathcal{L}^2 \left(\frac{\mathcal{L}^2}{4} - 1 - k_3^2 \right) \right] \right. \\
& \times ((\boldsymbol{\sigma}^1 \boldsymbol{\sigma}^2) - (\boldsymbol{\sigma}^1 \boldsymbol{e})(\boldsymbol{\sigma}^2 \boldsymbol{e})) \left. \right] \left[\boldsymbol{A}(ik_3) \boldsymbol{C}(ik_3) + \boldsymbol{B}(ik_3) \boldsymbol{B}(ik_3) \right] \\
& + 2\tau_i^1 \tau_j^1 \tau_j^2 \tau_i^2 \left[\left(\frac{\mathcal{L}^2}{2} - 1 - k_3^2 \right)^2 + \frac{1}{2} \mathcal{L}^2 \left(\frac{\mathcal{L}^2}{4} - 1 - k_3^2 \right) ((\boldsymbol{\sigma}^1 \boldsymbol{\sigma}^2) - (\boldsymbol{\sigma}^1 \boldsymbol{e})(\boldsymbol{\sigma}^2 \boldsymbol{e})) \right] \\
& \quad \times \boldsymbol{A}(ik_3) \boldsymbol{B}(ik_3) \\
& + 2(\boldsymbol{\tau}^1 \boldsymbol{\tau}^2)^2 \left[\left(\frac{\mathcal{L}^2}{2} - 1 - k_3^2 \right)^2 - \frac{1}{2} \mathcal{L}^2 \left(\frac{\mathcal{L}^2}{4} - 1 - k_3^2 \right) ((\boldsymbol{\sigma}^1 \boldsymbol{\sigma}^2) - (\boldsymbol{\sigma}^1 \boldsymbol{e})(\boldsymbol{\sigma}^2 \boldsymbol{e})) \right] \\
& \quad \times \boldsymbol{A}(ik_3) \boldsymbol{B}(ik_3) \\
& + \tau_i^1 \tau_j^1 \tau_j^2 \tau_i^2 \left[\left(\frac{\mathcal{L}^2}{2} - 1 - k_3^2 \right)^2 - \frac{1}{2} \mathcal{L}^2 \left(\frac{\mathcal{L}^2}{4} - 1 - k_3^2 \right) ((\boldsymbol{\sigma}^1 \boldsymbol{\sigma}^2) - (\boldsymbol{\sigma}^1 \boldsymbol{e})(\boldsymbol{\sigma}^2 \boldsymbol{e})) \right] \\
& \quad \times \left[\boldsymbol{A}(ik_3) \boldsymbol{A}(ik_3) + \boldsymbol{B}(ik_3) \boldsymbol{B}(ik_3) \right] \left. \right\}, \quad (2.8)
\end{aligned}$$

where $E = m - \mathcal{L}$, $2m$ and spin and isospin operator $X^1 Y^2$ means $(\alpha' | X^1 | \beta) (\beta' | Y^2 | \alpha)$. This matrix element can be written as a linear combination of $(\alpha' | X^1 | \alpha) (\beta' | Y^2 | \beta)$, which we simply write as XY . Then

$$\begin{aligned}
A_{1\pi}(E) = & \left(-\frac{3}{4} + \frac{1}{4} (\boldsymbol{\tau}\boldsymbol{\tau}) + \frac{3}{4} (\boldsymbol{\sigma}\boldsymbol{\sigma}) - \frac{1}{4} (\boldsymbol{\tau}\boldsymbol{\tau}) (\boldsymbol{\sigma}\boldsymbol{\sigma}) - \frac{3}{2} (\boldsymbol{\sigma}\boldsymbol{e}) (\boldsymbol{\sigma}\boldsymbol{e}) \right. \\
& \left. + \frac{1}{2} (\boldsymbol{\tau}\boldsymbol{\tau}) (\boldsymbol{\sigma}\boldsymbol{e}) (\boldsymbol{\sigma}\boldsymbol{e}) \right) \frac{f^2}{4\pi} \frac{2\pi^2}{m} \delta \left(E - m + \frac{1}{2m} \right) \quad (2.9)
\end{aligned}$$

and

$$\begin{aligned}
A_{2\pi}(E) = & A_{2\pi}^{(1)} + (\boldsymbol{\tau}\boldsymbol{\tau}) A_{2\pi}^{(2)} + (\boldsymbol{\sigma}\boldsymbol{\sigma}) A_{2\pi}^{(3)} + (\boldsymbol{\tau}\boldsymbol{\tau}) (\boldsymbol{\sigma}\boldsymbol{\sigma}) A_{2\pi}^{(4)} \\
& + (\boldsymbol{\sigma}\boldsymbol{e}) (\boldsymbol{\sigma}\boldsymbol{e}) A_{2\pi}^{(5)} + (\boldsymbol{\tau}\boldsymbol{\tau}) (\boldsymbol{\sigma}\boldsymbol{e}) (\boldsymbol{\sigma}\boldsymbol{e}) A_{2\pi}^{(6)}. \quad (2.10)
\end{aligned}$$

The final expression for $A_{2\pi}$ is given in the Appendix and numerical results are given in Table I. In general, the forward scattering amplitudes D and A can be expanded in the following way,

$$\begin{aligned}
D(E) = & D^{(1)} + (\boldsymbol{\tau}\boldsymbol{\tau}) D^{(2)} + (\boldsymbol{\sigma}\boldsymbol{\sigma}) D^{(3)} + (\boldsymbol{\tau}\boldsymbol{\tau}) (\boldsymbol{\sigma}\boldsymbol{\sigma}) D^{(4)} + (\boldsymbol{\sigma}\boldsymbol{e}) (\boldsymbol{\sigma}\boldsymbol{e}) D^{(5)} \\
& + (\boldsymbol{\tau}\boldsymbol{\tau}) (\boldsymbol{\sigma}\boldsymbol{e}) (\boldsymbol{\sigma}\boldsymbol{e}) D^{(6)}, \quad (2.11)
\end{aligned}$$

and

$$\begin{aligned}
A(E) = & A^{(1)} + (\boldsymbol{\tau}\boldsymbol{\tau}) A^{(2)} + (\boldsymbol{\sigma}\boldsymbol{\sigma}) A^{(3)} + (\boldsymbol{\tau}\boldsymbol{\tau}) (\boldsymbol{\sigma}\boldsymbol{\sigma}) A^{(4)} + (\boldsymbol{\sigma}\boldsymbol{e}) (\boldsymbol{\sigma}\boldsymbol{e}) A^{(5)} \\
& + (\boldsymbol{\tau}\boldsymbol{\tau}) (\boldsymbol{\sigma}\boldsymbol{e}) (\boldsymbol{\sigma}\boldsymbol{e}) A^{(6)}.
\end{aligned}$$

Each coefficient $D^{(i)}(E)$ and $A^{(i)}(E)$ separately satisfies dispersion relations¹⁾

$$D^{(i)}(E) = \frac{P}{\pi} \int_{-\infty}^{\infty} \frac{A^{(i)}(E')}{E' - E} dE' \quad (2.12)$$

$$= \frac{P}{\pi} \int_{m_0}^{\infty} \frac{A^{(i)}(E')}{E' - E} dE' + \frac{f^2}{4\pi} \frac{2\pi}{m} \frac{\lambda_i}{m - \frac{1}{2m} - E} - \frac{32\pi}{m^2} \frac{R^{-1}}{1 - R^{-1}\rho} \frac{\mu_i}{E - E_D} + \frac{1}{\pi} \int_{-\infty}^{m-2/m} \frac{A^{(i)}(E')}{E' - E} dE',$$

where

$$\lambda_1 = -\frac{3}{4}, \quad \lambda_2 = \frac{1}{4}, \quad \lambda_3 = \frac{3}{4}, \quad \lambda_4 = -\frac{1}{4}, \quad \lambda_5 = -\frac{3}{2}, \quad \lambda_6 = \frac{1}{2},$$

$$\mu_1 = \frac{3}{16}, \quad \mu_2 = -\frac{3}{16}, \quad \mu_3 = \frac{1}{16}, \quad \mu_4 = -\frac{1}{16}, \quad \mu_5 = \mu_6 = 0,$$

and R is the radius of the deuteron, $R^{-1} = \sqrt{mB}$, and $\rho (= \rho(-B, -B))$ is the effective range for spin triplet state, and $E_D = m - 2B + B^2/2m$, B being the binding energy of the deuteron.

For $E \geq m$, $A(E)$ can be related by the so-called optical theorem to the total scattering cross section $\sigma_{tot}(E)$ as

$$A(E) = \frac{q}{2E} \sigma_{tot}(E). \quad (2.13)$$

Table I.

A^*	$A_{2\pi}^{(1)}$	$A_{2\pi}^{(2)}$	$A_{2\pi}^{(3)}$	$A_{2\pi}^{(4)}$	$A_{2\pi}^{(5)}$	$A_{2\pi}^{(6)}$
2.0	0.00	0.00	0.00	0.00	0.00	0.00
2.05	-0.18	0.45	-0.10	0.39	0.08	-0.06
2.1	-0.31	0.55	-0.16	0.43	0.15	-0.12
2.2	-0.48	0.72	-0.21	0.49	0.27	-0.24
2.4	-0.70	0.97	-0.27	0.56	0.43	-0.41
2.6	-0.89	1.13	-0.32	0.60	0.57	-0.54
2.8	-1.05	1.25	-0.37	0.61	0.69	-0.64
3.0	-1.21	1.31	-0.42	0.60	0.80	-0.71

* $E = m - A^2/2m$

§ 3. Application to proton-proton scattering

Probably no subtraction will be necessary for the dispersion relation for two-nucleon scattering. In practical applications, however, it is more convenient to make one subtraction to suppress the high energy effect which is less known. In this paper we use the dispersion relation of the type

$$D(E) - D(m) = \frac{E - m}{\pi} P \left[\int_m^{\infty} + \int_{-\infty}^m \right] \frac{A(E') dE'}{(E' - m)(E' - E)} \quad (3.1)$$

to bring in the experimental data. The second integral of (3.1) is the unphysical

contribution, arising from the exchange of pions. In the last section we have calculated one- and two-pion exchange term, the latter starting from $E = E_{2\pi} = m - (2/m) = m - 40$ Mev to the left. For $E < -m$, the absorptive part A is related to the total nucleon-antinucleon cross section. Between $-m$ and $E_{3\pi} = m - (9/2m) = m - 90$ Mev, from which three-pion term $A_{3\pi}$ starts, the absorptive part is very complicated and hard to evaluate. However, since $A(E_{3\pi})$ and $A(-m)$ are nearly equal, an interpolation by a straight line would give a rough estimate for three- and more-pion contribution. This term turned out to be small for low and moderate energies (see Table II).

In this section, the relation is used for analyzing proton-proton scattering data up to 40 Mev. In Eq. (3.1) the low energy S -wave scattering gives large contribution to both sides. In order to eliminate this, we subtract

$$\frac{E-m}{\pi} P \int_m^{\infty} \frac{A_{eff}(E')}{(E'-m)(E'-E)} dE'$$

from both sides, where A_{eff} is the absorptive part calculated from the effective range formula. Scattering length and effective range used here are

$$\begin{aligned} {}^1a(pp) &= -15.6 \times 10^{-13} \text{ cm} \\ {}^1r(pp) &= 2.65 \times 10^{-13} \text{ cm}. \end{aligned} \quad (3.2)$$

Taking unpolarized p - p beam, $D = D^{(1)} + D^{(2)}$, and $A = A^{(1)} + A^{(2)}$. The final formula is

$$\begin{aligned} D(E) &= -\frac{2\pi}{m} \left(a + \frac{a^2}{\sqrt{a^2 - 2ar}} \right) + \frac{\sigma_{eff}(E)}{m} \frac{1 + \frac{ar}{2} q_c^2}{\sqrt{a^2 - 2ar}} \\ &+ \frac{(E-m)}{\pi} P \int_m^{\infty} \frac{q' (\sigma(E') - \sigma_{eff}(E'))}{(E'-E)(E'-m)} dE' \\ &+ \frac{E-m}{\pi} \int_{-\infty}^m \frac{A(E')}{(E'-E)(E'-m)} dE'. \end{aligned} \quad (3.3)$$

The right-hand side is compared with $D(E)$, the real part of forward scattering amplitude,

$$\begin{aligned} D(E) &= \frac{2\pi}{mq_c} (\sin \delta_s \cos \delta_s + \sin \delta_P^{J=0} \cos \delta_P^{J=0} + 3 \sin \delta_P^{J=1} \cos \delta_P^{J=1} \\ &+ 5 \sin \delta_P^{J=2} \cos \delta_P^{J=2} + 5 \sin \delta_D \cos \delta_D + \dots) \\ A(E) &= \frac{2\pi}{mq_c} (\sin^2 \delta_s + \sin^2 \delta_P^{J=0} + 3 \sin^2 \delta_P^{J=1} + 5 \sin^2 \delta_P^{J=2} \\ &+ 5 \sin^2 \delta_D + \dots), \quad E \geq m \end{aligned} \quad (3.4)^*$$

* We neglect the difference between the nuclear bar phase shifts used here and true nuclear phase shifts which we do not know.

The total cross section used are those cited in the review article by Hess⁵⁾ and the values cited in (3.2). Results are given in Table II and plotted in Fig. 3. In this energy region, the experimental data show no disagreement with the theory. It is hoped that the more data on phase shifts, both for p - p and n - p be obtained shortly.

Table II.

E_{lab} (Mev)	D	1st line in (2)	2nd line	3rd line			Sum
				$1-\pi$	2π	3- or more- π	
9.68	0.88	1.09	0.02	-.23	.01	.01	.90
9.73	0.74	1.08	0.02	-.23	.01	.01	.89
14.16	0.80	1.04	0.02	-.28	.02	.01	.81
18.2	1.04	1.02	0.03	-.32	.02	.01	.76
19.8	0.71	1.02	0.03	-.33	.02	.01	.75
31.8	0.38 (0.52)*	1.04	0.05	-.38	.02	.02	.75
39.4	0.83	1.06	0.12	-.40	.03	.02	.83

* Result from the phase shift analysis from which the 14° point is dropped.

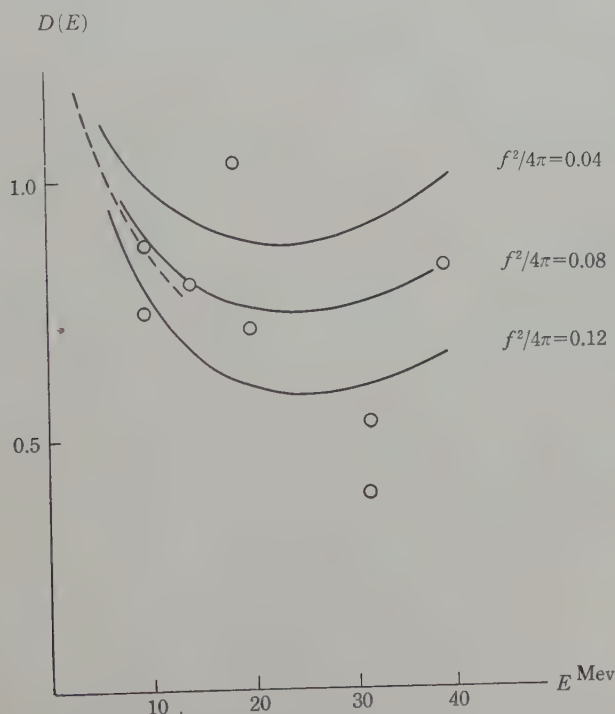


Fig. 3. Real part of the forward p - p scattering amplitude. Solid lines are theoretical curves (Eq. (3.3)) for various choices of $f^2/4\pi$. Circles are experimental points.³⁾ Dashed line is obtained with the effective range approximation.

§ 4. Application to nucleon-nucleon scattering at low energy*

In this section, we apply the dispersion relation to low energy nucleon-nucleon scattering. As proton-proton scattering has ambiguity due to Coulomb interaction, we concern ourselves with neutron-proton scattering in spin triplet, isospin singlet state which is experimentally well known. In this case,

$$D = D^{(1)} - 3D^{(2)} + D^{(3)} - 3D^{(4)} + \frac{1}{3}D^{(5)} - D^{(6)},$$

(4.1)

and

$$A = A^{(1)} - 3A^{(2)} + A^{(3)} - 3A^{(4)} + \frac{1}{3}A^{(5)} - A^{(6)}.$$

The relation to be used is

$$\begin{aligned} \lim_{E \rightarrow m} \frac{D(E) - D(m)}{E - m} &= \lim_{E \rightarrow m} \frac{P}{\pi} \int_m^\infty \frac{A(E') dE'}{(E' - E)(E' - m)} + \frac{1}{\pi} \int_{-\infty}^m \frac{A(E') dE'}{(E' - m)^2} \\ &\quad + \frac{32\pi R^{-1}}{(1 - R^{-1}\rho) m^2 (m - E_D)^2} \\ &= \lim_{E \rightarrow m} \frac{P}{\pi} \int_m^\infty \frac{\frac{q'}{2E'} \sigma(E') dE'}{(E' - E)(E' - m)} + \frac{1}{\pi} \int_{-\infty}^m \frac{A(E') dE'}{(E' - m)^2} \\ &\quad - \frac{f^2}{4\pi} \frac{2\pi}{m} \frac{1}{\left(E - m + \frac{1}{2m}\right)^2} + \frac{32\pi R^{-1}}{(1 - R^{-1}\rho) m^2 (m - E_D)^2}. \end{aligned} \quad (4.2)$$

The last term in the right-hand side comes from deuteron, and the third term comes from one-pion exchange process. The second term comes from two-pion exchange process which we have calculated in § 2, and from three- and more-pion exchange processes which we neglect here as its contribution may be small.

* The problem discussed in this section is the same as the one discussed by S. Matsuyama.⁶⁾ He retained only one-pion exchange neglecting multi-pion processes, while we included also two-pion contribution here. He discussed n - p scattering in spin triplet state, so

$$\begin{aligned} D &= D^{(1)} - D^{(2)} + D^{(3)} - D^{(4)} + \frac{1}{3}D^{(5)} - \frac{1}{3}D^{(6)}, \\ A &= A^{(1)} - A^{(2)} + A^{(3)} - A^{(4)} + \frac{1}{3}A^{(5)} - \frac{1}{3}A^{(6)}, \end{aligned} \quad (4.1)'$$

and the relation used was

$$\begin{aligned} D(E) - D(m) &= (E - m) \frac{P}{\pi} \int_m^\infty \frac{\frac{q'}{2E'} \sigma(E') dE'}{(E' - E)(E' - m)} - \frac{f^2}{4\pi} \frac{4\pi}{3m} \frac{(E - m)}{\left(E' - m + \frac{1}{2m}\right)^2} \\ &\quad + \frac{16\pi R^{-1}(E - m)}{(1 - R^{-1}\rho) m^2 (m - E_D)(E - E_D)}. \end{aligned} \quad (4.2)'$$

The second and third terms are functions of the pion-nucleon coupling constant, f^2 . The left-hand side and the first term in the right-hand side can be expressed in terms of observable quantities,

$$D(E_c) = D_{S\text{-wave}}(E_c) + D_{D\text{-wave}}(E_c) + \dots,$$

$$D_{S\text{-wave}}(E_c) = \frac{8\pi}{mq_c} \sin \delta_s \cos \delta_s,$$

.....,

and

$$A(E_c) = A_{S\text{-wave}}(E_c) + A_{D\text{-wave}}(E_c) + \dots,$$

$$A_{S\text{-wave}}(E_c) = \frac{8\pi}{mq_c} \sin^2 \delta_s,$$

.....,

and

$$q_c \cot \delta_s = -\frac{1}{a_s} + \frac{1}{2} r_s q_c^2 - P r_s^3 q_c^4 + \dots,$$

.....,

where the subscript c means "in the center-of-mass system".

The left-hand side of Eq. (4.2) contains only S -wave scattering length a_s , and S -wave effective range r_s ,

$$\lim_{E \rightarrow m} \frac{D(E) - D(m)}{E - m} = 4\pi \left(a_s^3 - \frac{1}{2} a_s^2 r_s \right).$$

The first term in the right-hand side of Eq. (4.2) contains the contribution from all even partial waves, but we can neglect all contributions except that from low energy S -wave scattering in good approximation,

$$\lim_{E \rightarrow m} \frac{P}{\pi} \int_m^\infty \frac{A(E') dE'}{(E' - m)(E' - E)} \doteq -2\pi a_s^2 \frac{2a_s^2 + |a_s r_s| - 2a_s r_s + 4|a_s r_s| \frac{r_s P_s}{a_s}}{\sqrt{a_s^2 - a_s r_s + |a_s r_s| + 4|a_s r_s| \frac{r_s P_s}{a_s}}}.$$

The last term,

$$\frac{32\pi R^{-1}}{(1 - R^{-1}\rho) m^2 (m - E_D)^2} = \frac{8\pi}{(1 - \gamma\rho) \gamma^3},$$

where

$$\frac{\gamma^2}{m} = B,$$

$$\gamma = 1/a_s + \frac{1}{2} \rho \gamma^2 - P \rho^3 \gamma^4,$$

$$r_s = \rho - 4P \rho^3 / a_s^2.$$

Thus (4.2) is

$$4\pi \left(a_s^3 - \frac{1}{2} a_s^2 r_s \right) + 4\pi a_s^2 \frac{a_s^2 + \frac{1}{2} |a_s r_s| - a_s r_s + 2 |a_s r_s| \frac{r_s P_s}{a_s}}{\sqrt{a_s^2 - a_s r_s + |a_s r_s| + 4 |a_s r_s| \frac{r_s P_s}{a_s}}} - \frac{8\pi}{(1-\gamma\rho)\gamma^3} = -\frac{f^2}{4\pi} \frac{2\pi}{m} \frac{1}{\left(E-m+\frac{1}{2m}\right)^2} + \frac{1}{\pi} \int_{-\infty}^{m-2/m} \frac{A(E')}{(E'-m)^2} dE' \quad (4.3)$$

Inserting the experimental value for a_s and B known at present,

$$a_s = (5.377 \pm 0.023) \times 10^{-13} \text{ cm}$$

$$B = 2.226 \pm 0.004 \text{ Mev}$$

and an assumption,

$$P = 0.00 \pm 0.03,$$

we see the left-hand side of Eq. (4.3) is

$$-17.6 \pm 2.2.^*$$

The right-hand side, the unphysical contribution, is

$$-11.7 - 2.6 = -14.3 \quad \text{for } f^2/4\pi = 0.07$$

$$-13.4 - 3.8 = -17.2 \quad \text{for } f^2/4\pi = 0.08$$

$$-15.1 - 4.8 = -19.9 \quad \text{for } f^2/4\pi = 0.09.$$

The first numbers in the left represent the one-pion exchange contribution, and the second numbers represent the two-pion exchange contribution.

Thus we find as the pion-nucleon coupling constant

$$f^2/4\pi = 0.082 \pm 0.008.$$

§ 5. Conclusion and discussions

The results of the previous sections confirm the correctness of dispersion theoretic approach to the two-nucleon problem. The conventional meson theory of nuclear forces is as follows. One calculates nuclear potential perhaps by adiabatic approximation or including recoil corrections by expanding in p/m . Then he solves the Schrödinger equation with this potential to see if the theoretical phase shifts agree with experiment. This way of approach, however, meets with several difficulties. First, it is difficult to define "potential" without the adiabatic approximation. Second, the Schrödinger equation is non-relativistic and the rela-

* If P is negative, the coupling constant decreases.

tivistic two-body equation is not well established. Finally, we have to introduce cutoff since meson potential is badly singular at the origin.

In our approach all of these difficulties are removed. It is free from the adiabatic approximation and it is possible to formulate the theory in a relativistic way although we did it non-relativistically. It is our feeling that no subtraction is necessary in two-nucleon dispersion relations so that no more additional constant is needed. Taketani⁷⁾ has developed with success a theory in which he divided nuclear forces into inner and outer regions and proposed to deal only with the outer region meson theoretically. Taketani's idea is reproduced here by making one subtraction thereby eliminating less known high energy effects.

In this paper we have been concerned only with the forward scattering. By including the momentum transfer as another variable more information can be obtained from dispersion relations and it may even be possible to use them for solving the two-nucleon problem.⁸⁾

This type of dispersion relations together with similar ones for n - p scattering could be used to determine the accurate value of the pion-nucleon coupling constant if the nucleon-nucleon scattering is measured with sufficient accuracy.

Appendix

Using the identities,

$$(\boldsymbol{\tau}^1 \boldsymbol{\tau}^2)^2 = 3 - 2(\boldsymbol{\tau}^1 \boldsymbol{\tau}^2),$$

and

$$\tau_i^1 \tau_j^1 \tau_j^2 \tau_i^2 = 3 + 2(\boldsymbol{\tau}^1 \boldsymbol{\tau}^2),$$

Eq. (2.8) becomes

$$\begin{aligned} A = & -\frac{1}{16\pi A} \int_{-k}^k dk_3 \\ & \left\{ \left(\frac{A^2}{2} - 1 - k_3^2 \right)^2 [3A(ik_3)C(ik_3) + 6B(ik_3)B(ik_3) \right. \\ & \quad \left. + 12A(ik_3)B(ik_3) + 3A(ik_3)A(ik_3)] \right. \\ & + (\boldsymbol{\tau}^1 \boldsymbol{\tau}^2) \left(\frac{A^2}{2} - 1 - k_3^2 \right)^2 [2A(ik_3)A(ik_3) - 2A(ik_3)C(ik_3)] \\ & + \frac{1}{2} A^2 \left(\frac{A^2}{4} - 1 - k_3^2 \right) ((\boldsymbol{\sigma}^1 \boldsymbol{\sigma}^2) - (\boldsymbol{\sigma}^1 e)(\boldsymbol{\sigma}^2 e)) [3A(ik_3)C(ik_3) \\ & \quad \left. - 3A(ik_3)A(ik_3)] \right. \\ & + \frac{1}{2} A^2 \left(\frac{A^2}{4} - 1 - k_3^2 \right) ((\boldsymbol{\sigma}^1 \boldsymbol{\sigma}^2) - (\boldsymbol{\sigma}^1 e)(\boldsymbol{\sigma}^2 e)) (\boldsymbol{\tau}^1 \boldsymbol{\tau}^2) \\ & \quad \times [-2A(ik_3)A(ik_3) - 4B(ik_3)B(ik_3) \\ & \quad \left. + 8A(ik_3)B(ik_3) - 2A(ik_3)C(ik_3)] \right\} \end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{16\pi A} \int_{-k}^k dk_3 \\
&\quad \left\{ 3 \left(\frac{D^2}{2} - 1 - k_3^2 \right)^2 \left[f^4 \frac{1}{k_3^2 + \varepsilon^2} + f^4 \frac{1}{(\varepsilon - ik_3)^2} \right. \right. \\
&\quad \left. \left. - \frac{16f^2}{9\pi} \frac{\varepsilon}{k_3^2 + \varepsilon^2} \int_0^\infty \frac{\sigma_{33}}{\omega^2 + k_3^2} dp + \frac{32}{81\pi^2} \left(\int_0^\infty \frac{\sigma_{33} dp}{\omega^2 + k_3^2} \right)^2 \right] \right. \\
&\quad + 2(\tau^1 \tau^2) \left(\frac{D^2}{2} - 1 - k_3^2 \right)^2 \left[-f^4 \frac{1}{k_3^2 + \varepsilon^2} + f^4 \frac{1}{(\varepsilon - ik_3)^2} \right. \\
&\quad \left. + \frac{8f^2}{9\pi} \int_0^\infty \frac{\sigma_{33} dp}{\omega(\omega^2 + k_3^2)} - \frac{8k_3^2}{81\pi^2} \left(\int_0^\infty \frac{\sigma_{33} dp}{\omega(\omega^2 + k_3^2)} \right)^2 \right] \\
&\quad + \frac{3}{2} D^2 \left(\frac{D^2}{4} - 1 - k_3^2 \right) ((\sigma^1 \sigma^2) - (\sigma^1 e)(\sigma^2 e)) \\
&\quad \times \left[f^4 \frac{1}{k_3^2 + \varepsilon^2} - f^4 \frac{1}{(\varepsilon - ik_3)^2} \right. \\
&\quad \left. - \frac{8f^2}{9\pi} \int_0^\infty \frac{\sigma_{33} dp}{\omega(\omega^2 + k_3^2)} + \frac{8k_3^2}{81\pi^2} \left(\int_0^\infty \frac{\sigma_{33} dp}{\omega(\omega^2 + k_3^2)} \right)^2 \right] \\
&\quad + D^2 \left(\frac{D^2}{4} - 1 - k_3^2 \right) ((\sigma^1 \sigma^2) - (\sigma^1 e)(\sigma^2 e)) (\tau^1 \tau^2) \\
&\quad \times \left[-\frac{f^4}{(\varepsilon - ik_3)^2} - \frac{f^4}{k_3^2 + \varepsilon^2} + \frac{4f^2}{9\pi} \frac{\varepsilon}{k_3^2 + \varepsilon^2} \int_0^\infty \frac{\sigma_{33}}{\omega^2 + k_3^2} dp \right. \\
&\quad \left. - \frac{2}{81\pi^2} \left(\int_0^\infty \frac{\sigma_{33} dp}{\omega^2 + k_3^2} \right)^2 \right] \Big\} \\
&= I_1(E) + I_2(E) (\tau^1 \tau^2) + I_3(E) ((\sigma^1 \sigma^2) - (\sigma^1 e)(\sigma^2 e)) \\
&\quad + I_4(E) ((\sigma^1 \sigma^2) - (\sigma^1 e)(\sigma^2 e)) (\tau^1 \tau^2) \\
&= I_1 \left(-\frac{1}{4} - \frac{1}{4} (\tau\tau) - \frac{1}{4} (\sigma\sigma) - \frac{1}{4} (\tau\tau) (\sigma\sigma) \right) \\
&\quad + I_2 \left(-\frac{3}{4} + \frac{1}{4} (\tau\tau) - \frac{3}{4} (\sigma\sigma) + \frac{1}{4} (\tau\tau) (\sigma\sigma) \right) \\
&\quad + I_3 \left(-\frac{1}{2} - \frac{1}{2} (\tau\tau) + \frac{1}{2} (\sigma e)(\sigma e) + \frac{1}{2} (\tau\tau) (\sigma e)(\sigma e) \right) \\
&\quad + I_4 \left(-\frac{3}{2} + \frac{1}{2} (\tau\tau) + \frac{3}{2} (\sigma e)(\sigma e) - \frac{1}{2} (\tau\tau) (\sigma e)(\sigma e) \right)
\end{aligned}$$

$$= A_{2\pi}^{(1)}(E) + (\tau\tau) A_{2\pi}^{(2)}(E) + (\sigma\sigma) A_{2\pi}^{(3)}(E) + (\tau\tau)(\sigma\sigma) A_{2\pi}^{(4)}(E) \\ + (\sigma e)(\sigma e) A_{2\pi}^{(5)}(E) + (\tau\tau)(\sigma e)(\sigma e) A_{2\pi}^{(6)}(E).$$

Thus

$$A_{2\pi}^{(1)}(E) = -\frac{1}{4}I_1(E) - \frac{3}{4}I_2(E) - \frac{1}{2}I_3(E) - \frac{3}{2}I_4(E),$$

$$A_{2\pi}^{(2)}(E) = -\frac{1}{4}I_1(E) + \frac{1}{4}I_2(E) - \frac{1}{2}I_3(E) + \frac{1}{2}I_4(E),$$

$$A_{2\pi}^{(3)}(E) = -\frac{1}{4}I_1(E) - \frac{3}{4}I_2(E),$$

$$A_{2\pi}^{(4)}(E) = -\frac{1}{4}I_1(E) + \frac{1}{4}I_2(E),$$

$$A_{2\pi}^{(5)}(E) = \frac{1}{2}I_3(E) + \frac{3}{2}I_4(E),$$

$$A_{2\pi}^{(6)}(E) = \frac{1}{2}I_3(E) - \frac{1}{2}I_4(E),$$

where

$$I_1 = -\frac{3}{16\pi A} \left[f^4 \left\{ 4m(2k^2+1) \tan^{-1} \left(\frac{2mk}{2k^2+1} \right) - \frac{2}{3}k^3 \right. \right. \\ \left. \left. + 4k(2k^2+1) + \frac{2k(2k^2+1)^2}{k^2 + (2k^2+1)^2/4m^2} \right\} \right. \\ \left. + f^2 \left\{ -\frac{32}{9\pi} (2k^2+1)^2 \tan^{-1} \left(\frac{2mk}{2k^2+1} \right) \int_0^\infty \frac{\sigma_{33} dp}{\omega^2} \right\} \right. \\ \left. + \frac{32}{81\pi^2} \int_{-k}^k (2k^2+1-x^2)^2 \left(\int_0^\infty \frac{\sigma_{33}}{\omega^2+x^2} dp \right)^2 dx \right], \\ I_2 = -\frac{1}{8\pi A} \left[f^4 \left\{ -4m(2k^2+1) \tan^{-1} \left(\frac{2mk}{2k^2+1} \right) - \frac{2}{3}k^3 \right. \right. \\ \left. \left. + 4k(2k^2+1) + \frac{2k(2k^2+1)^2}{k^2 + (2k^2+1)^2/4m^2} \right\} \right. \\ \left. + f^2 \left\{ \frac{8}{9\pi} \int_{-k}^k (2k^2+1-x^2)^2 \left(\int_0^\infty \frac{\sigma_{33} dp}{\omega(\omega^2+x^2)} \right) dx \right\} \right. \\ \left. - \frac{8}{81\pi^2} \int_{-k}^k (2k^2+1-x^2)^2 \left(\int_0^\infty \frac{x\sigma_{33} dp}{(\omega^2+x^2)\omega} \right)^2 dx \right],$$

$$\begin{aligned}
I_3 = & -\frac{3}{32\pi A} \left[f^4 \left\{ -16k(k^2+1) + 16m \frac{k^2(k^2+1)}{2k^2+1} \tan^{-1} \left(\frac{2mk}{2k^2+1} \right) \right\} \right. \\
& + f^2 \left\{ -\frac{32}{9\pi} (k^2+1) \int_{-k}^k (k^2-x^2) \left(\int \frac{\sigma_{33} dp}{\omega(\omega^2+x^2)} \right) dx \right\} \\
& \left. + \frac{32}{81\pi^2} (k^2+1) \int_{-k}^k (k^2-x^2) \left(\int_0^\infty \frac{x\sigma_{33} dp}{(\omega^2+x^2)\omega} \right) dx \right], \\
I_4 = & -\frac{1}{16\pi A} \left[f^4 \left\{ -16k(k^2+1) - 16m \frac{k^2(k^2+1)}{2k^2+1} \tan^{-1} \left(\frac{2mk}{2k^2+1} \right) \right\} \right. \\
& + f^2 \left\{ \frac{32}{9\pi} (k^2+1) k^2 \tan^{-1} \left(\frac{2mk}{2k^2+1} \right) \int_0^\infty \frac{\sigma_{33} dp}{\omega^2} \right\} \\
& \left. - \frac{8}{81\pi^2} (k^2+1) \int_{-k}^k (k^2-x^2) \left(\int_0^\infty \frac{\sigma_{33} dp}{\omega^2+x^2} \right)^2 dx \right],
\end{aligned}$$

and

$$k = \sqrt{D^2/4 - 1}.$$

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Note added in proof: Two new sets of phase shift were reported besides those tabulated in Table II. They fit closely to the theoretical curve.

E_{lab}	[Mev]	D	Reference
9.69		0.91	9)
25.63	0.08	0.78 0.02	10)

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Letters to the Editor

The opinions expressed in these columns do not necessarily reflect those of the Board of Editors. Communications should be submitted in duplicate and should be held to within 100 lines (pica type) on standard size letter paper (approx. 21×30 cm.), so that each letter may be arranged into two pages when printed. Do not forget to count in enough space for formulas, figures or tables.

On the Model of Elementary Particles

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December 22, 1959

The present field theories are formulated with the fundamental concept that elementary particles are created or annihilated. But how should we understand the concept that some elementary particles are suddenly created or annihilated? We have no such discontinuous phenomena in various macroscopic reactions. Why should we admit such discontinuous idea of abrupt creation or annihilation only in the reactions of elementary particles? Some people may answer that it is because the quantum mechanics controlling them is based on the principle of discontinuity. But is it not an unrealistic idea to insist upon such a mysterious idea as well as the idea of point-like particles?

From this standpoint it may be allowable to consider the reactions between elementary particles as continuous, disregarding abrupt creation and annihilation of particles. It is easy to think this way at least in our common sense.

According to this viewpoint, if we

examine various decay modes of elementary particles we find that there finally remain only proton, electron, neutrino and gamma, the stable particles. Now we assume that the fundamental particles are p (proton), e (electron), ν (neutrino) and γ (gamma), and all other "elementary" particles are compounded with their finally reaching decay products. But since the sum of masses of these decay products is less than that of the parent particle, we must add some energy term G to these masses as in Table I, where the G term should be determined in such a way that each elementary particle has its own character, i.e., the rest mass, the charge, the spin, the strangeness, etc., and each G term will therefore be different from one another. More concretely speaking, we may take the G term as "geon" derived by Wheeler,¹⁾ for instance, because the geon is constructed with the photon and the neutrino, i.e., the fundamental particles. The geon theory is, as Wheeler discussed, classical; we must therefore modify it in future by taking into account the quantum effects or some new mechanical effects and then the graviton will become one of the fundamental particles. Furthermore, in future we must discuss how the geon can have the strangeness. Then the various reactions between the "elemen-

tary" particles may be explained by the parting and meeting of these constructing particles but not by sudden annihilation and creation of elementary particles.

Table I

"elementary" particles	compound modes
μ^\pm	$e^\pm + \nu + \bar{\nu} + G$
π^0	$2\gamma + G$
π^+	$e^+ + 2\nu + \bar{\nu} + G$
θ^0	$e^+ + e^- + 3\nu + 3\bar{\nu} + G$
τ^+	$2e^+ + e^- + 5\nu + 4\bar{\nu} + G$
n	$p + e^- + \bar{\nu} + G$
Λ^0	$p + e^- + \nu + 2\bar{\nu} + G$
Σ^+	$p + e^- + e^+ + 2\nu + 2\bar{\nu} + G$
Σ^-	$p + 2e^- + \nu + 3\bar{\nu} + G$
Ξ^-	$p + 2e^- + 2\nu + 4\bar{\nu} + G$

Now, we regard the following facts as favourable to our model: (1) From the experiments of electron-proton scattering at Stanford²⁾ the following charge distributions of proton and neutron (Fig. 1) are now considered as the best. The neutron has the negative charge distribution for the outer part of its extension and this can be well explained by considering that n is constructed as in Table I with e rotating around p .

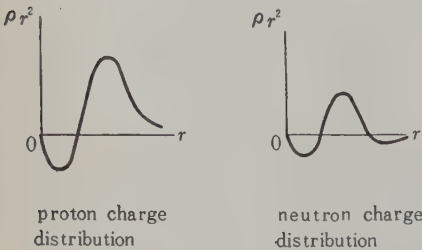


Fig. 1

(2) Sakata³⁾ has very recently proposed the new composite model that Λ , n and p are constructed of (μ^-, B^+) , (e^-, B^+) and (ν, B^+) respectively, in order to explain Gamba-Okubo-Marshak's symmetry⁴⁾—the weak interactions are symmetric with respect to the exchanges $\Lambda \leftrightarrow \mu^-$, $n \leftrightarrow e^-$ and $p \leftrightarrow \nu$,—where B^+ is an unknown boson with positive electric charge. But if we regard B^+ as a composite particle constructed of p , $\bar{\nu}$ and G , then Sakata's new model becomes similar to our model. But then in Sakata's new model the fundamental particles are p , e , ν and μ whereas in our model they are p , e , ν and γ . The mass difference between μ and e is obvious in our model.

We have discussed only the guiding principle and the frame of our model. Next problems are as follows: a) Why do the fundamental particles exist? What are their structures? b) By what mechanism are composite particles constructed of fundamental particles? c) Can this model explain the Nishijima-Gell-Mann theory and various selection rules?

The author would like to discuss these problems in future.

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An Example of Nonlocal Interaction

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January 11, 1960

In the theories of nonlocal interactions which have been discussed by many authors, the lengths expressing the extensions of interactions are assumed to be derived artificially, but first of all we must derive them naturally from the physical viewpoint for the discussions of nonlocal interactions.

From the recent experiments at Stanford¹⁾ the proton has been found to have an electric charge distribution. If we regard the electric charge as dynamical quantity derived by the interaction with the electromagnetic field, as in the other coupling constants, the extension of electric charge will mean that the domain of interaction between the proton and the electromagnetic field is not a point but some three-dimensional volume. Therefore, the magnitudes of charge distributions at some points mean the strengths of interactions between them at those points, and consequently, will connect with the probabilities that the point-like proton exists at those points. In fact, if the charge distribution $e(x)$ is observed, it will be given by the relation

$$e(x) = \rho(x)w(x), \quad (1)$$

where $w(x)$ is the probability of proton's being at x . Since $\rho(x)$ is conjectured as three-dimensional spherical symmetry from the Stanford experiments, Eq. (1) will be

$$e(r) = \rho(r)w(r), \quad (1')$$

where $r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$. If $e(r)$ is observed by any method, we get $w(r)$ from (1') but as the fragment of charge has not yet been observed, such a microscopic quantity as $e(r)$ could not be observed but only its integral, the total charge e . In this case, it is natural to define the observable position of proton as the mean of its positions in the microscopic region where its charge is distributed.

From this point of view, the coordinate of proton is found to be the mean value of the sum of X_μ and $\hat{\xi}_\mu$ with $w(r)$ as a weight function, where X_μ and $\hat{\xi}_\mu$ mean the coordinate of its center of mass and the relative coordinate to its center of mass respectively, and $r = (\hat{\xi}_1^2 + \hat{\xi}_2^2 + \hat{\xi}_3^2)^{1/2}$. Now $\hat{\xi}_0 = 0$ since the charge will have a three-dimensional distribution around its center of mass. In practice, since $w(r)$ is a function of only r , the mean values of $\hat{\xi}_\mu$ ($\mu = 1, 2, 3$) with $w(r)$ as a weight function are zero and the mean values of

$$q_\mu \equiv X_\mu + \hat{\xi}_\mu \quad (\mu = 1, 2, 3, 4), \quad \hat{\xi}_0 = 0 \quad (2)$$

are equal to X_μ . That is, the coordinate of proton are equal to those of its center of mass. But the mean values of q_μ^2 are

$$\overline{q_\mu^2} = X_\mu^2 + 2\overline{X_\mu \hat{\xi}_\mu} + \overline{\hat{\xi}_\mu^2} = X_\mu^2 + \overline{r^2}. \quad (3)$$

If we write

$$S^2 \equiv q_\mu^2, \quad S^2 \equiv X_\mu^2, \quad r^2 \equiv a^2, \quad (4)$$

this is the same as an example of indefinite metric (nonlocal interaction) discussed by Markov.²⁾ Consequently, the charge distributions of elementary particles mean that the singularities of

propagators, which have been regarded to lie on the light cone, lie on the hyperboloid

$$S'^2 \equiv S^2 + a^2 = R^2 - T^2 + a^2 = 0 \quad (5)$$

and that the Coulomb potential is not e/R but

$$e/(R^2 + a^2)^{1/2}, \quad (6)$$

where

$$a = (r^2)^{1/2} \\ = \left\{ \int_0^\infty w(r) r^4 dr / \int_0^\infty w(r) r^2 dr \right\}^{1/2}, \quad (7)$$

which has been derived artificially by Markov, is connected with the charge distribution from Eq. (1').

From above, the proton, in general the elementary particle, is considered to have a three-dimensional spherical inner distribution of position from the microscopic point of view and to have the nonlocal effect shown by Markov from the macroscopic point of view. This seems to be important because the nonlocal extensions of interactions, which have been introduced artificially, will be naturally introduced by the charge distribution and Markov's model will have a practical use.

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Some Considerations on the Parity-Non-Conserving Interactions in the Theory of Propagators

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January 25, 1960

Recently, it is generally believed that the parity is not conserved in the so-called weak interactions such as the Fermi-interaction. Those interactions are described by Lagrangians which are not invariant under the space reflection and the charge conjugation, and especially in cases of beta-decays special elementary particle named two component neutrino.

Though such interactions are weak in the low energy region, observed effects of such interactions may not be considered to be the effects of the lowest order in the sense of the perturbation theory, because those interactions belong to the second kind, accordingly the perturbation expansion might not be possible even as an asymptotic expansion.¹⁾

On the other hand, d'Espagnat and Prentki²⁾ and Sekine³⁾ have shown that the propagators of fields which have parity-nonconserving interactions, involve terms containing γ_5 , and the Z -factors have form of $Z + \gamma_5 Z'$. (In this note we consider only the renormalization factors of propagators and not those of vertex parts.) Interactions treated by them belong to the first kind, and perturbation-theoretical methods are used. However, realistic parity-noncon-

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serving interactions may belong to the second kind, then uses of such methods are not reasonable.

In this note we consider the case of the Fermi-interactions.

The purpose of this note is to show that the coexistence of the *STP*-type interaction and the *VA*-type interaction is not necessary for the disappearance of γ_5 in the propagators, if perturbation theory is avoided.

Now we consider the following interaction :

$$L_{int} = \sum_{i=P, N, e} g_i (\bar{\psi}_i O_i \psi_i) \times (\bar{\psi}_e O_i (a + b\gamma_5) \psi_\nu) + h.c. \quad (1)$$

where g_i , a and b are real constants.

It should be noticed that the total energy-momentum vector is not commutative with the operators of space reflection and charge conjugation.

On the other hand, when a Lagrangian is given, we can easily obtain the equations for the propagators by using the method of Schwinger.⁴⁾ When the perturbation theory is avoided, the vertex parts

$$\Gamma_{\alpha\beta, \gamma\delta} = G_\alpha^{-1} G_\beta^{-1} \langle 0 | T \psi_\alpha \psi_\beta \bar{\psi}_\gamma \bar{\psi}_\delta | 0 \rangle \times G_\gamma^{-1} G_\delta^{-1} \quad (2)$$

(where $\alpha, \beta, \gamma, \delta = P, N, e, \nu$) may be chosen as a boundary condition for the functional differential equations for the propagators.⁵⁾ Because of the noncommutativity of the total energy-momentum vector with the space reflection and the charge conjugation, the ratio between the scalar and the pseudoscalar parts is not determined by the form of the interaction Lagrangian and may depend on the momenta of participant particles.

In order to explain the experimental data on the asymmetry and the polarization of the electrons emitted in the processes of beta-decay, it is not necessary to assume the two component neutrino, but it is sufficient to choose the boundary condition $\Gamma_{P\bar{e}, N\nu}$ so that it satisfies the following condition :

$$\begin{aligned} & \Gamma_{P\bar{e}, N\nu}(\not{p}_P, \not{p}_e; \not{p}_N, \not{p}_\nu) \Big|_{p_e^2=m_e^2, p_\nu^2=0} \\ &= [(1+\gamma_5)\gamma_\mu]_{lepton} \\ & \times [F_\mu(\not{p}_P, \not{p}_e; \not{p}_N, \not{p}_\nu) \\ & + \gamma_5 F'_\mu(\not{p}_P, \not{p}_e; \not{p}_N, \not{p}_\nu)]_{nucleon} \quad (3) \end{aligned}$$

where F_μ and F'_μ are vector functions free from Dirac matrices referring to the leptons.

The equations for one-body propagators can be written as follows (in momentum representation) :

$$(i\gamma_\mu \not{p}_\mu + m_\alpha + \Sigma_\alpha^*(p)) G_\alpha(p) = 1, \quad (\alpha = P, N, e, \nu) \quad (4)$$

$$\begin{aligned} \Sigma_P^*(p) &= \sum_i g_i \int dk dq O_i G_N(k) \\ & \times G_e(k+q-p) O_i (a+b\gamma_5) \\ & \times G_\nu(q) \Gamma_{N\nu, le}(k, q; p, k+q-p), \end{aligned} \quad (5a)$$

$$\begin{aligned} \Sigma_N^*(p) &= \sum_i g_i \int dk dq O_i G_P(k) \\ & \times G_\nu(k+q-p) (a-b\gamma_5) O_i \\ & \times G_e(q) \Gamma_{Pe, N\nu}(k, q; p, k+q-p), \end{aligned} \quad (5b)$$

$$\begin{aligned} \Sigma_e^*(p) &= \sum_i g_i \int dk dq O_i (a+b\gamma_5) G_\nu(k) \\ & \times G_P(k+q-p) \\ & \times O_i G_N(q) \Gamma_{N\nu, Fe}(q, k; k+q-p, p), \end{aligned} \quad (5c)$$

$$\begin{aligned} \Sigma_\nu^*(p) &= \sum_i g_i \int dk dq (a-b\gamma_5) O_i G_e(k) \\ & \times G_N(k+q-p) \end{aligned}$$

$$\times O_i G_P(q) \Gamma_{re, N\nu}(q, k; k+q-p, p). \quad (5d)$$

In order that Σ_α^{*} 's have forms

$$\Sigma_\alpha^{*}(p) = i\gamma p \Sigma_\alpha^{*(1)}(p^2) + \Sigma_\alpha^{*(2)}(p^2), \quad (\alpha = P, N, e, \nu) \quad (6)$$

$$\Sigma_\nu^{*(2)}(p^2) = 0^*, \quad (7)$$

it is not necessary that b vanishes, but $b \neq \pm a$ is excluded. If $a = \pm b$, Σ_α^{*} and Σ_ν^{*} cannot have the form (6). Analogously we see that the equality is not necessary for the validity of (3). Then the requirement (3), (6) and (7) may be compatible in the general framework of the theory of propagators. (If $a = \pm b$, Γ must always involve the projection operator $1 \pm \gamma_5$ independently of the momenta of participant particles; accordingly (3) would be replaced by a more severe one.)

If the requirement (3) is replaced by a more severe one, for example

$$\begin{aligned} & \Gamma_{Fe, N\nu}(p_P, p_e; p_N, p_\nu)_{p_e^2 = m_e^2} \\ &= [(1 + \gamma_5) \gamma_\mu]_{lepton} [F_\mu(p_P, p_e; p_N, p_\nu) \\ &+ \gamma_5 F'_\mu(p_P, p_e; p_N, p_\nu)]_{nucleon} \quad (8) \end{aligned}$$

(6) can never take place, even if $a \neq \pm b$.

When Z -factors do not involve γ_5 , the particle picture of each field is not distorted, and the anticommutation relations of field operators at equal time are

$$[\psi_\alpha(\mathbf{x}, t), \bar{\psi}_\alpha(\mathbf{x}', t)]_+ = Z_\alpha^{-1} \gamma_4 \delta(\mathbf{x} - \mathbf{x}'). \quad (9)$$

The circumstances are analogous in the cases of boson-fermion interactions belonging to the second kind. Then it is an interesting problem to investigate the so-called τ - θ puzzle from the above standpoint.

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On the Universality of the Weak Interactions

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and Tetsuro Sakuma

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February 6, 1960

One of the most striking features of the weak decay interactions consists in their universality. By assuming that all decay interactions are of the second kind, Umezawa, Konuma and Nakagawa¹⁾ have shown that all relevant coupling constants are related to the length of approximately the same order of magnitude: $10^{-17} \sim 10^{-20}$ cm.

Universal interactions have, in general, many outstanding properties. The most familiar classical examples of the universal interaction are electromagnetic and gravitational interactions. In the case of electromagnetic interaction, Lorentz' law of the electromagnetic force $\mathbf{f} = \rho \mathbf{E} + 1/c \cdot \mathbf{j} \times \mathbf{H}$ is independent of the detailed properties of the matters and fields which produce the current vector $\mathbf{J}_\mu \equiv (\mathbf{j}, \rho)$. On account of this universality, it is guaranteed to introduce the electromagnetic interactions correctly

into quantum mechanical systems by the well-known formal procedure of replacing ∂_μ by $\partial_\mu - ie/\hbar c \cdot A_\mu$.

In this short note, we shall show that the approximate universality of the weak decay interactions also allows us to introduce correctly the weak interactions by a formal procedure similar to the substitution $\partial_\mu \rightarrow \partial_\mu - ie/\hbar c \cdot A_\mu$ in the electromagnetic case.

When all weak interactions are switched off, the total system of elementary particles is described by Lagrangian:

$$\mathcal{L} = \mathcal{L}_B^0 + \mathcal{L}_M^0 + \mathcal{L}_L^0 + \mathcal{L}_{BM}(0), \quad (1)$$

where, \mathcal{L}_B^0 , \mathcal{L}_M^0 and \mathcal{L}_L^0 are Lagrangians describing free baryons, bosons and leptons, respectively, and $\mathcal{L}_{BM}(0)$ is the Lagrangian describing the strong interactions of baryons and bosons:

$$-\mathcal{L}_{BM}(0) = ig' \sum_{\alpha=\tau,0,-} J_\alpha^* \pi_\alpha + ig' \sum_{s=+,0} \mathcal{J}_s^* K_s + h.c., \quad (2)$$

$$\left. \begin{aligned} J_\alpha &\equiv (\bar{N} \gamma_5 \tau_\alpha N) + (\bar{Y} \gamma_5 \tau_\alpha Y) + \dots, \\ \mathcal{J}_+ &\equiv (\bar{A} \gamma_5 p) + \dots, \text{ etc.} \end{aligned} \right\}. \quad (3)$$

Now, if we assume that the baryons have an internal structure of the order $r_0 \sim 10^{-20}$ cm, then the multipole interactions of the baryons and bosons should be taken into account, as mentioned in the previous papers.²⁾ Thus, the interaction (2) must be supplemented by these multipole terms, and (2) turns out to be

$$\begin{aligned} -\mathcal{L}_{BM}(r_0) &= ig' \sum_{\alpha} J_\alpha^* \pi_\alpha \\ &+ ig' r_0 \sum_{\alpha} \mathcal{J}_\mu^{\alpha*} \partial_\mu \pi_\alpha + \dots \\ &+ ig' \sum_s \mathcal{J}_s^* K_s \\ &+ ig' r_0 \sum_s J_\mu^{s*} \partial_\mu K_s + \dots \\ &+ h.c., \end{aligned} \quad (4)$$

where

$$\begin{aligned} \mathcal{J}_\mu^+ &\equiv (\bar{A} \gamma_\mu (1 + \gamma_5) p) + \dots, \\ J_\mu^+ &\equiv (\bar{n} \gamma_\mu (1 + \gamma_5) p) + \dots, \text{ etc.} \end{aligned} \quad (5)$$

It is just at this stage to introduce the universal lepton interactions in close analogy to the electromagnetic case. We propose that the lepton interaction may be correctly switched on by the following substitutions:

$$\left. \begin{aligned} \pi^+ &\rightarrow \pi^+; \quad \partial_\mu \pi^+ \rightarrow \partial_\mu \pi^+ \\ &\quad + l[(\bar{e} + \bar{\mu}) \gamma_\mu (1 + \gamma_5) \nu], \\ \pi^0 &\rightarrow \pi^0; \quad \partial_\mu \pi^0 \rightarrow \partial_\mu \pi^0, \\ \pi^- &\rightarrow \pi^-; \quad \partial_\mu \pi^- \rightarrow \partial_\mu \pi^- \\ &\quad - l[\bar{\nu} \gamma_\mu (1 + \gamma_5) (e + \mu)], \end{aligned} \right\} \quad (6)$$

and similar substitutions for K mesons, where the parameter l should have a dimension of length.

Applying this substitution to (4), we have

$$\begin{aligned} -\mathcal{L}_{BM}(r_0) &\rightarrow ig' \sum_{\alpha} J_\alpha^* \pi_\alpha \\ &+ ig' r_0 \sum_{\alpha} \mathcal{J}_\mu^{\alpha*} \partial_\mu \pi_\alpha + \dots \\ &+ ig' \sum_s \mathcal{J}_s^* K_s \\ &+ ig' r_0 \sum_s J_\mu^{s*} \partial_\mu K_s + \dots \\ &+ ig' r_0 l (\mathcal{J}_\mu^{+*} [(\bar{e} + \bar{\mu}) \gamma_\mu (1 + \gamma_5) \nu] \\ &\quad + \mathcal{J}_\mu^{-*} [\bar{\nu} \gamma_\mu (1 + \gamma_5) (e + \mu)]) \\ &+ ig' r_0 l (J_\mu^{+*} [(\bar{e} + \bar{\mu}) \gamma_\mu (1 + \gamma_5) \nu] \\ &\quad + J_\mu^{-*} [\bar{\nu} \gamma_\mu (1 + \gamma_5) (e + \mu)]) \\ &+ h.c. \end{aligned} \quad (7)$$

The additional terms introduced by these substitutions are enough to describe the leptonic decays of hyperons and the β -decays. Just as in the electromagnetic case, the substitution (6) also affects the Lagrangian \mathcal{L}_M^0 of free bosons. For

example, pi-meson part \mathcal{L}_π^0 of the \mathcal{L}_M^0 turns out to be

$$\begin{aligned}
 -\mathcal{L}_\pi^0 \rightarrow & \sum_{\alpha=1}^3 (\partial_\mu \pi_\alpha \cdot \partial_\mu \pi_\alpha + l^2 \pi_\alpha \pi_\alpha) \\
 & + l(\partial_\mu \pi^* [(\bar{e} + \bar{\mu}) \gamma_\mu (1 + \gamma_5) \nu] \\
 & - \partial_\mu \pi [\bar{\nu} \gamma_\mu (1 + \gamma_5) (e + \mu)]) \\
 & + l^2 [(\bar{e} + \bar{\mu}) \gamma_\mu (1 + \gamma_5) \nu]^* \\
 & \times [(\bar{e} + \bar{\mu}) \gamma_\mu (1 + \gamma_5) \nu]. \quad (8)
 \end{aligned}$$

Among the resulting terms, the terms proportional to l describe the leptonic decays of pi-meson, and the term proportional to l^2 contains the interaction responsible for the μ - e decay. The same procedure applied to K -meson part \mathcal{L}_K^0 is enough to derive the interactions responsible for the leptonic decays of K -meson. In order to guarantee the well-known equality of the μ - and β -decay coupling constants, we should prefer the value of the parameter l to be the same order of magnitude as the dipole moment $g'r_0$, i.e.

$$l \sim g'r_0 \sim (10^{-17} \sim 10^{-20}) \text{ cm}. \quad (9)$$

In this way, almost all the known weak interactions are derived qualitatively by the substitution (6), whereby any unwanted interactions do not come out. Unfortunately, the quantitative agreement of our universal interaction with experiment is rather poor. This point will be discussed by K. Iwata.

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Macroscopic Causality and Analyticity of Electromagnetic Form Factor

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February 20, 1960

Recently the present author has investigated the analytic properties of the scattering amplitude on the basis of the macroscopic causality. In the present note we shall apply the previous method to electron-nucleon scattering and discuss the analytic property of the electromagnetic form factor of the nucleon in the relativistic theory. In order to avoid unnecessary complications in the following discussions, we shall neglect the dependence on the spin and iso-spin.

If the electron has an unknown structure or one photon exchange does not give a good approximation, we cannot directly obtain information on the electromagnetic structure of the nucleon from the electron-nucleon scattering. Therefore, we shall assume that the electron is a structureless point particle and one photon exchange gives a good approximation.

From the above assumptions, we can easily see that the scattering amplitude of electron-nucleon scattering is written as

$$t = F(\mathcal{A}^2) \cdot (\mathcal{A}^2)^{-1}, \quad (1)$$

where \mathcal{A}^2 is the square of the invariant momentum transfer. $(\mathcal{A}^2)^{-1}$ and $F(\mathcal{A}^2)$ are the photon propagator and the electromagnetic form factor of the nucleon respectively. The analyticity of

the latter is the main object of the present note. It should be noted that the scattering amplitude of electron-nucleon scattering has no dependence on the square of the total energy momentum W . In the case where the scattering amplitude has no dependence on W , it is concluded that, as will be discussed later, from the requirement for the macroscopic causality, the scattering amplitude is regular in the upper half plane of Δ^2 except for the particular case. Then we can say that from (1) the electromagnetic form factor is regular in the upper half plane of Δ^2 .

In order to explain in a more detail, let us introduce the Fourier transform defined by

$$t(\Delta^2) = \int_{-\infty}^{\infty} d\alpha \exp i\alpha \Delta^2 T(\alpha). \quad (2)$$

Although $T(\alpha)$ is not uniquely determined as we have seen in the previous paper, we do not repeat the discussion on this problem. The restriction on $\mathbf{p}, \mathbf{p}', \mathbf{R}, \mathbf{r}, \mathbf{R}', \mathbf{r}'$ for getting a non-vanishing transition amplitude (3.1) in the previous paper are

$$\mathbf{R}' - \mathbf{R} - V_A(\mathbf{p}') t \approx [V_A(\mathbf{p}) - V_A(\mathbf{p}')] T, \quad (3)$$

$$\begin{aligned} \mathbf{R}' - \mathbf{r}' - \mathbf{R} + 2(\mathbf{p} - \mathbf{p}')\alpha - V_B(-\mathbf{p}')t \\ \approx [V_A(\mathbf{p}) - V_B(-\mathbf{p}')] T, \end{aligned} \quad (4)$$

and

$$\mathbf{r} + 2(\mathbf{p} - \mathbf{p}')\alpha \approx [V_A(\mathbf{p}) - V_B(-\mathbf{p})] T. \quad (5)$$

From (3) and (4), we get

$$\begin{aligned} \mathbf{r}' \approx [V_A(\mathbf{p}') - V_B(-\mathbf{p}')] (t - T) \\ + 2(\mathbf{p} - \mathbf{p}')\alpha. \end{aligned} \quad (6)$$

In the limit of $|\mathbf{p} - \mathbf{p}'| \rightarrow \infty$ we obtain

$$\mathbf{r}' \approx -\mathbf{r} \approx 2(\mathbf{p} - \mathbf{p}')\alpha. \quad (7)$$

Consider the case where $\mathbf{p} \approx -\mathbf{p}'$. In this case if α is negative (7) shows the direction and the sense of \mathbf{r} and \mathbf{r}' approximately agree with those of \mathbf{p} and \mathbf{p}' respectively. Then there hold the discussions similar to the ones in section IV in the previous paper, and we reach the conclusion that, in order to satisfy the macroscopic causality, it should be able to make $T(\alpha)$ zero for $\alpha < 0$ by a suitable choice of the ambiguity of $T(\alpha)$ except for the particular case as discussed in the end of section IV of reference 1). Accordingly, if the scattering amplitude has no dependence on W , the scattering amplitude is regular in the upper half plane as a function of Δ^2 .

The argument so far stated can be easily extended to the case where the scattering amplitude t has a W -dependence but the variation of W can be regarded to be so small that Eqs. (3) ~ (5) remain unchanged.

The author would like to express his sincere thanks to Prof. R. Utiyama for his continual encouragement and discussions.

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Possible Interconnection between Nucleon-Structure and Multiple Production in Cosmic Ray Energy Regions

Hideaki Nagai and Daisuke Itô

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March 7, 1960

The characteristic features of the multiple production of particles in cosmic ray energy regions have recently been successfully described by the so-called "two center model" proposed by Niu¹⁾ and others.²⁾ In this model, particles are produced instantaneously in two small regions around the colliding nucleons, and then, they fly away isotropically in the rest systems of each "fire balls".

Previously, one of the present authors³⁾ has also shown that the main features of the phenomena are also well described by "shaking off" proper fields around the nucleons. In this model, the extensions of the proper fields play the same rôle as the fire balls in Niu's model. Therefore, it seems to us highly probable that Niu's fire ball may have some intimate connection with the structure of nucleon.

In this short note, we estimate the spatial extensions of the fire balls by analyzing the distributions of transverse momenta of secondary particles. If we assume a spherical distribution of the momenta $P(k)$ of secondary particles in the rest systems of each fire balls, then the distribution of the transverse momenta is given by

$$P(k_{\perp}) dk_{\perp} = \frac{k_{\perp} \int_{-\infty}^{+\infty} dk_{\parallel} P(\sqrt{k_{\parallel}^2 + k_{\perp}^2 + \mu^2})}{\int_0^{\infty} k_{\perp} dk_{\perp} \int_{-\infty}^{+\infty} dk_{\parallel} P(\sqrt{k_{\parallel}^2 + k_{\perp}^2 + \mu^2})} dk_{\perp}. \quad (1)$$

On the other hand, the Fourier transform

$$\rho(\mathbf{r}) = \int P(k) e^{i\mathbf{k} \cdot \mathbf{r}} d\mathbf{k} \quad (2)$$

of $P(k)$ may be regarded as the spatial distribution of the particle-source (fire ball) in its rest system. For simplicity, now let us assume Gaussian distribution

$$\rho(r) = \rho(0) \exp\left[-\left(\frac{r}{a}\right)^2\right]$$

$$\text{or } P(k) = P(0) \exp\left[-\frac{k^2 a^2}{4}\right], \quad (3)$$

then the distribution of transverse momenta $P(k_{\perp})$ is readily calculated as

$$P(k_{\perp}) = 0.5 a^2 k_{\perp} \exp\left[-\frac{a^2}{4} k_{\perp}^2\right]. \quad (4)$$

As is shown in Fig 1, this distribution reproduces fairly well the results observed by Japanese E. C. C. group⁴⁾ and Bristol group,⁵⁾ if we choose the value of a as

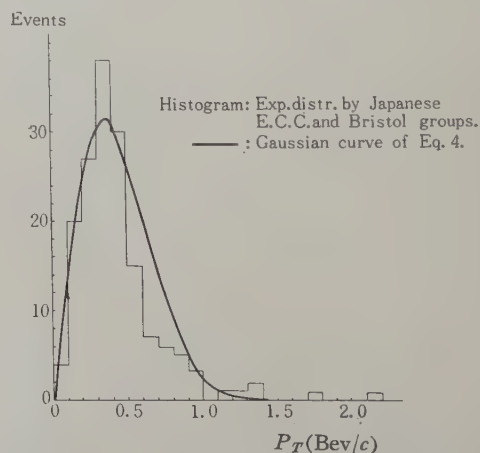


Fig. 1. The distribution of transverse momenta

$$\frac{2\hbar c}{a} \approx 0.5 \text{ BeV, i. e. } a \approx 0.75 \times 10^{-13} \text{ cm.} \quad (5)$$

The root mean square radius $\langle r^2 \rangle^{1/2}$ is readily obtained by expanding $P(k)$ in powers of k^2 :

$$P(k) = P(0) \left[1 - \frac{k^2}{6} \langle r^2 \rangle + \dots \right]. \quad (6)$$

For our Gaussian distribution, we have

$$\langle r^2 \rangle^{1/2} = \sqrt{\frac{3}{2}} a \approx 0.92 \times 10^{-13} \text{ cm} \quad (7)$$

which is very similar to the mean square radius of nucleon-core measured by the electron scattering experiments. From the above result, we conjecture that Niu's fire ball may have some intimate connections with the nucleon-

core structure.

We express our cordial thanks to Professor Y. Ôno and to Professor I. Miura of the Research Institute for Nuclear Study for their kind interest. After completion of this work, we were informed of a similar result obtained by Prof. S. Takagi and his collaborators. We are grateful for their sending a preprint.

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Errata

Cluster Sums and Related Coefficients of the Ising

Model

Shigetoshi KATSURA

Prog. Theor. Phys. **20** (1958), 192

"Non-crossing chains" written in Abstract in page 192, line 11 in page 193, and line 13 in page 199 should be read "two-end chains which have no junctions with odd branches".

Errata

Ground State of a System Consisting of Two Oppositely Charged Particles in Coulomb Field

Mitio INOKUTI, Kanji KATSUURA and Hiroshi MIMURA

Prog. Theor. Phys. **23** (1960), 186-187,

p. 187, r., lines 16-17 $\zeta=2.19765$, $\eta=0.728095$ should read:
 $\zeta=0.728095$, $\eta=2.19765$

line 23 attempted should read: attempted

line 34 Bd. 35 should read: Bd. 34.

line 38 *Phys. Rev.* **99** (1955) 653. should read: *Phys. Rev.* **99** (1955) 635.

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Collective Excitations of Electrons in Degenerate Bands. I*—Spin Waves in Stoner's Model of Ferromagnetism—*

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Research Institute for Fundamental Physics, Kyoto University, Kyoto

(Received October 21, 1959)

Spin waves in the collective electron model of ferromagnetism are derived in a completely similar manner to that adopted in deriving exciton waves in insulators. The internal motion of an electron-hole pair forming the spin wave with a long wavelength is shown to be localized in the ordinary space. The frequency of the spin wave with long wavelength coincides with the result obtained by K. Yosida and T. Kasuya in the case where all electron spins are pointed toward the same direction in the ground state. It is concluded generally that spin waves break down unless there is a sufficiently large difference between the number of the electrons with up spin and the number of those with down spin.

In the many-electron theory developed by Tomonaga,¹⁾ Bohm-Pines,²⁾ and Sawada³⁾ one has neglected the fact that the energy bands of electrons are degenerate. In this case one obtains only the oscillating states of sound waves as the bound states of an electron-hole pair. In this and forthcoming papers we will point out possibilities of obtaining other kinds of collective oscillations in the many-electron system for which the degeneracies of the energy bands of electrons are important.

In this first paper we consider the spin-degeneracy which leads to the possibility of obtaining spin waves. Although the concept of spin waves has already been well established,⁴⁾⁻⁸⁾ our treatment will be instructive for our later development of introducing new collective waves. Further, it will clearly show in a band theoretical language that the cause of the spin waves is identical with that of excitons in insulators. Here we adopt Stoner's model⁹⁾ of ferromagnetism in which the energy band of the α -electrons (i.e. electrons with α spin) is taken to be different from that of the β -electrons because of the exchange interaction between electrons. This idea may be formulated mathematically as below.

We start from the Hamiltonian

$$H = \sum_{\mathbf{k}} \epsilon(\mathbf{k}) n_{\mathbf{k}\sigma} + \frac{1}{2} \sum_{\mathbf{k} \neq 0} J(\mathbf{k}) \rho_{\mathbf{k}} \rho_{-\mathbf{k}}, \quad (1)$$

where $n_{\mathbf{k}\sigma}$ and $\rho_{\mathbf{k}}$ are given respectively by

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and

$$n_{k\sigma} = a_{k\sigma}^* a_{k\sigma}$$

$$\rho_k = \sum_l^\sigma a_{l+k\sigma}^* a_{l\sigma},$$

$a_{k\sigma}$ being the destruction operator for the electron with wave vector \mathbf{k} and spin σ . In the above, $J(\mathbf{k})$ is the Fourier coefficient of the interaction potential between two electrons. In the interaction part of Hamiltonian (1), we turn our attention to the term

$$\frac{1}{2} \sum_{\mathbf{k} \neq 0} \sum_l^\sigma J(\mathbf{k}) \cdot a_{l+k\sigma}^* a_{l\sigma}^* a_{l\sigma} a_{l+k\sigma}.$$

This may be written as

$$\frac{N}{2} \sum_{\mathbf{k} \neq 0} J(\mathbf{k}) - \frac{1}{2} \sum_{\mathbf{k} \neq 0} \sum_l^\sigma J(\mathbf{k}) \cdot n_{l+k\sigma} n_{l\sigma}. \quad (2)$$

If we evaluate $\langle \Phi | H | \Phi \rangle$ where

$$\Phi = \prod_{|\mathbf{k}| \leq k_{F\alpha}} a_{k\alpha}^* \prod_{|\mathbf{k}| \leq k_{F\beta}} a_{k\beta}^* \Phi_0, \quad (3)$$

Φ_0 being the state vector of the real vacuum, the interaction terms which can give contributions are limited to those given by (2). In the above,

$$\frac{V}{(2\pi)^3} \sum_\sigma \frac{4\pi}{3} k_{F\sigma}^3 = N, \quad (4)$$

N being the total number of electrons. We will determine $k_{F\sigma}$ by the following requirements: We minimize $\langle \Phi | H | \Phi \rangle$ in the framework of the trial function (3) in which $k_{F\sigma}$ is the variational parameter restricted by condition (4). Here we consider the case $k_{F\alpha} < k_{F\beta}$ which is illustrated by Fig. 1.

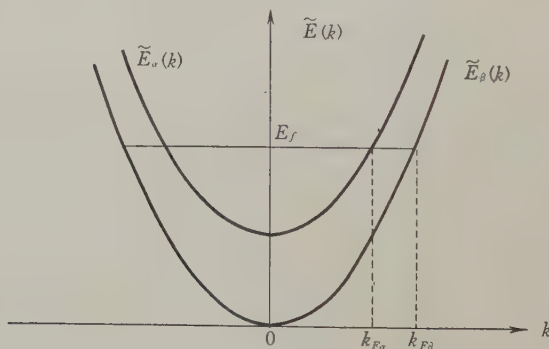


Fig. 1.

Thus we give up fulfilling the requirement that Φ corresponding to the ferromagnetic ground state should satisfy the exact spin multiplicity.

It might be impossible to obtain such a ferromagnetic ground state as given above in the single band model adopted here, if we take into account the correlation

effect or the screening effect of our electron assembly. Indeed, several authors^{10),11)} assert that this screening effect is strong enough to prevent the appearance of the ferromagnetism. Although we cannot accept their assertion without doubt because they have treated the correlation effect under certain approximations leading to a stronger screening than the actual one, the disappearance of the ferromagnetism in the single band model might be inevitable because of the following reason. (1) The loss of the kinetic energy in converting the paramagnetic configuration to the ferromagnetic one. (2) Between two electrons there is the short-range correlation in Brueckner-Bethe's sense which weakens the effect of the Pauli principle. (3) The loss of the long-range correlation energy due to the decrease of the screening effect: In the ferromagnetic configuration the screening effect is weaker than that in the paramagnetic one, since in the former the polarizability of the electron medium is weaker than the latter. Further, it is generally accepted that the electron assembly will form a lattice when r_s is sufficiently large. In this case also the Stoner Model will break down. We will not enter into these problems in more detail, because in this paper we consider a fictitious system for which the Stoner Model prevails, assuming an appropriate $E(\mathbf{k})$ and an appropriate $J(\mathbf{k})$.

Now, introducing the notation

$$n_{\mathbf{k}\sigma}^0 \begin{cases} = 1 & \text{if } k \leq k_{F\sigma} \\ = 0 & \text{if } k > k_{F\sigma} \end{cases}$$

and neglecting a constant, we rewrite (2) as follows:

$$\begin{aligned} & -\frac{1}{2} \sum_{\mathbf{k} \neq 0} \sum_l^\sigma J(\mathbf{k}) n_{l+\mathbf{k}\sigma} n_{l\sigma} \\ & = -\sum_{\mathbf{k} \neq 0} \sum_l^\sigma J(\mathbf{k}) n_{l+\mathbf{k}\sigma}^0 n_{l\sigma} \\ & + \frac{1}{2} \sum_{\mathbf{k} \neq 0} \sum_l^\sigma J(\mathbf{k}) n_{l+\mathbf{k}\sigma}^0 n_{l\sigma}^0 \\ & - \frac{1}{2} \sum_{\mathbf{k} \neq 0} \sum_l^\sigma J(\mathbf{k}) \cdot (n_{l+\mathbf{k}\sigma} - n_{l+\mathbf{k}\sigma}^0) (n_{l\sigma} - n_{l\sigma}^0). \end{aligned}$$

The second term of the right side of the above equation is merely a constant and is omitted in our discussions because we are now concerning with excitation energies. The last term may be neglected because we here consider the excitations of one-pair states. Then we may write Hamiltonian (1) as follows:

$$\left. \begin{aligned} H &= H_0 + H', \\ H_0 &= \sum_{\mathbf{k}} \widetilde{E}_\sigma(\mathbf{k}) \cdot n_{\mathbf{k}\sigma}, \\ H' &= (1/2) \sum_{\mathbf{k} \neq 0} \sum_{(l,\sigma) \neq (l'-\mathbf{k},\sigma')} \sum_{\sigma\sigma'} J(\mathbf{k}) \\ & \quad \times a_{l+\mathbf{k}\sigma}^* a_{l\sigma} a_{l'-\mathbf{k}\sigma'}^* a_{l'\sigma'}, \end{aligned} \right\} \quad (5)$$

where

$$\tilde{E}_\sigma(\mathbf{k}) \equiv E(\mathbf{k}) - \sum_l J(l) n_{\mathbf{k}+\mathbf{l}\sigma}^0. \quad (6)$$

If we neglect H' , we obtain the continuum of excitation energies of one-pair states as shown in Fig. 2, Fig. 3 and Fig. 4. The one-pair states may be classified into three groups. One of them is composed of the one-pair states given by α -electrons and α -holes and those given by β -electrons and β -holes (cf. Fig. 2). The second group is composed of the one-pair states given by β -electrons and α -holes (cf. Fig. 3). In this group one can never find the pair states whose wave vectors are smaller than $(k_{\beta\beta} - k_{\beta\alpha})$ in absolute values, as shown in Fig. 3. The third group is composed of the pair states given by α -electrons and β -holes (cf. Fig. 4). Note that these groups are completely separated in determining the one-pair states by means of New Tamm-Dancoff's Approximation. In order to study spin waves with long wavelengths, we here consider only the third group, because the first group gives merely plasma oscillations as the bound states of the electron-hole pair and the second group can never give any long wavelength oscillations.

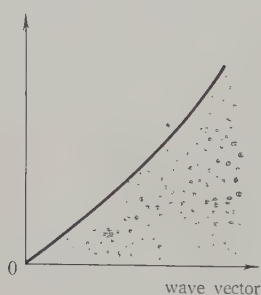


Fig. 2.
Excitation energies of the one-pair states composed of σ -electrons and σ -holes.

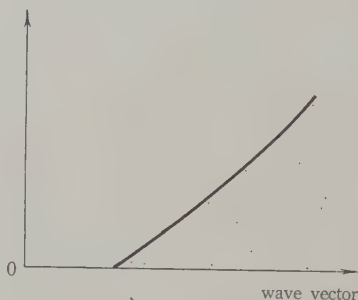


Fig. 3.
Excitation energies of the one-pair states composed of β -electrons and α -holes.

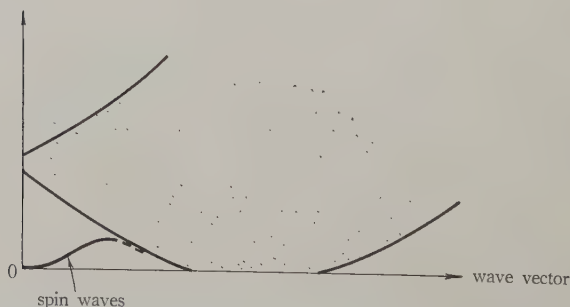


Fig. 4. 1. The bottom of α -band is lower than the Fermi level. Excitation energies of the one-pair states composed of α -electrons and β -holes.

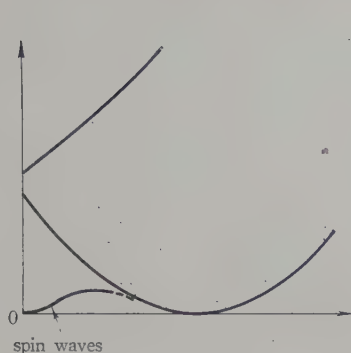


Fig. 4. 2. The bottom of α -band coincides with the Fermi energy level.

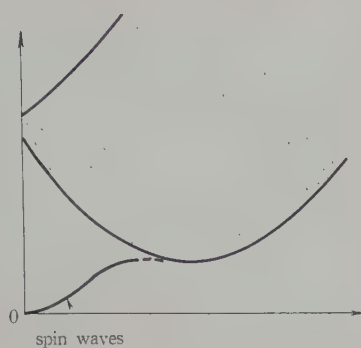


Fig. 4. 3. The bottom of α -band is higher than the Fermi level.

If we neglect H' , we get energy gaps in the excitation energies of the one-pair states belonging to the third group as shown in Fig. 4. However, if we take H' into account, we obtain bound states of these pairs just as in the case of exciton states in insulators because of the attractive interactions between the electrons and the holes. Hereafter we will study these bound states whose excitation energies are lying in the energy gap mentioned above.

Denoting the space orbital corresponding to the operator a_k by $\varphi_k(\mathbf{x})$, we define

$$\left. \begin{aligned} \psi(x) &= \sum a_{k\alpha} \varphi_k(\mathbf{x}) \exp[-i\tilde{E}_\alpha(k)t/\hbar] \\ \phi^*(x) &= \sum a_{k\beta} \varphi_k(\mathbf{x}) \exp[-i\tilde{E}_\beta(k)t/\hbar] \end{aligned} \right\} \quad (7)$$

Then the distribution amplitude of a pair state Ψ_n describing the electron-hole pair is given by

$$\chi_n(1, 2) = (\Phi, T\{\psi(1)\phi(2)\}\Psi_n), \quad (8)$$

where Φ is given by (3) and other notations are standard. As shown by Gell-Mann and Low,¹²⁾

$$\chi_q(1, 2) = \exp[i\mathbf{q} \cdot \mathbf{X} - i\omega(\mathbf{q})t/\hbar] f_q(x), \quad (9)$$

for the pair state with wave vector \mathbf{q} . In the above,

$$\mathbf{X} \equiv \frac{\mathbf{x}_1 + \mathbf{x}_2}{2}, \quad t \equiv \frac{t_1 + t_2}{2} \quad \text{and} \quad x \equiv x_1 - x_2.$$

For the bound state Ψ_q , we get

$$\begin{aligned} \chi_q(1, 2) &= - \int d^4x_3 d^4x_4 d^4x_5 d^4x_6 \\ &\times G_\alpha(1:3) G_\beta(2:4) \Gamma(3, 4:5, 6) \chi_q(5, 6), \end{aligned} \quad (10)$$

which was derived by Gell-Mann and Low¹²⁾ from Bethe-Salpeter's equation for

two-body Green Function describing our pair :

$$G(1, 2 : 3, 4) = (\Phi, T \{ \psi(1) \phi(2) \psi^*(3) \phi^*(4) \} \Phi).$$

In the above, Γ , the interaction function,¹²⁾ is given by the sum of all diagrams which cannot be decomposed into simpler diagrams connected by one ψ and one ϕ line, and

$$G_\alpha(1 : 2) \equiv (\Phi, T \{ \psi(1) \psi^*(2) \} \Phi),$$

$$G_\beta(1 : 2) \equiv (\Phi, T \{ \phi(1) \phi^*(2) \} \Phi).$$

Now, the most important interactions between an electron and a hole for the appearance of the exciton states was shown¹³⁾ to be the ladder type interactions as shown in Fig. 5. Thus, we here adopt only the ladder type interactions shown in Fig. 5. Then¹⁴⁾

$$\Gamma(3, 4 : 5, 6)$$

$$= \frac{1}{\hbar} J(\mathbf{x}_3 - \mathbf{x}_4) \delta(t_3 - t_4) \delta^4(x_5 - x_3) \delta^4(x_6 - x_4).$$

Substituting this in Eq. (10) and using (9) or

$$\chi_q(\mathbf{x}_1 t, \mathbf{x}_2 t) = \exp[i\mathbf{q} \cdot \mathbf{X} - i w(q) t / \hbar] f_q(\mathbf{x}),$$

we obtain

$$\begin{aligned} f_q(\mathbf{x}) = & - \sum' [w(q) - \widetilde{E}_\alpha(\mathbf{k} + \mathbf{q}) + \widetilde{E}_\beta(\mathbf{k})]^{-1} \\ & |\mathbf{k}| \leq k_{F\beta} \cap |\mathbf{k} + \mathbf{q}| > k_{F\alpha} \\ & \times \int d^3x' \exp \left[i \left(\mathbf{k} + \frac{\mathbf{q}}{2} \right) \cdot (\mathbf{x} - \mathbf{x}') \right] J(\mathbf{x}') f_q(\mathbf{x}'). \end{aligned} \quad (11)$$

Instead of solving Eq. (11) directly, we consider its Fourier transform. Then Eq. (11) is written as

$$\begin{aligned} [w(q) - \widetilde{E}_\alpha(\mathbf{p} + \mathbf{q}) + \widetilde{E}_\beta(\mathbf{p})] f_q(\mathbf{p}) \\ = - \sum_{\mathbf{l}} J(\mathbf{l}) f_q(\mathbf{p} + \mathbf{l}) \end{aligned} \quad (12)$$

where

$$f_q(\mathbf{p}) = \int \exp \left[-i \left(\mathbf{p} + \frac{\mathbf{q}}{2} \right) \cdot \mathbf{x} \right] f_q(\mathbf{x}) d^3x. \quad (13)$$

It should be noted that $f_q(\mathbf{p})$ has non-vanishing value only if \mathbf{p} satisfies the following condition :

$$|\mathbf{p}| \leq k_{F\beta} \cap |\mathbf{p} + \mathbf{q}| > k_{F\alpha}. \quad (14)$$

Eq. (12) tells that $w(q)$ is pushed down from the continuum by a finite distance if $f_q(\mathbf{p})$ has no node in momentum space and is a smooth function of \mathbf{p}



Fig. 5.

in the region given by (14). This is certainly satisfied when $|\mathbf{q}|$ is small as compared with $(k_{F\beta} - k_{F\alpha})$. If $|\mathbf{q}|$ is comparable with $(k_{F\beta} - k_{F\alpha})$, on the other hand, $f_q(\mathbf{p})$ would not be a smooth function of \mathbf{p} , unless the bottom of the α -band is extremely higher than the Fermi energy, and would be overwhelmingly large for values of \mathbf{p} in the neighborhood of \mathbf{p}_n corresponding to the bottom of the continuum. Even in this case $w(q)$ might certainly be separated from the continuum by a finite distance, if $J(l)$ had sufficiently strong singularity at $l=0$. Thus there arises the possibility that $w(q)$ becomes negative for comparable values of $|\mathbf{q}|$ with $(k_{F\beta} - k_{F\alpha})$ if we assume an incomplete ferromagnetic configuration (corresponding to the case $k_{F\alpha} \neq 0$, cf. Fig. 4. 1 or Fig. 4. 2) as well as

$$\lim_{l \rightarrow 0} J(l) \propto l^{-2}.$$

The appearance of such an unstable spin wave would be deeply connected with the fact that we cannot obtain an incomplete ferromagnetic configuration with a lower energy than the energy given by the complete ferromagnetic configuration (corresponding to the case $k_{F\alpha} = 0$) if we assume an electron gas model and adopt the Hartree-Fock approximation.¹⁵⁾

Hereafter we will examine the solution of Eq. (12) for sufficiently small values of $|\mathbf{q}|$ as compared with $(k_{F\beta} - k_{F\alpha})$. Now, substituting (6) into (12), we obtain

$$\begin{aligned} & \{w - E(\mathbf{k} + \mathbf{q}) + E(\mathbf{k})\} f_q(\mathbf{k}) \\ &= \sum_l J(l) (n_{\mathbf{k}+\mathbf{l}\beta}^0 - n_{\mathbf{k}+\mathbf{l}+\mathbf{q}\alpha}^0) f_q(\mathbf{k}) \\ & \quad - \sum_l J(l) \cdot f_q(\mathbf{k} + \mathbf{l}) \\ &= \sum_l J(l) \{f_q(\mathbf{k}) - f_q(\mathbf{k} + \mathbf{l})\} \\ & \quad (|\mathbf{k} + \mathbf{l}| < k_{F\beta} \cap |\mathbf{k} + \mathbf{l} + \mathbf{q}| > k_{F\alpha}) \\ & \quad - \sum_l J(l) \cdot f_q(\mathbf{k}) \\ & \quad (|\mathbf{k} + \mathbf{l}| > k_{F\beta} \cap |\mathbf{k} + \mathbf{l} + \mathbf{q}| < k_{F\alpha}). \end{aligned}$$

The second term in the last expression does not appear unless $|\mathbf{q}|$ is larger than $(k_{F\beta} - k_{F\alpha})$. Therefore, we get

$$\begin{aligned} & \{w - E(\mathbf{k} + \mathbf{q}) + E(\mathbf{k})\} f_q(\mathbf{k}) \\ &= \sum_{\mathbf{k}'} \S J(\mathbf{k}' - \mathbf{k}) \{f_q(\mathbf{k}) - f_q(\mathbf{k}')\}. \end{aligned} \quad (15)$$

In the above, the notation \S in the summation $\sum_{\mathbf{k}'} \S$ means that \mathbf{k}' should satisfy the following condition;

$$|\mathbf{k}'| \leq k_{F\beta} \cap |\mathbf{k}' + \mathbf{q}| > k_{F\alpha}.$$

In the special case of $k_{F\alpha} = 0$, i.e. all electron spins are pointed toward the same direction, Eq. (15) coincides with the result obtained by K. Yosida and T. Kasuya.⁸⁾ The

procedure necessary for solving Eq. (15) in the case of $k_{Fa}=0$ has been given by them. Following them, we take a summation of both sides of Eq. (15) over \mathbf{k} . Then we obtain

$$\sum_{\mathbf{k}}^{\S} \left\{ w - \frac{\hbar^2}{m} (\mathbf{q} \cdot \mathbf{k}) - \frac{\hbar^2}{2m} q^2 \right\} f_q(\mathbf{k}) = 0$$

or

$$w = \frac{\hbar^2 q^2}{2m} + q \left(\frac{\hbar^2}{m} \right) \left\{ \sum_{\mathbf{k}}^{\S} \left(\frac{\mathbf{q}}{q} \right) \cdot \mathbf{k} f_q(\mathbf{k}) / \sum_{\mathbf{k}}^{\S} f_q(\mathbf{k}) \right\}, \tag{16}$$

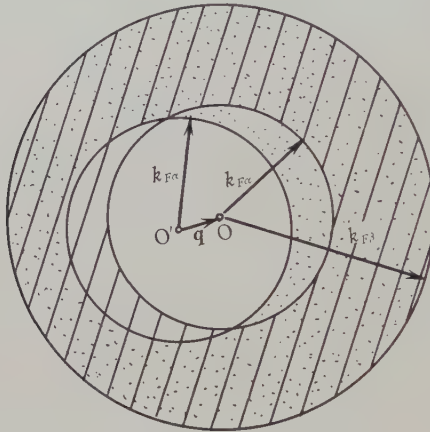


Fig. 6.

where we have assumed that $E(k) = \hbar^2 k^2 / 2m$, m being the effective mass of our band.

Now the domain of \mathbf{k} in the summation $\sum_{\mathbf{k}}^{\S}$ is the dotted part in Fig. 4 and should be deformed into another domain if we take another value of \mathbf{q} . As we have assumed that

$$|\mathbf{q}| \ll |k_{F\beta} - k_{Fa}|$$

and are now concerned with the lowest order term of w with respect to q , we sometimes approximate the domain by the shadowed part in Fig. 6, or we sometimes replace $\sum_{\mathbf{k}}^{\S}$ by

$$\sum_{\mathbf{k}}^{\dagger} \equiv \sum_{\mathbf{k}} (|\mathbf{k}| \leq k_{F\beta} \cap |\mathbf{k}| > k_{Fa}).$$

Further, we will use the following notation :

$$\sum^* \equiv \sum^{\S} - \sum^{\dagger}.$$

Now, $f_q(\mathbf{k})$ can have non-vanishing values only in the dotted region in Fig. 6, while $f_q(\mathbf{k})$ vanishes discontinuously when \mathbf{k} passes through the boundary of this region. In order to obtain a more compact expression of (16), we define a func-

tion $\tilde{f}_q(\mathbf{k})$ as follows:

$$\tilde{f}_q(\mathbf{k}) \begin{cases} \equiv f_q(\mathbf{k}) & \text{at points in the dotted region (§).} \\ \equiv \text{defined by means of the analytical continuation from the above} \\ & \text{function at points outside the above region.} \end{cases}$$

Then (16) may be expressed as

$$\begin{aligned} w = & \frac{\hbar^2 q^2}{2m} + q \left(\frac{\hbar^2}{m} \right) \left\{ \sum_{\mathbf{k}}^* \left(\frac{\mathbf{q}}{q} \right) \cdot \mathbf{k} \tilde{f}_q(\mathbf{k}) / \sum_{\mathbf{k}} \tilde{f}_q(\mathbf{k}) \right\} \\ & + q \left(\frac{\hbar^2}{m} \right) \left\{ \sum_{\mathbf{k}}^{\dagger} \left(\frac{\mathbf{q}}{q} \right) \cdot \mathbf{k} \tilde{f}_q(\mathbf{k}) / \sum_{\mathbf{k}} \tilde{f}_q(\mathbf{k}) \right\}. \end{aligned} \quad (16')$$

Further we may adopt the following expansion used by K. Yosida and T. Kasuya,⁸⁾

$$\tilde{f}_q(\mathbf{k}) = f^0(\mathbf{k}) + \mathbf{q} \cdot \mathbf{f}^1(\mathbf{k}) + \dots$$

Thus, if w is expanded in powers of q , the zeroth term must vanish, i.e.

$$w = q \cdot w^{(1)} + q^2 w^{(2)} + \dots$$

Accordingly, we get from Eq. (15)

$$\sum_{\mathbf{k}'} J(\mathbf{k}' - \mathbf{k}) \{f^0(\mathbf{k}) - f^0(\mathbf{k}')\} = 0.$$

From this equation we obtain the conclusion that $f^0(\mathbf{k})$ is independent of \mathbf{k} . This conclusion together with the following fact,

$$\sum_{\mathbf{k}}^{\dagger} \mathbf{q} \cdot \mathbf{k} = 0,$$

assures that

$$\begin{aligned} w = & \frac{\hbar^2 q^2}{2m} + q \left(\frac{\hbar^2}{m} \right) \left(\sum_{\mathbf{k}}^* k_z f^0 / \sum_{\mathbf{k}} \tilde{f}_q(\mathbf{k}) \right) \\ & + q^2 \left(\frac{\hbar^2}{m} \right) \left(\sum_{\mathbf{k}}^{\dagger} k_z f^1(\mathbf{k}) / \sum_{\mathbf{k}} f^0 \right) + O(q^3), \end{aligned} \quad (16'')$$

where k_z is the z -component of the vector \mathbf{k} , the z -axis being taken in the direction of the vector \mathbf{q} . As

$$\sum_{\mathbf{k}}^* k_z = q N_{\alpha},$$

the second term of the right-hand side of (16'') becomes

$$q^2 \left(\frac{\hbar^2}{m} \right) \frac{N_{\alpha} f^0}{\sum_{\mathbf{k}} f^0} + O(q^3) = q^2 \left(\frac{\hbar^2}{m} \right) \frac{N_{\alpha}}{N_{\beta} - N_{\alpha}} + O(q^3),$$

where $N_{\sigma} \equiv (V/(2\pi)^3) \cdot (4\pi/3) k_{F\sigma}^3$ is the number of σ -electrons. Thus we obtain

$$\begin{aligned} w = & w^{(2)} \cdot q^2 + O(q^3), \\ w^{(2)} = & \frac{\hbar^2}{2m} \left(\frac{N_{\beta} + N_{\alpha}}{N_{\beta} - N_{\alpha}} \right) + \frac{\hbar^2}{m} \left(\frac{1}{N_{\beta} - N_{\alpha}} \right) \sum_{\mathbf{k}}^{\dagger} k_z \varphi(\mathbf{k}). \end{aligned} \quad (17)$$

In the above, $\varphi(\mathbf{k}) \equiv f^1(\mathbf{k})/f^0$ is determined by the following equation:

$$-\frac{\hbar^2}{m} k_z = \sum_{\mathbf{k}'} J(\mathbf{k}-\mathbf{k}') \{ \varphi(\mathbf{k}) - \varphi(\mathbf{k}') \}, \quad (18)$$

which is derived from Eq. (15). Then the second term on the right-hand side of Eq. (17) becomes

$$-\frac{1}{2} \left(\frac{1}{N_\beta - N_\alpha} \right) \sum_{\mathbf{k}}^\dagger \sum_{\mathbf{k}'}^\dagger J(\mathbf{k}-\mathbf{k}') [\varphi(\mathbf{k}) - \varphi(\mathbf{k}')]^2.$$

Note that the above expression is always negative provided that $J(k) > 0$. Therefore, from Eq. (17) it is concluded that $w^{(2)}$ should have the following form,

$$w^{(2)} = \left(\frac{N_\beta + N_\alpha}{N_\beta - N_\alpha} \right) \frac{\hbar^2}{2m} \left\{ 1 - \frac{C(N_\alpha, N_\beta)}{r_s} \right\}, \quad (19)$$

if we assume $J(k) = V^{-1}(4\pi e^2/k^2)$, i.e. the completely unscreened interaction. In the above, $C(N_\alpha, N_\beta)$ is a *positive* dimensionless constant which depends only on N_α and N_β , and $r_s \equiv r_{0f}/(\hbar^2/m e^2)$, where the volume per electron in our system is set equal to $(4/3)\pi r_0^3$. Thus spin waves are certainly stable when r_s is sufficiently large, though they certainly break down when r_s is sufficiently small. (If we adopt an effective or screened interaction for $J(k)$, $C(N_\alpha, N_\beta)$ in (19) should depend on r_s . Since the screening effect grows stronger as r_s increases, $\varphi(\mathbf{k})$ given by (17) becomes larger as r_s increases. Then C should also become larger. Thus we can never obtain stable spin waves if we adopt a very strongly screened interaction.)

Now, Eq. (18) can be expressed as

$$\varphi(\mathbf{k}) = -\mathcal{A}^{-1}(\hbar^2/m) k_z + \mathcal{A}^{-1} \sum_{\mathbf{k}'}^\dagger J(\mathbf{k}-\mathbf{k}') \varphi(\mathbf{k}'), \quad (18')$$

where

$$\mathcal{A} \equiv \sum_{\mathbf{k}'}^\dagger J(\mathbf{k}-\mathbf{k}').$$

Substituting (18') into (17), we get

$$w^{(2)} = w^{(2)'} + w^{(2)''}$$

$$w^{(2)'} \equiv \frac{\hbar^2}{m} \left(\frac{1}{N_\beta - N_\alpha} \right) \left\{ \frac{N}{2} - (\hbar^2/m) \sum_{\mathbf{k}}^\dagger \mathcal{A}^{-1} k_z^2 \right\},$$

$$w^{(2)''} \equiv \left(\frac{1}{N_\beta - N_\alpha} \right) (\hbar^2/m) \sum_{\mathbf{k}}^\dagger \sum_{\mathbf{k}'}^\dagger \mathcal{A}^{-1} k_z J(\mathbf{k}-\mathbf{k}') \varphi(\mathbf{k}').$$

Hereafter we will investigate $w^{(2)}$ in the limiting case of $N_\alpha \rightarrow N_\beta$. We here introduce the following small (positive) number,

$$\epsilon \equiv (k_{F\beta} - k_{F\alpha}/k_F) \text{ or } \text{const} \times (N_\beta - N_\alpha/N),$$

where $(V/(2\pi)^3) \times (4\pi k_F^3/3) = (N/2)$. It is easily shown that $w^{(2)'}$ remains finite as $\epsilon \rightarrow 0$, because

In the above, we have used the following relations:

$$\frac{d\eta}{ds} = \cos \theta, \quad u = \eta \cos \theta. \quad (21)$$

Further, we obtain

$$\begin{aligned} \int_{s_0} dS J(\mathbf{k}-\mathbf{x}) \varphi(\mathbf{x}) &= 2\pi C \int u_z^* J(\eta) ds \\ &= 2\pi C k_z \int_0^{2k_F} \eta J(\eta) \left(1 - \frac{\eta^2}{2k_F^2}\right) d\eta. \end{aligned} \quad (22)$$

In the above, we have used

$$\begin{aligned} \zeta &= k_z - \eta \sin \theta \cos \chi = k_z \left(1 - \frac{\eta \sin \theta}{k_F}\right) \\ &= k_z \left(1 - \frac{\eta^2}{2k_F^2}\right) \end{aligned}$$

as well as Eq. (21). Substituting (20) and (22) into (18''), and omitting $O(\epsilon)$ in (18''), we get

$$C = -\frac{(\hbar^2/m)}{\Gamma_2 \epsilon},$$

where

$$\Gamma_2 = \frac{V}{(2\pi)^3} \times 2\pi k_F \int_0^{2k_F} \frac{\eta^3}{2k_F^2} J(\eta) d\eta.$$

Then we get

$$\sum_{\mathbf{k}'} J(\mathbf{k}-\mathbf{k}') \varphi(\mathbf{k}') = -\frac{(\hbar^2/m) \Gamma_1}{\Gamma_2} k_z \{1 + O(\epsilon)\},$$

where

$$\Gamma_1 = \frac{V}{(2\pi)^3} \times 2\pi k_F \int_0^{2k_F} \eta J(\eta) \left(1 - \frac{\eta^2}{2k_F^2}\right) d\eta.$$

Thus we obtain

$$w^{(2)''} = -\frac{(\hbar^2/m)^2}{(N_\beta - N_\alpha)(E_\beta - E_\alpha)} \times \frac{\Gamma_1}{\Gamma_2} \sum_{\mathbf{k}} k_z^2 \{1 + O(\epsilon)\}. \quad (23)$$

When $J(\eta) = \text{const}$, $w^{(2)''} = 0$ or $\Gamma_1 = 0$. Then $w^{(2)}$ remains finite as $\epsilon \rightarrow 0$. On the other hand, if $J(\eta)$ (> 0) is a monotonic decreasing function of η , it is easily shown that $\Gamma_1 > 0$. Obviously $\Gamma_2 > 0$, provided that $J(\eta) > 0$. Then we may conclude from (23) that $w^{(2)''} < 0$ as long as ϵ is sufficiently small. It should be noted that the main term on the right-hand side of (23) is $O(1, \epsilon)$. In actual problems $J(\eta)$ is usually positive and monotonic decreasing function of η . Thus

it is concluded that the spin waves break down unless there is a sufficiently large difference between the number of α -electrons and the one of β -electrons.

The reason why $\lim_{q \rightarrow 0} \omega = 0$ is that there are many states which are degenerate to our ground state. In this respect spin waves are different from exciton waves in insulators. In other respects both kinds of waves are much alike in character. For example, the propagation of the spin polarization corresponds to the propagation of the orbital polarization. Further, the internal motion of our electron-hole pair forming a spin wave with a long wavelength is localized in a finite region. This is analogous to the localized character of the internal motion of the electron-hole pair forming an exciton wave in an insulator. The internal motion in our case is described by $f_q(\mathbf{x})$. This is given by (13) or

$$f_q(\mathbf{x}) \propto \int_{k_{F\alpha} < p < k_{F\beta}} d^3p \exp[i(\mathbf{p} + \frac{\mathbf{q}}{2}) \cdot \mathbf{x}] f_q(\mathbf{p}).$$

We may evaluate this up to the lowest order term with respect to q . Then we get

$$f_q(\mathbf{x}) \propto \frac{e^{i(q/2) \cdot \mathbf{x}}}{r} \{k_{F\beta}^2 j_1(k_{F\beta} r) - k_{F\alpha}^2 j_1(k_{F\alpha} r)\},$$

where $r = |\mathbf{x}|$ and j_1 is the spherical Bessel function. Thus we find

$$f_q(\mathbf{x}) \xrightarrow{r \rightarrow \infty} \text{const. } g_q(\mathbf{x}), \quad r^2,$$

where $|g_q(\mathbf{x})| \leq 1$. Therefore $f_q(\mathbf{x})$ can be normalized and accordingly we see that the electron-hole pair of the spin wave is bounded literally.* Such a localized character of the internal motion leads to a spin polarization in a localized region and would be essential to a classical image of a spin wave. If the internal motion were infinitely spread over the whole space as in the case of scattering states, we would merely obtain a nearly uniform decrease of the β -spin density throughout the whole space.

In our treatment we have not specified the form of $J(\mathbf{k})$. This may be regarded as a Fourier coefficient of an effective interaction discussed by Hubbard¹⁰⁾ and others. In our treatment, however, the retardation effect of the effective interaction is not taken into consideration.

The results obtained here in the ladder approximation can be reproduced by a technique¹³⁾ developed in the case of excitons in insulators: In the ladder approximation we may replace H' by

$$H' \rightarrow \mathcal{H}' \equiv -\frac{1}{2} \sum_{\mu} \sum_{l'l'}^{\sigma\sigma'} J(\mu) \\ \times \{\vartheta^*(l'\sigma' | l + \mu\sigma) \vartheta(l' - \mu\sigma' | l\sigma)\}$$

* Accordingly, it seems that our spin waves cannot be accelerated by an external electric field.

$$+\vartheta^*(\mathbf{l}\sigma|\mathbf{l}'-\mu\sigma')\vartheta(\mathbf{l}+\mu\sigma|\mathbf{l}'\sigma')\}. \quad (24)$$

In the above,

$$\vartheta(\mathbf{k}_1\sigma_1|\mathbf{k}_2\sigma_2)\equiv b(\mathbf{k}_1\sigma_1)a(\mathbf{k}_2\sigma_2),$$

where

$$a(\mathbf{k}\sigma)\begin{cases} \equiv a_{k\sigma} & \text{if } |\mathbf{k}| > k_{F\sigma}, \\ \equiv 0 & \text{if } |\mathbf{k}| \leq k_{F\sigma}, \end{cases}$$

and

$$b(\mathbf{k}\sigma)\begin{cases} \equiv a_{k\sigma}^* & \text{if } |\mathbf{k}| \leq k_{F\sigma}, \\ \equiv 0 & \text{if } |\mathbf{k}| > k_{F\sigma}. \end{cases}$$

Further, we may use the following commutation rule:

$$\begin{aligned} &[\vartheta(\mathbf{p}_1\sigma_1|\mathbf{p}_2\sigma_2), \vartheta^*(\mathbf{p}_1'\sigma_1'|\mathbf{p}_2'\sigma_2')] \\ &= \delta(\mathbf{p}_1|\mathbf{p}_1')\delta(\mathbf{p}_2|\mathbf{p}_2')\delta(\sigma_1|\sigma_1')\delta(\sigma_2|\sigma_2'), \end{aligned} \quad (25)$$

and

$$[\vartheta(\mathbf{p}_1\sigma_1|\mathbf{p}_2\sigma_2), \vartheta(\mathbf{p}_1'\sigma_1'|\mathbf{p}_2'\sigma_2')]=0.$$

Then we have bound states each of which is represented by the state vector

$$\left. \begin{aligned} \Psi_q &= \mathfrak{A}_q \Phi \\ \mathfrak{A}_q &= \sum_{\mathbf{k}} f_q(\mathbf{k}) \vartheta^*(\mathbf{k}|\beta|\mathbf{k}+\mathbf{q}\alpha). \end{aligned} \right\} \quad (26)$$

The excitation energy of this bound state which is denoted by w is given by the following equation of motion,

$$\frac{\hbar}{i} \dot{\mathfrak{A}}_q = w \mathfrak{A}_q = [H, \mathfrak{A}_q]. \quad (27)$$

Substituting (26) into (27) and using (24) and (25), we obtain Eq. (12). However, the internal motion of our pair would not be defined unambiguously in the Tamm-Dancoff method. This is the reason why we have adopted Gell-Mann and Low's method.

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Polarization of the Recoil Nucleon from the Photoproduction of Pion

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The transverse polarization of the recoil proton from $\gamma + p \rightarrow \pi^0 + p$ is estimated for $E_\gamma = 260, 300, 320, 350$, and 400 Mev in a phenomenological way. In § 2 calculation of the polarization is carried out for unpolarized incident particles, where the contribution from the electric quadrupole radiation in the $p_{3/2}$ state is neglected. Here only the experimental angular distributions of $\gamma + p \rightarrow \pi^0 + p$ and $\gamma + p \rightarrow \pi^+ + n$, and six phase shifts of the pion-nucleon scattering are used as known quantities. Six transition amplitudes of the photopion production, corresponding to $s_{1/2}$, $p_{1/2}$, and $p_{3/2}$ final states with isotopic spin $1/2$ and $3/2$, can be expressed in terms of these known quantities, after we solve six simultaneous quadratic equations. Then 10 to 20% polarization is theoretically expected over a wide range of angles in the center of mass system. § 3 is devoted to review the transition amplitudes obtained in § 2.

§ 1. Introduction

Polarization of the recoil proton from the photopion production has recently been measured¹⁾ concerning the parity of the second resonance.²⁾ It would, therefore, be worth while to estimate the polarization in lower energy regions.

In the present paper the polarization is calculated irrespective of any special theory, though we are helped by the Chew-Low theory in course of computations. Experimental angular distributions of $\gamma + p \rightarrow \pi^0 + p$ and $\gamma + p \rightarrow \pi^+ + n$, and experimental phase shifts of the pion-nucleon scattering are taken as the basis of our calculation. We try to express the polarization in terms of these quantities as much as possible.

As is well known, the predominant contribution to $\gamma + p \rightarrow \pi^0 + p$ comes from the magnetic dipole radiation with the $p_{3/2}$ final state, and the contribution from other multipole transitions is small in the energy region $E_\gamma = 260 \text{ Mev} \sim 350 \text{ Mev}$, or so. This fact suggests that the polarization should be small, because the polarization comes from interference of different partial waves. This is realized in § 2.

Any discrepancy between future experiment and our result might provide interesting information on the interpretation of the photopion process. For example, contribution of the d -wave, or behaviors of the s -wave would be clarified from this information.

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§ 2. Calculation

The differential cross sections of $\gamma + p \rightarrow \pi^0 + p$ and $\gamma + p \rightarrow \pi^+ + n$ are written down in terms of six transition amplitudes, $M_3, M_1, M_{31}, M_{11}, M_{33}$, and M_{13} , where the subscripts should be understood as the same with those of the pion scattering phase shifts. We neglect the contribution from the electric quadrupole radiation in the $p_{3/2}$ state throughout the present work. The phase of M is, as is well known, related to the scattering phase shifts,³⁾ namely

$$M = Re^{i\delta}, \quad (2.1)$$

where R is a positive or negative real number. The unavoidable ambiguity in sign is, therefore, absorbed in R .

The differential cross sections of the respective processes are written as

$$4k^2 \frac{d\sigma^0}{d\Omega} = A^0 + B^0 \cos \theta + C^0 \cos^2 \theta, \quad (2.2)$$

and

$$4k^2 \frac{d\sigma^+}{d\Omega} = A^+ + B^+ \cos \theta + C^+ \cos^2 \theta, \quad (2.3)$$

where k is the photon momentum. A, B , and C are given by combinations of six R 's and six δ 's. For instance, one gets

$$\begin{aligned} C^0 = & -\frac{1}{2} R_{33}^2 - \frac{1}{4} R_{13}^2 + \frac{\sqrt{2}}{2} R_{33} R_{13} \cos (\delta_{33} - \delta_{13}) \\ & - R_{31} R_{33} \cos (\delta_{31} - \delta_{33}) + \frac{\sqrt{2}}{2} R_{31} R_{13} \cos (\delta_{31} - \delta_{13}) \\ & + \frac{\sqrt{2}}{2} R_{11} R_{33} \cos (\delta_{33} - \delta_{11}) - \frac{1}{2} R_{11} R_{13} \cos (\delta_{11} - \delta_{13}). \end{aligned} \quad (2.4)$$

Table I

Gamma-ray energy in lab. E_γ (Mev)	Coefficients of π^0 angular distribution (10^{-4}) ^a			Coefficients of π^+ angular distribution (10^{-4}) ^b			Pion scattering phase shifts (degree) ^{c, d}					
	A^0	B^0	C^0	A^+	B^+	C^+	δ_3	δ_1	δ_{31}	δ_{11}	δ_{33}	δ_{13}
260	61	0	-40	79	-15	-38	-11	7	5	-3	26	2
300	133	0	-89	113	-7	-44	-13	12	5	-2	50	2
320	164	17	-96	123	-2	-48	-14	7	3	7	65	-1
350	160	20	-98	112	8	-40	-15	11	-3	2	94	-3
400	119	25	-77	82	20	-20	-19	16	-4	8	121	-2

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Others are immediately found in our review work.⁴⁾

We are now going to solve six simultaneous quadratic equations of the unknown R , using experimental values of the A , B , C , and δ , which are shown in Table I.

In solving the equations, the following set is taken as approximate solutions⁵⁾:

$$\begin{aligned}\bar{R}_3 &= \frac{4ef}{\sqrt{3}} \frac{(kq)^{1/2}}{\mu}, \\ \bar{R}_1 &= \sqrt{2} \bar{R}_3, \\ \bar{R}_{33} &= -\frac{4}{\sqrt{3}} \frac{e}{f} \left(\frac{k}{q}\right)^{3/2} \frac{g_p - g_n}{4M/\mu} \sin \delta_{33}, \\ \bar{R}_{13} &= \bar{R}_{31} = \bar{R}_{11} = 0,\end{aligned}\tag{2.5}$$

where $e^2 = 1/137$, $f^2 = 0.08$, k is the incident photon momentum, q is the pion momentum, M and μ are the nucleon and pion mass, respectively, and $g_p - g_n = 2.78 - 1.91$ nuclear magnetons.

Now the quadratic equations are reduced to linear equations of the new set of unknown $\delta R = R - \bar{R}$, whose square is considered to be small enough to be disregarded. These linear equations are, of course, easy to solve, and the solutions are displayed in Table II.

Table II

E_γ (Mev)	R_3	R_1	R_{31}	R_{11}	R_{33}	R_{13}
260	0.062	0.070	-0.020	-0.020	-0.089	-0.006
300	0.077	0.060	-0.003	0.020	-0.132	-0.018
320	0.070	0.090	-0.006	-0.010	-0.136	0.013
350	0.080	0.055	0.000	-0.020	-0.134	-0.029
400	0.070	0.101	0.009	-0.016	-0.109	-0.015

The polarization of the recoil proton from $\gamma + p \rightarrow \pi^0 + p$ for unpolarized incident particles is now readily calculated, by inserting R into the following formula⁶⁾:

$$P = n \frac{dP}{d\Omega} \bigg/ \frac{d\sigma}{d\Omega},\tag{2.6}$$

where

$$\begin{aligned}4k^2 \frac{dP^0}{d\Omega} &= \sin \theta \left[\frac{2}{3} R_3 R_{31} \sin (\delta_3 - \delta_{31}) + \frac{1}{3} R_1 R_{11} \sin (\delta_1 - \delta_{11}) \right. \\ &\quad - \frac{\sqrt{2}}{3} R_3 R_{11} \sin (\delta_3 - \delta_{11}) - \frac{\sqrt{2}}{3} R_1 R_{31} \sin (\delta_1 - \delta_{31}) \\ &\quad - \frac{1}{3} R_{33} R_3 \sin (\delta_{33} - \delta_3) - \frac{1}{6} R_{13} R_1 \sin (\delta_{13} - \delta_1) \\ &\quad \left. + \frac{\sqrt{2}}{6} R_{33} R_1 \sin (\delta_{33} - \delta_1) + \frac{\sqrt{2}}{6} R_{13} R_3 \sin (\delta_{13} - \delta_3) \right]\end{aligned}$$

$$\begin{aligned}
& + \sin \theta \cos \theta \left[R_{33} R_{31} \sin (\delta_{33} - \delta_{31}) + \frac{1}{2} R_{13} R_{11} \sin (\delta_{13} - \delta_{11}) \right. \\
& \left. - \frac{\sqrt{2}}{2} R_{33} R_{11} \sin (\delta_{33} - \delta_{11}) - \frac{\sqrt{2}}{2} R_{13} R_{31} \sin (\delta_{13} - \delta_{31}) \right], \quad (2.7)
\end{aligned}$$

and

$$\mathbf{n} = (\mathbf{k} \times \mathbf{q}) / |\mathbf{k} \times \mathbf{q}|. \quad (2.8)$$

The results are plotted in Fig. 1, which shows that about 10 to 20% polarization is expected over a wide range of angles.

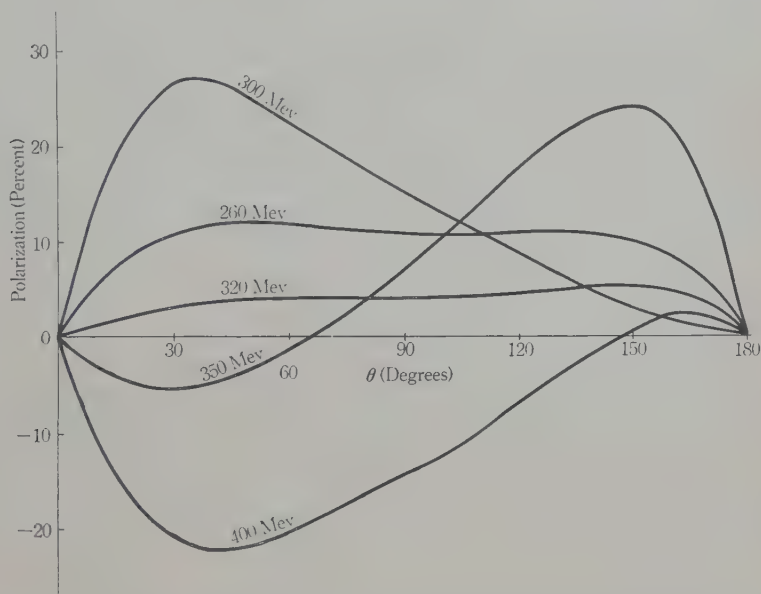


Fig. 1. Polarization of the recoil proton from $\gamma + p \rightarrow \pi^0 + p$ for gamma-ray energies 260, 300, 320, 350, and 400 Mev. The direction of the polarization is chosen to be parallel to $\mathbf{k} \times \mathbf{q}$.

§ 3. Discussion

In this section, the transition amplitudes R obtained in § 2 are criticized, and some comments are added.

a) Inaccuracy of R

One sees unreasonable fluctuations of R vs. E_γ in Table II. This is mostly due to the inaccuracy of A , B , C , and δ . For example, if we assume empirical linear momentum dependence for δ_3 and δ_1 , and cubic momentum dependence for the p phase shifts, more smooth energy dependence is obtained for R . In addition, the change of C^+ of the photopion production for various energies, for instance, is not smooth, as it is found in Table I. Accordingly, the values of R in Table II should not be taken so seriously. The polarization thus obtained is, therefore, reliable

only in order of magnitude. For $E_\gamma=260$ Mev and 320 Mev the approximate solutions (2.5) are found to be quite good, so that the polarization is more reliable than other cases.

b) Relative sign of R

In our framework, we do not have any method to determine the relative sign of R . As is shown in reference 3), R_s and R_l must have the same sign. The approximate solutions \bar{R}_{33} and \bar{R}_3 have, on the other hand, the opposite sign.

Since (2.5) is not a good approximation for 300 and 350 Mev, we have tried to solve quadratic equations. In this case, we just choose the solutions in which all or most of the p -wave amplitudes are negative. This is, of course, not justified.

c) d -wave contribution

Stein¹⁾ has obtained positive polarization for $E_\gamma=550$ Mev at $\theta=90^\circ$, while the present calculation shows negative one for E_γ higher than the first resonance. Since the interference between p - and d -waves gives rise to positive polarization,²⁾ it is possible that the resultant is positive. The disagreement between the present calculation and experiment might provide information on the d -wave contribution.

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On the Representation of the Canonical Commutation Relation of Bose Fields

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A representation of the canonical commutation relation of Bose fields is given in a way which is independent of the choice of the bases of the test functions and covariant with respect to the Euclidean transformation of the coordinate system. It is shown that the representation is characterized by an integral on the conjugate space L^* of the space L of the test functions and a real function on $\Sigma \otimes L^*$ where Σ is the group of the transformations $f \rightarrow u^{-1}f + \varphi$; $f \in L^*$, $\varphi \in L$ and u is a Euclidean transformation of L^* . The conditions for the irreducibility of a representation and the unitary equivalence of the representations and the existence of unique vacuum state are given. An example of the inequivalent Euclidean covariant irreducible representations containing unique vacuum state is given.

Introduction

When we treat quantum-mechanically a system having infinite degrees of freedom, we meet situations very different from those for the case of finite degrees of freedom. The most striking difference appears in the structure of the Hilbert space or in the representation of the canonical commutation relation. In the case of finite degrees of freedom, it is well known that the canonical commutation relation determines the structure of the Hilbert space in an essentially unique way. The irreducible representation of the canonical commutation relation is given uniquely by the well-known Schrödinger representation within the class of the unitary equivalence.¹⁾ However, in the case of infinite degrees of freedom, such is not the case. There are infinitely (at least continuously) many unitarily inequivalent irreducible representations.²⁾

This is a very interesting phenomenon in connection with the divergence difficulty of quantum field theory. The peculiar orthogonality found by Van Hove³⁾ and discussed by Friedrichs⁴⁾ and others⁵⁾ is closely connected with the appearance of the inequivalent representations. It was shown that in a simple model, i.e. neutral scalar meson with fixed source, the free field and the interacting field belong to mutually inequivalent representations.⁶⁾

It seems very probable that this is also the case in more realistic theories such as quantum electrodynamics. Recent analysis by Landau et al.⁷⁾ showed that Z_3 factors in quantum electrodynamics are zero. This is a similar situation to that found in the above model.³⁾ However, no rigorous proof of this conjecture has been obtained, at present.

Another interesting question is whether there are inequivalent representations satisfying physically necessary conditions such as the relativistic covariance and the existence of unique vacuum state. A result due to Hall and Wightman⁸⁾ indicates that the possibility of the existence of a physically plausible canonical theory of interacting fields depends on the existence of such inequivalent representations (which are inequivalent to the representation containing the free vacuum state).

In order to approach these problems it is necessary to develop a representation theory of the canonical commutation relation of fields which is independent of the choice of the basis of the test function.

In this paper we shall give a mathematical theory of the basis-independent and Euclidean covariant representation of the canonical commutation relation of Bose fields. As for other mathematical theories we refer to the works of Garding-Wightman⁹⁾ and Segal.¹⁰⁾ The method of the former authors is not basis-independent. Our method resembles in many respects to that of Segal.

§ 1. Preliminaries

Since a complex Bose field can be split into two real Bose fields it suffices to consider real fields. Let $Q_s(r, t)$ and $P_s(r, t)$ be the field operators of the fields and their canonical conjugate operators respectively where the index s denotes the tensor index and the kind of the fields. These operators themselves have only a symbolical meaning and the actually meaningful operators are their averages $Q(\varphi)$ and $P(\varphi)$ with respect to a real test function $\varphi \equiv \varphi_s(r)$:

$$Q(\varphi) = \sum_s \int Q_s(r, t) \varphi_s(r) d^3 r, \quad P(\varphi) = \sum_s \int P_s(r, t) \varphi_s(r) d^3 r. \quad (1.1)$$

$Q(\varphi)$ and $P(\varphi)$, which are self-adjoint operators, satisfy the canonical commutation relations

$$[Q(\varphi), P(\psi)] = i(\varphi, \psi), \quad [Q(\varphi), Q(\psi)] = [P(\varphi), P(\psi)] = 0, \quad (1.2)$$

where the scalar product (φ, ψ) is defined by

$$(\varphi, \psi) = (\psi, \varphi) = \sum_s \int \varphi_s(r) \psi_s(r) d^3 r. \quad (1.3)$$

As is well known, the operators $Q(\varphi)$ and $P(\varphi)$ satisfying (1.2) are unbounded operators. In order to avoid mathematical complications due to the unboundedness of $Q(\varphi)$ and $P(\varphi)$ we introduce the unitary operators $S(\varphi)$ and $T(\varphi)$:

$$S(\varphi) = e^{iQ(\varphi)}, \quad T(\varphi) = e^{iP(\varphi)}. \quad (1.4)$$

Then they satisfy

$$\begin{aligned} T(\varphi) S(\psi) T^{-1}(\varphi) &= e^{i(\varphi, \psi)} S(\psi), \\ S(\varphi) S(\psi) &= S(\varphi + \psi), \quad T(\varphi) T(\psi) = T(\varphi + \psi). \end{aligned} \quad \begin{cases} (1.5) \\ (1.5)' \end{cases}$$

Let L be a real linear topological space of the test functions. The topology of L is usually referred to as the norm topology induced by the scalar product (1.3). In this case L is a Hilbert space. However, in some cases L is to be taken as the space of the test functions of Schwartz. So we do not specify the topology of L and only assume the following properties for L : (1) $L^* \supset L$, where L^* is the conjugate space of L . (2) The linear functional (f, φ) ; $f \in L^*$, $\varphi \in L$ has the property that if $f \in L$ then it reduces to the scalar product (1.3) (we use the same symbol for the linear functional as the scalar product). In the both cases quoted above these conditions for L are satisfied.

The representation of the canonical commutation relation of Bose fields is a realization of $S(\varphi)$ and $T(\varphi)$ as the unitary operators on a Hilbert space which satisfy (1.5) and are strongly continuous with respect to φ . Then $Q(\varphi)$ and $P(\varphi)$ are the self-adjoint generators of the one parameter families of the unitaries $S(t\varphi)$ and $T(t\varphi)$ respectively.

Corresponding to every Euclidean coordinate transformation (rotation ω and translation a) the test functions suffer a transformation u :

$$u\varphi \equiv \sum_{s'} C_{ss'}(\omega) \varphi_{s'}(\omega r + a), \quad (1.6)$$

where $C_{ss'}(\omega)$ denotes the transformation coefficient of the tensor index. This u obviously satisfies

$$(u\varphi, u\phi) = (\varphi, \phi).$$

We denote the group of all Euclidean transformation u of L by E . The Euclidean transformation u can be extended on L^* by

$$(uf, \varphi) = (f, u^{-1}\varphi). \quad (1.7)$$

Speaking more correctly, uf must be written as $u^{*-1}f$ where u^* is the conjugate operator of u . However, no confusion will arise by writing $u^{*-1}f$ as uf , so we use this notation for the sake of brevity.

A representation is called "Euclidean covariant" if there is a unitary operator $T(u)$ for every u in E such that

$$\begin{aligned} S(u\varphi) &= T(u)S(\varphi)T^{-1}(u), & T(u\varphi) &= T(u)T(\varphi)T^{-1}(u), & \left\{ \begin{array}{l} (1.8) \\ (1.8)' \end{array} \right. \\ T(u)T(v) &= T(uv). \end{aligned}$$

In Euclidean covariant representations the linear and the angular momentum operators can be defined as the infinitesimal generators of $T(u)$.

Define $T(u, \varphi)$ by

$$T(u, \varphi) = T(u)T(\varphi), \quad (1.9)$$

then from (1.5) and (1.8)

$$\begin{aligned} T(u, \varphi)S(\psi)T^{-1}(u, \varphi) &= e^{i(\varphi, \psi)} S(u\psi), & \left\{ \begin{array}{l} (1.10) \\ (1.10)' \end{array} \right. \\ T(u, \varphi)T(v, \psi) &= T(uv, v^{-1}\varphi + \psi). \end{aligned}$$

Denote the transformation

$$S(\psi) \rightarrow e^{i(\sigma\psi)} S(u\psi) \quad (1.11)$$

by $\sigma(u, \varphi)$. (1.10) shows that $\sigma(u, \varphi)$ forms a group with the multiplication rule

$$\sigma(u, \varphi) \sigma(v, \psi) = \sigma(uv, v^{-1}\varphi + \psi) \quad (1.12)$$

and in Euclidean covariant representations $\sigma(u, \varphi)$ is induced by a unitary operator $T(\sigma(u, \varphi)) = T(u)T(\varphi)$. We denote the group of all $\sigma(u, \varphi)$; $u \in E, \varphi \in L$ by Σ .

Conversely, if we have a representation of the Abelian operators $S(\varphi)$; $\varphi \in L$ in which the transformation $\sigma(u, \varphi)$ is induced by a unitary operator $T(u, \varphi)$, then we have a Euclidean covariant representation of the canonical commutation relation. This is the key of our construction method of the Euclidean covariant representations.

§ 2. Representation of $S(\varphi)$

Let $F(f)$ be a complex valued bounded continuous function of L^* such that, by choosing a suitable finite sequence $\varphi_1 \cdots \varphi_n$ of the elements in L , $F(f)$ is expressible in the form

$$F(f) = F((f, \varphi_1), \cdots (f, \varphi_n)). \quad (2.1)$$

Such function $F(f)$ is called "tame function." Let the set of all tame functions be \mathfrak{T} . For the set \mathfrak{T} we may apply the theory of Daniell integral.¹¹⁾ Our terminology is after Loomis.¹¹⁾ Let X be a bounded integral. Let \mathfrak{S}_X be the Hilbert space of all X -square summable Baire functions $\mathcal{V}(f)$. \mathfrak{T} is dense in \mathfrak{S}_X . Note that any X -square summable function $\mathcal{V}(f)$ is X -summable since $|X(\mathcal{V})|^2 \leq X(1)X(|\mathcal{V}|^2)$. The scalar product in \mathfrak{S}_X is given by

$$(\mathcal{V}, \Phi)_X = X(\overline{\mathcal{V}}\Phi) \quad (2.2)$$

where $\overline{\mathcal{V}}$ is the complex conjugate of \mathcal{V} .

Let $F(f)$ be an X -essentially bounded function. Then define the operator $M(F)$ on \mathfrak{S}_X by

$$M(F)\mathcal{V}(f) = F(f)\mathcal{V}(f). \quad (2.3)$$

Every tame function is essentially bounded for any X .

Now we have a representation of $S(\varphi)$ on \mathfrak{S}_X by

$$S(\varphi) = M(e^{i(f, \varphi)}), \quad (2.4)$$

namely

$$S(\varphi)\mathcal{V}(f) = e^{i(f, \varphi)}\mathcal{V}(f). \quad (2.4)'$$

Then the transformation $\sigma(u, \varphi)$ induces the transformation

$$M(F) \rightarrow M(F_{\sigma(u, \varphi)}) \quad (2.5)$$

where

$$F_{\sigma(u, \varphi)}(f) = F(u^{-1}f + \varphi). \quad (2.6)$$

Thus Σ is isomorph to the group of one to one mappings of L^*

$$f\sigma(u, \varphi) = u^{-1}f + \varphi. \quad (2.7)$$

§ 3. Euclidean covariant representation

Let X and Y be bounded integrals. If X is absolutely continuous with respect to Y we denote it in symbol $X \ll Y$. If $X \ll Y$ and $Y \ll X$ we denote it by $X \sim Y$. If $X \ll Y$, then by a theorem of Radon-Nikodim⁽¹⁾ there is a real non-negative Y -summable function $D(f)$ such that for any Y -summable function $\Psi(f)$, $X(\Psi) = (D\Psi)$. This $D(f)$ is denoted by $(dX/dY)(f)$. If $X \ll Y \ll Z$, then

$$\frac{dX}{dY} \cdot \frac{dY}{dZ} = \frac{dX}{dZ}. \quad (3.1)$$

Let G be any group of one to one mappings α of L^* . Then for every $\alpha \in G$

$$\begin{aligned} X_\alpha(\Psi) &= X(\Psi_{\alpha^{-1}}) \\ \Psi_\alpha(f) &= \Psi(f_\alpha) \end{aligned} \quad (3.2)$$

defines the integral X_α . An integral X is called "G-quasi-invariant" if $X_\alpha \ll X$ for every α in G . Since G is a group this implies that $X_\alpha \sim X$ for every α in G . A G-quasi-invariant integral X is called "G-ergodic" if there is no G-quasi-invariant integral Y such that $Y \ll X$ but $Y \sim X$ or $Y \equiv 0$.

Theorem 1. Let X be a bounded integral and the representation of $S(\varphi)$ on \mathfrak{S}_X be given by (2.4). Then there are the unitary operators $T(\sigma(u, \varphi)) = T(u, \varphi)$ satisfying (1.10) if and only if X is Σ -quasi-invariant.

If $T(\sigma); \sigma \in \Sigma$ exist, then they are given in the following form:

$$T(\sigma)\Psi(f) = \left[\frac{dX_\sigma}{dX}(f) \right]^{1/2} e^{iJ(\sigma, f)} \Psi(f\sigma) \quad (3.3)$$

where $J(\sigma, f)$ is a real function on $\Sigma \otimes L^*$ satisfying

$$J(\sigma, f) + J(\rho, f\sigma) = J(\sigma\rho, f). \quad (3.4)$$

Proof: Ad the "only if" part: Since X is bounded 1 is in \mathfrak{S}_X . Let $\chi^\sigma(f) \in \mathfrak{S}_X$ be the mapping image of 1 under the operation of $T(\sigma)$; $\chi^\sigma(f) = T(\sigma)1$. Then for any tame function $F(f)$ we have

$$\begin{aligned} X_\sigma(F) &= X(F_{\sigma^{-1}}) = (1, M(F_{\sigma^{-1}})1)_X = (T(\sigma)1, M(F)T(\sigma)1)_X \\ &= (\chi^\sigma, M(F)\chi^\sigma)_X = (\chi^\sigma, F\chi^\sigma)_X = X(|\chi^\sigma|^2 F). \end{aligned} \quad (3.5)$$

This shows that $X_\sigma \ll X$ for every σ in Σ . From (3.5) we see

$$|\chi^\sigma(f)|^2 = \frac{dX_\sigma}{dX}(f),$$

hence $\chi^\sigma(f)$ is in the form

$$\chi^\sigma(f) = \left[\frac{dX_\sigma}{dX}(f) \right]^{1/2} e^{iJ(\sigma, f)}. \quad (3.6)$$

Since X is bounded and $X_\sigma \sim X$ for every $\sigma \in \Sigma$, dX_σ/dX is X -essentially bounded. So that from $T(\sigma\rho) = T(\sigma)T(\rho)$ we have

$$\chi^{\sigma\rho}(f) = T(\sigma\rho)1 = T(\sigma)\chi^\rho(f) = T(\sigma)M(\chi^\rho)T^{-1}(\sigma)T(\sigma)1 = \chi^\rho(f\sigma)\chi^\sigma(f). \quad (3.7)$$

On the other hand,

$$\begin{aligned} X\left(\frac{dX_{\sigma\rho}}{dX}F\right) &= X_{\sigma\rho}(F) = X(F_{\rho^{-1}\sigma^{-1}}) = X_\rho(F_{\sigma^{-1}}) = X\left(\frac{dX_\rho}{dX}F_{\sigma^{-1}}\right) \\ &= X_\sigma\left(\left[\frac{dX_\rho}{dX}\right]_\sigma F\right) = X\left(\frac{dX_\sigma}{dX}\left[\frac{dX_\rho}{dX}\right]_\sigma F\right), \end{aligned}$$

hence

$$\frac{dX_{\rho\sigma}}{dX}(f) = \frac{dX_\sigma}{dX}(f) \frac{dX_\rho}{dX}(f\sigma). \quad (3.8)$$

From (3.6), (3.7) and (3.8) we have (3.4).

Ad the "if" part: Define $T(\sigma)$ by

$$T(\sigma)\Psi(f) = \left[\frac{dX_\sigma}{dX}(f) \right]^{1/2} \Psi(f\sigma).$$

Then from (3.8) $T(\sigma)$ satisfies $T(\sigma)T(\rho) = T(\sigma\rho)$.

$$\begin{aligned} T(\sigma)S(\psi)T^{-1}(\sigma)\Psi(f) &= T(\sigma)S(\psi)\left[\frac{dX_{\sigma^{-1}}}{dX}(f)\right]^{1/2}\Psi(f\sigma^{-1}) \\ &= T(\sigma)e^{i(f,\psi)}\left[\frac{dX_{\sigma^{-1}}}{dX}(f)\right]\Psi(f\sigma^{-1}) \\ &= e^{i(f\sigma,\psi)}\left[\frac{dX_{\sigma^{-1}}}{dX}(f\sigma)\right]^{1/2}\Psi(f\sigma\sigma^{-1}) \\ &= e^{i(f\sigma,\psi)}\Psi(f) = e^{i(u^{-1}f+\psi,\psi)}\Psi(f) = e^{i(\psi,\psi)}e^{i(f,u\psi)}\Psi(f). \end{aligned}$$

This shows $T(\sigma(u\psi))S(\psi)T^{-1}(\sigma(u\psi)) = e^{i(\psi,\psi)}S(u\psi)$.

$$(T(\sigma)\Psi, T(\sigma)\Psi)_X = X\left(\frac{dX_\sigma}{dX}|\Psi_\sigma|^2\right) = X_\sigma(|\Psi_\sigma|^2) = X(|\Psi|^2) = (\Psi, \Psi)_X$$

This shows the unitarity of $T(\sigma)$. Q.E.D.

From the theorem 1 we have a Euclidean covariant representation of the canonical commutation relation :

$$\left. \begin{aligned} S(\varphi) \Psi(f) &= e^{iJ(\varphi, \varphi)} \Psi(f) \\ T(\varphi) \Psi(f) &= \left[\frac{dX_\varphi}{dX}(f) \right]^{1/2} e^{iI(\varphi, f)} \Psi(f + \varphi) \\ T(u) \Psi(f) &= \left[\frac{dX_u}{dX}(f) \right]^{1/2} e^{iK(u, f)} \Psi(u^{-1}f) \end{aligned} \right\} \quad (3.9)$$

where $I(\varphi, f) = J(\sigma(1, \varphi), f)$, $K(u, f) = J(\sigma(u, 0), f)$, and they satisfy

$$\left. \begin{aligned} I(\varphi, f) + I(\psi, f + \varphi) &= I(\varphi + \psi, f) \\ K(u, f) + K(v, u^{-1}f) &= K(uv, f) \\ I(u^{-1}\varphi, u^{-1}f) - I(\varphi, f) &= K(u, f + \varphi) - K(u, f) \end{aligned} \right\} \quad (3.10)$$

From (3.9) we have a representation of $Q(\varphi)$ and $P(\varphi)$:

$$\left. \begin{aligned} Q(\varphi) \Psi(f) &= (f, \varphi) \Psi(f), \\ P(\varphi) \Psi(f) &= -i\partial_\varphi \Psi(f) - \left(\hat{\xi}(\varphi, f) + \frac{i}{2} \eta(\varphi, f) \right) \Psi(f), \end{aligned} \right\} \quad (3.11)$$

where

$$\begin{aligned} \partial_\varphi \Psi(f) &= \lim_{t \rightarrow 0} (\Psi(f + t\varphi) - \Psi(f)) / t, \\ \hat{\xi}(\varphi, f) &= \lim_{t \rightarrow 0} (I(t\varphi, f) - 1) / t, \\ \eta(\varphi, f) &= \lim_{t \rightarrow 0} \left(\frac{dX_{t\varphi}}{dX}(f) - 1 \right) / t. \end{aligned} \quad (3.12)$$

They have the properties

$$\begin{aligned} \partial_{\varphi+\psi} \Psi(f) &= \partial_\varphi \Psi(f) + \partial_\psi \Psi(f), \quad \hat{\xi}(\varphi + \psi, f) = \hat{\xi}(\varphi, f) + \hat{\xi}(\psi, f) \\ \eta(\varphi + \psi, f) &= \eta(\varphi, f) + \eta(\psi, f) \end{aligned} \quad (3.13)$$

and

$$X(\eta(\varphi, f) \Psi(f)) = -X(\partial_\varphi \Psi(f)). \quad (3.14)$$

(3.14) secures the self-adjointness of $P(\varphi)$. (3.11) is a generalization of the well-known Schrödinger representation of the canonical commutation relation.

In the theorem 1 we have treated the Euclidean covariant representation. If we take L instead of Σ in the theorem 1 we have the theorem corresponding to the not necessarily Euclidean covariant representation. Since $L \subset \Sigma$ (regarding L as an addition group), L -quasi-invariance of an integral X does not necessarily imply Σ -quasi-invariance of X . So that there may be representations of the canonical commutation relation in which $T(u)$ satisfying (1.8) do not exist.

Theorem 2. The representation (3.9) is irreducible, i.e. there is no operator on \mathfrak{H}_X which commutes with all of $S(\varphi)$ and $T(\varphi)$; $\varphi \in L$ but the scalar multiples of 1, if X is L -ergodic.

To prove this theorem we need the following lemma.

Lemma 1. Let \mathfrak{A} be the set of all operators $M(F)$ where F is X -essentially bounded and \mathfrak{A}' be the set of all bounded operators on \mathfrak{S}_X which commute with all operators in \mathfrak{A} . Then $\mathfrak{A} = \mathfrak{A}'$.

Proof: Since \mathfrak{A} is Abelian, $\mathfrak{A} \subset \mathfrak{A}'$ is obvious. So it suffices to show $\mathfrak{A}' \subset \mathfrak{A}$. Let B be any operator in \mathfrak{A}' . Then by the assumption

$$M(F)B\mathcal{T}(f) = BM(F)\mathcal{T}(f) = BF(f)\mathcal{T}(f). \quad (3.15)$$

Let $K(f)$ be the mapping image of 1 under the operation of B ; $B1 = K(f)$. From (3.15) we have

$$M(F)K(f) = F(f)K(f) = BF(f). \quad (3.16)$$

Let $F(f)$ be a tame function. Since \mathfrak{T} is dense in \mathfrak{S}_X , (3.16) shows for any $\mathcal{T}(f)$ in \mathfrak{S}_X

$$B\mathcal{T}(f) = K(f)\mathcal{T}(f).$$

Since B is bounded, $K(f)$ must be bounded, hence we have

$$B = M(K). \quad \text{Q.E.D.}$$

Proof of theorem 2: Let \mathfrak{R}' be the set of all bounded operators which commute with all of $S(\varphi)$ and $T(\varphi)$; $\varphi \in L$. Then it is obvious that $\mathfrak{R}' \subset \mathfrak{A}'$, hence by the lemma 1 $\mathfrak{R}' \subset \mathfrak{A}$; so any operator B in \mathfrak{R}' is in the form of $M(K)$. Let B be a projection in \mathfrak{R}' . Then from $B^2 = B$ it follows $K^2(f) = K(f)$. By assumption $T(\varphi)B = BT(\varphi)$ for all $\varphi \in L$; so we have $K(f + \varphi) = K(f)$ for all $\varphi \in L$. Define the integral Y by

$$Y(F) = X(KF).$$

Then $Y \ll X$ and Y is L -quasi-invariant. For

$$\begin{aligned} Y_\varphi(F) &= Y(F_{-\varphi}) = X(KF_{-\varphi}) = X_\varphi(K_\varphi F) = X_\varphi(KF) \\ &= X\left(\frac{dX_\varphi}{dX}KF\right) = Y\left(\frac{dX_\varphi}{dX}F\right). \end{aligned}$$

By the assumption of L -ergodicity of X , $Y \equiv 0$ or $Y \sim X$. If $Y \equiv 0$, then $K(f) \equiv 0$. If $Y \sim X$, then $dX/dY = (dY/dX)^{-1} = K^{-1}$ exists too. This and $K^2 = K$ imply $K(f) \equiv 1$, hence any projection in \mathfrak{R}' is either 1 or 0. Q.E.D.

Theorem 3. Let both X and X' be Σ -quasi-invariant integrals, and $S(\varphi)$, $T(\sigma)$ and $S'(\varphi)$, $T'(\sigma)$ be Euclidean covariant representations on \mathfrak{S}_X and $\mathfrak{S}_{X'}$ given in the form of (3.9) respectively. Then there is an isometrical mapping V from \mathfrak{S}_X to $\mathfrak{S}_{X'}$ such that

$$S'(\varphi) = VS(\varphi)V^{-1}, \quad T'(\sigma) = VT(\sigma)V^{-1} \quad (3.17)$$

if and only if $X \sim X'$ and there is a real function $C(f)$ on L^* such that

$$C(f\sigma) - C(f) = J(\sigma, f) - J'(\sigma, f). \quad (3.18)$$

Proof: Ad the "only if" part: Let $V \cdot 1 = \chi'(f) \in \mathfrak{S}_{X'}$ and $V^{-1} \cdot 1 = \chi(f) \in \mathfrak{S}_X$. Then for any tame function $F(f)$

$$X(F) = (1, M(F)1)_X = (1, V^{-1}M'(F)V1)_X = (\chi', M'(F)\chi')_{X'} = X'(|\chi'|^2 F). \quad (3.19)$$

This shows $X \leq X'$. Similarly, we have

$$X'(F) = X(|\chi|^2 F), \quad (3.19)'$$

so that $X' \leq X$, hence $X \sim X'$. Since for any tame function $F(f)$

$$VF(f) = VM(F)1 = VM(F)V^{-1}V1 = M'(F)\chi'(f) = F(f)\chi'(f)$$

and \mathfrak{I} is dense in \mathfrak{S}_X , we have

$$V\psi(f) = \chi'(f)\psi(f). \quad (3.20)$$

Similarly, we have

$$V^{-1}\psi(f) = \chi(f)\psi(f). \quad (3.20)'$$

From $VV^{-1} = 1$ and (3.20), (3.20)' we have $\chi(f)\chi'(f) = 1$. From this and (3.19), (3.19)' χ and χ' are in the form

$$\chi'(f) = \left[\frac{dX}{dX'}(f) \right]^{1/2} e^{iC(f)}, \quad \chi(f) = \left[\frac{dX'}{dX}(f) \right]^{1/2} e^{-iC(f)}. \quad (3.21)$$

By assumption $T'(\sigma) = VT(\sigma)V^{-1}$. From this and (3.20), (3.20)', (3.21)

$$\begin{aligned} VT(\sigma)V^{-1}\psi(f) &= VT(\sigma) \left[\frac{dX'}{dX}(f) \right]^{1/2} e^{-iC(f)}\psi(f) \\ &= V \left[\frac{dX_\sigma}{dX}(f) \cdot \frac{dX'}{dX}(f\sigma) \right]^{1/2} e^{-iC(f\sigma) + iJ(\sigma, f)}\psi(f\sigma) \\ &= \left[\frac{dX_\sigma}{dX}(f) \frac{dX'_\sigma}{dX_\sigma}(f) \frac{dX}{dX}(f) \right]^{1/2} \exp[i(J(\sigma, f) - C(f\sigma) + C(f))]\psi(f\sigma) \\ &= \left[\frac{dX'_\sigma}{dX'}(f) \right]^{1/2} \exp[i(J(\sigma, f) - C(f\sigma) + C(f))]\psi(f\sigma) \\ &= \left[\frac{dX'_\sigma}{dX'}(f) \right]^{1/2} e^{iJ'(\sigma, f)}\psi(f\sigma), \end{aligned} \quad (3.22)$$

hence we have $J(\sigma, f) - J'(\sigma, f) = C(f\sigma) - C(f)$.

Ad the "if" part: Define V by

$$\begin{aligned} V\psi(f) &= \left[\frac{dX}{dX'}(f) \right]^{1/2} e^{iC(f)}\psi(f), \\ V^{-1}\psi(f) &= \left[\frac{dX'}{dX}(f) \right]^{1/2} e^{-iC(f)}\psi(f). \end{aligned}$$

Then $VS(\varphi)V^{-1}=S'(\varphi)$ is obvious. Proof of $VT(\sigma)V^{-1}=T'(\sigma)$ is a tautology of (3.22). Isometry of V is proved as

$$(V\mathcal{F}, V\mathcal{F})_{X'} = X' \left(-\frac{dX}{dX'} |\mathcal{F}|^2 \right) = X(|\mathcal{F}|^2) = (\mathcal{F}, \mathcal{F})_X. \quad \text{Q. E. D.}$$

The similar theorem for the not necessarily Euclidean covariant representation is given by replacing Σ by L , $T(\sigma)$ by $T(\varphi)$ and $J(\sigma, f)$ by $I(\varphi, f)$ in theorem 3.

Theorem 3 shows that two types of unitary inequivalence of the representation may occur, i.e. from the inequivalence of the integrations and from the non-existence of the function $C(f)$ satisfying (3.18).

§ 4. The vacuum state

A vacuum state ϕ is a state in a Euclidean covariant representation which is invariant under the operation of $T(u)$ for any u in E .

$$T(u)\phi = \phi. \quad (4.1)$$

Theorem 4. Let X be Σ -quasi-invariant integral and consider a Euclidean covariant representation on \mathfrak{H}_X . Then there is a vacuum state ϕ if and only if there is an E -invariant integral Y such that $Y \ll X$ and a real function $N(f)$ on L^* such that

$$K(u, f) = N(u^{-1}f) - N(f). \quad (4.2)$$

Proof: Ad the "if" part: Let $\phi(f)$ be given by

$$\phi(f) = e^{-iN(f)} \left[\frac{dY}{dX}(f) \right]^{1/2}. \quad (4.3)$$

Then from (3.9) and (4.2) we have for every $u \in E$

$$\begin{aligned} T(u)\phi(f) &= \left[\frac{dX_u}{dX}(f) \cdot \frac{dY}{dX}(u^{-1}f) \right]^{1/2} e^{iK(u, f) - iN(u^{-1}f)} \\ &= \left[\frac{dX_u}{dX}(f) \frac{dY_u}{dX_u}(f) \right]^{1/2} e^{-iN(f)} = \left[\frac{dY}{dX}(f) \right]^{1/2} e^{-iN(f)} = \phi(f). \end{aligned}$$

Therefore $\phi(f)$ is a vacuum state. Ad the "only if" part: Let ϕ be a vacuum state. Then from $T(u)\phi(f) = \phi(f)$ and (3.9)

$$\phi(f) = e^{iK(u, f)} \left[\frac{dX_u}{dX}(f) \right]^{1/2} \phi(u^{-1}f). \quad (4.4)$$

Let Y be an integral defined by

$$Y(F) = X(|\phi|^2 F). \quad (4.5)$$

Then from (4.4)

$$Y(F_u) = X(|\phi|^2 F_u) = X\left(\frac{dX_u}{dX} |\phi_u|^2 F_u\right) = X_u(|\phi_u|^2 F_u) = X(|\phi|^2 F) = Y(F). \quad (4.6)$$

By definition $Y \ll X$ and (4.6) shows that Y is E -invariant. By definition

$$\frac{dY}{dX}(f) = |\phi(f)|^2,$$

hence $\phi(f)$ is of the form of (4.3). From this and (4.4)

$$\begin{aligned} \left[\frac{dY}{dX}(f)\right]^{1/2} e^{-iN(f)} &= e^{iK(u,f) - iN(u^{-1}f)} \left[\frac{dX_u}{dX}(f) \frac{dY}{dX}(u^{-1}f)\right]^{1/2} \\ &= e^{iK(u,f) - iN(u^{-1}f)} \left[\frac{dY}{dX}(f)\right]^{1/2}. \end{aligned}$$

Therefore we have $K(u, f) = N(u^{-1}f) - N(f)$. Q.E.D.

Theorem 5. The vacuum state is unique if X is E -ergodic.

Proof: Let $\phi(f)$ be a vacuum state. Then the integral Y defined by (4.5) is E -invariant. By the assumption of E -ergodicity of X , $Y \sim X$ or $Y = 0$. If $Y = 0$, then $\phi(f) = 0$ so that this case may be eliminated, hence $Y \sim X$. Since $\phi(f)$ is of the form of (4.3) and $N(f)$ in (4.3) satisfies (4.2) by theorem 3, we may replace X by Y and $K(u, f)$ by 0 by a suitable unitary transformation from \mathfrak{H}_X to \mathfrak{H}_Y . By this transformation $\phi(f)$ is mapped to 1. And in the representation in \mathfrak{H}_Y , $T(u)$ is given by

$$T(u)\psi(f) = \psi(u^{-1}f). \quad (4.7)$$

Let $\phi'(f)$ be another vacuum state in \mathfrak{H}_Y . Then from (4.7) for every $u \in E$

$$\phi'(f) = \phi'(u^{-1}f). \quad (4.8)$$

Consider the operator $M(\phi')$, then by (4.8) $M(\phi')$ commutes with all of $T(u)$ and *a fortiori* to all of $S(\varphi)$. By the same argument as in the proof of theorem 2 we can show that any operator which commutes with all of $T(u)$ and $S(\varphi)$ is a scalar multiple of the identity if X is E -ergodic. This shows that $\phi'(f) = \text{const.}$ Q.E.D.

§ 5. Examples

We give lastly some examples of the inequivalent irreducible Euclidean covariant representations.

(1) Let L be such that L^* is properly larger than L ; $L^* \supset L$ and $L^* \neq L$, and X be any Σ -quasi-invariant and L -ergodic integral. Then consider the representations on \mathfrak{H}_X in which $I(\varphi, f) \equiv 0$ and $I'(\varphi, f) = 2(f, \varphi) + (\varphi, \varphi)$ respectively. These representations are inequivalent, since there is no function $C(f)$ satisfying

$$C(f + \varphi) - C(f) = I'(\varphi, f) - I(\varphi, f) = 2(f, \varphi) + (\varphi, \varphi). \quad (5.1)$$

Formally (5.1) is satisfied by

$$C(f) = (f, f).$$

However, this $C(f)$ does not exist everywhere on L^* if $L^* \supset L$ and $L^* \neq L$.

(2) Canonical normal representations:¹²⁾ Let L be a Hilbert space. Then $L^* = L$. Let $\varphi_1, \varphi_2, \dots$ be an arbitrary orthonormal complete set in L . Let $F(f)$ be a tame function such that

$$F(f) = \prod_{i=1}^{\infty} F_i((f\varphi_i)) \quad (5.2)$$

where $F_i(x) = 1$ but for finite members of i 's. Every tame function can be approximated by a finite sum of the tame functions in the form of (5.2). Let X^c be an integral such that for the tame functions in the form of (5.2)

$$X^c(F) = \prod_{i=1}^{\infty} X_i^c(F_i), \quad (5.3)$$

where

$$X_i^c(F_i) = \frac{c}{\sqrt{\pi}} \int_{-\infty}^{\infty} F_i(x) e^{-c^2 x^2} dx. \quad (5.4)$$

Since $X_i^c(1) = (c/\sqrt{\pi}) \int_{-\infty}^{\infty} e^{-c^2 x^2} dx = 1$, (5.3) is actually a finite product for the tame functions in the form of (5.2). (5.3) with (5.4) determines an integral X^c . If $c \neq c'$, then X^c and $X^{c'}$ are not equivalent. From the definition of dX/dY and (5.3) if $X^c \sim X^{c'}$, $dX^c/dX^{c'}$ must be given by

$$\frac{dX^c}{dX^{c'}}(f) = \prod_{i=1}^{\infty} [ce^{-c^2(f, \varphi_i)^2}/c'e^{-c'^2(f, \varphi_i)^2}] = \prod_{i=1}^{\infty} \left[\frac{c}{c'} e^{-(c^2 - c'^2)(f, \varphi_i)^2} \right]. \quad (5.5)$$

However, (5.5) is 0 or ∞ for any f if $c \neq c'$. Hence $X^c \not\sim X^{c'}$ if $c \neq c'$. X^c is Σ -quasi-invariant, for

$$\frac{dX_{\sigma(u, \varphi)}^c}{dX^c}(f) = \prod_{i=1}^{\infty} [e^{-c^2((u^{-1}f + \varphi), \varphi_i)^2}/e^{-c^2(f, \varphi_i)^2}] = \exp[-c^2(\varphi, \varphi) - 2c^2(u^{-1}f, \varphi)]. \quad (5.6)$$

Now let us show that any representation on \mathfrak{H}_X is irreducible. To prove this we need the following lemma.

Lemma 2. Let \mathfrak{R} be a ring of von Neumann¹³⁾ on a Hilbert space \mathfrak{H} and $[\mathfrak{R}\mathcal{V}]$ be the closed linear manifold spanned by the elements of \mathfrak{H} in the form of $A\mathcal{V}$, $A \in \mathfrak{R}$. Suppose that \mathcal{V} is such that $[\mathfrak{R}\mathcal{V}] = \mathfrak{H}$. Then \mathfrak{R} coincides with the ring \mathfrak{B} of all bounded operators on \mathfrak{H} if and only if the projection $E_{\mathcal{V}}$ onto is in \mathfrak{R} .

Proof: Ad the "only if" part: Obvious. Ad the "if" part: By assumption, for any $\phi \in \mathfrak{H}$ there is a sequence of operators $A_1, A_2, \dots \in \mathfrak{R}$ such that $A_n\mathcal{V}$ converges to ϕ . Define the projections E_n by

$$E_n = A_n E_{\Psi} A_n^* / (\Psi, A_n^* A_n \Psi).$$

Then E_n converges weakly to E_θ , for, for any $\chi, \chi' \in \mathfrak{H}$,

$$(\chi', E_n \chi) = (\chi', A_n \Psi) (A_n \Psi, \chi) / (A_n \Psi, A_n \Psi),$$

so that $\lim (\chi', E_n \chi) = (\chi', \Phi) (\Phi, \chi) / (\Phi, \Phi) = (\chi', E_\theta \chi)$; accordingly, if E_Ψ is in \mathfrak{R} , E_θ is in \mathfrak{R} for every $\Phi \in \mathfrak{H}$. This means $\mathfrak{R} = \mathfrak{B}$.

Proof of the statement: Let Ψ and \mathfrak{R} in the above lemma be the identity function in \mathfrak{H}_{X^c} and the ring generated by $S(\varphi)$; $\varphi \in L$ respectively. Since $\mathfrak{R} \supset \mathfrak{H}$, $[\mathfrak{R} 1] = \mathfrak{H}_{X^c}$. Let $\mathfrak{H}_{X_i^c}$ be the Hilbert spaces of the functions $F_i(x)$ whose scalar product $(F_i, G_i)_i$ is given by

$$(F_i, G_i)_i = \frac{c}{\sqrt{\pi}} \int_{-\infty}^{\infty} \bar{F}_i(x) G_i(x) e^{-c^2 x^2} dx.$$

Then \mathfrak{H}_{X^c} is the incomplete infinite direct product space $\prod_{i=0}^{\infty} \mathfrak{H}_{X_i^c}$ of von Neumann¹⁴⁾ which contains the identity function. Then we may consider the representation of $S(\varphi_i)$ and $T(\varphi_i)$ on $\mathfrak{H}_{X_i^c}$ which is the restriction of $S(\varphi_i)$ and $T(\varphi_i)$ on \mathfrak{H}_{X^c} onto $\mathfrak{H}_{X_i^c}$. Since any representation of $S(\varphi_i)$ and $T(\varphi_i)$ on $\mathfrak{H}_{X_i^c}$ is irreducible in $\mathfrak{H}_{X_i^c}$, the projection E_i onto the identity function in $\mathfrak{H}_{X_i^c}$ is in \mathfrak{R}_i where \mathfrak{R}_i is the ring generated by $S(\varphi_i)$ and $T(\varphi_i)$ on $\mathfrak{H}_{X_i^c}$. Since \mathfrak{R} is generated by $\mathfrak{R}_i, i=1, 2, \dots$, and $E = \prod_{i=1}^{\infty} E_i$, $E_i \in \mathfrak{R}_i$ implies $E \in \mathfrak{R}$, where E is the projection onto the identity function in \mathfrak{H}_{X^c} , hence by lemma 2 we have $\mathfrak{R} = \mathfrak{B}$. Q.E.D.

Thus we have seen that there are at least continuously many inequivalent Euclidean covariant irreducible representations.

Now, consider the representation of $S(\varphi)$ and $T(\varphi)$ on \mathfrak{H}_{X^c} in which $I(\varphi, f)$ and $K(u, f)$ in (3.9) is zero. Then from (5.6), $\eta(\varphi, f)$ in (3.12) is given by

$$\eta(\varphi, f) = -2c^2(f, \varphi), \quad (5.7)$$

accordingly, from (3.11) we have

$$(P(\varphi) - ic^2 Q(\varphi))1 = 0. \quad (5.8)$$

By introducing the annihilation-creation operators $a(\varphi)$ and $a^*(\varphi)$ by

$$a(\varphi) = \frac{1}{\sqrt{2}} (Q(\varphi) + iP(\varphi)), \quad a^*(\varphi) = \frac{1}{\sqrt{2}} (Q(\varphi) - iP(\varphi)), \quad (5.9)$$

(4.8) is rewritten as

$$[(c^2 + 1)a(\varphi) + (c^2 - 1)a^*(\varphi)]1 = 0. \quad (5.10)$$

If $c=1$, then (5.10) shows that the identity function in \mathfrak{H}_{X^1} must be identified with the free vacuum state; so the representation in \mathfrak{H}_{X^1} is identical with the representation of free field. For $c \neq 1$, by the transformations

$$Q(\varphi) \rightarrow Q(c\varphi), \quad P(\varphi) \rightarrow P(c^{-1}\varphi), \quad \mathcal{F}(f) \rightarrow \mathcal{F}(c^{-1}f) \quad (5.11)$$

the representation in \mathfrak{H}_{X^c} is transformed into the representation in \mathfrak{H}_{X^1} .

Finally, from (5.6) we have

$$dX_u^c/dX^c=1, \quad (5.12)$$

hence X^c is E -invariant. From the 3rd equation in (3.9) and (5.12) we have $T(u)1=1$ for every $u \in E$, hence the identity function in \mathfrak{H}_{X^c} is the vacuum state. It is well known that in the free field representation the vacuum state is unique. Since the representation in \mathfrak{H}_{X^c} is connected with the representation in \mathfrak{H}_{X^1} by transformation (5.11), it is obvious that the vacuum state in \mathfrak{H}_{X^c} is also unique. It is of considerable interest whether there are inequivalent Euclidean covariant representations containing unique vacuum state other than the ones given here. We have no definite result about this question.

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Note added in proof :

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A New Approach to the Theory of Classical Fluids. I

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An exact integral equation is found for the pair distribution function. The integral equation is of somewhat different nature from the usual ones known in the theory of classical fluids, in the point that it involves an infinite series. The Helmholtz free energy is expressed as a series expansion which may be more rapidly convergent than the usual one. It is shown that the integral equation can be derived also by means of a variational principle from the expression for the free energy. It is pointed out that the theory of classical fluids may be constructed with the knowledge of the pair distribution function alone, even if a form of the pair interaction potential is not known.

§ 1. Introduction

In the classical statistical mechanics, the thermodynamic quantities and pair distribution function of an imperfect gas are expressed as a form of series expansion in powers of the density.^{1),2),3)} As such series expansions necessarily converge badly at high densities, many attempts have been made to overcome such a difficulty.

One of these attempts is based on the integral equation method for deriving the pair distribution function. Since the integral equation for the pair distribution function involves the triple distribution function, it is necessary to introduce the assumption of a suitable closure in order to solve the integral equation. The closure which has been widely used is the Kirkwood superposition approximation, leading to the well-known integral equations of Kirkwood⁴⁾ and Yvon-Born-Green;^{5),6),7)} one of the present authors has pointed out that there exists an inconsistency in the solution of these integral equations.⁸⁾

Other attempts are based on partial summation of particular terms appearing in the series expansions. This method has originated from the work of Montroll and Mayer⁹⁾ for the pair distribution function. Remarkable progress has been made after that.^{10),11)} It is hoped that this method is able to give a formalism which may be useful at high densities, especially, at the densities of liquid state.

Following the latter method, one of the present authors has proposed the *hyper-netted chain approximation* to consider by means of the Fourier transformation as many terms as possible in the series expansion formulas for the Helmholtz free energy and the pair distribution function.¹²⁾ He has shown also that the pair distribution function in this approximation satisfies an integral equation similar to the Yvon-Born-Green integral equation.^{13),14)}

In the preliminary report¹⁵⁾ we have shown that the technique used in reference 14) can be applied to get an exact integral equation for the pair distribution function. It has also been shown that the hyper-netted chain approximation appears as the zeroth approximation to solve the integral equation. Recently, van Leeuwen, Groeneveld and de Boer¹⁶⁾ have derived the integral equation which is identical with the one derived by us.

The purpose of the present paper is to present the details of the preliminary report and to propose a new expansion scheme of the thermodynamic quantities which may be useful at high densities.

In the case of the hyper-netted chain approximation, one of the authors has derived the expressions for the pair distribution function, the chemical potential and the free energy, independently.^{12), 13), 14)} The calculation was simple for the pair distribution function but it was very cumbersome for the chemical potential and the free energy. Because of this fact, the other of the authors has noticed that the calculation will be much simplified if we first get the expression for the pair distribution function and then derive the expression for the free energy by means of the method which is equivalent to integrating the expression for the internal energy in terms of the pair distribution function just obtained.¹⁷⁾ In this way we have two methods in deriving the expression for the free energy. That is to say, the one is to use the series expansion formula for the free energy and the other is to use the derived expression for the pair distribution function.

When we derive the expressions for the free energy and the pair distribution function in some approximation, the latter method is applicable in a consistent way only if some compatibility condition is satisfied in that approximation.^{8), 17), 18)} When we do not know whether this compatibility condition is satisfied in that approximation, we have to derive the expressions for the pair distribution function and for the free energy, independently. Therefore it was meaningful to derive the expressions for the pair distribution function and for the free energy, independently, in the case of the hyper-netted chain approximation; though in this case it was confirmed later that the above compatibility condition was satisfied and so the two methods gave an identical result.

The situation is much simpler when no approximation is introduced into the theory, because in this case there cannot be any discrepancy among the expressions for the free energy derived by different methods. It is sufficient to derive the expression by the simplest method. In the text of the present paper we shall use the latter method in deriving the expression for the free energy. For the sake of completeness and technical interest, we shall derive the expression in the Appendix by means of the former method.

The contents of the present paper are as follows. In § 2 we present the definition of the functions to be used in the following sections. In § 3 we derive the exact integral equation for the pair distribution function by starting from the series expansion of the pair distribution function. The results of § 3 have been

reported previously¹⁵⁾ and obtained independently by de Boer *et al.*¹⁶⁾ In § 4 we derive the expression for the Helmholtz free energy with the aid of the results of § 3. In § 5 we show that the integral equation derived in § 3 can be obtained also from the free energy in § 4 by means of a variational principle. The stationary character of the free energy is shown to be convenient in the derivation of the expressions for some thermodynamic quantities. In § 6 we give a summary of our results and some remarks. In the Appendix we derive the integral equation for the pair distribution function as well as the expressions for the chemical potential and the free energy by means of the *graphical method*.¹⁴⁾ The derivation of the integral equation presented there may be more easily understood than that in § 3, for a reader who is familiar with the graphical method.

§ 2. Definition of functions $w(r)$, $x(r)$, $z_s(r)$, $z(r)$ and $v(r)$

Let us consider a classical fluid of volume V and temperature T composed of N identical particles interacting with central forces. The pair distribution function $g(r)$ is defined by

$$g(r_{12}) = \lim_{\substack{N \rightarrow \infty \\ V \rightarrow \infty \\ \rho = N/V = \text{const.}}} \frac{V^2 \int \cdots \int d\mathbf{r}_3 \cdots d\mathbf{r}_N \exp \left[-\frac{1}{kT} \sum \phi(r_{ij}) \right]}{\int \cdots \int d\mathbf{r}_1 \cdots d\mathbf{r}_N \exp \left[-\frac{1}{kT} \sum \phi(r_{ij}) \right]}, \quad (2.1)$$

where $\phi(r_{ij})$ is the pair interaction potential between particles i and j . The definition of the pair distribution has been chosen in such a way that $g(r)$ becomes rigorously equal to 1 for values of r much larger than the range of $\phi(r)$.

We shall start with considering the function $w(r)$, defined by

$$g(r) = \exp \left[-\frac{\phi(r)}{kT} + w(r) \right]. \quad (2.2)$$

It is known that $w(r)$ can be expressed as a series expansion in powers of particle number density $N/V = \rho$ in the following form:¹⁹⁾

$$w(r_{12}) = \sum_{m=1}^{\infty} \frac{\rho^m}{m!} \int \cdots \int d\mathbf{r}_3 \cdots d\mathbf{r}_{m+2} \sum_{m+2 \geq i > j \geq 1}^{(W)} \prod b(r_{ij}), \quad (2.3)$$

where

$$b(r_{ij}) \equiv e^{-\phi(r_{ij})/kT} - 1. \quad (2.4)$$

In order to explain the meaning of $\sum^{(W)}$ in (2.3), we shall adopt a graphical representation of one of the products $\prod b(r_{ij})$ occurring in (2.3). Particles 1 and 2 are represented by a white circle respectively. Particles 3, 4, \cdots and $m+2$, over which the integrations are to be performed, are represented by numbered points (or numbered black circles). A factor $b(r_{ij})$ is represented by a line (b -bond) drawn between two points i and j . Then a product $\prod b(r_{ij})$ is represented by a bond diagram. Two particles i and j are said to be directly connected if $b(r_{ij})$

appears in the product. Obviously, two or more particles may also be connected indirectly, through other particles.

Futhermore, we shall introduce an important notion of "s-point", which is a point by which the diagram can be separated into two independent parts. In other words, it may be said to be a point through which all possible paths going from particle 1 to particle 2 must pass. Such a point has been called a "node" by de Boer *et. al.*¹⁶⁾ In Fig. 1 the s-points are represented by black circles.

For convenience of later considerations, we introduce following four restrictions which are to be imposed on the summation of diagrams:

- | | |
|---|-------|
| (I): Each particle of the set 3, 4, ..., m+2 is independently connected to particles 1 and 2.
(II): Particles 3, 4, ..., m+2 are connected among themselves independently of particles 1 and 2.
(III): Particles 1 and 2 are not directly connected.
(IV): The diagram has no s-point. | (2.5) |
|---|-------|

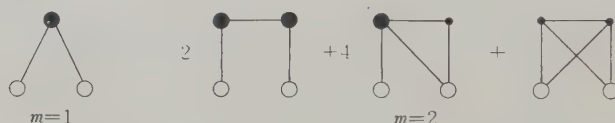


Fig. 1. Diagrams appearing in $\Sigma^{(W)}$ for $m=1$ and $m=2$.

Black circles represent s-points.

Now $\Sigma^{(W)}$ occurring in (2.3) is defined in such a way that the sum is taken over all diagrams being possible under restrictions (I), (II) and (III) (see Fig. 1). We shall define functions $x(r)$, $z_s(r)$, $z(r)$ and $v(r)$ as follows:

$$x(r_{12}) = \sum_{m=2}^{\infty} \frac{\rho^m}{m!} \int \cdots \int d\mathbf{r}_3 \cdots d\mathbf{r}_{m+2} \sum_{m+2 \geq i > j \geq 1} \Sigma^{(X)} \Pi b(r_{ij}), \quad (2.6)$$

where $\Sigma^{(X)}$ is the sum over all diagrams being possible under restrictions (I), (II), (III) and (IV).

$$z_s(r_{12}) = \sum_{m=1}^{\infty} \frac{\rho^m}{m!} \int \cdots \int d\mathbf{r}_3 \cdots d\mathbf{r}_{m+2} \sum_{m+2 \geq i > j \geq 1} \Sigma^{(Z_s)} \Pi b(r_{ij}), \quad (2.7)$$

where $\Sigma^{(Z_s)}$ is the sum over all diagrams which are possible under restriction (I) and also the restriction that there exists at least one s-point. Such diagrams are automatically under restrictions (II) and (III).

$$z(r_{12}) = \sum_{m=0}^{\infty} \frac{\rho^m}{m!} \int \cdots \int d\mathbf{r}_3 \cdots d\mathbf{r}_{m+2} \sum_{m+2 \geq i > j \geq 1} \Sigma^{(Z)} \Pi b(r_{ij}), \quad (2.8)$$

where $\Sigma^{(Z)}$ is the sum over all diagrams being possible under restrictions (I) and (IV).

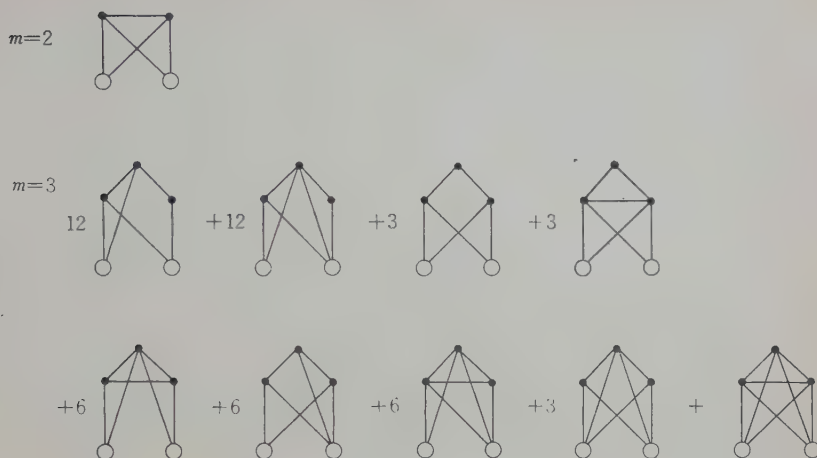


Fig. 2. Diagrams appearing in $\Sigma^{(X)}$ for $m=2$ and $m=3$.

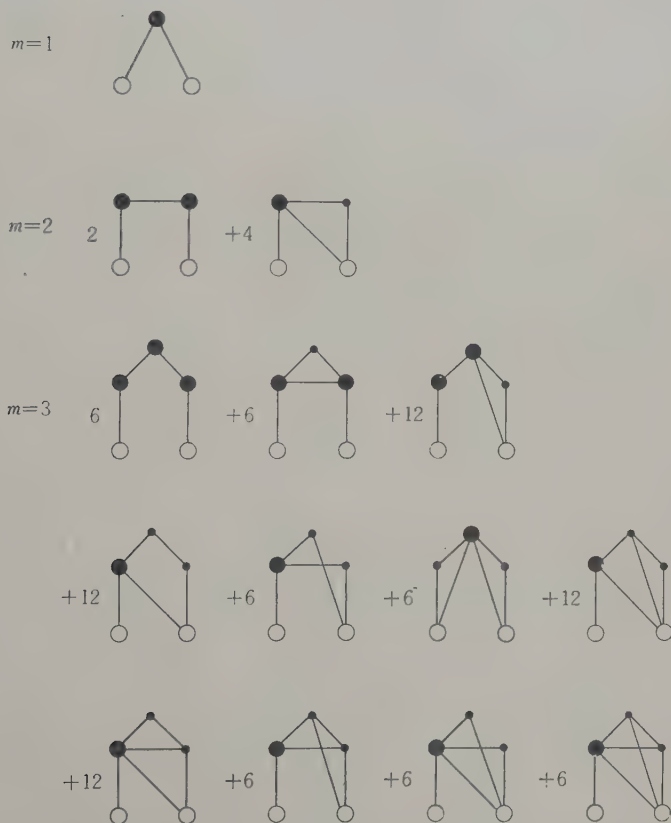


Fig. 3. Diagrams appearing in $\Sigma^{(Z_S)}$ for $m=1, 2$ and 3 . Black circles represent s -points.

Fig. 4. Diagrams appearing in $\Sigma^{(Z)}$ for $m=0, 1$ and 2 .

$$v(r_{12}) = \sum_{m=0}^{\infty} \frac{\rho^m}{m!} \int \cdots \int d\mathbf{r}_3 \cdots d\mathbf{r}_{m+2} \sum_{m+2 \geq i > j \geq 1}^{(\mathcal{V})} \prod b(r_{ij}), \quad (2.9)$$

where $\Sigma^{(\mathcal{V})}$ is the sum over all diagrams being possible under restriction (I) alone.

Some diagrams appearing in $\Sigma^{(X)}$, $\Sigma^{(Z_S)}$ and $\Sigma^{(Z)}$ are shown in Figs. 2, 3 and 4, respectively.

It is also known that the pair distribution function is expressed in a form^{2), 19)}

$$g(r_{12}) = e^{-\phi(r_{12})/kT} \left[1 + \sum_{m=1}^{\infty} \frac{\rho^m}{m!} \int \cdots \int d\mathbf{r}_3 \cdots d\mathbf{r}_{m+2} \sum_{m+2 \geq i > j \geq 1}^{(\mathcal{G})} \prod b(r_{ij}) \right], \quad (2.10)$$

where $\Sigma^{(\mathcal{G})}$ is the sum over all diagrams being possible under restrictions (I) and (III). The equivalence of (2.10) to (2.2) with (2.3) has been proved by Meeron¹⁹⁾ and also by de Boer *et al.*¹⁶⁾

In the following the Fourier transforms of $w(r)$, $x(r)$, $z_s(r)$, $z(r)$ and $v(r)$ will be denoted as $W(k)$, $X(k)$, $Z_s(k)$, $Z(k)$ and $V(k)$, respectively. Namely, for example,

$$z(r) = \frac{1}{V} \sum_k Z(k) e^{ikr} \quad (2.11)$$

$$Z(k) = \int d\mathbf{r} z(r) e^{-ikr}.$$

The functions defined in this section are not independent of each other. The relations among them will be discussed in the next section.

§ 3. Integral equation for the pair distribution function

We shall derive several relations among the functions defined in the preceding section. The relations will be seen to give an integral equation for the pair distribution function.

Let us begin with the consideration of diagrams in $\Sigma^{(W)}$, which has been defined

below (2.5). The diagrams appearing in $\Sigma^{(W)}$ can be divided into two groups. One group consists of the diagrams having no s -point, which are just those appearing in $\Sigma^{(X)}$, as is seen from (2.6). The other group consists of the diagrams having at least one s -point, which are just those appearing in $\Sigma^{(ZS)}$, as is seen from (2.7). Therefore we get

$$w(r) = x(r) + z_s(r). \quad (3.1)$$

In quite the same way, we have

$$v(r) = z(r) + z_s(r). \quad (3.2)$$

Diagrams appearing in $\Sigma^{(ZS)}$ can be grouped together by the number of s -points. Let us consider a diagram having n s -points which is shown in Fig. 5. Let r_1 particles be between particle 1 and s -point s_1 , r_2 particles between s_1 and s_2 , and so on. Obviously we have

$$\sum_{i=1}^{n+1} r_i + n = m.$$



Fig. 5. A diagram appearing in $\Sigma^{(ZS)}$ which has n s -points.

The number of ways in which n particles may be selected from m particles and arranged in order on n s -points and the remaining $(m-n)$ particles may be divided into $(n+1)$ groups each containing r_1, r_2, \dots, r_{n+1} particles is

$$\frac{m!}{(m-n)!} \times \frac{(m-n)!}{\prod_{i=1}^{n+1} r_i!} = \frac{m!}{\prod_{i=1}^{n+1} r_i!}.$$

Therefore the diagrams having n s -points contribute

$$\sum_{\substack{n+1 \\ \sum_{i=1}^{n+1} r_i = m-n}} \rho^n \int d\mathbf{r}_{s_1} \cdots d\mathbf{r}_{s_n} \left[\frac{\rho^{r_1}}{r_1!} \int d\{r_1\} \Sigma^{(Z)} \prod b(r_{ij}) \right] \left[\frac{\rho^{r_2}}{r_2!} \int d\{r_2\} \Sigma^{(Z)} \prod b(r_{ij}) \right] \\ \times \cdots \times \left[\frac{\rho^{r_{n+1}}}{r_{n+1}!} \int d\{r_{n+1}\} \Sigma^{(Z)} \prod b(r_{ij}) \right]$$

to $z_s(r_{12})$, where the first sum is taken over all sets $\{r_1, r_2, \dots, r_{n+1}\}$ satisfying $\sum_{i=1}^{n+1} r_i = m-n$, and $\int d\{r_i\}$ is the integral over r_i particles. In the above expression $\Sigma^{(Z)}$ appears, because the diagrams which are to appear between neighboring s -points are free from restrictions (II) and (III) of (2.5) but must not have an s -point (see (2.8)). When the above contributions are summed with respect to m , the summations over r_1, r_2, \dots and r_{n+1} can be performed independently. As the result we get, with the aid of (2.8),

$$z_s(r_{12}) = \sum_{n=1}^{\infty} \rho^n \int d\mathbf{r}_3 d\mathbf{r}_4 \cdots d\mathbf{r}_{n+2} z(r_{13}) z(r_{34}) \cdots z(r_{n+1, n+2}) z(r_{n+2, 2}). \quad (3.3)$$

By the use of Fourier transforms, (3.3) is rewritten in a compact form:

$$\begin{aligned} Z_s(k) &= \rho Z^2(k) + \rho^2 Z^3(k) + \cdots \\ &= \frac{\rho Z^2(k)}{1 - \rho Z(k)}. \end{aligned} \quad (3.3')$$

Next we shall consider diagrams appearing in $\Sigma^{(Z)}$. These diagrams can be grouped together by the number of parts into which the diagrams are divided when particles 1 and 2 are removed. Let us consider a diagram which is to be divided into n parts (see Fig. 6). Let the parts contain r_1, r_2, \cdots and r_n particles respectively. We first suppose that particles 1 and 2 are not directly connected in the diagram, namely $b(r_{12})$ does not appear. The number of ways in which m particles may be divided into n groups each containing r_1, r_2, \cdots, r_n particles is

$$m! / \prod_{i=1}^n r_i!,$$

where it is noted that $\sum_{i=1}^n r_i = m$. Therefore the diagrams which are to be divided into n parts contribute

$$\sum_{\substack{n \\ \sum_{i=1}^n r_i = m}} \frac{1}{n!} \prod_{i=1}^n \left[\frac{\rho^{r_i}}{r_i!} \int d\{r_i\} \Sigma^{(W)} \prod b(r_{ij}) \right]$$

to $z(r_{12})$, where a factor $1/n!$ has been introduced because the first sum is to be taken independently over r_1, r_2, \cdots and r_n satisfying $\sum_{i=1}^n r_i = m$. The reason why $\Sigma^{(W)}$ appears in the above expression is that the diagrams appearing in each part are to be restricted by (I) and (II) of (2.5) and have been assumed tentatively not to contain $b(r_{12})$. However, $\Sigma^{(W)}$ must be replaced by $\Sigma^{(Z)}$ when $n=1$, since diagrams appearing in $\Sigma^{(Z)}$ must not contain an s -point. When the summation over m is performed, the summation over r_1, r_2, \cdots and r_n in the above expression can be performed independently. Therefore the contribution of diagrams not containing $b(r_{12})$ to $z(r_{12})$ is found to be, by the use of (2.6) and (2.3),

$$x(r_{12}) + \sum_{n=2}^{\infty} \frac{1}{n!} w^n(r_{12}) = x(r_{12}) + e^{w(r_{12})} - 1 - w(r_{12}).$$

So far, we have considered the diagrams in which $b(r_{12})$ does not appear. In a similar way as above, the contribution of diagrams containing $b(r_{12})$ to $z(r_{12})$ can be shown to be

$$b(r_{12}) + b(r_{12}) \sum_{n=1}^{\infty} \frac{1}{n!} w^n(r_{12}) = b(r_{12}) e^{w(r_{12})}.$$

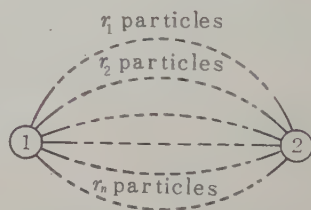


Fig. 6. A diagram appearing in $\Sigma^{(Z)}$ which has n parts.

Adding this contribution to the former one, we can write $z(r)$ in a form:

$$z(r) = [b(r) + 1]e^{w(r)} - 1 + x(r) - w(r). \quad (3.4)$$

In the preliminary report,¹⁵⁾ (3.3) (or (3.3')) and (3.4) have been expressed symbolically as

$$z_S = \text{---} z \text{---} = \text{---} z \text{---} z \text{---} + \text{---} z \text{---} z \text{---} z \text{---} + \dots \quad (3.3f)$$

and

$$\begin{aligned} z &= b + \text{---} b \text{---} w + \text{---} b \text{---} w \text{---} w + \dots \\ &+ x + \text{---} w \text{---} w + \text{---} w \text{---} w \text{---} w + \dots \end{aligned} \quad (3.4f)$$

The meaning of (3.3f) is obvious if we compare (3.3f) with (3.3) or (3.3'). The meaning of (3.4f) is also obvious if we remember two expressions above (3.4). Conversely if we wish to derive (3.3) or (3.3') from (3.3f), we may make a diagram occurring on the right-hand side of (3.3f) correspond to the integral, the integrand of which is the product of z -functions having the factor ρ to the power of the number of black circles in the diagram. For example,

$\text{---} z \text{---} z \text{---} z \text{---}$ corresponds to the integral $\rho^3 \int d\mathbf{r}_3 d\mathbf{r}_4 z(r_{13}) z(r_{34}) z(r_{42})$. To

derive (3.4) from (3.4f) we may make a diagram correspond to the product of functions occurring in the diagram, divided by the symmetry number of the diagram.

For example $\text{---} b \text{---} w$ and $\text{---} w \text{---} w$ correspond to $1/2! \cdot b(r) w^2(r)$ and $1/3! \cdot w^3(r)$,

respectively. In fact such an approach has been developed in reference 14) and will be used in the Appendix of the present paper.

Next we shall show that the sum $\Sigma^{(X)}$ appearing in the definition (2.6) of $x(r)$ can be reduced to a more restricted sum $\Sigma^{(X')}$ if one uses v -bonds in place of b -bonds. Let a diagram appearing in $\Sigma^{(X)}$ have a pair of points by which the diagram is divided into two parts in the following way. The one is the part that includes particles 1 and 2 and cannot be divided any more by that pair of points. The other is the part consisting of the rest. Diagrams to be included in the latter part are restricted by restriction (I) of (2.5) alone, namely they belong to $\Sigma^{(r)}$ (see (2.9)). Therefore the latter part can be represented by a v -bond inserted between that pair of points, the original diagram in $\Sigma^{(X)}$ being reduced to a simpler form. Such a reduction can be continued until any such pair of points does not appear. After all (2.6) is reduced to

$$x(r_{12}) = \sum_{m=2}^{\infty} \frac{\rho^m}{m!} \int \dots \int d\mathbf{r}_3 \dots d\mathbf{r}_{m+2} \sum_{m+2 \geq i > j \geq 1} \Sigma^{(X')} \prod v(r_{ij}), \quad (3.5)$$

where $\Sigma^{(X')}$ is the sum over all diagrams, composed of v -bonds, being possible

under the following restriction (V) as well as (I), (II), (III) and (IV) of (2.5).*

(V): *There exists no part which is only connected to the rest of the diagram by means of two points.*

In Fig. 7 we show some diagrams appearing in $\Sigma^{(X')}$, where the bold lines denote v -bonds. Comparing Fig. 7 with Fig. 2, one sees that a set of diagrams appearing in $\Sigma^{(X')}$ is in fact a subset of those appearing in $\Sigma^{(X)}$.

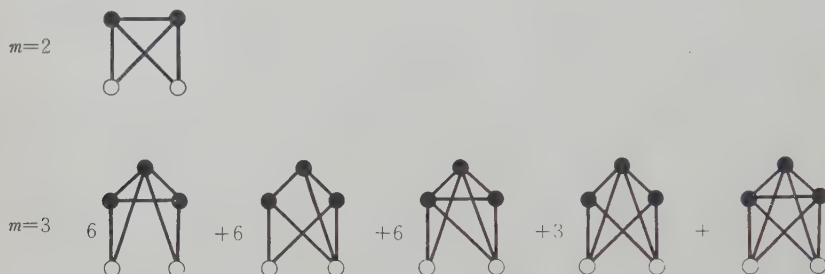


Fig. 7. Diagrams appearing in $\Sigma^{(X')}$ for $m=2$ and $m=3$.
Bold lines denote v -bonds.

A set of (3.1), (3.2), (3.3') and (3.4) can be arranged, by eliminating $z_s(r)$ or $Z_s(k)$, as follows:

$$v(r) = [b(r) + 1]e^{w(r)} - 1 \quad (3.6)$$

$$1 - \rho Z(k) = \frac{1}{1 + \rho V(k)} \quad (3.7)$$

$$w(r) = x(r) + v(r) - z(r). \quad (3.8)$$

It is to be remembered that $Z(k)$ and $V(k)$ are the Fourier transforms of $z(r)$ and $v(r)$ respectively. The function $v(r)$ is related to the pair distribution function $g(r)$ by

$$v(r) = g(r) - 1, \quad (3.9)$$

which is readily seen with the aid of (2.2), (2.4) and (3.6).** A set of (3.5), (3.6), (3.7) and (3.8) determines in principle the functions $v(r)$, $w(r)$, $z(r)$ and $x(r)$, and so the pair distribution function $g(r)$. Therefore it may be said that this set of equations is the integral equation for the pair distribution function. It should be noted that this integral equation is an exact one.

The set of equations may be solved for example by the iteration as follows. Introduce a tentative form of $w(r)$ in (3.6) to calculate $v(r)$. Next use thus calculated $v(r)$ in (3.5) and (3.7) to calculate $x(r)$ and $z(r)$ respectively. Then


* It is noted that the original formula (2.6) can be reproduced if we introduce the formula (2.9) for $v(r)$ into (3.5).

** Relation (3.9) may be derived also by comparing (2.10) with (2.9).

one gets an improved form of $w(r)$ by using thus calculated $v(r)$, $x(r)$ and $z(r)$ in (3.8). Repeating such a procedure, we shall obtain in principle the exact forms of $w(r)$, $v(r)$, $x(r)$ and $z(r)$ and so the exact form of $g(r)$, if the tentative form $w(r)$ is suitably chosen and the iteration converges. If one starts with $w(r)=0$, the above procedure will give the series expansions of $w(r)$, $v(r)$, $x(r)$ and $z(r)$ in powers of ρ which have been shown in (2.3), (2.9), (2.6) and (2.8) respectively. When the pair interaction potential $\phi(r)$ has the Fourier transform, one may start with $v(r)=\phi(r)/kT$ in the above procedure. Then we shall obtain the *giant cluster expansion* which has been proposed by Abé.¹¹⁾

Although the set of (3.5), (3.6), (3.7) and (3.8) determines in principle the pair distribution function exactly, we must in practice restrict the types of diagrams appearing in (3.5) within some special types. If we take $x(r)=0$ as the zeroth approximation, the set of equations is reduced to

$$\left. \begin{aligned} z(r) &= [b(r) + 1] e^{w(r)} - 1 - w(r) \\ \text{and} \quad W(k) &= \frac{\rho Z^2(k)}{1 - \rho Z(k)} \end{aligned} \right\}, \quad (3.10)$$

which are just the equations in the hyper-netted chain approximation proposed by one of the authors.^{13), 14)} It is noted that in this approximation $w(r)$ is equal to $z_s(r)$. The next approximation will be to approximate $x(r)$ by the contribution arising from the diagram  (see Fig. 7).

The above-derived set of equations is equivalent to the set of equations which have been derived by van Leeuwen, Groeneveld and de Boer.¹⁶⁾ We shall show the correspondence of our notations to theirs in Table I. Diagrams appearing in $\Sigma^{(X')}$ (see (3.5)) have been called "basic elementary diagrams" by de Boer *et al.*

Table I. The correspondence of the present notations to those of de Boer *et al.*¹⁶⁾
The upper row shows the present notations and the lower row those of de Boer *et al.*

$b(r)$	$w(r)$	$x(r)$	$z_s(r)$	$z(r)$	$v(r)$	$[b(r) + 1] e^{w(r)} - 1$
$f(r)$	$S(r)$	$E(r)$	$N(r)$	$X(r)$	$G(r)$	$F(r)$

§ 4. Expression for the Helmholtz free energy

As is well-known, the part of the Helmholtz free energy due to the presence of interactions is expressed in a form:¹⁾

$$\frac{A_1}{NkT} = -\frac{1}{\rho} \sum_{m=2}^{\infty} \frac{\rho^m}{m!} \cdot \frac{1}{V} \int \dots \int d\mathbf{r}_1 \dots d\mathbf{r}_m \sum_{m \geq i > j \geq 1} \prod b(r_{ij}). \quad (4.1)$$

All products which are more than singly connected.

It is possible to reduce (4.1) in a similar way as in the preceding section. In fact it will be undertaken in the Appendix. Since it is somewhat complicated and tedious, an alternative procedure will be used to derive the expression for the free energy in the present section.

One of the authors has shown that the part of the free energy due to the presence of interactions is in general expressed as¹⁷⁾

$$A_1 = N \frac{\rho}{2} \int d\hat{\xi} \int d\mathbf{r} \phi(r) g(r; \hat{\xi}), \quad (4.2)$$

where $g(r; \hat{\xi})$ is the pair distribution function in such a case that the pair interaction potential is $\hat{\xi}\phi(r)$. As far as a classical fluid is concerned, relation (4.2) is equivalent to the thermodynamical relation

$$E_1 = \frac{\partial(A_1/T)}{\partial(1/T)}, \quad (4.3)$$

since the interaction part E_1 of the internal energy is expressed in terms of $g(r)$ as⁷⁾

$$E_1 = N \frac{\rho}{2} \int d\mathbf{r} \phi(r) g(r). \quad (4.4)$$

When the pair interaction potential is $\hat{\xi}\phi(r)$, the functions $b(r)$, $w(r)$, and so on, are also dependent on $\hat{\xi}$. In particular $b(r)$ is

$$b(r; \hat{\xi}) = e^{-\hat{\xi}\phi(r)/kT} - 1. \quad (4.5)$$

Taking notice of (2.2) and (4.5), we rewrite (4.2) in a form:

$$\frac{A_1}{NkT} = -\frac{\rho}{2} \int d\hat{\xi} \int d\mathbf{r} \frac{\partial b(r; \hat{\xi})}{\partial \hat{\xi}} e^{w(r; \hat{\xi})}. \quad (4.2')$$

In the following, $\hat{\xi}$ occurring in $b(r)$, $w(r)$, and so on, will not be written explicitly, so far as the confusion does not occur.

Differentiation of (3.6) with respect to $\hat{\xi}$ leads to

$$\frac{\partial b(r)}{\partial \hat{\xi}} e^{w(r)} = \frac{\partial v(r)}{\partial \hat{\xi}} - \frac{\partial w(r)}{\partial \hat{\xi}} - v(r) \frac{\partial w(r)}{\partial \hat{\xi}},$$

which is rewritten, by the use of (3.8), as

$$\begin{aligned} \frac{\partial b(r)}{\partial \hat{\xi}} e^{w(r)} = & \frac{\partial}{\partial \hat{\xi}} \left[v(r) - w(r) - w(r)v(r) + \frac{1}{2}v^2(r) - z(r)v(r) \right] \\ & + v(r) \frac{\partial z(r)}{\partial \hat{\xi}} + x(r) \frac{\partial v(r)}{\partial \hat{\xi}}. \end{aligned} \quad (4.6)$$

Furthermore we obtain, with the aid of (3.7),

$$V(k) \frac{\partial Z(k)}{\partial \xi} = -\frac{1}{\rho^2} \frac{\partial}{\partial \xi} [\log \{1 - \rho Z(k)\} + \rho Z(k)]. \quad (4.7)$$

When (4.6) is introduced into (4.2') and the term arising from $v(r) \cdot \partial z(r) / \partial \xi$ is replaced by (4.7) with the aid of property of the Fourier transform, A_1 is expressed as follows:

$$\begin{aligned} \frac{A_1}{NkT} = & -\frac{\rho}{2} \int d\mathbf{r} \left[v(r) - w(r) - w(r)v(r) + \frac{1}{2}v^2(r) - z(r)v(r) \right] \\ & + \frac{1}{2\rho} \frac{1}{V} \sum_k [\log \{1 - \rho Z(k)\} + \rho Z(k)] + \frac{A_1'}{NkT}, \end{aligned} \quad (4.8)$$

where

$$\frac{A_1'}{NkT} = -\frac{\rho}{2} \int d\xi \int d\mathbf{r} x(r) \frac{\partial v(r)}{\partial \xi}. \quad (4.9)$$

In deriving (4.8), we have used the fact that $v(r)$, $w(r)$ and $Z(k)$ vanish at $\xi=0$.

A_1' can be expressed as a sum over diagrams similar to (4.1):

$$\frac{A_1'}{NkT} = -\frac{1}{\rho} \sum_{m=4}^{\infty} \frac{\rho^m}{m!} \frac{1}{V} \int \dots \int d\mathbf{r}_1 \dots d\mathbf{r}_m \sum_{m \geq i > j \geq 1} \Sigma^{(A')} \Pi v(r_{ij}). \quad (4.10)$$

In this expression $\Sigma^{(A')}$ denotes the sum over the diagrams which are obtained by introducing $v(r_{12})$ -bond into the diagrams appearing in $\Sigma^{(X')}$. If we take notice of the restrictions imposed on the diagrams in $\Sigma^{(X')}$ (see below (3.5)), it can be proved that $\Sigma^{(A')}$ is the sum over all diagrams which are *more than doubly connected*, though the proof will be omitted here. Fig. 8 shows some examples of the diagrams appearing in $\Sigma^{(A')}$, which is to be compared with Fig. 7.

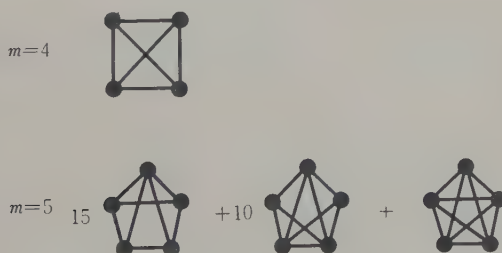


Fig. 8. Diagrams appearing in $\Sigma^{(A')}$ for $m=4$ and $m=5$.

The proof of (4.10) is as follows. Let us consider the integral

$$\int \dots \int d\mathbf{r}_1 d\mathbf{r}_2 \dots d\mathbf{r}_{m+2} \frac{\partial v(r_{12})}{\partial \xi} \sum_{m+2 \geq i > j \geq 1} \Sigma^{(X')} \Pi v(r_{ij}),$$

which can be written, with the aid of the definition of $\Sigma^{(A')}$, as

$$\frac{2}{(m+2)(m+1)} \frac{\partial}{\partial \xi} \int \cdots \int d\mathbf{r}_1 \cdots d\mathbf{r}_{m+2} \sum_{m+2 \geq i > j \geq 1}^{(A')} \Pi v(r_{ij}).$$

The factor $2/(m+2)(m+1)$ enters because the total of diagrams appearing in $\Sigma^{(A')}$ is symmetrical in m particles $3, 4, \dots$ and $m+2$ while the total of diagrams appearing in $\Sigma^{(A')}$ is symmetrical in $m+2$ particles $1, 2, 3, \dots$ and $m+2$. Therefore (4.9) is transformed in the following way (see (3.5)).

$$\begin{aligned} \frac{A_1'}{NkT} &= -\frac{\rho}{2} \int_0^1 d\xi \frac{1}{V} \iint d\mathbf{r}_1 d\mathbf{r}_2 \frac{\partial v(r_{12})}{\partial \xi} x(r_{12}) \\ &= -\frac{\rho}{2} \int_0^1 d\xi \frac{1}{V} \sum_{m=2}^{\infty} \frac{\rho^m}{m!} \int \cdots \int d\mathbf{r}_1 \cdots d\mathbf{r}_{m+2} \frac{\partial v(r_{12})}{\partial \xi} \sum_{m+2 \geq i > j \geq 1}^{(A')} \Pi v(r_{ij}) \\ &= -\frac{\rho}{2} \int_0^1 d\xi \frac{1}{V} \sum_{m=2}^{\infty} \frac{\rho^m}{m!} \frac{2}{(m+2)(m+1)} \frac{\partial}{\partial \xi} \int \cdots \int d\mathbf{r}_1 \cdots d\mathbf{r}_{m+2} \sum_{m+2 \geq i > j \geq 1}^{(A')} \Pi v(r_{ij}) \\ &= -\frac{1}{\rho} \sum_{m=4}^{\infty} \frac{\rho^m}{m!} \frac{1}{V} \int \cdots \int d\mathbf{r}_1 \cdots d\mathbf{r}_m \sum_{m \geq i > j \geq 1}^{(A')} \Pi v(r_{ij}), \end{aligned}$$

where we have used again that $v(r)$ vanishes at $\xi=0$. The proof has been completed.

For convenience of the following deductions, we shall rewrite (4.8) by the use of (3.6), in a form:

$$\begin{aligned} \frac{A_1}{NkT} &= -\frac{\rho}{2} \int d\mathbf{r} \left[\{b(r)+1\} e^{w(r)} - 1 - w(r) - w(r)v(r) + \frac{1}{2} v^2(r) \right] \\ &\quad + \frac{1}{2\rho} \frac{1}{V} \sum_k [\log \{1 - \rho Z(k)\} + \rho Z(k) + \rho^2 Z(k) V(k)] + \frac{A_1'}{NkT}. \quad (4.11) \end{aligned}$$

If we drop A_1'/NkT , replace $w(r)$ by $z_s(r)$ and notice of (3.2), we obtain the expression for the free energy in the hyper-netted chain approximation¹⁴⁾ (see (3.10)).

§ 5. Variational principle for the free energy

We shall begin with the definition of a functional derivative. Let P be a functional of $\alpha(r)$. When $\alpha(r)$ is varied by $\delta\alpha(r)$, the corresponding first order variation of P is denoted by δP . Then the functional derivative $\delta P/\delta\alpha(r)$ is defined by*

$$\delta P = \int d\mathbf{r} \frac{\delta P}{\delta\alpha(r)} \delta\alpha(r). \quad (5.1)$$

* This definition is shown to be equivalent to the one given before.¹⁸⁾

If $\alpha(r)$ is a functional of other function $\beta(r)$, the following relation is readily proved.

$$\frac{\delta P}{\delta \beta(r)} = \int d\mathbf{r}' \frac{\delta P}{\delta \alpha(r')} \frac{\delta \alpha(r')}{\delta \beta(r)}. \quad (5.2)$$

As is seen from (4.10), A_1'/NkT is a functional of $v(r)$. Applying the functional derivative to it, we obtain

$$x(r) = -\frac{2}{\rho} \frac{\delta}{\delta v(r)} \left(\frac{A_1'}{NkT} \right). \quad (5.3)$$

The proof of (5.3) can be done by reversing the reasonings used in the derivation of (4.10) from (4.9), though the detail is omitted here.

Now we consider the first order variation of the free energy when the functions $w(r)$, $v(r)$ and $z(r)$ are varied *independently* in (4.11). The variation is written, by the use of (5.3), as

$$\begin{aligned} \delta \left(\frac{A_1}{NkT} \right) = & -\frac{\rho}{2} \int d\mathbf{r} [\{b(r) + 1\} e^{w(r)} - 1 - v(r)] \delta w(r) \\ & + \frac{1}{2\rho} \frac{1}{V} \sum_k \left[-\frac{1}{1 - \rho Z(k)} + 1 + \rho V(k) \right] \delta \rho Z(k) \\ & - \frac{\rho}{2} \int d\mathbf{r} [v(r) - w(r) - z(r) + x(r)] \delta v(r). \end{aligned} \quad (5.4)$$

The coefficients of $\delta w(r)$, $\delta \rho Z(k)$ and $\delta v(r)$ in the above expression are seen to vanish if there exist relations (3.6), (3.7) and (3.8) among $w(r)$, $v(r)$ and $z(r)$. Therefore we have an interesting and useful fact that the free energy (4.11) is stationary with respect to the functions satisfying (3.6), (3.7) and (3.8). Conversely, this fact may be expressed in such a way that the set of equations (3.6), (3.7) and (3.8) is derived by means of a variational principle that the free energy (4.11) is stationary with respect to the variations of $w(r)$, $v(r)$ and $z(r)$. Such a stationary character of the free energy has been found also in the hyper-netted chain approximation.¹⁴⁾ The above-mentioned stationary character of the free energy may give us a practical method to determine the approximate forms of $w(r)$, $v(r)$ and $z(r)$ and so of the pair distribution function. If we take suitable functional forms for $w(r)$, $v(r)$ and $z(r)$ which contain some parameters, the best values of the parameters can be determined so that (4.11) becomes stationary. The free energy itself can be determined approximately by such a procedure.

The stationary character of (4.11) is useful in calculating other quantities by means of thermodynamical relations, because we may neglect in (4.11) the dependence of $w(r)$, $v(r)$ and $z(r)$ upon density and temperature. As is seen from (4.10), A_1'/NkT does not depend explicitly on temperature T . Therefore it is sufficient to consider the temperature dependence of $b(r)$ alone in the right-hand side of (4.11), as far as the temperature dependence is concerned. We accordingly

obtain the expression for the interaction part of the internal energy, by the use of relation (4.3), as

$$\begin{aligned} E_1 &= -Nk \frac{\rho}{2} \int d\mathbf{r} \frac{\partial b(r)}{\partial (1/T)} e^{w(r)} = N \frac{\rho}{2} \int d\mathbf{r} \phi(r) e^{-\phi(r)/kT + w(r)} \\ &= N \frac{\rho}{2} \int d\mathbf{r} \phi(r) g(r). \end{aligned} \quad (5.5)$$

The last expression of (5.5) is just (4.4). Such a result is to be expected naturally if we remember that (4.2) or (4.2') is equivalent to (4.3).

The expression for the interaction part of the pressure is also derived by differentiating (4.11) and (4.10) with respect to explicitly appearing ρ . The result is

$$\begin{aligned} \frac{p_1}{\rho kT} &= \rho \frac{\partial}{\partial \rho} \left(\frac{A_1}{NkT} \right) = - \frac{\rho}{2} \int d\mathbf{r} \left[v(r) - w(r) - w(r)v(r) + \frac{1}{2} v^2(r) \right] \\ &\quad - \frac{1}{2\rho} \frac{1}{V} \sum_k [\log \{1 - \rho Z(k)\} + \rho V(k) - \rho^2 Z(k) V(k)] \\ &\quad - \frac{1}{\rho} \sum_{m=4}^{\infty} (m-1) \frac{\rho^m}{m!} \frac{1}{V} \int \cdots \int d\mathbf{r}_1 \cdots d\mathbf{r}_m \sum_{\substack{m \geq i > j \geq 1}}^{(A')} \Pi v(r_{ij}), \end{aligned} \quad (5.6)$$

where we have used (3.6) and (3.7). On the other hand, it is well-known that the pressure is expressed in terms of $g(r)$ in a form:⁷⁾

$$p_1 = - \frac{\rho^2}{6} \int d\mathbf{r} r \frac{d\phi(r)}{dr} g(r). \quad (5.7)$$

It can be proved directly that (5.6) is equivalent to (5.7), though the proof is omitted here.

The expression for the interaction part of the chemical potential is derived as follows:

$$\begin{aligned} \frac{\mu_1}{kT} &= \frac{A_1}{NkT} + \frac{p_1}{\rho kT} = -\rho \int d\mathbf{r} \left[v(r) - w(r) - w(r)v(r) + \frac{1}{2} v^2(r) \right] \\ &\quad - \frac{1}{2\rho} \frac{1}{V} \sum_k [\rho V(k) - \rho Z(k) - 2\rho^2 Z(k) V(k)] \\ &\quad - \frac{1}{\rho} \sum_{m=4}^{\infty} \frac{\rho^m}{(m-1)!} \frac{1}{V} \int \cdots \int d\mathbf{r}_1 \cdots d\mathbf{r}_m \sum_{\substack{m \geq i > j \geq 1}}^{(A')} \Pi v(r_{ij}), \end{aligned} \quad (5.8)$$

which will be derived also in the Appendix by the procedure similar to the one used in § 3.

One of the authors has shown that the pair distribution function is in general related to the free energy by¹⁸⁾

$$g(r) = \frac{1}{N} \frac{2}{\rho} \frac{\partial A_1}{\partial \phi(r)}, \quad (5.9)$$

which can be rewritten in our case, with the aid of (5.2), and (2.2), as

$$e^{w(r)} = -\frac{2}{\rho} \frac{\delta}{\delta b(r)} \left(\frac{A_1}{NkT} \right). \quad (5.9')$$

As we may neglect in (4.11) the functional dependence of $w(r)$, $v(r)$ and $z(r)$ on $b(r)$, we can readily confirm that relation (5.9') is valid in our case as it should be.

§ 6. Summary and concluding remarks

The results which have been obtained are summarized as follows. The interaction part of the Helmholtz free energy is expressed in a form:

$$\begin{aligned} \frac{A_1}{NkT} = & -\frac{\rho}{2} \int d\mathbf{r} \left[\{b(r) + 1\} e^{w(r)} - 1 - w(r) - w(r)v(r) + \frac{1}{2}v^2(r) \right] \\ & + \frac{1}{2\rho} \frac{1}{V} \sum_k [\log \{1 - \rho Z(k)\} + \rho Z(k) + \rho^2 Z(k) V(k)] \\ & - \frac{1}{\rho} \sum_{m=4}^{\infty} \frac{\rho^m}{m!} \frac{1}{V} \int \dots \int d\mathbf{r}_1 \dots d\mathbf{r}_m \sum_{m \geq i > j \geq 1} \Pi v(r_{ij}), \end{aligned} \quad (6.1)$$

All products which are more than *doubly* connected.

where

$$b(r) = e^{-\phi(r)/kT} - 1. \quad (6.2)$$

The functions $w(r)$, $v(r)$ and $z(r)$ ($V(k)$ and $Z(k)$ are the Fourier transforms of $v(r)$ and $z(r)$, respectively) are to be determined in such a way that (6.1) is stationary with respect to the variations of those functions. Namely the functions satisfy a set of the following equations:

$$v(r) = [b(r) + 1] e^{w(r)} - 1, \quad (6.3)$$

$$1 - \rho Z(k) = \frac{1}{1 + \rho V(k)}, \quad (6.4)$$

and

$$w(r) = x(r) + v(r) - z(r). \quad (6.5)$$

The function $x(r)$ is defined by

$$x(r_{12}) = \sum_{m=2}^{\infty} \frac{\rho^m}{m!} \int \dots \int d\mathbf{r}_3 \dots d\mathbf{r}_{m+2} \sum_{m+2 \geq i > j \geq 1} \Sigma^{(X')} \Pi v(r_{ij}), \quad (6.6)$$

where $\Sigma^{(X')}$ means the sum over all the different diagrams which are to be obtained if one removes a bond $v(r_{12})$ from the diagrams appearing in the last term on the right-hand side of (6.1). The pair distribution function $g(r)$ is related to $v(r)$ by

$$g(r) = v(r) + 1, \quad (6.7)$$

so the structure factor $S(k)$ being determined directly by X -rays scattering experiment is related to $V(k)$ by

$$S(k) = 1 + \rho V(k). \quad (6.7')$$

The above results have been obtained from a standpoint that the form of pair interaction potential is known. Such a standpoint is usual in the molecular theory of fluids. The pair interaction potential is, however, not a quantity which is obtained directly by means of experiment, while the pair distribution function or the structure factor can be directly determined by X -rays or neutron scattering experiment. Therefore the above results may be used in the following way.

The set of (6.3)–(6.7) can be used to determine $b(r)$ or the pair interaction potential from the knowledge of the pair distribution function. It is a new method to determine the pair interaction potential. It may be said that the method is superior to the methods used in the past, since in the latter methods a suitable functional form of the pair interaction potential has to be chosen beforehand while in the above method there appears no need for such a choice. From the knowledge of the pair distribution function one can calculate $v(r)$, $z(r)$ and $w(r)$ with the aid of (6.4)–(6.7). Consequently the free energy can be calculated by means of the knowledge of the pair distribution function alone, by the use of the expression which is obtained by eliminating $b(r)$ from (6.1) with the aid of (6.3). In this way, we have reached to a formalism of the theory of classical fluids based upon the knowledge of the pair distribution function alone. It should be stressed that throughout the present paper the total interaction potential is assumed to be the sum of the pair interaction potentials.

The above situation may be stated as follows. If we know the pair distribution function rather than the pair interaction potential, the original series expansion for the pair distribution function may be considered as the integral equation determining $b(r)$ or the pair interaction potential. The set of equations (6.3)–(6.7) may be considered as the solution of this integral equation. Conversely, if we know the pair interaction potential, the set of equations (6.3)–(6.7) is the integral equation for the pair distribution function and the original series expansion is the solution of this.

In concluding the present paper, we shall consider about the region of validity of the results so far obtained. The region will of course be limited within the region where the original series expansions such as (2.3) and (4.1) or their analytical continuations, if exist, may be valid. The present authors hope that the original series expansions or their analytic continuations may be valid in the liquid region as well as in the gaseous regions, though it is an open question how far they are valid. Furthermore, we hope that the series expansions appearing in (6.1) and (6.6) may be more rapidly convergent than the original series expansions and that the results in this paper may be useful in the theory of condensation.

In forthcoming papers of this series the results obtained in this paper will be

generalized to the case of multi-component fluids and will also be applied to simple systems such as one-dimensional systems in order to investigate how far they are valid.

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Appendix

Derivation by means of the "graphical method"

In this appendix we try to derive the expressions for the pair distribution function, the chemical potential and the free energy by means of the *graphical method*. The diagrams have been used in the text to derive the expression for the pair distribution function. However, in this Appendix diagrams are used in a sense somewhat different from the one in the text. For this reason we shall include the derivation of the expression for the pair distribution function. For the detail of the graphical method we refer to reference 14). Here the derivation of the expressions will be given in a concise way as much as possible.

In the graphical method the starting formulas for the free energy and other quantities are expressed as^{14)*}

$$-\frac{A_1}{kT} = \frac{1}{2} \bullet \overset{b}{\text{---}} \bullet + \text{Sum of all the topologically different diagrams which consist of three or more black circles } (\bullet) \text{ and } \overset{b}{\text{---}} \text{ and which are more than singly connected.} \quad (\text{A} \cdot 1)$$

$$-\frac{\mu_1}{kT} = \circ \overset{b}{\text{---}} \bullet + \text{Sum of all the topologically different diagrams which consist of one white circle } (\circ) \text{ and two or more black circles and } \overset{b}{\text{---}} \text{ and for which each black circle is connected to the white circle by at least two independent paths and the black circles are connected to each other independently of the white circle.} \quad (\text{A} \cdot 2)$$

$$w(r_{12}) = \text{Sum of all the topologically different diagrams which consist of two white circles, } \circ_1 \text{ and } \circ_2, \text{ and one or more black circles and several } \overset{b}{\text{---}} \text{ and which are under the restriction } (W) \text{ where } (W) \text{ means (I), (II) and (III) in (2.5).} \quad (\text{A} \cdot 3)$$

A propagator is defined in such a way that it is a part of a diagram which is connected to another part of the diagram by only two points. If these two points are considered to be particles 1 and 2, the propagator is a diagram appearing in $\Sigma^{(V)}$ in the text.

An *s-point* is defined in the same way as in the text, namely it is such a

* Here the suffices attached to lines are written in small letters instead of large letters.

point (or a circle) that the propagator is to be separated into two parts if one removes it.

The total of propagators can be grouped into two classes, *propagator Z* and *propagator Z_s*:

Propagator *Z* is the total of the propagators which have no *s*-point.

Propagator *Z_s* is the total of the propagators which have at least one *s*-point.

Furthermore propagator *Z_s* can be grouped together by the number of *s*-points in terms of propagator *Z*, namely propagator *Z_s* has the structure

$$\begin{array}{c} \circ_1 \xrightarrow{z_s} \circ_2 \end{array} = \begin{array}{c} \circ_1 \xrightarrow{z} \bullet \xrightarrow{z} \circ_2 \end{array} + \begin{array}{c} \circ_1 \xrightarrow{z} \bullet \xrightarrow{z} \bullet \xrightarrow{z} \circ_2 \end{array} + \dots \quad (\text{A} \cdot 4)$$

Propagator *Z* can be grouped together into three classes. The first is *B* itself. The second consists of all the propagators which are to be separated into two or more parts if we remove the two points. The third consists of the rest which is denoted as propagator *X*. Therefore propagator *Z* has the structure

$$\begin{array}{c} \circ_1^1 \\ | \\ \circ_2 \end{array} = \begin{array}{c} \circ_1^1 \\ | \\ \circ_2 \end{array} b + \begin{array}{c} \circ_1^1 \\ | \\ \circ_2 \end{array} x + \begin{array}{c} \circ_1^1 \\ | \\ \circ_2 \end{array} + b \begin{array}{c} \circ_1^1 \\ | \\ \circ_2 \end{array} w + b \begin{array}{c} \circ_1^1 \\ | \\ \circ_2 \end{array} w w + \dots \\ + w \begin{array}{c} \circ_1^1 \\ | \\ \circ_2 \end{array} w + w \begin{array}{c} \circ_1^1 \\ | \\ \circ_2 \end{array} w w + \dots, \quad (\text{A} \cdot 5)$$

where $\begin{array}{c} \circ_1^w \\ | \\ \circ_2 \end{array}$ is $w(r_{12})$ (see (A·3)).

Propagator *X* introduced above turns out to be expressed as

$$\begin{array}{c} \circ_1^x \\ | \\ \circ_2 \end{array} = \text{Sum of all the propagators which are under the restriction (X)} \\ \text{where (X) means (I), (II), (III) and (IV) in (2.5).} \quad (\text{A} \cdot 6)$$

In other words, $\begin{array}{c} \circ_1^x \\ | \\ \circ_2 \end{array}$ is equal to $x(r_{12})$ in the text. It is readily seen, in the same way as in the text, that $\begin{array}{c} \circ_1^w \\ | \\ \circ_2 \end{array}$ has the structure

$$\begin{array}{c} \circ_1^w \\ | \\ \circ_2 \end{array} = \begin{array}{c} \circ_1^x \\ | \\ \circ_2 \end{array} + \begin{array}{c} \circ_1^{z_s} \\ | \\ \circ_2 \end{array}. \quad (\text{A} \cdot 7)$$

If we group the diagrams in (A·6) in an analogous way as was done above (3.5) in the text, $\begin{array}{c} \circ_1^x \\ | \\ \circ_2 \end{array}$ can be expressed as

$$\begin{array}{c} \circ_1^x \\ | \\ \circ_2 \end{array} = \text{Sum of all the topologically different diagrams which consist of} \\ \text{two white circles, } \circ_1 \text{ and } \circ_2, \text{ and two or more black circles and several} \\ \frac{z+z_s}{v} \equiv \text{---} \text{ and which are under the restriction (X')} \\ \text{where (X') means all the restrictions stated below (3.5) in the text.} \quad (\text{A} \cdot 8)$$

A set of (A·4), (A·5), (A·7) and (A·8) determines $v(r) = z(r) + z_s(r)$, hence the pair distribution function $g(r) = 1 + v(r)$.

It is necessary to introduce a process which will be called identification, in order to derive the expressions for the chemical potential and the free energy in terms of $w(r)$, $x(r)$, $z(r)$ and $z_s(r)$ (see references 12) and 14)).

In the following we represent ---^b by a simple line --- , removing the subscript b . Let us consider a propagator which is composed of simple lines and which is under the restriction (X'). Such a propagator will hereafter be represented by a wavy line --- . Furthermore, several kinds of propagators are defined as follows:

$$\left. \begin{aligned} \text{---}^b &\equiv \text{---} + \text{---} \\ \text{---}^s &\equiv \text{---}^b + \text{---}^b + \dots \\ \text{---}^{s'} &\equiv \text{---} + \text{---}^s \end{aligned} \right\} \quad (\text{A} \cdot 9)$$

Then we call the propagators having such a structure as shown in Fig. A·1 to be *identifiable*. The process of replacing an identifiable propagator by a simple line is called *identification*. The *number of times of identification* is defined

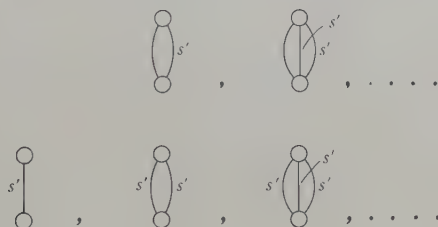


Fig. A·1. Identifiable propagators.

in such a way that all the identifiable parts in a diagram are replaced by simple lines at a time. When a part of a diagram needs larger times of identification than the other, the former is called of *higher order* than the latter. The *lower* or *the same order* is defined in a similar way.

The chemical potential. Diagrams appearing in (A·2) can be grouped according to the types of irreducible diagrams, which are obtained after the applications of possible identifications to the diagrams in (A·2) and to which the identification cannot be applied any more *in a unique manner*. The irreducible diagrams are shown in Fig. A·2.

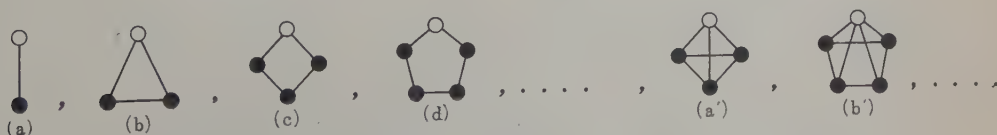
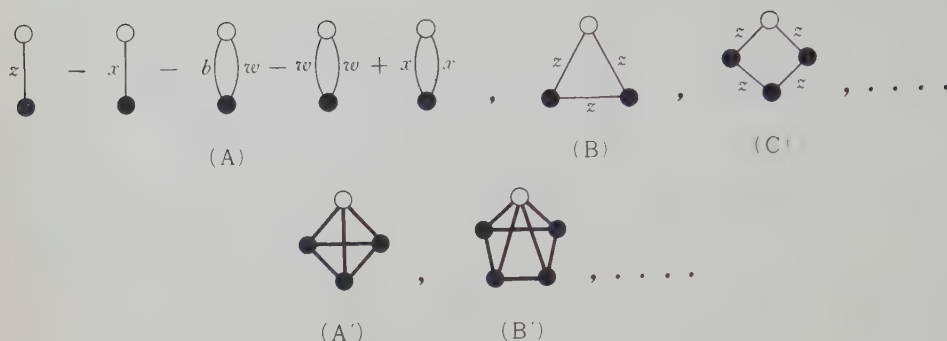


Fig. A·2. The irreducible diagrams for the chemical potential.

Fig. A.3. Bold lines represent the propagators $Z+Z_S$.

The diagrams in (A.2) belonging to the irreducible diagrams (a), (b), etc., in Fig. A.2 are always contained in (A), (B), etc., respectively, in Fig. A.3. However, the converse is not true; for example, the diagrams in (A) do not necessarily belong to (a). Therefore the contribution, of the diagrams belonging to (a), to the chemical potential is equal to that of (A) minus that of the diagrams not belonging to (a) among (A). The contributions of the diagrams belonging to (b), (c), etc., are calculated in an analogous way.

Among (A), the diagrams not belonging to (a) can be shown to be those in which one edge w is of higher order than the other edges when we write the diagrams as in (A.5). That is, these diagrams are

$$\sum_{z'} \sum_{w'}^{(h+s)} z' \bigcirc w' - b \bigcirc w - \sum_{z'''} \sum_{w'}^{(h+s)} x'' \bigcirc w' + \sum_{z'''} \sum_{z'''}^{(h)} x'' \bigcirc x'''. \quad (\text{A} \cdot 10)$$

Here z' , w' and x'' (x''') are the diagrams belonging to propagators z , w and x , respectively. $\sum_{w'}^{(h+s)}$ means the sum over all those diagrams, belonging to propagator W , which are of higher or same order than the other edge.

The diagrams, the contributions of which are to be subtracted from those of (B), (C), ..., are seen to be the diagrams in which one of the edges neighboring to the white circle is of higher order than the other edges. The contributions of such diagrams turn out to be summed up as

$$\sum_{z'_S} \sum_{z'}^{(h)} z' \bigcirc z'_S \equiv \sum_{z'} \sum_{z'_S}^{(l)} z' \bigcirc z'_S. \quad (\text{A} \cdot 11)$$

The contributions to be subtracted from those of (A'), (B'), ..., can be shown, in an analogous way to the above, to be

$$\sum_{z'} \sum_{z'''}^{(l)} z' \bigcirc x'' + \sum_{z'''} \sum_{z'''}^{(s)} x'' \bigcirc x''' + \sum_{z'''} \sum_{z'_S}^{(h+s)} z'_S \bigcirc x'''. \quad (\text{A} \cdot 12)$$

Therefore the contributions to be subtracted from those of Fig. A·3, that is the sum of (A·10), (A·11) and (A·12), are obtained as

$$z \begin{array}{c} \circ \\ \text{---} \\ \bullet \end{array} w - b \begin{array}{c} \circ \\ \text{---} \\ \bullet \end{array} w, \quad (\text{A} \cdot 13)$$

where we have used (A·7). After all the chemical potential is expressed as

$$\begin{aligned} -\frac{\mu_1}{kT} = & \left[z \begin{array}{c} \circ \\ \text{---} \\ \bullet \end{array} - x \begin{array}{c} \circ \\ \text{---} \\ \bullet \end{array} - b \begin{array}{c} \circ \\ \text{---} \\ \bullet \end{array} w - w \begin{array}{c} \circ \\ \text{---} \\ \bullet \end{array} w + x \begin{array}{c} \circ \\ \text{---} \\ \bullet \end{array} x + \begin{array}{c} z \quad z \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ z \quad z \end{array} + \begin{array}{c} z \quad z \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ z \quad z \end{array} + \dots \\ & + \begin{array}{c} \circ \quad \circ \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \circ \quad \circ \end{array} + \begin{array}{c} \circ \quad \circ \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \circ \quad \circ \end{array} + \dots \Big] \\ & - \left[z \begin{array}{c} \circ \\ \text{---} \\ \bullet \end{array} w - b \begin{array}{c} \circ \\ \text{---} \\ \bullet \end{array} w \right], \quad (\text{A} \cdot 14) \end{aligned}$$

which is rewritten as

$$\begin{aligned} -\frac{\mu_1}{kT} = & \rho Z(0) - \rho X(0) + \frac{1}{2} z_s(0) - \frac{\rho}{2} \int d\mathbf{r} [\{z(r) + z_s(r) + x(r)\}^2 - x^2(r)] \\ & + \begin{array}{c} \circ \quad \circ \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \circ \quad \circ \end{array} + \begin{array}{c} \circ \quad \circ \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \circ \quad \circ \end{array} + \dots \quad (\text{A} \cdot 14') \end{aligned}$$

Note here that the first and the second rows of (A·14') are equal respectively to the first two and the last rows of the last expression of (5·8) in the text.

The free energy. The reduction of (A·1) will be made in quite a similar way as that of (A·2). The diagrams in (A·1) are grouped together according to the irreducible diagrams that are shown in Fig. A·4. The contributions of the diagrams belonging to (a), (b), etc., in Fig. A·4 are equal respectively to those of (A), (B), etc., in Fig. A·5 minus those of the diagrams not belonging to (a), (b), etc., respectively.

The contributions to be subtracted are those of the diagrams in which one edge is of higher order than the others in "(A) written in the form of (A·5)", (B), (C), etc. These contributions among (A) and (B), (C), ..., and (A'),

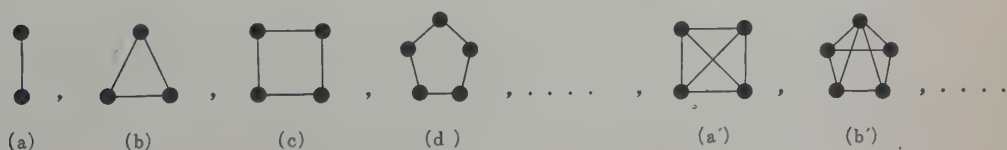


Fig. A·4. The irreducible diagrams for the free energy.

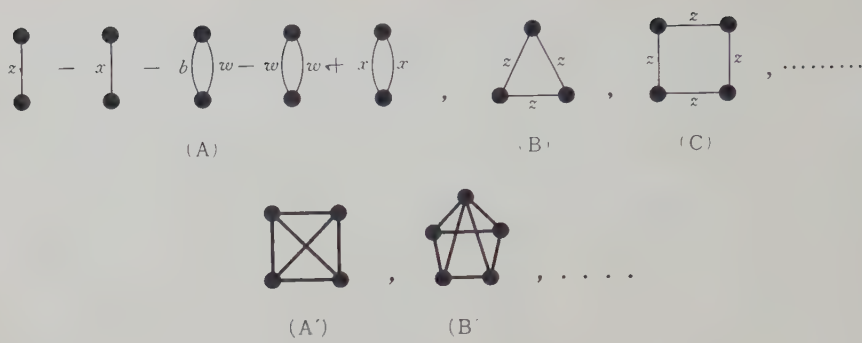


Fig. A-5.

(B'), ..., are shown to be equal to (A·10), (A·11) and (A·12), respectively, if the white circles are replaced by the black circles. Therefore their sum is

$$z \text{ loop } w - b \text{ loop } w \dots \tag{A-15}$$

After all the free energy is obtained as

$$\begin{aligned} -\frac{A_1}{VkT} = & \left[z - x - b \text{ loop } w - w \text{ loop } w + x \text{ loop } x + \text{triangle} + \text{square} + \dots \right. \\ & \left. + \text{square with diagonals} + \text{pentagon} + \dots \right] \\ & - \left[z \text{ loop } w - b \text{ loop } w \right] \tag{A-16} \end{aligned}$$

$$\begin{aligned} = & \frac{\rho^2}{2} Z(0) - \frac{\rho^2}{2} X(0) + \frac{1}{2} \sum_k [-\log \{1 - \rho Z(k)\} - \rho Z(k)] \\ & - \frac{\rho^2}{4} \int d\mathbf{r} [\{z(r) + z_s(r) + x(r)\}^2 - x^2(r)] \\ & + \text{square with diagonals} + \text{pentagon} + \dots \tag{A-16'} \end{aligned}$$

It is to be noted that (A·16') is identical with (4·8) and (4·10) in the text.

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Ambiguity of $\partial D_{s.c.}(x)/\partial x^2$ and Causality

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In the definition of the derivatives of the causal function $D_c(x)$ a new type of ambiguity is pointed out. It can be removed by causality requirements, but not by renormalization. Or in other words, an effect of causality requirements in the graphical calculation is made explicit.

§ 1. Introduction

While the causal "function" $D_c(x)$, $x=(x_0, x_1, x_2, x_3)$, of a massless bose field and its derivatives with respect to $x^2=x_0^2-x_1^2-x_2^2-x_3^2$ are well-defined as Schwartz, distributions of a single variable x^2 , they cannot directly be defined as distributions of four variables x_0, x_1, x_2 and x_3 . This situation is brought about by the singularity at the common tip of the past and future parts of the light-cone.¹⁾ In order to construct a distribution having a singularity at the tip we must define it as a finite part in some limit process consisting, for example, of a series of distributions which are zero at the tip¹⁾ or of a series of indefinitely differentiable functions such as $1/\{i(2\pi)^2(x^2-i\varepsilon)\}$, $\varepsilon>0$. The results of various definitions may differ from each other, and this ambiguity in the case of $D_c(x)$ will turn out to interfere with some physical postulates since $\partial D_c(x)/\partial x^2$ is essentially equal to $D_c(x)D_c(x)^{2),3),4)}$ which in turn corresponds to a self-energy Feynman diagram. Without the condition "massless", clear-cut results cannot be obtained, but the essential features are not altered because the most singular part of the causal function is independent of its mass.

§ 2. Definitions of $\partial D_c(x)/\partial x^2$

Let us investigate first an indefinitely differentiable function with a parameter z

$$D_{s.c.}(x, z) = -\frac{i}{(2\pi)^2} \cdot \frac{1}{x^2+z} \quad \text{for } \text{Im } z < 0 \quad (1a)$$

and its limit

$$D_{s.c.}(x, z) = -\frac{i}{(2\pi)^2} \left\{ \text{Pf} \frac{1}{x^2+z} + i\pi\delta(x^2+z) \right\} \quad \text{for } \text{Im } z = 0 \text{ and } z \neq 0, \quad (1b)$$

which is a well-defined distribution of the four variables. Roughly speaking, (1)

is the element of analytic continuation* at x^2 of a distribution $D_c(x)$. In any case as long as z does not vanish, we can treat $D_c(x, z)$ as a "function" of one variable x^2 without any oversight of its structure on the tip. Thus, using the formula

$$i \int_0^\infty d\alpha e^{-i\alpha(x^2+z)} = \begin{cases} \frac{1}{x^2+z} & \text{for } \text{Im } z < 0 \\ \text{Pf} \frac{1}{x^2+z} + i\pi\delta(x^2+z) & \text{for } \text{Im } z = 0, \end{cases} \quad (2)$$

we obtain as the Fourier transform of $D_{s,c}(x, z)$

$$D_{s,c}(k, z) = \frac{1}{(2\pi)^2} \int d^4x \int_0^\infty d\alpha e^{-ikx - i\alpha x^2 - i\alpha z} = \frac{i}{4} \int_0^\infty \frac{d\alpha}{\alpha^2} e^{-i\alpha z + ik^2/(4\alpha)} = i \int_0^\infty d\beta e^{i\beta k^2 - iz/(4\beta)},$$

where the formula

$$\int d^4x e^{-i\alpha x^2 - ikx} = \frac{i\pi^2}{\alpha|\alpha|} e^{ik^2/(4\alpha)} \quad (3)$$

is used and $\beta = 1/4\alpha$. For a closed expression an integral representation of the modified Hankel function is required:

$$K_1(e^{-i\pi/2}\omega\lambda) = \frac{1}{2\lambda} \int_0^\infty e^{i\omega(\beta + \lambda^2/\beta)/2} d\beta \quad (4)$$

for

$$0 < \arg \omega < \pi \quad \text{and} \quad -\frac{\arg \omega}{2} \leq \arg \lambda \leq \frac{\pi}{2} - \frac{\arg \omega}{2}.$$

The validity of this representation is guaranteed by the analyticity of both sides in the stated domain and the validity in special cases.⁵⁾

Now, for $\text{Im } k^2 > 0$

$$D_{s,c}(k, z) = -\left(\frac{z}{k^2}\right)^{1/2} K_1\{(k^2)^{1/2}(z^2)^{1/2}\} \quad (5)$$

where $-\pi/2 \leq \arg(z)^{1/2} \leq 0$ and $0 < \arg(k^2)^{1/2} < \pi/2$. With the aid of the formula

$$K_1(t) = \sum_{n=0}^\infty \frac{t^{2n+1}}{n!(n+1)!2^{2n+1}} \ln\left(\frac{\gamma t}{2}\right) - \frac{1}{2} \sum_{n=0}^\infty \frac{t^{2n+1}}{n!(n+1)!2^{2n+1}} \times \left(2 \sum_{m=1}^n \frac{1}{m} + \frac{1}{m+1}\right) + \frac{1}{t}, \quad (6)$$

where $\ln \gamma$ is Euler's constant, the series expansion of $D_{s,c}(k, z)$ is obtained:

* See ref. 4). The theory of hyperfunctions as boundary values of analytic functions is thoroughly investigated by M. Sato whose early papers are cited in ref. 4).

$$D_{s.c.}(k, z) = -\frac{1}{k^2} + z \left(\frac{1}{4} - \frac{1}{2} \ln \frac{\gamma(k^2)^{1/2}}{2} \right) - \frac{1}{4} z \ln z + O(z^2 \ln z)$$

for $0 < \arg(k^2)^{1/2} < \frac{\pi}{2}$

(7)

and its boundary value on the real k^2 -axis

$$D_{s.c.}(k, z) = -\text{Pf} \frac{1}{k^2} + i\pi\delta(k^2) + z \left\{ \frac{1}{4} - \frac{1}{2} \ln \frac{\gamma|k^2|^{1/2}}{2} - \frac{i\pi}{4} \theta(-k^2) \right\}$$

$$-\frac{1}{4} z \ln z + O(z^2 \ln z) \quad \text{for } k^2 \text{ real.}$$
(8)

This result shows first of all that $D_{s.c.}(k, z)$ converges to $D_c(k) \equiv i\pi\delta(k^2) - \text{Pf}(1/k^2)$ independent of $\arg z$. Next we must set out to define $\partial D_c(x)/\partial x^2$. As long as z remains different from zero, $\partial D_c(x, z)/\partial x^2 = \partial D_c(x, z)/\partial z$ is well-defined and for $\text{Im } z < 0$, in particular, it is equal to the ordinary product $\{(2\pi)^2/i\} D_c(x, z) D_c(x, z)$. Therefore we may define a distribution $\partial D_c(x)/\partial x^2$ of four variables as the finite part of $\partial D_c(x, z)/\partial z$ for $z \rightarrow 0$. Taking e^{-ikx} as a test function, we get

$$\int d^4x \frac{\partial D_{s.c.}(x)}{\partial x^2} e^{-ikx} = \text{Pf}(z \rightarrow 0) \int d^4x \frac{\partial D_{s.c.}(x, z)}{\partial z} e^{-ikx} = \text{Pf}(z \rightarrow 0) \frac{\partial}{\partial z} D_{s.c.}(k, z)$$

$$= \text{Pf}(z \rightarrow 0) \left\{ -\frac{1}{2} \ln \frac{\gamma|k^2|^{1/2}}{2} - \frac{i\pi}{4} \theta(-k^2) - \frac{1}{4} \ln z + O(z \ln z) \right\}$$

$$= -\frac{1}{2} \ln \frac{\gamma|k^2|^{1/2}}{2} - \frac{i}{4} \{\pi\theta(-k^2) + \arg z\},$$
(9)

where the subtracted infinite part is $-\ln|z|/4$, a real logarithmically diverging quantity, corresponding to $(-\ln|z|/4)\delta(x)$ and the arbitrary $(-i/4)\arg z$ corresponds to $(-i/4)(\arg z)\delta(x)$. Three representative values of $\arg z$ are $-\pi$, $-\pi/2$ and 0 ; $-\pi/2$ is the one usually written in the theory of quantized fields, while $-\pi$ and 0 mean approaching to the light-cone, especially to the tip, from within and from without respectively. In these three cases

$$\int d^4x \frac{\partial D_{s.c.}(x)}{\partial x^2} e^{-ikx} = -\frac{1}{2} \ln \frac{\gamma|k^2|^{1/2}}{2} + \begin{cases} -\frac{i\pi}{4} \theta(k^2) & \text{for } \arg z = -\pi \\ \frac{i\pi}{8} \varepsilon(k^2) & \text{for } \arg z = -\frac{\pi}{2} \\ -\frac{i\pi}{4} \theta(-k^2) & \text{for } \arg z = 0, \end{cases} \quad (9)'$$

the second of which has already been given by W. Güttinger.²⁾ The factor $\{\pi\theta(-k^2) + \arg z\}$ in (9) originates from $\ln(\gamma t/2)$ in (6), a glance at which tells us that the higher order derivatives of $D_c(x)$ also contain the same factor in their imaginary parts and have the same "threshold property" as the one in (9).

It is interesting to introduce along with $D_c(k, z)$ another function $D_{AC}(k, z)$:

$$D_{Ac}(k, z) = - \left(\frac{z}{k^2} \right)^{1/2} K_1 \{ (k^2)^{1/2} (z)^{1/2} \}$$

$$\text{with } \begin{cases} 0 \leq \arg(z)^{1/2} \leq \frac{\pi}{2} \text{ and } -\frac{\pi}{2} < \arg(k^2)^{1/2} < 0 \\ \text{or } -\pi \leq \arg(z)^{1/2} \leq \frac{\pi}{2} \text{ and } \frac{\pi}{2} < \arg(k^2)^{1/2} < \pi, \end{cases}$$

whose boundary value is the Fourier transform of $D_{Ac}(x, z)$:

$$D_{Ac}(x, z) = \begin{cases} -\frac{i}{(2\pi)^2} \cdot \frac{1}{x^2 + z} & \text{for } \text{Im } z > 0 \\ -\frac{i}{(2\pi)^2} \left\{ \text{Pf} \frac{1}{x^2 + z} - i\pi \delta(x^2 + z) \right\} & \text{for } \text{Im } z = 0 \text{ and } z \neq 0. \end{cases}$$

It is evident that only for $\arg z = -\pi$ or 0 $D_c(k, z)$ and $D_{Ac}(k, z)$ can be united into an analytic function regular in the k^2 -plane cut on the real positive or negative axis respectively. This fact has appeared as the threshold property in the boundary value on the real k^2 -axis shown in (9) and (9)'.

§ 3. Threshold property of a self-energy part

The threshold property of a self-energy part is well-known, yet it shall be reviewed briefly. In order to avoid an inessential complication in the course of renormalization the mass of an intermediate particle is kept from zero at first. The regulators of Pauli-Villars⁶⁾ are applied, or in other words

$$\Sigma(k, M^2) = \frac{1}{(2\pi)^4} \int d^4 p \text{reg } \mathcal{A}_c(p) \text{reg } D_c(k-p) \quad (10)$$

with

$$\left. \begin{aligned} \text{reg } \mathcal{A}_c(p) &= \text{Pf} \frac{1}{-p^2 + m^2} + i\pi \delta(-p^2 + m^2) - \left\{ \text{Pf} \frac{1}{-p^2 + M^2} + i\pi \delta(-p^2 + M^2) \right\}, \\ \text{reg } D_c(k-p) &= \text{Pf} \frac{1}{-(k-p)^2} + i\pi \delta((k-p)^2) \\ &\quad - \left\{ \text{Pf} \frac{1}{-(k-p)^2 + M^2} + i\pi \delta(-(k-p)^2 + M^2) \right\}, \end{aligned} \right\} \quad (11)$$

is first evaluated, and then after subtracting suitable terms the auxiliary mass M is made to tend towards infinity. Applying (2) and (3), we find

$$\Sigma(k, M^2) = \frac{i}{16\pi^2} \int_0^1 d\xi \int_0^\infty \frac{d\lambda}{\lambda} e^{i\lambda \xi(1-\xi)k^2} (1 - e^{-i\xi \lambda M^2}) (e^{-i(1-\xi)\lambda m^2} - e^{-i(1-\xi)\lambda M^2}).$$

The integration with respect to λ gives zero for $\xi=0$ or 1 . For $0 < \xi < 1$, the

differentiation with respect to k^2 , the integration with respect to λ with the aid of (2) and finally the integration with respect to k^2 in view of the formula

$$\frac{d \ln t}{dt} = \text{Pf} \frac{1}{t} + i\pi \delta(t) \quad \text{for } \arg t = 0 \text{ or } -\pi, \quad (12)$$

lead to

$$\begin{aligned} & \ln \xi(M^2 - \xi k^2) - \ln(m^2 - \xi k^2) - \ln \{\xi M^2 - \xi^2(1 - \xi)k^2\} \\ & + \ln \{\xi M^2 + (1 - \xi)m^2 - \xi(1 - \xi)k^2\}, \end{aligned}$$

where the arguments of the quantities under the \ln signs are 0 or $-\pi$, and the constant of integration with respect to k^2 vanish. In order only to get rid of divergence there can be infinitely many ways of subtraction, each of which defines a multiplication of $D_c(x)$ and $\mathcal{A}_c(x)$.⁷⁾ Here two cases are to be written. One is the mass renormalization at $k^2=0$; the other is the finite part in the manner of the previous section. In the latter case m^2 is reduced to zero without new complication and the whole expression is divided by $i/(2\pi)^2$ for the purpose of the immediate comparison with the expression of (9)'. The results are

$$\sum_{ren}(k) = \frac{i}{16\pi^2} \int_0^1 d\xi \ln \frac{m^2}{m^2 - \xi k^2} \quad \text{with } \arg(m^2 - \xi k^2) = 0 \text{ or } -\pi, \quad (13a)$$

$$\frac{(2\pi)^2}{i} \sum_{rf}(k) = -\frac{1}{2} \ln |k^2|^{1/2} + \frac{i\pi}{4} \theta(k^2), \quad (13b)$$

where the subtracted terms are $(i/16\pi^2)(\ln M^2/m^2 - 1)$ and $(1/4)\ln M^2$ respectively, and the former multiplied by i , pure real, may be interpreted as the self-energy. (13b) coincides with (9)' for $\arg z = -\pi$ except for $(-1/2)\ln \gamma/2$ resulting from the difference of the ways of subtraction and corresponding to $(-1/2)(\ln \gamma/2)\delta(x)$.

The comparison becomes complete, however, if we vary $\arg M^2$ continuously from 0 to $-\pi$, i.e. in (10) we substitute

$$\left. \begin{aligned} \text{reg } \mathcal{A}_c(p) &= \text{Pf} \frac{1}{-p^2 + m^2} + i\pi \delta(-p^2 + m^2) - \frac{1}{-p^2 + M^2} \\ \text{reg } D_c(k-p) &= \text{Pf} \frac{1}{-(k-p)^2} + i\pi \delta((k-p)^2) - \frac{1}{-(k-p)^2 + M^2} \end{aligned} \right\} \quad \text{for } \text{Im } M^2 < 0, \quad (11)'$$

or their limits* (11) where M^2 is real and now can be negative as well as positive. It is obvious from the results of the previous section with k and x interchanged that these complex and negative values of M^2 maintain the power of

* These cases have been argued by N. Nakanishi, Prog. Theor. Phys. **17** (1957), 401, especially a footnote on p. 409, with Feynman cutoff method.

regulation. In fact, the same way of evaluation as the above leads to

$$\Sigma(k, M^2) = \frac{i}{16\pi^2} \left\{ - \int_0^1 d\xi \ln \frac{m^2 - \xi k^2}{\xi} + \ln M^2 + O\left(\frac{1}{M^2}\right) \right\}$$

with $\arg(m^2 - \xi k^2) = 0$ or $-\pi$ and $-\pi \leq \arg M^2 \leq 0$, (14)

whence

$$\Sigma_{ren}(k) = \frac{i}{16\pi^2} \left(\int_0^1 d\xi \frac{m^2}{m^2 - \xi k^2} + i \arg M^2 \right)$$

with $\arg(m^2 - \xi k^2) = 0$ or $-\pi$ and $-\pi \leq \arg M^2 \leq 0$ (15a)

and by placing $m^2 = 0$

$$\frac{(2\pi)^2}{i} \Sigma_{reg}(k) = -\frac{1}{2} \ln |k^2|^{1/2} + \frac{i}{4} \left\{ \pi \theta(k^2) + \arg M^2 \right\}. \quad (15b)$$

Here $(i/4) \{ \pi \theta(k^2) + \arg M^2 \}$ gives rise to a wrong threshold property as $(-i/4) \times \{ \pi \theta(-k^2) + \arg z \}$ did in (9). This $\arg M^2$, however, cannot be subtracted for it would make the self-energy complex. Thus, two quite different classes of approximations, (1) and (11), have the same range of ambiguity concerning the threshold property.

§ 4. Summary

Apart from the perturbation theory, each of the two general postulates, the commutativity of two local field operators at a space-like distance and the positive-definiteness of the squared four momentum operator, confirms the conclusion that various Green-like and non-Green-like "functions" of a variable k^2 can be built up of the two boundary values on the real k^2 -axis of a single analytic function regular in the k^2 -plane cut on the real positive axis.⁸⁾ In view of § 2, therefore, the approximation $D_c(x, z) = \{-i/(2\pi)^2\} \{ \text{Pf } 1/(x^2 + z) + i\pi \delta(x^2 + z) \}$ with $z < 0$ to $D_c(x)$ is distinguished by meeting the causality requirements from any other approximation of the same type with the argument of z arbitrary. And in view of § 3 the real auxiliary M in the regulator method possesses a certain physical meaning although the interpretation as a mass cannot be carried out owing to imaginary coupling constants.

In the local field theory an accumulation of two local field operators at a space-time point is inevitable, and this accumulation, corresponding to the tip of the light-cone in § 2, gives rise not only to the well-known ambiguity which can be removed only by comparison with experiment or by renormalization,^{6),7)} but, in the graphical calculation, also to a new type of ambiguity which was latent. By developing this ambiguity the present paper has made explicit how the causality requirements operate in the graphical calculation.

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Pion-Nucleon Interaction, Anomalous Magnetic Moment of Nucleon and Composite Model for Pion

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The effective pion-nucleon interaction and the anomalous magnetic moment of the nucleon are calculated on the basis of the composite model for the pion in which the fundamental interaction is assumed to be an adequate linear combination of scalar, tensor and pseudoscalar, or vector and pseudovector Fermi type couplings. The results are qualitatively in agreement with experiment.

§ 1. Introduction

Since the problems related to the low energy p -wave pion-nucleon interaction were solved by the meson theory with fixed extended source, the phenomena in which the effect of the s -wave pion-nucleon interaction plays an important role, e.g. s -wave pion-nucleon scattering, the anomalous magnetic moment of the nucleon, the radius of the nucleon core, etc. have been studied by many authors. Perhaps the virtual nucleon-antinucleon pair plays the important role in these phenomena, but the consistent explanation about them as a whole is not yet obtained, as we have not a precise treatment concerning the nucleon pair.

In the previous paper,¹⁾ hereafter referred to as I, we used the composite model for the pion²⁾ to treat the nucleon pair, because in this model the pion is regarded as a bound state of one nucleon pair and the nucleon pair can be treated more suitably by the method based on this model. In I we assumed that the fundamental interaction is the direct Fermi type interaction and its coupling type is pseudoscalar. The assumption is the simplest one which gives a pseudoscalar pion. Then the effective pion-nucleon interaction is equivalent to the usual meson theory.

Now the fundamental interaction is assumed to be an arbitrary linear combination of scalar (s), tensor (t) and pseudoscalar (p), or vector (v) and pseudovector (a) Fermi type couplings.* We hope that our results are equivalent to the usual meson theory in the region mainly related to the p -wave pion-nucleon interaction, and also are consistent with experiment in the region in which the s -wave interactions give the major contributions. The actual treatments (the calculation and the approximation) are same as I. In section 2 we consider the eigenvalue problem

* We consider here only the special combination³⁾ of s , v , t , a and p for the convenience of calculation.

for the bound state of one nucleon pair by the new Tamm-Dancoff method and regard this state as the pion. Then we obtain a normalized wave function for this state. In section 3 we get an effective pion-nucleon interaction by the canonical transformation using the above wave function. In section 4 the anomalous magnetic moment of the nucleon is calculated perturbation-theoretically, by using this model. In section 5 we search an adequate combination of s , t and p , or of v and a , for our results obtained in sections 2, 3 and 4 to be consistent with experiment.

§ 2. Eigenvalue problem for bound state of nucleon-antinucleon pair

The interaction Hamiltonian between nucleons and antinucleons is assumed to have the following form :

$$H' = \sum_k \sum_{i=1}^3 \frac{g_{1k}}{M^2} \int (\bar{\psi}(\mathbf{r}) O_k \tau_i \psi(\mathbf{r})) (\bar{\psi}(\mathbf{r}) O_k \tau_i \psi(\mathbf{r})) d\mathbf{r} \\ + \sum_k \frac{g_{0k}}{M^2} \int (\bar{\psi}(\mathbf{r}) O_k \psi(\mathbf{r})) (\bar{\psi}(\mathbf{r}) O_k \psi(\mathbf{r})) d\mathbf{r} \quad (1)$$

where $\psi = \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix}$ is the nucleon field operator. τ_i 's are the isotopic spin operators and O_k ($k=s, v, t, a, p$) are Dirac matrices which correspond to one of the five invariant interaction types, scalar, vector, tensor, pseudovector and pseudoscalar, respectively, and

$$O_s = 1 \\ O_v = i\gamma_\mu \\ O_t = \frac{1}{2\sqrt{2}i} (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu) \\ O_a = i\gamma_5 \gamma_\mu \\ O_p = \gamma_5. \quad (2)$$

Now we represent (1) in terms of the creation and annihilation operators and make the same approximation as I. Then we get the following interaction Hamiltonian corresponding to (I.3) :

$$H' \simeq \frac{1}{(2\pi)^3} \sum_{p_i} \frac{1}{M^2} \int d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{p}_3 d\mathbf{p}_4 \\ \left[g_{1p} \{ (a_{p_3}^{p_3*} a_{p_1}^{p_1*} b_{p_2}^{p_2*} b_{p_4}^{p_4*} B_{p_1 p_2}^i B_{p_3 p_4}^i + b_{p_3}^{p_3} b_{p_1}^{p_1} a_{p_2}^{p_2} a_{p_4}^{p_4} C_{p_1 p_2}^i C_{p_3 p_4}^i) \delta(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4) \right. \\ \left. - 2a_{p_1}^{p_1*} b_{p_2}^{p_2*} b_{p_3}^{p_3} a_{p_4}^{p_4} B_{p_1 p_2}^i C_{p_3 p_4}^i \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) \} \right. \\ + g_{0p} \{ (a_{p_3}^{p_3*} a_{p_1}^{p_1*} b_{p_2}^{p_2*} b_{p_4}^{p_4*} B_{p_1 p_2} B_{p_3 p_4} + b_{p_3}^{p_3} b_{p_1}^{p_1} a_{p_2}^{p_2} a_{p_4}^{p_4} C_{p_1 p_2} C_{p_3 p_4}) \delta(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4) \\ \left. - 2a_{p_1}^{p_1*} b_{p_2}^{p_2*} b_{p_3}^{p_3} a_{p_4}^{p_4} B_{p_1 p_2} C_{p_3 p_4} \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) \} \right. \\ \left. + g_{1s} \{ -2a_{p_1}^{p_1*} b_{p_3}^{p_3*} b_{p_4}^{p_4} a_{p_2}^{p_2} E_{p_1 p_2}^i E_{p_3 p_4}^i \delta(\mathbf{p}_1 - \mathbf{p}_2 + \mathbf{p}_3 - \mathbf{p}_4) \} \right]$$

$$\begin{aligned}
& +g_{08} \times 2a_{p_1}^{p_1*} b_{p_3}^{p_3*} b_{p_4}^{p_4} a_{p_2}^{p_2} E_{p_1 p_2} E_{p_3 p_4} \delta(\mathbf{p}_1 - \mathbf{p}_2 + \mathbf{p}_3 - \mathbf{p}_4) \\
& +g_{1z} \{ (a_{p_3}^{p_3*} a_{p_1}^{p_1*} b_{p_2}^{p_2*} b_{p_4}^{p_4*} S_{p_1 p_2}^i S_{p_3 p_4}^i + b_{p_3}^{p_3*} b_{p_1}^{p_1*} a_{p_2}^{p_2} a_{p_4}^{p_4} T_{p_1 p_2}^i T_{p_3 p_4}^i) \delta(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4) \\
& \quad + 2(a_{p_1}^{p_1*} b_{p_3}^{p_3*} b_{p_4}^{p_4} a_{p_2}^{p_2} O_{p_1 p_3}^i O_{p_3 p_4}^i \delta(\mathbf{p}_1 - \mathbf{p}_2 + \mathbf{p}_3 - \mathbf{p}_4) \\
& \quad + a_{p_1}^{p_1*} b_{p_2}^{p_2*} b_{p_3}^{p_3*} a_{p_4}^{p_4} S_{p_1 p_2}^i T_{p_3 p_4}^i \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4)) \} \\
& +g_{0r} \{ (a_{p_3}^{p_3*} a_{p_1}^{p_1*} b_{p_2}^{p_2*} b_{p_4}^{p_4*} S_{p_1 p_2} S_{p_3 p_4} + b_{p_3}^{p_3*} b_{p_1}^{p_1*} a_{p_2}^{p_2} a_{p_4}^{p_4} T_{p_1 p_2} T_{p_3 p_4}) \delta(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4) \\
& \quad + 2(-a_{p_1}^{p_1*} b_{p_3}^{p_3*} b_{p_4}^{p_4} a_{p_2}^{p_2} O_{p_1 p_2} O_{p_3 p_4} \delta(\mathbf{p}_1 - \mathbf{p}_2 + \mathbf{p}_3 - \mathbf{p}_4) \\
& \quad + a_{p_1}^{p_1*} b_{p_2}^{p_2*} b_{p_3}^{p_3*} a_{p_4}^{p_4} S_{p_1 p_2} T_{p_3 p_4} \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4)) \} \\
& -g_{1r} \{ (a_{p_3}^{p_3*} a_{p_1}^{p_1*} b_{p_2}^{p_2*} b_{p_4}^{p_4*} I_{p_1 p_2}^i I_{p_3 p_4}^i + b_{p_3}^{p_3*} b_{p_1}^{p_1*} a_{p_2}^{p_2} a_{p_4}^{p_4} J_{p_1 p_2}^i J_{p_3 p_4}^i) \delta(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4) \\
& \quad + 2(a_{p_1}^{p_1*} b_{p_3}^{p_3*} b_{p_4}^{p_4} a_{p_2}^{p_2} K_{p_1 p_2}^i K_{p_3 p_4}^i \delta(\mathbf{p}_1 - \mathbf{p}_2 + \mathbf{p}_3 - \mathbf{p}_4) \\
& \quad + a_{p_1}^{p_1*} b_{p_2}^{p_2*} b_{p_3}^{p_3*} a_{p_4}^{p_4} I_{p_1 p_2}^i J_{p_3 p_4}^i \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4)) \} \\
& -g_{0z} \{ (a_{p_3}^{p_3*} a_{p_1}^{p_1*} b_{p_2}^{p_2*} b_{p_4}^{p_4*} I_{p_1 p_2} I_{p_3 p_4} + b_{p_3}^{p_3*} b_{p_1}^{p_1*} a_{p_2}^{p_2} a_{p_4}^{p_4} J_{p_1 p_2} J_{p_3 p_4}) \delta(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4) \\
& \quad + 2(-a_{p_1}^{p_1*} b_{p_3}^{p_3*} b_{p_4}^{p_4} a_{p_2}^{p_2} K_{p_1 p_2} K_{p_3 p_4} \delta(\mathbf{p}_1 - \mathbf{p}_2 + \mathbf{p}_3 - \mathbf{p}_4) \\
& \quad + a_{p_1}^{p_1*} b_{p_2}^{p_2*} b_{p_3}^{p_3*} a_{p_4}^{p_4} I_{p_1 p_2} J_{p_3 p_4} \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4)) \} \\
& -g_{1z} \{ (a_{p_3}^{p_3*} a_{p_1}^{p_1*} b_{p_2}^{p_2*} b_{p_4}^{p_4*} Y_{p_1 p_2}^i Y_{p_3 p_4}^i + b_{p_3}^{p_3*} b_{p_1}^{p_1*} a_{p_2}^{p_2} a_{p_4}^{p_4} Z_{p_1 p_2}^i Z_{p_3 p_4}^i) \delta(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4) \\
& \quad + 2(-a_{p_1}^{p_1*} b_{p_3}^{p_3*} b_{p_4}^{p_4} a_{p_2}^{p_2} U_{p_1 p_2}^i U_{p_3 p_4}^i \delta(\mathbf{p}_1 - \mathbf{p}_2 + \mathbf{p}_3 - \mathbf{p}_4) \\
& \quad + a_{p_1}^{p_1*} b_{p_2}^{p_2*} b_{p_3}^{p_3*} a_{p_4}^{p_4} Y_{p_1 p_2}^i Z_{p_3 p_4}^i \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4)) \} \\
& -g_{0r} \{ (a_{p_3}^{p_3*} a_{p_1}^{p_1*} b_{p_2}^{p_2*} b_{p_4}^{p_4*} Y_{p_1 p_2} Y_{p_3 p_4} + b_{p_3}^{p_3*} b_{p_1}^{p_1*} a_{p_2}^{p_2} a_{p_4}^{p_4} Z_{p_1 p_2} Z_{p_3 p_4}) \delta(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4) \\
& \quad + 2(a_{p_1}^{p_1*} b_{p_3}^{p_3*} b_{p_4}^{p_4} a_{p_2}^{p_2} U_{p_1 p_2} U_{p_3 p_4} \delta(\mathbf{p}_1 - \mathbf{p}_2 + \mathbf{p}_3 - \mathbf{p}_4) \\
& \quad + a_{p_1}^{p_1*} b_{p_2}^{p_2*} b_{p_3}^{p_3*} a_{p_4}^{p_4} Y_{p_1 p_2} Z_{p_3 p_4} \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4)) \} \quad (3)
\end{aligned}$$

where

$$\begin{aligned}
B^i &= (\tau_i \tau_2) \sigma_2 & B &= C = \tau_2 \sigma_2 \\
C^i &= (\tau_2 \tau_i) \sigma_2 \\
E^i &= \tau_i & E &= 1 \\
O^i &= \tau_i \sigma_k & O &= \sigma_k \\
S^i &= (\tau_i \tau_2) (\sigma_k \sigma_2) & S &= \tau_2 (\sigma_k \sigma_2) \\
T^i &= -(\tau_2 \tau_i) (\sigma_2 \sigma_k) & T &= -\tau_2 (\sigma_2 \sigma_k) \\
I^i &= -i(\tau_i \tau_2) (\sigma_k \sigma_2) & I &= -i\tau_2 (\sigma_k \sigma_2) \\
J^i &= -i(\tau_2 \tau_i) (\sigma_2 \sigma_k) & J &= -i\tau_2 (\sigma_2 \sigma_k) \\
K^i &= \tau_i & K &= 1 \\
U^i &= -i\tau_i \sigma_k & U &= -i\sigma_k
\end{aligned}$$

$$\begin{aligned}
 Y^i &= (\tau_i \tau_2) \sigma_2 & Y &= \tau_2 \sigma_2 \\
 Z^i &= (\tau_2 \tau_i) \sigma_2 & Z &= \tau_2 \sigma_2.
 \end{aligned} \tag{4}$$

If we calculate the commutator between Hamiltonian and $-a_{p+k/2}^{\nu_1*} B_{\rho_1 \rho_2}^j b_{-(p-k/2)}^{\nu_2*}$ or $b_{-(p+k/2)}^{\nu_1} C_{\rho_1 \rho_2}^j a_{p-k/2}^{\nu_2}$, using the same approximation as I, we have the following commutators (corresponding to (I.5) and (I.6)).

$$\begin{aligned}
 & [-a_{p+k/2}^{\nu_1*} B_{\rho_1 \rho_2}^j b_{-(p-k/2)}^{\nu_2*}, H'] \\
 &= -\frac{1}{(2\pi)^3 M^2} \sum_{\rho_3 \rho_4} \int d\mathbf{p}' \left\{ -(8g_{1p} + 2g_{1s} - 2g_{0s} + 6g_{1t} - 6g_{0t} \right. \\
 &\quad + 2g_{1v} - 2g_{0v} + 2g_{1a} - 6g_{0a}) a_{p'+k/2}^{\rho_3*} B_{\rho_3 \rho_4}^j b_{-(p'-k/2)}^{\rho_4*} \\
 &\quad + (10g_{1p} - 2g_{0p} + 6g_{1t} - 6g_{0t} \\
 &\quad \left. + 6g_{1v} - 6g_{0v} - 10g_{1a} + 2g_{0a}) b_{-(p'+k/2)}^{\rho_3} C_{\rho_3 \rho_4}^j a_{p'+k/2}^{\rho_4} \right\}, \tag{5}
 \end{aligned}$$

$$\begin{aligned}
 & [b_{-(p+k/2)}^{\nu_1} C_{\rho_1 \rho_2}^j a_{p-k/2}^{\nu_2}, H'] \\
 &= -\frac{1}{(2\pi)^3 M^2} \sum_{\rho_3 \rho_4} \int d\mathbf{p}' \left\{ -(10g_{1p} - 2g_{0p} + 6g_{1t} - 6g_{0t} \right. \\
 &\quad + 6g_{1v} - 6g_{0v} - 10g_{1a} + 2g_{0a}) a_{p'+k/2}^{\rho_3*} B_{\rho_3 \rho_4}^j b_{-(p'-k/2)}^{\rho_4*} \\
 &\quad + (8g_{1p} + 2g_{1s} - 2g_{0s} + 6g_{1t} - 6g_{0t} \\
 &\quad \left. + 2g_{1v} - 2g_{0v} + 2g_{1a} - 6g_{0a}) b_{-(p'+k/2)}^{\rho_3} C_{\rho_3 \rho_4}^j a_{p'+k/2}^{\rho_4} \right\}. \tag{6}
 \end{aligned}$$

The eigenvalue equation for the one pair excited state is directly obtained with the use of the commutator (5) and (6), as derived in I.

Case I (s, t, p):

$$\begin{aligned}
 1 &= \frac{1}{(2\pi)^3 M^2} \{8g_{1p} + 2(g_{1s} - g_{0s}) + 6(g_{1t} - g_{0t})\} \\
 &\quad \times \int d\mathbf{p} \left(\frac{1}{E_{p+k/2} + E_{p-k/2} - \omega_k} + \frac{1}{E_{p+k/2} + E_{p-k/2} + \omega_k} \right) \tag{7a}
 \end{aligned}$$

under the condition

$$\begin{aligned}
 8g_{1p} + 2(g_{1s} - g_{0s}) + 6(g_{1t} - g_{0t}) &= 10g_{1p} - 2g_{0p} + 6(g_{1t} - g_{0t}) \\
 (g_{1p} - g_{0p} &= g_{1s} - g_{0s}). \tag{8a}
 \end{aligned}$$

Case II (v, a):

$$\begin{aligned}
 1 &= \frac{1}{(2\pi)^3 M^2} \{2(g_{1v} - g_{0v}) + 2(g_{1a} - 3g_{0a})\} \\
 &\quad \times \int d\mathbf{p} \left(\frac{1}{E_{p+k/2} + E_{p-k/2} - \omega_k} + \frac{1}{E_{p+k/2} + E_{p-k/2} + \omega_k} \right) \tag{7b}
 \end{aligned}$$

under the condition

$$\begin{aligned} 2(g_{1v}-g_{0v})+2g_{1\alpha}-6g_{0\alpha} &= 6(g_{1v}-g_{0v})-10g_{1\alpha}+2g_{0\alpha} \\ (g_{1v}-g_{0v} &= 3g_{1\alpha}-2g_{0\alpha}). \end{aligned} \quad (8b)$$

This eigenvalue equation has two types of solutions, as stated in I, and one of them $\omega_k \sim \sqrt{k^2 + \mu^2}$ (μ is the pion mass) is obtained if we cut the integral in (7) at the nucleon mass and take coupling constants as follows:

Case I:

$$4g_{1p} + (g_{1s} - g_{0s}) + 3(g_{1t} - g_{0t}) = 30. \quad (9a)$$

Case II:

$$g_{1v} - g_{0v} + g_{1\alpha} - 3g_{0\alpha} = 30. \quad (9b)$$

The explicit form of the wave function representing this bound state (pion) has been obtained in I.

$$\Psi_{\pi}^i(k) = A_k^{i*} \Psi_0 \quad (I.13)$$

A_k^{i*} and its Hermite conjugate A_k^i are

$$A_k^{i*} \cong \frac{1}{\sqrt{2\omega_k}} \sum_{p_1 p_2} \int d\mathbf{p} \left(\frac{a_{p+k/2}^{p_1*} B_{p_1 p_2}^i b_{-(p-k/2)}^{p_2*}}{E_{p+k/2} + E_{p-k/2} - \omega_k} + \frac{b_{-(p+k/2)}^{p_1} C_{p_1 p_2}^i a_{p-k/2}^{p_2}}{E_{p+k/2} + E_{p-k/2} + \omega_k} \right), \quad (I.23)$$

$$A_k^i \cong \frac{1}{\sqrt{2\omega_k}} \sum_{p_1 p_2} \int d\mathbf{p} \left(\frac{a_{p-k/2}^{p_1*} B_{p_1 p_2}^i b_{-(p+k/2)}^{p_2*}}{E_{p+k/2} + E_{p-k/2} + \omega_k} + \frac{b_{-(p-k/2)}^{p_1} C_{p_1 p_2}^i a_{p+k/2}^{p_2}}{E_{p+k/2} + E_{p-k/2} - \omega_k} \right). \quad (I.24)$$

§ 3. Effective Hamiltonian for pion-nucleon interaction

We search for the effective Hamiltonian for the pion-nucleon system by using the pion wave function derived in I. We perform the following canonical transformation,

$$e^{iS} O e^{-iS} = O + [iS, O] + \frac{1}{2} [iS, [iS, O]] + \dots, \quad (I.26)$$

where

$$\begin{aligned} S = \frac{f}{(2\pi)^{3/2}} \sum_j \int d\mathbf{k} d\mathbf{p} \frac{1}{\sqrt{2\omega_k}} \left[\left(\frac{a_{p+k/2}^{p_1*} B_{p_1 p_2}^j b_{-(p-k/2)}^{p_2*}}{E_{p+k/2} + E_{p-k/2} - \omega_k} + \frac{b_{-(p+k/2)}^{p_1} C_{p_1 p_2}^j a_{p-k/2}^{p_2}}{E_{p+k/2} + E_{p-k/2} + \omega_k} \right) A_k^{j*} \right. \\ \left. + \left(\frac{a_{p-k/2}^{p_1*} B_{p_1 p_2}^j b_{-(p+k/2)}^{p_2*}}{E_{p+k/2} + E_{p-k/2} + \omega_k} + \frac{b_{-(p-k/2)}^{p_1} C_{p_1 p_2}^j a_{p+k/2}^{p_2}}{E_{p+k/2} + E_{p-k/2} - \omega_k} \right) A_k^{j*} \right] \end{aligned} \quad (I.27)$$

on the total Hamiltonian $H = H_0 + H'$, and the subsidiary condition:

$$A_k^i \Psi_0 = 0. \quad (I.21)$$

Then the transformed form of (I.21) becomes

$$\int d\mathbf{p} \frac{1}{\sqrt{2\omega_k}} \left(\frac{a_{\mu-k/2}^{p1*} B_{\rho_1 \rho_2}^j b_{-(\mu+k/2)}^{p2*}}{E_{\mu+k/2} + E_{\mu-k/2} + \omega_k} + \frac{b_{-(\mu-k/2)}^{p1} C_{\rho_1 \rho_2}^j a_{\mu+k/2}^{p2}}{E_{\mu+k/2} + E_{\mu-k/2} - \omega_k} \right) |F'_0 = 0 \quad (\text{I} \cdot 29)$$

by using $f \sim 15.4$ and the cutoff as before.

Next we transform the total Hamiltonian by (I·26). Before we perform it, we rewrite H' as follows,

$$H' \sim \frac{1}{(2\pi)^3 M^2} \sum_{i\rho} \int d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{p}_3 d\mathbf{p}_4 \\ \times \{[\text{I}] + [\text{II}] + [\text{III}] + 2([\text{IV}] + [\text{V}] + [\text{VI}] + [\text{VII}])\} \quad (10)$$

where

Case I

$$\begin{aligned} [\text{I}] &= a_{p_3}^{p3*} a_{p_1}^{p1*} a_{p_2}^{p2} a_{p_4}^{p4} \delta(\mathbf{p}_1 - \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4) \{g_{1p} D_{\rho_1 \rho_2}^i D_{\rho_3 \rho_4}^i + g_{0p} D_{\rho_1 \rho_2} D_{\rho_3 \rho_4} \\ &\quad + g_{1s} E_{\rho_1 \rho_2}^i E_{\rho_3 \rho_4}^i + g_{0s} E_{\rho_1 \rho_2} E_{\rho_3 \rho_4} + g_{1t} (O_{\rho_1 \rho_2}^i O_{\rho_3 \rho_4}^i + R_{\rho_1 \rho_2}^i R_{\rho_3 \rho_4}^i) \\ &\quad + g_{0t} (O_{\rho_1 \rho_2} O_{\rho_3 \rho_4} + R_{\rho_1 \rho_2} R_{\rho_3 \rho_4})\} \\ [\text{II}] &= a_{p_3}^{p3*} a_{p_1}^{p1*} b_{p_2}^{p2*} b_{p_4}^{p4*} \delta(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4) \{g_{1p} B_{\rho_1 \rho_2}^i B_{\rho_3 \rho_4}^i + g_{0p} B_{\rho_1 \rho_2} B_{\rho_3 \rho_4} \\ &\quad + g_{1s} F_{\rho_1 \rho_2}^i F_{\rho_3 \rho_4}^i + g_{0s} F_{\rho_1 \rho_2} F_{\rho_3 \rho_4} + g_{1t} (P_{\rho_1 \rho_2}^i P_{\rho_3 \rho_4}^i + S_{\rho_1 \rho_2}^i S_{\rho_3 \rho_4}^i) \\ &\quad + g_{0t} (P_{\rho_1 \rho_2} P_{\rho_3 \rho_4} + S_{\rho_1 \rho_2} S_{\rho_3 \rho_4})\} \\ [\text{III}] &= b_{p_3}^{p3} b_{p_1}^{p1} a_{p_2}^{p2} a_{p_4}^{p4} \delta(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4) \{g_{1p} C_{\rho_1 \rho_2}^i C_{\rho_3 \rho_4}^i + g_{0p} C_{\rho_1 \rho_2} C_{\rho_3 \rho_4} \\ &\quad + g_{1s} G_{\rho_1 \rho_2}^i G_{\rho_3 \rho_4}^i + g_{0s} G_{\rho_1 \rho_2} G_{\rho_3 \rho_4} + g_{1t} (Q_{\rho_1 \rho_2}^i Q_{\rho_3 \rho_4}^i + T_{\rho_1 \rho_2}^i T_{\rho_3 \rho_4}^i) \\ &\quad + g_{0t} (Q_{\rho_1 \rho_2} Q_{\rho_3 \rho_4} + T_{\rho_1 \rho_2} T_{\rho_3 \rho_4})\} \\ [\text{IV}] &= a_{p_1}^{p1*} a_{p_3}^{p3*} b_{p_4}^{p4*} a_{p_2}^{p2} \delta(\mathbf{p}_1 - \mathbf{p}_2 + \mathbf{p}_3 - \mathbf{p}_4) \{-g_{1p} D_{\rho_1 \rho_2}^i B_{\rho_3 \rho_4}^i - g_{0p} D_{\rho_1 \rho_2} B_{\rho_3 \rho_4} \\ &\quad - g_{1s} E_{\rho_1 \rho_2}^i F_{\rho_3 \rho_4}^i - g_{0s} E_{\rho_1 \rho_2} F_{\rho_3 \rho_4} + g_{1t} (O_{\rho_1 \rho_2}^i P_{\rho_3 \rho_4}^i + R_{\rho_1 \rho_2}^i S_{\rho_3 \rho_4}^i) \\ &\quad + g_{0t} (O_{\rho_1 \rho_2} P_{\rho_3 \rho_4} + R_{\rho_1 \rho_2} S_{\rho_3 \rho_4})\} \\ [\text{V}] &= a_{p_1}^{p1*} b_{p_3}^{p3} a_{p_4}^{p4} a_{p_2}^{p2} \delta(\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) \{g_{1p} D_{\rho_1 \rho_2}^i C_{\rho_3 \rho_4}^i + g_{0p} D_{\rho_1 \rho_2} C_{\rho_3 \rho_4} \\ &\quad + g_{1s} E_{\rho_1 \rho_2}^i G_{\rho_3 \rho_4}^i + g_{0s} E_{\rho_1 \rho_2} G_{\rho_3 \rho_4} + g_{1t} (O_{\rho_1 \rho_2}^i Q_{\rho_3 \rho_4}^i + R_{\rho_1 \rho_2}^i T_{\rho_3 \rho_4}^i) \\ &\quad + g_{0t} (O_{\rho_1 \rho_2} Q_{\rho_3 \rho_4} + R_{\rho_1 \rho_2} T_{\rho_3 \rho_4})\} \\ [\text{VI}] &= a_{p_1}^{p1*} b_{p_3}^{p3*} b_{p_4}^{p4} a_{p_2}^{p2} \delta(\mathbf{p}_1 - \mathbf{p}_2 + \mathbf{p}_3 - \mathbf{p}_4) \{-g_{1p} D_{\rho_1 \rho_2}^i D_{\rho_3 \rho_4}^i + g_{0p} D_{\rho_1 \rho_2} D_{\rho_3 \rho_4} \\ &\quad - g_{1s} E_{\rho_1 \rho_2}^i E_{\rho_3 \rho_4}^i + g_{0s} E_{\rho_1 \rho_2} E_{\rho_3 \rho_4} + g_{1t} (O_{\rho_1 \rho_2}^i O_{\rho_3 \rho_4}^i + R_{\rho_1 \rho_2}^i R_{\rho_3 \rho_4}^i) \\ &\quad - g_{0t} (O_{\rho_1 \rho_2} O_{\rho_3 \rho_4} + R_{\rho_1 \rho_2} R_{\rho_3 \rho_4})\} \\ [\text{VII}] &= a_{p_1}^{p1*} b_{p_2}^{p2*} b_{p_3}^{p3} a_{p_4}^{p4} \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) \{-g_{1p} B_{\rho_1 \rho_2}^i C_{\rho_3 \rho_4}^i - g_{0p} B_{\rho_1 \rho_2} C_{\rho_3 \rho_4} \\ &\quad - g_{1s} F_{\rho_1 \rho_2}^i G_{\rho_3 \rho_4}^i - g_{0s} F_{\rho_1 \rho_2} G_{\rho_3 \rho_4} + g_{1t} (P_{\rho_1 \rho_2}^i Q_{\rho_3 \rho_4}^i + S_{\rho_1 \rho_2}^i T_{\rho_3 \rho_4}^i) \\ &\quad + g_{0t} (P_{\rho_1 \rho_2} Q_{\rho_3 \rho_4} + S_{\rho_1 \rho_2} T_{\rho_3 \rho_4})\} \end{aligned}$$

$$B^i = (\tau_i \tau_2) \left(\sigma_2 - \frac{\sigma \mathbf{p}}{E_p + M} - \frac{\sigma \mathbf{q}}{E_q + M} \sigma_2 \right) D_p D_q,$$

$$B = \tau_2 \left(\sigma_2 - \frac{\sigma \mathbf{p}}{E_p + M} - \frac{\sigma \mathbf{q}}{E_q + M} \sigma_2 \right) D_p D_q$$

$$C^i = (\tau_2 \tau_i) \left(\sigma_2 - \sigma_2 \frac{\sigma \mathbf{p}}{E_p + M} - \frac{\sigma \mathbf{q}}{E_q + M} \right) D_p D_q,$$

$$C = \tau_2 \left(\sigma_2 - \sigma_2 \frac{\sigma \mathbf{p}}{E_p + M} - \frac{\sigma \mathbf{q}}{E_q + M} \right) D_p D_q$$

$$D^i = \tau_i \left(\frac{\sigma \mathbf{p}}{E_p + M} - \frac{\sigma \mathbf{q}}{E_q + M} \right) D_p D_q,$$

$$D = \left(\frac{\sigma \mathbf{p}}{E_p + M} - \frac{\sigma \mathbf{q}}{E_q + M} \right) D_p D_q$$

$$E^i = \tau_i \left(1 - \frac{\sigma \mathbf{p}}{E_p + M} - \frac{\sigma \mathbf{q}}{E_q + M} \right) D_p D_q,$$

$$E = \left(1 - \frac{\sigma \mathbf{p}}{E_p + M} - \frac{\sigma \mathbf{q}}{E_q + M} \right) D_p D_q$$

$$F^i = (\tau_i \tau_2) \left(\frac{\sigma \mathbf{p}}{E_p + M} - \frac{\sigma \mathbf{q}}{E_q + M} \right) \sigma_2 D_p D_q,$$

$$F = \tau_2 \left(\frac{\sigma \mathbf{p}}{E_p + M} - \frac{\sigma \mathbf{q}}{E_q + M} \right) \sigma_2 D_p D_q$$

$$G^i = (\tau_2 \tau_i) \sigma_2 \left(\frac{\sigma \mathbf{p}}{E_p + M} - \frac{\sigma \mathbf{q}}{E_q + M} \right) D_p D_q,$$

$$G = \tau_2 \sigma_2 \left(\frac{\sigma \mathbf{p}}{E_p + M} - \frac{\sigma \mathbf{q}}{E_q + M} \right) D_p D_q$$

$$O^i = \tau_i \left(\sigma_k - \frac{\sigma \mathbf{p}}{E_p + M} \sigma_k - \frac{\sigma \mathbf{q}}{E_q + M} \right) D_p D_q,$$

$$O = \left(\sigma_k - \frac{\sigma \mathbf{p}}{E_p + M} \sigma_k - \frac{\sigma \mathbf{q}}{E_q + M} \right) D_p D_q$$

$$P^i = (\tau_i \tau_2) \left(-\frac{\sigma \mathbf{p}}{E_p + M} - \sigma_k + \sigma_k \frac{\sigma \mathbf{q}}{E_q + M} \right) \sigma_2 D_p D_q,$$

$$P = \tau_2 \left(-\frac{\sigma \mathbf{p}}{E_p + M} - \sigma_k + \sigma_k \frac{\sigma \mathbf{q}}{E_q + M} \right) \sigma_2 D_p D_q$$

$$Q^i = (\tau_2 \tau_i) \sigma_2 \left(\frac{\sigma \mathbf{p}}{E_p + M} - \sigma_k - \sigma_k \frac{\sigma \mathbf{q}}{E_q + M} \right) D_p D_q,$$

$$Q = \tau_2 \sigma_2 \left(\frac{\sigma \mathbf{p}}{E_p + M} - \sigma_k - \sigma_k \frac{\sigma \mathbf{q}}{E_q + M} \right) D_p D_q$$

$$\begin{aligned}
R^i &= \tau_i \left(-\frac{\sigma \mathbf{p}}{E_p + M} \sigma_k + \sigma_k \frac{\sigma \mathbf{q}}{E_q + M} \right) D_p D_q, \\
R &= \left(-\frac{\sigma \mathbf{p}}{E_p + M} \sigma_k + \sigma_k \frac{\sigma \mathbf{q}}{E_q + M} \right) D_p D_q \\
S^i &= (\tau_i \tau_2) \left(\sigma_k - \frac{\sigma \mathbf{p}}{E_p + M} \sigma_k \frac{\sigma \mathbf{q}}{E_q + M} \right) \sigma_2 D_p D_q, \\
S &= \tau_2 \left(\sigma_k - \frac{\sigma \mathbf{p}}{E_p + M} \sigma_k \frac{\sigma \mathbf{q}}{E_q + M} \right) \sigma_2 D_p D_q \\
T^i &= (\tau_2 \tau_i) \sigma_2 \left(-\sigma_k + \frac{\sigma \mathbf{p}}{E_p + M} \sigma_k \frac{\sigma \mathbf{q}}{E_q + M} \right) D_p D_q, \\
T &= \tau_2 \sigma_2 \left(-\sigma_k + \frac{\sigma \mathbf{p}}{E_p + M} \sigma_k \frac{\sigma \mathbf{q}}{E_q + M} \right) D_p D_q.
\end{aligned} \tag{11a}$$

Case II

$$\begin{aligned}
[\text{I}] &= -a_{p3}^{\rho 3*} a_{p1}^{\rho 1*} a_{p2}^{\rho 2} a_{p4}^{\rho 4} \delta(\mathbf{p}_1 - \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4) \{g_{1v}(H_{p1 p2}^i H_{p3 p4}^i + K_{p1 p2}^i K_{p3 p4}^i) \\
&\quad + g_{0v}(H_{p1 p2} H_{p3 p4} + K_{p1 p2} K_{p3 p4}) + g_{1\alpha}(U_{p1 p2}^i U_{p3 p4}^i + X_{p1 p2}^i X_{p3 p4}^i) \\
&\quad + g_{0\alpha}(U_{p1 p2} U_{p3 p4} + X_{p1 p2} X_{p3 p4})\} \\
[\text{II}] &= -a_{p3}^{\rho 3*} a_{p1}^{\rho 1*} b_{p2}^{\rho 2*} b_{p4}^{\rho 4*} \delta(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4) \{g_{1v}(I_{p1 p2}^i I_{p3 p4}^i + L_{p1 p2}^i L_{p3 p4}^i) \\
&\quad + g_{0v}(I_{p1 p2} I_{p3 p4} + L_{p1 p2} L_{p3 p4}) + g_{1\alpha}(V_{p1 p2}^i V_{p3 p4}^i + Y_{p1 p2}^i Y_{p3 p4}^i) \\
&\quad + g_{0\alpha}(V_{p1 p2} V_{p3 p4} + Y_{p1 p2} Y_{p3 p4})\} \\
[\text{III}] &= -b_{p3}^{\rho 3} b_{p1}^{\rho 1} a_{p2}^{\rho 2} a_{p4}^{\rho 4} \delta(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4) \{g_{1v}(J_{p1 p2}^i J_{p3 p4}^i + M_{p1 p2}^i M_{p3 p4}^i) \\
&\quad + g_{0v}(J_{p1 p2} J_{p3 p4} + M_{p1 p2} M_{p3 p4}) + g_{1\alpha}(W_{p1 p2}^i W_{p3 p4}^i + Z_{p1 p2}^i Z_{p3 p4}^i) \\
&\quad + g_{0\alpha}(W_{p1 p2} W_{p3 p4} + Z_{p1 p2} Z_{p3 p4})\} \\
[\text{IV}] &= -a_{p1}^{\rho 1*} a_{p3}^{\rho 3*} b_{p4}^{\rho 4*} a_{p2}^{\rho 2} \delta(\mathbf{p}_1 - \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4) \{g_{1v}(H_{p1 p2}^i I_{p3 p4}^i + K_{p1 p2}^i L_{p3 p4}^i) \\
&\quad + g_{0v}(H_{p1 p2} I_{p3 p4} + K_{p1 p2} L_{p3 p4}) + g_{1\alpha}(U_{p1 p2}^i V_{p3 p4}^i + X_{p1 p2}^i Y_{p3 p4}^i) \\
&\quad + g_{0\alpha}(U_{p1 p2} V_{p3 p4} + X_{p1 p2} Y_{p3 p4})\} \\
[\text{V}] &= -a_{p1}^{\rho 1*} b_{p3}^{\rho 3} a_{p4}^{\rho 4} a_{p2}^{\rho 2} \delta(\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) \{g_{1v}(H_{p1 p2}^i J_{p3 p4}^i + K_{p1 p2}^i M_{p3 p4}^i) \\
&\quad + g_{0v}(H_{p1 p2} J_{p3 p4} + K_{p1 p2} M_{p3 p4}) + g_{1\alpha}(U_{p1 p2}^i W_{p3 p4}^i + X_{p1 p2}^i Z_{p3 p4}^i) \\
&\quad + g_{0\alpha}(U_{p1 p2} W_{p3 p4} + X_{p1 p2} Z_{p3 p4})\} \\
[\text{VI}] &= -a_{p1}^{\rho 1*} b_{p3}^{\rho 3*} b_{p4}^{\rho 4} a_{p2}^{\rho 2} \delta(\mathbf{p}_1 - \mathbf{p}_2 + \mathbf{p}_3 - \mathbf{p}_4) \{g_{1v}(H_{p1 p2}^i H_{p3 p4}^i + K_{p1 p2}^i K_{p3 p4}^i) \\
&\quad - g_{0v}(H_{p1 p2} H_{p3 p4} + K_{p1 p2} K_{p3 p4}) - g_{1\alpha}(U_{p1 p2}^i U_{p3 p4}^i + X_{p1 p2}^i X_{p3 p4}^i) \\
&\quad + g_{0\alpha}(U_{p1 p2} U_{p3 p4} + X_{p1 p2} X_{p3 p4})\}
\end{aligned}$$

$$\begin{aligned}
 [\text{VII}] = & -a_{p1}^{p1*} b_{p2}^{p2*} b_{p3}^{p3} a_{p4}^{p4} \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) \{ g_{1v} ((I_{p1 p2}^i J_{p3 p4}^i + L_{p1 p2}^i M_{p3 p4}^i) \\
 & + g_{0v} (I_{p1 p2} J_{p3 p4} + L_{p1 p2} M_{p3 p4}) + g_{1\alpha} (V_{p1 p2}^i W_{p3 p4}^i + Y_{p1 p2}^i Z_{p3 p4}^i) \\
 & + g_{0\alpha} (V_{p1 p2} W_{p3 p4} + Y_{p1 p2} Z_{p3 p4}) \}
 \end{aligned}$$

$$H^i = -i\tau_i \left(\frac{\boldsymbol{\sigma} \mathbf{p}}{E_p + M} \sigma_k + \sigma_k \frac{\boldsymbol{\sigma} \mathbf{q}}{E_q + M} \right) D_p D_q,$$

$$H = -i \left(\frac{\boldsymbol{\sigma} \mathbf{p}}{E_p + M} \sigma_k + \sigma_k \frac{\boldsymbol{\sigma} \mathbf{q}}{E_q + M} \right) D_p D_q$$

$$I^i = -i(\tau_i \tau_2) \left(\sigma_k + \frac{\boldsymbol{\sigma} \mathbf{p}}{E_p + M} \sigma_k \frac{\boldsymbol{\sigma} \mathbf{q}}{E_q + M} \right) \sigma_2 D_p D_q,$$

$$I = -i\tau_2 \left(\sigma_k + \frac{\boldsymbol{\sigma} \mathbf{p}}{E_p + M} \sigma_k \frac{\boldsymbol{\sigma} \mathbf{q}}{E_q + M} \right) \sigma_2 D_p D_q$$

$$J^i = -i(\tau_2 \tau_i) \sigma_2 \left(\sigma_k + \frac{\boldsymbol{\sigma} \mathbf{p}}{E_p + M} \sigma_k \frac{\boldsymbol{\sigma} \mathbf{q}}{E_q + M} \right) D_p D_q,$$

$$J = -i\tau_2 \sigma_2 \left(\sigma_k + \frac{\boldsymbol{\sigma} \mathbf{p}}{E_p + M} \sigma_k \frac{\boldsymbol{\sigma} \mathbf{q}}{E_q + M} \right) D_p D_q$$

$$K^i = \tau_i \left(1 + \frac{\boldsymbol{\sigma} \mathbf{p}}{E_p + M} \frac{\boldsymbol{\sigma} \mathbf{q}}{E_q + M} \right) D_p D_q,$$

$$K = \left(1 + \frac{\boldsymbol{\sigma} \mathbf{p}}{E_p + M} \frac{\boldsymbol{\sigma} \mathbf{q}}{E_q + M} \right) D_p D_q$$

$$L^i = (\tau_i \tau_2) \left(\frac{\boldsymbol{\sigma} \mathbf{p}}{E_p + M} + \frac{\boldsymbol{\sigma} \mathbf{q}}{E_q + M} \right) \sigma_2 D_p D_q,$$

$$L = \tau_2 \left(\frac{\boldsymbol{\sigma} \mathbf{p}}{E_p + M} + \frac{\boldsymbol{\sigma} \mathbf{q}}{E_q + M} \right) \sigma_2 D_p D_q$$

$$M^i = (\tau_2 \tau_i) \sigma_2 \left(\frac{\boldsymbol{\sigma} \mathbf{p}}{E_p + M} + \frac{\boldsymbol{\sigma} \mathbf{q}}{E_q + M} \right) D_p D_q,$$

$$M = \tau_2 \sigma_2 \left(\frac{\boldsymbol{\sigma} \mathbf{p}}{E_p + M} + \frac{\boldsymbol{\sigma} \mathbf{q}}{E_q + M} \right) D_p D_q$$

$$U^i = -i\tau_i \left(\sigma_k + \frac{\boldsymbol{\sigma} \mathbf{p}}{E_p + M} \sigma_k \frac{\boldsymbol{\sigma} \mathbf{q}}{E_q + M} \right) D_p D_q,$$

$$U = -i \left(\sigma_k + \frac{\boldsymbol{\sigma} \mathbf{p}}{E_p + M} \sigma_k \frac{\boldsymbol{\sigma} \mathbf{q}}{E_q + M} \right) D_p D_q$$

$$V^i = -i(\tau_i \tau_2) \left(\frac{\boldsymbol{\sigma} \mathbf{p}}{E_p + M} \sigma_k + \sigma_k \frac{\boldsymbol{\sigma} \mathbf{q}}{E_q + M} \right) \sigma_2 D_p D_q,$$

$$V = -i\tau_2 \left(\frac{\boldsymbol{\sigma} \mathbf{p}}{E_p + M} \sigma_k + \sigma_k \frac{\boldsymbol{\sigma} \mathbf{q}}{E_q + M} \right) \sigma_2 D_p D_q$$

$$\begin{aligned}
W^i &= -i(\tau_2 \tau_i) \sigma_2 \left(\frac{\sigma \mathbf{p}}{E_p + M} \sigma_k + \sigma_k \frac{\sigma \mathbf{q}}{E_q + M} \right) D_p D_q, \\
W &= -i\tau_2 \sigma_2 \left(\frac{\sigma \mathbf{p}}{E_p + M} \sigma_k + \sigma_k \frac{\sigma \mathbf{q}}{E_p + M} \right) D_p D_q \\
X^i &= \tau_i \left(\frac{\sigma \mathbf{p}}{E_p + M} + \frac{\sigma \mathbf{q}}{E_q + M} \right) D_p D_q, \\
X &= \left(\frac{\sigma \mathbf{p}}{E_p + M} + \frac{\sigma \mathbf{q}}{E_q + M} \right) D_p D_q \\
Y^i &= (\tau_i \tau_2) \left(1 + \frac{\sigma \mathbf{p}}{E_p + M} \frac{\sigma \mathbf{q}}{E_q + M} \right) \sigma_2 D_p D_q, \\
Y &= \tau_2 \left(1 + \frac{\sigma \mathbf{p}}{E_p + M} \frac{\sigma \mathbf{q}}{E_q + M} \right) \sigma_2 D_p D_q \\
Z^i &= (\tau_2 \tau_i) \sigma_2 \left(1 + \frac{\sigma \mathbf{p}}{E_p + M} \frac{\sigma \mathbf{q}}{E_q + M} \right) D_p D_q, \\
Z &= \tau_2 \sigma_2 \left(1 + \frac{\sigma \mathbf{p}}{E_p + M} \frac{\sigma \mathbf{q}}{E_q + M} \right) D_p D_q. \\
D_p &= \sqrt{(E_p + M)/2E_p}. \tag{11b}
\end{aligned}$$

Now we transform (10) by (I.26) and pick up the terms corresponding to the free Hamiltonian of the pion, the p -wave interaction and the s -wave interaction, as derived in I. Making the approximation $E_p \sim M$ and using $f=15.4$ and cutting off integrals with respect to \mathbf{p} and \mathbf{p}' at the nucleon mass, we get the following. The free Hamiltonian of the pion H_π is derived from the terms in

$$\frac{1}{2} \left[iS, \left[iS, H_0 + \frac{1}{(2\pi)^3 M^2} \sum_{i,p} \int d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{p}_3 d\mathbf{p}_4 \{ [\text{II}] + [\text{III}] + 2([\text{VI}] + [\text{VII}]) \} \right] \right],$$

Case I

$$\begin{aligned}
H_\pi &\sim \sum_j \int d\mathbf{k} \left(2 \left[1 - \frac{1}{60} \{ 4g_{1p} + (g_{1s} - g_{0s}) + 3(g_{1t} - g_{0t}) \} \right] \omega_k \right. \\
&\quad \left. + 2 \left[4 - \frac{1}{15} \{ 9g_{1p} - g_{0p} + (g_{1s} - g_{0s}) + 6(g_{1t} - g_{0t}) \} \right] \frac{M^2}{\omega_k} \right) A_k^{j*} A_k^j \tag{12a} \\
&= \sum_j \int d\mathbf{k} \omega_k A_k^{j*} A_k^j
\end{aligned}$$

by using (8a) and (9a).

Case II

$$\begin{aligned}
H_\pi &\sim \sum_j \int d\mathbf{k} \left(2 \left\{ 1 - \frac{1}{60} (g_{1v} - g_{0v} + g_{1a} + 3g_{0a}) \right\} \omega_k \right. \\
&\quad \left. + 8 \left\{ 1 - \frac{1}{15} (g_{1v} - g_{0v} - g_{1a} + g_{0a}) \right\} \frac{M^2}{\omega_k} \right) A_k^{j*} A_k^j. \tag{12b}
\end{aligned}$$

If we put

$$2 \left\{ 1 - \frac{1}{60} (g_{1v} - g_{0v} + g_{1a} + 3g_{0a}) \right\} = 1 \quad (13b)$$

and

$$1 - \frac{1}{15} (g_{1v} - g_{0v} - g_{1a} + g_{0a}) = 0, \quad (14b)$$

we get

$$H_\pi \sim \sum_j \int d\mathbf{k} \omega_k A_k^{j*} A_k^j. \quad (15)$$

The p -wave interaction Hamiltonian H_{pv} is obtained from the terms in

$$\left[iS, \frac{1}{(2\pi)^3 M^2} \sum_{\mathbf{p}} \int d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{p}_3 d\mathbf{p}_4 2([IV] + [V]) \right].$$

Case I

$$\begin{aligned} H_{pv} &\sim \frac{if}{6\pi^2} \frac{1}{M} \{4g_{1p} + (g_{1s} - g_{0s}) + 3(g_{1t} - g_{0t})\} \\ &\times \frac{1}{(2\pi)^{3/2}} \sum_j \int d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{k} \frac{1}{\sqrt{2\omega_k}} a_{p_1}^* \tau_j(\sigma\mathbf{k}) \{-A_k^j \delta(\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{k}) \\ &+ A_k^{j*} \delta(\mathbf{p}_1 - \mathbf{p}_2 + \mathbf{k})\} a_{p_2}. \end{aligned}$$

Using (9a) and $f \sim 15.4$, we obtain

$$\begin{aligned} H_{pv} &\sim \frac{iG}{\mu} \frac{1}{(2\pi)^{3/2}} \sum_j \int d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{k} \frac{1}{\sqrt{2\omega_k}} a_{p_1}^* \tau_j(\sigma\mathbf{k}) \{-A_k^j \delta(\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{k}) \\ &+ A_k^{j*} \delta(\mathbf{p}_1 - \mathbf{p}_2 + \mathbf{k})\} a_{p_2}, \end{aligned} \quad (16)$$

where

$$\frac{G^2}{4\pi} \sim 0.09. \quad (17a)$$

Case II

$$\begin{aligned} H_{pv} &\sim \frac{if}{6\pi^2} \frac{1}{M} \{(g_{1v} - g_{0v}) + (g_{1a} + 7g_{0a})\} \frac{1}{(2\pi)^{3/2}} \sum_j \int d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{k} \\ &\times \frac{1}{\sqrt{2\omega_k}} a_{p_1}^* \tau_j(\sigma\mathbf{k}) \{-A_k^j \delta(\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{k}) + A_k^{j*} \delta(\mathbf{p}_1 - \mathbf{p}_2 + \mathbf{k})\} a_{p_2} \\ &= \frac{iG}{\mu} \frac{1}{(2\pi)^{3/2}} \sum_j \int d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{k} \frac{1}{\sqrt{2\omega_k}} a_{p_1}^* \tau_j(\sigma\mathbf{k}) \\ &\times \{-A_k^j \delta(\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{k}) + A_k^{j*} \delta(\mathbf{p}_1 - \mathbf{p}_2 + \mathbf{k})\} a_{p_2} \end{aligned} \quad (16)$$

where

$$\frac{G^2}{4\pi} = \frac{1}{4\pi} \left[\frac{f}{6\pi^2} \frac{\mu}{M} \{(g_{1v} - g_{0v}) + (g_{1a} + 7g_{0a})\} \right]^2. \quad (17b)$$

The effective s -wave interaction Hamiltonian H_s is derived from the terms in

$$\begin{aligned} & \left[iS, H_0 + \frac{1}{(2\pi)^3 M^2} \sum_{i_p} \int d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{p}_3 d\mathbf{p}_4 \{ [\text{II}] + [\text{III}] + 2([\text{VI}] + [\text{VII}]) \} \right] \\ & + \frac{1}{2} \left[iS, \left[iS, H_0 + \frac{1}{(2\pi)^3 M^2} \sum_{i_p} \int d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{p}_3 d\mathbf{p}_4 \{ [\text{I}] + [\text{II}] + [\text{III}] \right. \right. \\ & \qquad \qquad \qquad \left. \left. + 2([\text{VI}] + [\text{VII}]) \} \right] \right]. \\ H_s \cong & \frac{f^2}{(2\pi)^3} \sum_{i_j} \int d\mathbf{p} d\mathbf{p}' d\mathbf{k} d\mathbf{k}' \frac{1}{\sqrt{2\omega_k}} \frac{1}{\sqrt{2\omega_{k'}}} \frac{1}{2M} \delta(\mathbf{p}' + \mathbf{k}' - \mathbf{p} - \mathbf{k}) \\ & \times a_{p'}^* [\hat{\epsilon} + \boldsymbol{\eta}(\boldsymbol{\tau} \cdot \boldsymbol{\omega})] a_p A_{k'}^{i*} A_k^j \end{aligned} \quad (18)$$

where

Case I

$$\begin{aligned} \hat{\epsilon} = & -2 + \frac{1}{60} \{ 6(5g_{1p} - g_{0p}) + 2(5g_{0s} - g_{1s}) - 12(g_{1t} + 3g_{0t}) \} \\ & + 4 \left[1 - \frac{1}{60} \{ 9g_{1p} - g_{0p} + (g_{1s} - g_{0s}) + 6(g_{1t} - g_{0t}) \} \right]^2 \\ & + \left(\frac{\omega_k}{2M} \right)^2 \left\{ -6 + \frac{1}{60} \{ 10(5g_{1p} - g_{0p}) + 6(5g_{0s} - g_{1s}) - 12(5g_{1t} + 7g_{0t}) \} \right. \\ & \quad \left. + 8 \left[1 - \frac{1}{60} \{ 9g_{1p} - g_{0p} + (g_{1s} - g_{0s}) + 6(g_{1t} - g_{0t}) \} \right] \right. \\ & \quad \left. \times \left[1 - \frac{1}{60} \{ (g_{0p} - g_{1p}) + (g_{1s} - g_{0s}) \} \right] \right. \\ & \quad \left. + 4 \left[1 - \frac{1}{60} \{ (g_{0p} - g_{1p}) + (g_{1s} - g_{0s}) \} \right]^2 \right\} \\ = & 1 + \frac{1}{30} \{ (5g_{0s} - g_{1s}) - 3(5g_{1t} + 3g_{0t}) \} \\ & + \left(\frac{\omega_k}{2M} \right)^2 \left[3 + \frac{1}{10} \{ (5g_{0s} - g_{1s}) - 3(5g_{1t} + 3g_{0t}) \} \right] \end{aligned} \quad (19a)$$

$$\begin{aligned} \eta = & 4 \left(\frac{\omega_k}{2M} \right) \left[-1 + \frac{1}{60} \{ (7g_{1p} + g_{0p}) + 3(g_{1s} - g_{0s}) + 6(g_{1t} - g_{0t}) \} \right. \\ & + 2 \left\{ 1 - \frac{1}{60} (9g_{1p} - g_{0p} + g_{1s} - g_{0s} + 6(g_{1t} - g_{0t})) \right\} \\ & \times \left\{ 1 - \frac{1}{60} (g_{0p} - g_{1p} + g_{1s} - g_{0s}) \right\} \\ & \left. + \left\{ 1 - \frac{1}{60} (9g_{1p} - g_{0p} + g_{1s} - g_{0s} + 6(g_{1t} - g_{0t})) \right\}^2 \right] \end{aligned}$$

$$\begin{aligned}
& +8 \left(\frac{\omega_k}{2M} \right)^3 \left[-1 + \frac{1}{60} \{ (7g_{1p} + g_{0p}) + 3(g_{1s} - g_{0s}) + 6(g_{1t} - g_{0t}) \} \right. \\
& \quad \left. + \frac{1}{2} \left\{ 1 - \frac{1}{60} (g_{0p} - g_{1p} + g_{1s} - g_{0s}) \right\}^2 \right] \\
& = 4 \left(\frac{\omega_k}{2M} \right)^3 \tag{20a}
\end{aligned}$$

by using (8a) and (9a).

Case II

$$\begin{aligned}
\xi &= -2 + \frac{2}{15} \{ (5g_{1v} - g_{0v}) - (5g_{1a} - g_{0a}) \} + 4 \left[1 - \frac{1}{15} \{ g_{1v} - g_{0v} - (g_{1a} - g_{0a}) \} \right]^2 \\
& + \left(\frac{\omega_k}{2M} \right)^2 \left[-6 + \frac{4}{15} (6g_{1v} - 5g_{1a} + g_{0a}) \right. \\
& \quad + 8 \left\{ 1 - \frac{1}{15} (g_{1v} - g_{0v} - g_{1a} + g_{0a}) \right\} \times \left\{ 1 + \frac{1}{30} (g_{1v} - g_{0v} - 3g_{1a} - g_{0a}) \right\} \\
& \quad \left. + 4 \left\{ 1 + \frac{1}{30} (g_{1v} - g_{0v} - 3g_{1a} - g_{0a}) \right\}^2 \right] \\
& = -2 + \frac{2}{15} (4g_{1v} - 2g_{1a} - g_{0a}) \\
& \quad + \left(\frac{\omega_k}{2M} \right)^2 \left\{ -6 + \frac{4}{15} (6g_{1v} - 5g_{1a} + g_{0a}) + 4 \left(1 - \frac{1}{10} g_{0a} \right)^2 \right\} \tag{19b}
\end{aligned}$$

$$\begin{aligned}
\eta &= 4 \left(\frac{\omega_k}{2M} \right) \left[-1 + \frac{1}{15} (g_{1v} - g_{0v} + g_{1a} + g_{0a}) \right. \\
& \quad + 2 \left\{ 1 - \frac{1}{15} (g_{1v} - g_{0v} - g_{1a} + g_{0a}) \right\} \times \left\{ 1 + \frac{1}{30} (g_{1v} - g_{0v} - 3g_{1a} - g_{0a}) \right\} \\
& \quad \left. + \left\{ 1 - \frac{1}{15} (g_{1v} - g_{0v} - g_{1a} + g_{0a}) \right\}^2 \right] \\
& + 8 \left(\frac{\omega_k}{2M} \right)^3 \left[-1 + \frac{1}{15} (g_{1v} - g_{0v} + g_{1a} + g_{0a}) \right. \\
& \quad \left. + \frac{1}{2} \left\{ 1 + \frac{1}{30} (g_{1v} - g_{0v} - 3g_{1a} - g_{0a}) \right\}^2 \right] \\
& = 4 \left(\frac{\omega_k}{2M} \right) \left\{ -1 + \frac{1}{15} (4g_{1a} - g_{0a}) \right\} \\
& + 8 \left(\frac{\omega_k}{2M} \right)^3 \left\{ -1 + \frac{1}{15} (4g_{1a} - g_{0a}) + \frac{1}{2} \left(1 - \frac{1}{10} g_{0a} \right)^2 \right\} \tag{20b}
\end{aligned}$$

by using (8b) and (14b).

§ 4. Anomalous magnetic moment of nucleon*

We calculate the anomalous magnetic moment of the nucleon by the second order perturbation, i.e. up to order eg^2 .** The covariant calculation is not applied to using the cutoff in the momentum space here as well as in the previous sections. We use a similar method to what have been adopted in the analysis of the magnetic moment of the nucleon by Goto.⁶⁾

The interaction Hamiltonian of the nucleon with the electromagnetic field is given by

$$H_{N\gamma} = -ie \int \bar{\psi}(\mathbf{r}) \gamma_\mu \frac{1+\tau_3}{2} \psi(\mathbf{r}) A_\mu(\mathbf{r}) d\mathbf{r}. \quad (21)$$

The time-ordered graphs contributing to the anomalous magnetic moment are illustrated in Fig. 1. The contributions from the diagrams (A1), (C)'s and (F)'s are equal to those from (A2), (D)'s and (G)'s respectively, because both diagrams can be taken by the time inversion from each other.

The contribution from the diagram (A1) in Fig. 1, for example, is obtained in the case of pseudoscalar coupling as follows.

$$\begin{aligned} (A1)_p &= \frac{-e}{(2\pi)^{3/2}} \frac{2}{(2\pi)^3} \sum_i \frac{g_{1p}}{M^2} \int d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{p}_3 d\mathbf{p}_4 d\mathbf{k} \delta(\mathbf{p}_1 - \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4) \\ &\quad \left[a_{p_1}^* \{ u^*(\mathbf{p}_1) (\beta \gamma_5 \tau_i) u(\mathbf{p}_2) \} a_{p_2} \text{Sp} \left\{ u^*(\mathbf{p}_3) (\beta \gamma_5 \tau_i) (-C v^*(\mathbf{p}_4)) v(\mathbf{p}_4) C (\alpha \mathbf{A}) \right. \right. \\ &\quad \times \frac{1+\tau_3}{2} u(\mathbf{p}_3) \left. \left. \right\} \delta(\mathbf{p}_3 + \mathbf{p}_4 + \mathbf{k}) - a_{p_1}^* \left\{ u^*(\mathbf{p}_1) (\beta \gamma_5 \tau_i) (-C v^*(\mathbf{p}_4)) v(\mathbf{p}_4) C \right. \right. \\ &\quad \times (\alpha \mathbf{A}) \frac{1+\tau_3}{2} u(\mathbf{p}_3) (\beta \gamma_5 \tau_i) u(\mathbf{p}_2) \left. \left. \right\} a_{p_2} \delta(\mathbf{p}_3 + \mathbf{p}_4 + \mathbf{k}) \right] (E_{p_2} - E_{p_1} - E_{p_3} - E_{p_4})^{-1} \\ &= -\frac{e}{(2\pi)^{9/2}} \sum_i \frac{2g_{1p}}{M^2} \int d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{p}_3 d\mathbf{p}_4 d\mathbf{k} \delta(\mathbf{p}_1 - \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4) \delta(\mathbf{p}_3 + \mathbf{p}_4 + \mathbf{k}) \\ &\quad \left[a_{p_1}^* \{ u^*(\mathbf{p}_1) (\beta \gamma_5 \tau_i) u(\mathbf{p}_2) \} a_{p_2} \text{Sp} \left\{ (\beta \gamma_5 \tau_i) \frac{E_{p_4} + (\alpha \mathbf{p}_4) - M\beta}{2E_{p_4}} (\alpha \mathbf{A}) \frac{1+\tau_3}{2} \right. \right. \\ &\quad \times \frac{E_{p_3} + (\alpha \mathbf{p}_3) + M\beta}{2E_{p_3}} \left. \left. \right\} - a_{p_1}^* \left\{ u^*(\mathbf{p}_1) (\beta \gamma_5 \tau_i) \frac{E_{p_4} + (\alpha \mathbf{p}_4) - M\beta}{2E_{p_4}} (\alpha \mathbf{A}) \frac{1+\tau_3}{2} \right. \right. \\ &\quad \times \frac{E_{p_3} + (\alpha \mathbf{p}_3) + M\beta}{2E_{p_3}} (\beta \gamma_5 \tau_i) u(\mathbf{p}_2) \left. \left. \right\} a_{p_2} \right] (E_{p_2} - E_{p_1} - E_{p_3} - E_{p_4})^{-1} \quad (22) \end{aligned}$$

* Already Iso⁴⁾ attempted to discuss the anomalous magnetic moment of the nucleon on this model.

** If the higher order contributions play the essential role, the conclusion from the present calculation will be meaningless.

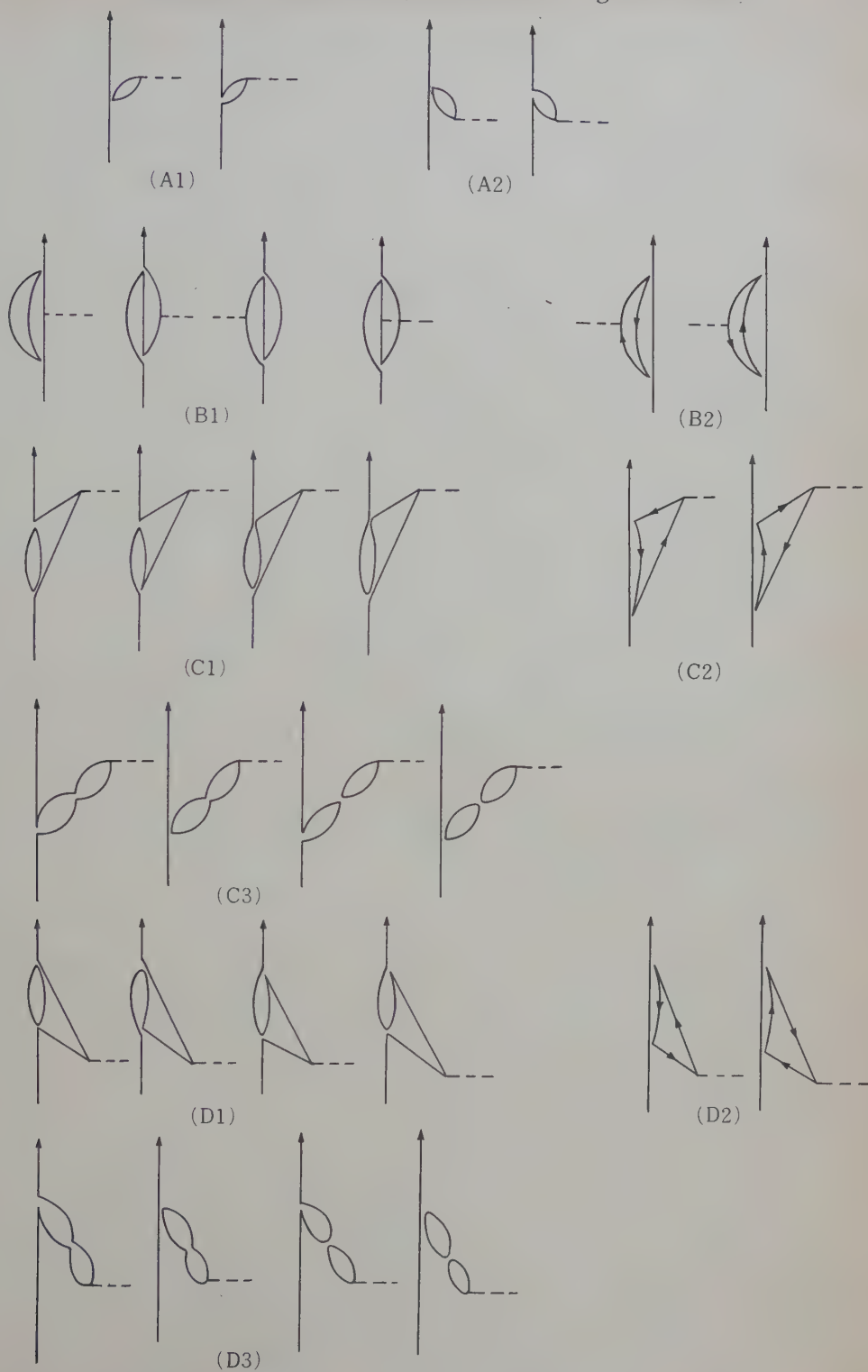


Fig. 1.

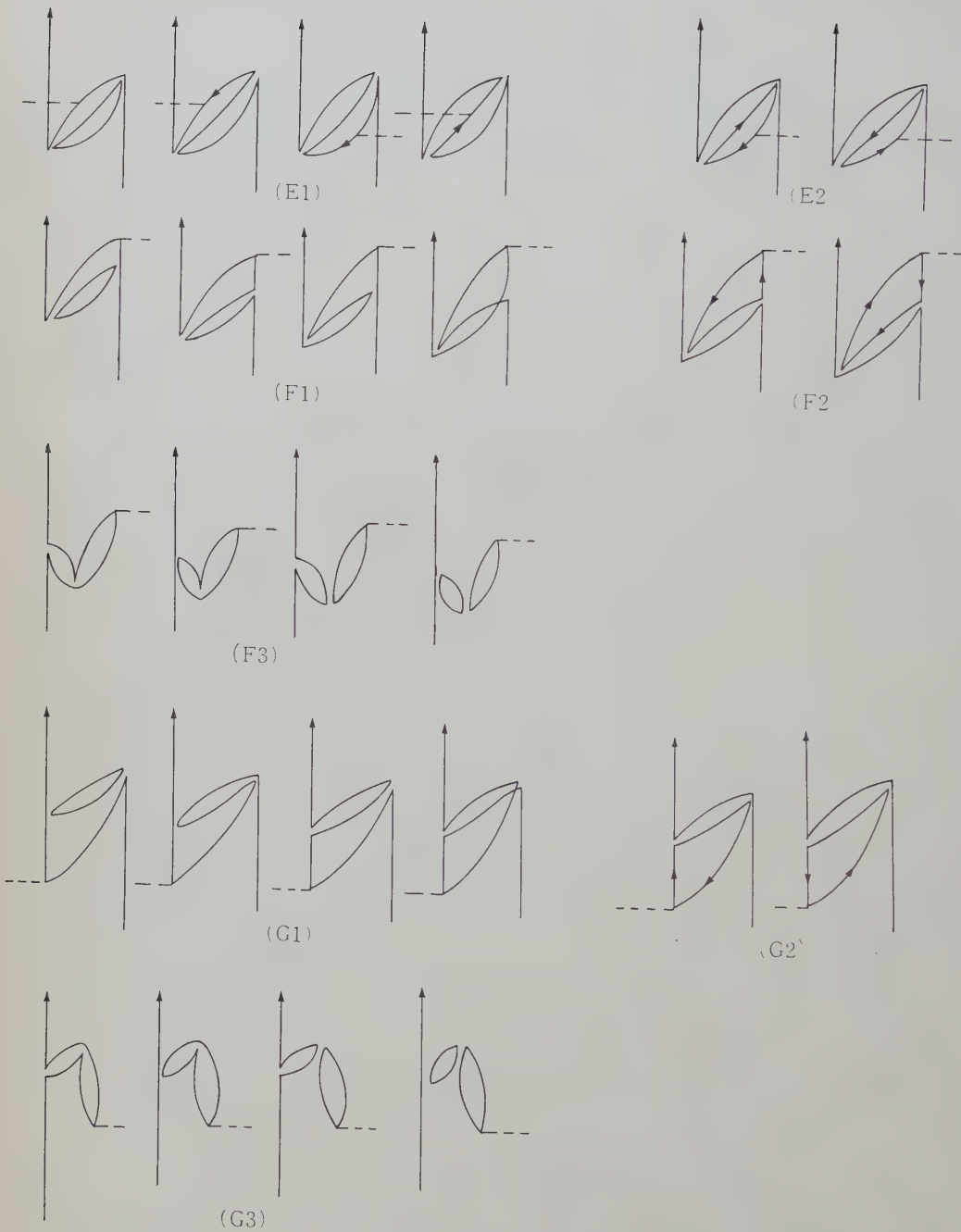


Fig. 1. (cont' α).

where $u(\mathbf{p})$ and $v(\mathbf{p})$ are spinors for the nucleon and the antinucleon respectively and

$$u(\mathbf{p})u^*(\mathbf{p}) = (E_p + (\boldsymbol{\alpha}\mathbf{p}) + M\beta)/2E_p, \quad (23)$$

$$-Cv^*(\mathbf{p})v(\mathbf{p})C = (E_p + (\boldsymbol{\alpha}\mathbf{p}) - M\beta)/2E_p. \quad (24)$$

The first term in (22) vanishes because the spur in it is zero. Using the relations

$$(\boldsymbol{\alpha}\mathbf{p} + M\beta)u(\mathbf{p}) = E_p u(\mathbf{p}), \quad (25)$$

$$u^*(\mathbf{p})(\boldsymbol{\alpha}\mathbf{p} + M\beta) = E_p u^*(\mathbf{p}), \quad (26)$$

we get

$$\begin{aligned} (A1)_p \sim & \frac{-e}{(2\pi)^{9/2}} \cdot \frac{2g_{1p}}{M^2} \int d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{p}_3 d\mathbf{p}_4 d\mathbf{k} \delta(\mathbf{p}_1 - \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4) d\mathbf{q} \delta\left(\mathbf{p}_3 + \mathbf{q} + \frac{\mathbf{k}}{2}\right) \\ & \times \delta\left(\mathbf{p}_4 - \mathbf{q} + \frac{\mathbf{k}}{2}\right) a_{p_1}^* u^*(\mathbf{p}_1) \left\{ (E_{p_1} + E_{p_4}) (\boldsymbol{\alpha}\mathbf{A}) (\boldsymbol{\alpha} \cdot \mathbf{p}_3 - \mathbf{p}_2) \right. \\ & \left. + (E_{p_3} - E_{p_2}) (\boldsymbol{\alpha} \cdot \mathbf{p}_1 + \mathbf{p}_4) (\boldsymbol{\alpha}\mathbf{A}) \right\} (4E_{p_3} E_{p_4})^{-1} \frac{3 - \tau_3}{2} u(\mathbf{p}_2) a_{p_2} \\ & \times (E_{p_2} - E_{p_1} - E_{p_3} - E_{p_4})^{-1}. \end{aligned} \quad (27)$$

Since the terms proportional to $(\boldsymbol{\alpha} \cdot \mathbf{A})$ and $(\boldsymbol{\alpha} \cdot \mathbf{p}_1 + \mathbf{p}_4) (\boldsymbol{\alpha} \cdot \mathbf{A}) (\boldsymbol{\alpha} \cdot \mathbf{p}_3 - \mathbf{p}_2)$ contribute the charge renormalization and the higher moments and do not contribute to the magnetic moment, we omit them in (27). We use the following relations,⁴⁾

$$(\boldsymbol{\alpha} \cdot \mathbf{a}) (\boldsymbol{\alpha} \cdot \mathbf{b}) = (\mathbf{a} \cdot \mathbf{b}) + i(\boldsymbol{\sigma} \cdot [\mathbf{a} \times \mathbf{b}]), \quad (28)$$

$$\int d\Omega_k (\mathbf{k} \cdot \mathbf{c}) (\mathbf{k} \cdot \mathbf{d}) = \frac{4\pi}{3} k^2 (\mathbf{c} \cdot \mathbf{d}), \quad (29)$$

$$(\mathbf{A} \cdot \mathbf{p}_1 + \mathbf{p}_2) \cong i\boldsymbol{\sigma} \cdot [\mathbf{A} \times (\mathbf{p}_1 - \mathbf{p}_2)], \quad (30)$$

$$(\mathbf{A} \cdot \mathbf{p}_1 - \mathbf{p}_2) = 0 \quad (31)$$

and make the approximation $E_{p_1} \sim E_{p_2} \sim M$. Further we expand $E_{q+k/2}$ and $E_{q-k/2}$ in (27) into power series of \mathbf{k} and pick up the magnetic moment terms; then we get

$$\begin{aligned} (A1)_p \sim & \frac{-e}{(2\pi)^{3/2}} \int d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{k} \delta(\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{k}) \\ & \times a_{p_1}^* u(\mathbf{p}_1) \left\{ \frac{g_{1p}}{(2\pi)^2} \frac{1}{2M} \int \frac{q^2 d\mathbf{q}}{E_q^3} i\boldsymbol{\sigma} \cdot [\mathbf{A} \times \mathbf{k}] \frac{3 - \tau_3}{2} \right\} u(\mathbf{p}_2) a_{p_2}. \end{aligned} \quad (32)$$

Carrying out the similar procedure for the other graphs in Fig. 1 and cutting off integrals on \mathbf{q} at the nucleon mass, we obtain the anomalous magnetic moment of the nucleon μ_N as follows,

$$\mu_N = \mu_s + \mu_v \tau_3, \quad (33)$$

where

Case I

$$\begin{aligned}\mu_s = & \frac{1}{(2\pi)^2} \frac{1}{15} \left\{ -(3g_{1p} + g_{0p}) - (3g_{1s} + g_{0s}) + 2(3g_{1t} + 5g_{0t}) \right\} \\ & + \frac{1}{(2\pi)^4} \frac{1}{45} (0.75g_{1p}^2 + 40.5g_{1p}g_{0p} + 13.75g_{0p}^2 + 12g_{1s}^2 - 24g_{1s}g_{0s} - 4g_{0s}^2 \\ & + 15.75g_{1t}^2 - 385.5g_{1t}g_{0t} - 443.25g_{0t}^2 + 24g_{1p}g_{1s} - 24g_{1p}g_{0s} \\ & - 24g_{0p}g_{1s} - 8g_{0p}g_{0s} - 312g_{1s}g_{1t} + 48g_{0s}g_{1t} + 204g_{1s}g_{0t} \\ & - 20g_{0s}g_{0t} + 322.5g_{1p}g_{1t} - 58.5g_{0p}g_{1t} + 97.5g_{1p}g_{0t} + 120.5g_{0p}g_{0t})\end{aligned}\quad (34a)$$

$$\begin{aligned}\mu_v = & \frac{1}{(2\pi)^2} \frac{1}{15} \left\{ (g_{1p} - g_{0p}) + (g_{1s} - g_{0s}) + 2(3g_{1t} + g_{0t}) \right\} \\ & + \frac{1}{(2\pi)^4} \frac{1}{45} (112.75g_{1p}^2 + 8.5g_{1p}g_{0p} + 9.75g_{0p}^2 + 60g_{1s}^2 + 8g_{1s}g_{0s} - 4g_{0s}^2 \\ & + 1087.75g_{1t}^2 - 93.5g_{1t}g_{0t} - 19.25g_{0t}^2 - 64g_{1p}g_{1s} - 24g_{0p}g_{1s} \\ & + 8g_{1p}g_{0s} - 8g_{0p}g_{0s} - 164g_{1s}g_{1t} - 52g_{0s}g_{1t} - 72g_{1s}g_{0t} \\ & + 16g_{0s}g_{0t} - 569.5g_{1p}g_{1t} + 117.5g_{0p}g_{1t} + 89.5g_{1p}g_{0t} - 19.5g_{1p}g_{0t})\end{aligned}\quad (35a)$$

Case II

$$\begin{aligned}\mu_s = & \frac{1}{(2\pi)^4} \frac{1}{45} (-23.25g_{1v}^2 + 4.5g_{1v}g_{0v} - 6.25g_{0v}^2 - 634.5g_{1v}g_{1\alpha} \\ & - 85.5g_{0v}g_{1\alpha} - 85.5g_{1v}g_{0\alpha} - 268.5g_{0v}g_{0\alpha} \\ & - 20.25g_{1\alpha}^2 + 82.5g_{1\alpha}g_{0\alpha} + 20.75g_{0\alpha}^2),\end{aligned}\quad (34b)$$

$$\begin{aligned}\mu_v = & \frac{1}{(2\pi)^4} \frac{1}{45} (720.75g_{1v}^2 - 15.5g_{1v}g_{0v} - 6.25g_{0v}^2 - 42.5g_{1v}g_{1\alpha} \\ & - 225.5g_{0v}g_{1\alpha} - 297.5g_{1v}g_{0\alpha} - 28.5g_{0v}g_{0\alpha} \\ & + 723.75g_{1\alpha}^2 + 86.5g_{1\alpha}g_{0\alpha} + 20.75g_{0\alpha}^2).\end{aligned}\quad (35b)$$

§ 5. Results

We search for the coupling constants g_{1k} and g_{0k} to satisfy the assumptions (8) and (9) (and further (13b) and (14b) in Case II), and so that H_s has the desirable form and μ_N gives the experimental value.

Case I

We assume the following relations:

$$g_{1p} - g_{0p} = g_{1s} - g_{0s}, \quad (8a)$$

$$4g_{1p} + (g_{1s} - g_{0s}) + 3(g_{1t} - g_{0t}) = 30, \quad (9a)$$

$$1 + \frac{1}{30} \left\{ (5g_{0s} - g_{1s}) - 3(5g_{1t} + 3g_{0t}) \right\} \sim 0, \quad (36a)$$

$$\mu_s \sim 0, \quad (37a)$$

where (36a) and (37a) are derived from (19a) and (34a) respectively. H_s is obtained by using (36a)

$$H_s \cong -\frac{f^2}{(2\pi)^3} \sum_{ij} \int d\mathbf{p} d\mathbf{p}' d\mathbf{k} d\mathbf{k}' \frac{1}{\sqrt{2\omega_k}} \frac{1}{\sqrt{2\omega_{k'}}} \delta(\mathbf{p}' + \mathbf{k}' - \mathbf{p} - \mathbf{k}) \frac{1}{2M} a_{p'}^* \times \frac{1}{2} \left(\frac{\omega_k}{M} \right)^3 (\boldsymbol{\tau} \boldsymbol{\omega}) a_p A_k^{i*} A_k^j. \quad (38a)$$

As the coupling constants are not decided from (8a), (9a), (36a) and (37a), we determine them only in the following three cases: (1) $g_{1p} = g_{0p}$ and $g_{1t} = g_{0t}$, (2) $g_{1p} = g_{0p} = -g_{1s}$ and (3) $g_{1p} = -g_{0p} = g_{1s}$. The numerical values are shown in Table I.

Table I.

	g_{1p}	g_{0p}	g_{1s}	g_{0s}	g_{1t}	g_{0t}	μ_v
(1)	7.50	7.50	-12.66	-12.66	-0.86	-0.86	0.37
(2)	10.18	10.18	-10.18	-10.18	-1.72	1.72	0.56
	1.30	1.30	-1.30	-1.30	4.14	-4.14	0.26
(3)	5.40	-5.40	5.40	-5.40	-0.40	0.40	0.11
	-3.37	3.37	-3.37	3.37	8.37	-8.37	1.47

Case II

Assuming the following relations:

$$g_{1v} - g_{0v} = 3g_{1a} - 2g_{0a}, \quad (8b)$$

$$g_{1v} - g_{0v} + g_{1a} - 3g_{0a} = 30, \quad (9b)$$

$$1 - \frac{1}{15} (g_{1v} - g_{0v} - g_{1a} + g_{0a}) = 0, \quad (14b)$$

$$-2 + \frac{2}{15} (4g_{1v} - 2g_{1a} - g_{0a}) \sim 0, \quad (36b)$$

where (36b) is derived from (19b), we get

$$g_{1v} = 7.5, \quad g_{0v} = -15, \quad g_{1a} = 7.5, \quad g_{0a} = 0.$$

By using these values, (13b) is satisfied, $G^2/4 \sim 0.09$ from (17b), the second term in (19b) is equal to zero and H_s is obtained as follows,

$$H_s \cong -\frac{f^2}{(2\pi)^3} \sum_{ij} \int d\mathbf{p} d\mathbf{p}' d\mathbf{k} d\mathbf{k}' \frac{1}{\sqrt{2\omega_k}} \frac{1}{\sqrt{2\omega_{k'}}} \delta(\mathbf{p}' + \mathbf{k}' - \mathbf{p} - \mathbf{k}) \frac{1}{2M} a_{p'}^* \times \left\{ 2 \left(\frac{\omega_k}{M} \right) + \frac{3}{2} \left(\frac{\omega_k}{M} \right)^3 \right\} (\boldsymbol{\tau} \boldsymbol{\omega}) a_p A_k^{i*} A_k^j. \quad (38b)$$

Moreover, we get from (33), (34b) and (35b)

$$\mu_N = -0.43 + 1.49\tau_3,$$

that is, $\mu_p = 1.06$, $\mu_n = -1.92$.

Thus, it seems to the author that the results obtained above are qualitatively in agreement with experiment in Case I rather than in Case II. However, we do not discuss about them quantitatively as they depend on the cutoff. Henceforward we must also consider the effects of Λ -particle in the phenomena related to the s -wave pion-nucleon interaction.⁶⁾

The author is grateful to the members of Yukawa Laboratory for their interest in this work.

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Origin of the Magnetic Anisotropy Energy of Cobalt Ferrite

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The magnetic anisotropy of cobalt ferrite is considered to arise from the cobaltous ions in the crystalline field of a low symmetry. The crystalline field due to the averaged-out charge distribution of Co^{2+} and Fe^{3+} ions in the octahedral sites gives the lowest-lying twofold degenerate orbital level of the Co^{2+} ion and to this level are associated four spin levels, corresponding to $S=3/2$, as each spin is subject to exchange field. Each of these orbitally doubly degenerate levels is further split into two by the low symmetry field arising from the difference of charges of Co^{2+} and Fe^{3+} ions and spin-orbit coupling. The magnitude of this energy splitting depends on the direction of the exchange field, thus on the direction of the magnetization, and from this origin the anisotropy energy arises. The relation between the magnitude of the anisotropy energy and the strength of the low symmetry field is discussed. Two kinds of configuration, A and B, of the Co^{2+} and Fe^{3+} ions neighbouring each Co^{2+} ion are considered as having low energy values. By properly choosing the relative numbers of Co^{2+} ions in these A and B configurations and the strengths of the respective low symmetry fields, the calculated temperature dependence of the cubic anisotropy constant K_1 can be fitted with that measured by Shenker. The values of the parameters thus determined seem to be reasonable, as they can be compared with those calculated on the assumption of the point charge model. The magnetic moment consisting of the spin and orbital moments is calculated to be $3.4\sim 3.5 \mu_B$ per Co^{2+} ion. Finally, the dependence of the anisotropy constant of the mixed Fe-Co ferrite on the concentration of the cobaltous ions is discussed.

§ 1. Introduction

The easy axis of magnetization of cobalt ferrite lies in the [100] direction and its anisotropy energy is very large compared with those of other ferromagnetic ferrites, such as Mn, Fe, and Ni ferrites, the easy direction of which lies in [111].

Theoretical investigations concerning the magnetic anisotropy of ferrites have been done by Yosida-Tachiki,¹⁾ Wolf²⁾ and Slonczewski.³⁾ Yosida and Tachiki have shown that the anisotropy energy of Mn and Ni ferrites comes from the fine structure coupling of Fe^{3+} ions in the A and B sites, and the anisotropy energy of magnetite comes from that of Fe^{3+} ions in the A and B sites and Fe^{2+} ions in the B sites. For cobalt ferrite, the contribution of Fe^{3+} ions to the anisotropy energy ($10^{-2}\sim 10^{-1} \text{ cm}^{-1}$) is negligible compared with the observed value, which is $1.6\sim 7.2 \text{ cm}^{-1}/\text{molecule}^{4,5)}$ at the absolute zero of temperature. The anisotropy energy due to the second order effect of the magnetic dipolar interaction makes the easy direction of magnetization [100], but its magnitude is also too small ($10^{-3}\sim 10^{-2} \text{ cm}^{-1}/\text{molecule}$) as shown by Yosida and Tachiki. Consequently, they con-

cluded that the anisotropy energy of cobalt ferrite might come from Co^{2+} ions.

On the other hand, it has been reported by Bickford et al.^{6,7)} and van der Burgt⁸⁾ that substitution of a small amount of cobalt for the divalent metallic ions in ferrites causes the change of easy direction of magnetization from $[111]$ to $[100]$ direction. Slonczewski³⁾ theoretically studied the state of a Co^{2+} ion in magnetite contaminated with a small amount of cobalt ferrite and succeeded in explaining the large increase of cubic anisotropy constant with the cobalt concentration in magnetite. Also, he could explain the magnitude of the non-cubic anisotropy energy induced by cooling in magnetic field. In his paper, he assumed that Co^{2+} ions are in the crystalline field of threefold symmetry, because the electron transfer from one iron ion to another in the B sites is so fast that Co^{2+} ions are effectively in the crystalline field produced by the averaged charge distribution. However, Co^{2+} ions in pure cobalt ferrite are in the crystalline field of a lower symmetry produced by Fe^{3+} and Co^{2+} ions distributed randomly in the B sites. As the result, a Co^{2+} ion in cobalt ferrite has to have a smaller anisotropy energy than a Co^{2+} ion in cobalt substituted magnetite.

In Section 2 of this paper, we discuss the energy levels of a Co^{2+} ion in the crystalline field of lower symmetry in cobalt ferrite. In Sections 3 and 4, we calculate the anisotropy energy and the magnetic moment of a Co^{2+} ion in this lower symmetry field and compare them with the experimental results. In Section 5, we discuss how the magnitude of the anisotropy energy of mixed ferrite $\text{Co}_x\text{Fe}_{3-x}\text{O}_4$ depends on the concentration of cobalt ions.

§ 2. Energy levels

Cobalt ferrite has the inverse spinel structure, namely Co^{2+} ions are in the B sites or octahedral sites. The B sites form a face centred cubic lattice with half of the positions empty and construct tetrahedrons linked to each other at the corner Co^{2+} ions as shown in Fig. 1. If the charges of Co^{2+} and Fe^{3+} ions in the B sites are averaged, Co^{2+} ions will find themselves at the centre of a charge distribution of threefold symmetry, whose axis is distributed along $[111]$, $[1\bar{1}\bar{1}]$, $[\bar{1}1\bar{1}]$, and $[\bar{1}\bar{1}1]$. In the cubic crystalline field, the ground level of a Co^{2+} ion (4F) splits into three orbital levels, namely Γ_4 , Γ_5 and Γ_2 , and the lowest one is the threefold degenerate level (Γ_4). This threefold degenerate level splits into a twofold degenerate level and a singlet level by the trigonal field due to the averaged charge distribution of the metallic ions in the B sites, Fe^{3+} ions in the A sites and the oxygen ions slightly distorted from the octahedral arrangement. It is assumed that the doublet level lies lower.³⁾ Furthermore, the orbital degeneracy will be removed by the crystalline field of lower symmetry arising from the difference of charges of Co^{2+} and Fe^{3+} ions in the B sites. The magnitude of this field of lower symmetry will be assumed to be smaller than that of the field of trigonal symmetry. We shall later distinguish between two configurations A and B shown by Fig. 1.

The Hamiltonian of a Co^{2+} ion in cobalt ferrite can be written as follows:

$$H = W_F + V_C + V_T + W_{ec} + W_{L,S} + V', \quad (2.1)$$

where W_F is the Hamiltonian of the free ion (without spin-orbit coupling), V_C the cubic crystalline field, V_T the trigonal field, W_{ec} the exchange energy, $W_{L,S}$ the spin-orbit coupling, and V' the crystalline field of lower symmetry. With this Hamiltonian, we shall calculate the low-lying levels. W_F and V_C ($\gtrsim 10^4 \text{ cm}^{-1}$) are assumed to be so large compared with the other parts that we can confine our consideration to the lowest three levels which are degenerate in the cubic field. As Abragam and Pryce⁹⁾ have shown, these three states correspond to the eigenstates of the z -component of the fictitious angular momentum \mathbf{l} of magnitude one, where z is the direction of the axis of trigonal symmetry. Namely, if we take the eigenfunctions of L_z , denoted by φ_m , which satisfy the following relation:

$$L_z \varphi_m = m \varphi_m \quad (m=1, 0, -1), \quad (2.2)$$

the doublet state corresponds to φ_1 and φ_{-1} and the singlet state to φ_0 . Then, the trigonal part of the Hamiltonian can be expressed by

$$V_T = \Delta(1 - L_z^2). \quad (2.3)$$

Here Δ is the energy separation between the doublet φ_1 and φ_{-1} and the singlet φ_0 , and it is positive. This positive sign can be confirmed by the numerical calculation of the trigonal field on the point charge model, and it is also consistent with the experiment as Slonczewski³⁾ has shown.

Using the molecular field approximation, the exchange energy can be expressed by

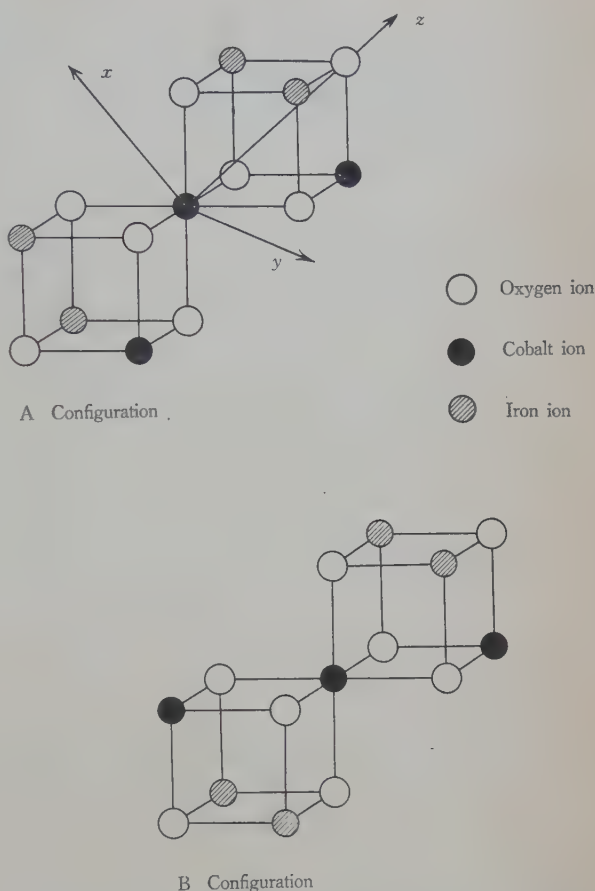


Fig. 1. Low energy configurations of cations neighbouring a Co^{2+} ion in the B sites of the spinel lattice.

$$W_{ex} = 2\mu_B l_z^2 S_z H_e + 2\mu_B (1 - l_z^2) S_z H_e', \quad (2.4)$$

where the z axis is parallel to the magnetization and H_e and H_e' are the exchange field for the doublet and singlet.

According to Abragam and Pryce,⁹⁾ the real angular momentum \mathbf{L} is equivalent to

$$L_{x,y} = -\alpha' l_{x,y}, \quad L_z = -\alpha l_z, \quad (2.5)$$

in the subspace consisting of the lowest three levels, where z indicates the axis parallel to one of the $\langle 111 \rangle$ directions and x and y the axes perpendicular to it, and α and α' are constants which are approximately equal to 3.2. Therefore, the L - S coupling can be expressed by

$$W_{L,S} = \lambda \mathbf{L} \cdot \mathbf{S} = -\alpha \lambda l_z S_z - \alpha' \lambda (l_x S_x + l_y S_y)$$

$$= \begin{vmatrix} \varphi_1 & \varphi_{-1} & \varphi_0 \\ -\alpha \lambda S_z & 0 & -\frac{\alpha' \lambda S^-}{\sqrt{2}} \\ 0 & \alpha \lambda S_z & -\frac{\alpha' \lambda S^+}{\sqrt{2}} \\ -\frac{\alpha' \lambda S^+}{\sqrt{2}} & -\frac{\alpha' \lambda S^-}{\sqrt{2}} & 0 \end{vmatrix}, \quad (2.6)$$

where S^+ and S^- represent $S_x + iS_y$ and $S_x - iS_y$.

The low symmetry crystalline potential may be expressed quite generally by

$$V' = E(l_x^2 - l_y^2) + F(l_x l_y + l_y l_x) + G(l_y l_z + l_z l_y) + H(l_z l_x + l_x l_z), \quad (2.7)$$

because the magnitude of \mathbf{l} is unity. The matrix elements of this V' can be expressed in terms of two parameters a and b as

$$V' = \begin{vmatrix} \varphi_1 & \varphi_{-1} & \varphi_0 \\ 0 & a & b \\ a^* & 0 & -b^* \\ b^* & -b & 0 \end{vmatrix}. \quad (2.8)$$

We shall calculate a and b in the Appendix.

Thus, we have formulated the effective Hamiltonian $V_T + W_{ex} + W_{L,S} + V'$ in the subspace consisting of the lowest doublet and singlet. The next problem is to obtain the eigenvalues of this Hamiltonian. Since the off-diagonal elements between φ_0 and φ_1 , namely $\alpha' \lambda$ and b , are expected to be considerably smaller than Δ ($\alpha' \lambda \sim 100 \sim 200 \text{ cm}^{-1}$, $b \sim 100 \text{ cm}^{-1}$ and $\Delta \sim 1000 \text{ cm}^{-1}$), we will first calculate the energy eigenvalues in the two-dimensional subspace of φ_1 and φ_{-1} . In this subspace, the Hamiltonian reduces to

$$H = 2\mu_B H_e S_z - \alpha \lambda S_z \sigma_z + \frac{a + a^*}{2} \sigma_x + i \frac{a - a^*}{2} \sigma_y, \quad (2.9)$$

where σ_x , σ_y and σ_z represent Pauli's spin matrices. If we take the direction of the vectors, $2\mu_B H_e \mathbf{e}_\zeta \pm \alpha \lambda \mathbf{e}_z$, as ζ' and ζ'' , where \mathbf{e}_ζ and \mathbf{e}_z are respectively the unit vectors along the ζ and z axes, and denote the components of the spin vector along ζ' and ζ'' by $S_{\zeta'}$ and $S_{\zeta''}$ as shown in Fig. 2, the matrix of Hamiltonian (2.9) with respect to φ_1 and φ_{-1} becomes

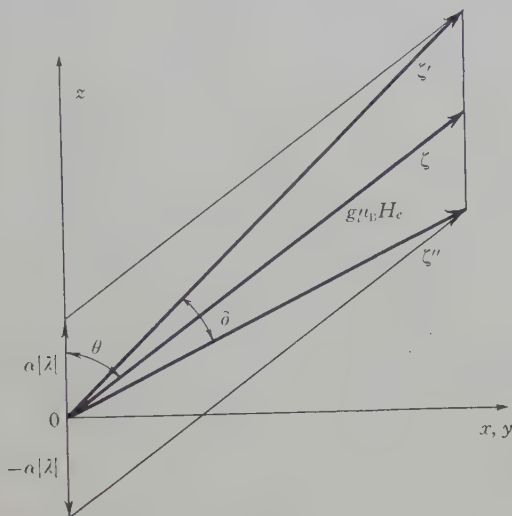


Fig. 2. Definition of the directions ζ' and ζ'' and the angles θ and δ used in (2.10) and (2.11). z and ζ indicate the direction of the trigonal axis and that of the magnetization.

$$H = \begin{bmatrix} S_{\zeta'} \sqrt{(2\mu_B H_e)^2 + (\alpha\lambda)^2} + 2(2\mu_B H_e) \alpha |\lambda| \cos \theta & aR \\ a^* \tilde{R} & S_{\zeta''} \sqrt{(2\mu_B H_e)^2 + (\alpha\lambda)^2} - 2(2\mu_B H_e) \alpha |\lambda| \cos \theta \end{bmatrix}, \quad (2.10)$$

where θ is the angle between the z and ζ axes. The matrix element $R_{MM'}$ of R is defined by the inner product of the two eigenfunctions of spin components $S_{\zeta'}$ and $S_{\zeta''}$, namely $R_{MM'} = \langle \chi_{S_{\zeta'}=M} | \chi_{S_{\zeta''}=M'} \rangle = \tilde{R}_{M'M}$. This rotation matrix R can be calculated in the case of $S=3/2$ as follows¹⁰⁾:

$$R = \begin{bmatrix} M=-\frac{3}{2} & M=-\frac{1}{2} & M=\frac{1}{2} & M=\frac{3}{2} \\ t^3 & \sqrt{3}t^2s & \sqrt{3}ts^2 & s^3 \\ -\sqrt{3}t^2s & t^3-2ts^2 & 2t^2s-s^3 & \sqrt{3}ts^2 \\ \sqrt{3}ts^2 & -2t^2s+s^3 & t^3-2ts^2 & \sqrt{3}t^2s \\ -s^3 & \sqrt{3}ts^2 & -\sqrt{3}t^2s & t^3 \end{bmatrix}, \quad (2.11)$$

where $s = \sin \delta/2$, $t = \cos \delta/2$. δ is the angle between the ζ' and ζ'' axes and can be expressed as $\delta = (\alpha |\lambda| / \mu_B H_e) \sin \theta$. Noting that δ is small and that a is smaller

than the exchange energy $2\mu_B H_e$, we take out the elements of (2.10) which refer to the same eigenvalues of S_z' and S_z'' and diagonalize them with respect to σ . Then, we obtain

$$E_M = \frac{1}{2} \left[M \left\{ \sqrt{(2\mu_B H_e)^2 + (\alpha\lambda)^2} + 2(2\mu_B H_e) \alpha |\lambda| \cos \theta \right. \right. \\ \left. \left. + \sqrt{(2\mu_B H_e)^2 + (\alpha\lambda)^2} - 2(2\mu_B H_e) \alpha |\lambda| \cos \theta \right\} \right. \\ \left. \pm \{ M^2 \left(\sqrt{(2\mu_B H_e)^2 + (\alpha\lambda)^2} + 2(2\mu_B H_e) \alpha |\lambda| \cos \theta \right. \right. \\ \left. \left. - \sqrt{(2\mu_B H_e)^2 + (\alpha\lambda)^2} - 2(2\mu_B H_e) \alpha |\lambda| \cos \theta \right)^2 \right. \\ \left. + 4a^* a R_{MM} R_{MM} \}^{1/2} \right] \cdot \left(M = \pm \frac{3}{2}, \pm \frac{1}{2} \right). \quad (2.12)$$

This equation may be expanded in powers of $\lambda/\mu_B H_e$ as

$$E_M = M(2\mu_B H_e) \mp [(M\alpha\lambda \cos \theta)^2 + a^* a]^{1/2} \\ + \frac{M(\alpha\lambda \sin \theta)^2}{2(2\mu_B H_e)} \pm \frac{(\alpha\lambda \sin \theta)^2 [(M\alpha\lambda \cos \theta)^2 + \kappa a^* a]}{2(2\mu_B H_e)^2 [(M\alpha\lambda \cos \theta)^2 + a^* a]^{1/2}} \\ + 2\mu_B H_e O \left[\left(\frac{\alpha\lambda}{2\mu_B H_e} \right)^4 \right], \\ \kappa = 3 \quad \text{for } M = \pm \frac{3}{2}, \\ \kappa = 7 \quad \text{for } M = \pm \frac{1}{2}. \quad (2.13)$$

The contribution from the off-diagonal elements of $R(R_{MM'}, M \neq M')$ to the energy eigenvalues is $\sim \{a^* a (\alpha\lambda)^2 / (\mu_B H_e)^3\} \cdot \sin^2 \theta$ and this can be neglected. If the two levels represented by (2.13), for example, the upper level of $M = -3/2$ and the lower level of $M = -1/2$, accidentally degenerate, these levels repel each other on account of the off-diagonal elements of R and separate by an amount of $\sim (|\alpha\lambda a| / \mu_B H_e) \cdot \sin \theta$. However, this will have little effect on the anisotropy energy, since this does not concern with the ground spin-orbit level. The contribution from the matrix elements connecting φ_1 and φ_{-1} to φ_0 is also not so important for the anisotropy energy, but it turns out to be responsible for the magnetic moment. We shall discuss the latter effect in Section 4.

§ 3. Anisotropy energy

The anisotropy energy measured isothermally is the free energy. Therefore, we shall now calculate the free energy of Co^{2+} ions. Assuming that cobaltous ions are equally populated in the four kinds of sites corresponding to four trigonal axes $\langle 111 \rangle$, we can express the free energy as

$$F = -(N/4)kT \sum_i \ln Z_i, \quad (3.1)$$

$$Z_i = \sum_j \exp[-E_j(\theta_i)/kT],$$

where $E_j(\theta_i)$ is the j -th energy level of a Co^{2+} ion given by (2.13) in the i -th position of the B sites. It can be shown that the contributions to the anisotropy energy from the third and higher order terms of (2.13) are negligibly small, and that those from the second order term give rise to an isotropic part and a small anisotropic part which is also negligible. Accordingly, taking only the first and second terms of (2.13), the anisotropic part of (3.1) becomes

$$F_{an} = -\frac{N}{4}kT \sum_i \ln \sum_{M=1/2, 3/2} \cosh(M2\mu_B H_e/kT) \\ \times \cosh[(M\alpha\lambda \cos\theta_i)^2 + \alpha^*a]^{1/2}/kT. \quad (3.2)$$

In analysing the experimental results, the anisotropy energy has usually been assumed to have a form of

$$K_1(\alpha_1^2\alpha_2^2 + \alpha_2^2\alpha_3^2 + \alpha_3^2\alpha_1^2) + K_2\alpha_1^2\alpha_2^2\alpha_3^2, \quad (3.3)$$

where α_i are the direction cosines of the magnetization with respect to the cubic axes and K_1 and K_2 are the anisotropy constants. However, F_{an} given by (3.2) cannot be accurately expressed in the form of (3.3), especially for small values of $|a/\alpha\lambda|$ and for low temperatures. In the actual case, $|a/\alpha\lambda|$ turns out to be larger than 0.3, and in such a case the anisotropy energy of (3.2) can be approximated by the form of (3.3) fairly well even at the absolute zero of temperature. Therefore, we determine K_1 and K_2 by putting F_{an} of (3.2) equal to (3.3) in the three directions, [100], [111], and [110], as done by Slonczewski.³⁾ Fig. 3 shows how K_1 and K_2 determined in this way depend on the values of $|a/(3/2)\alpha\lambda|$. For the value of $\alpha|\lambda|$, we took

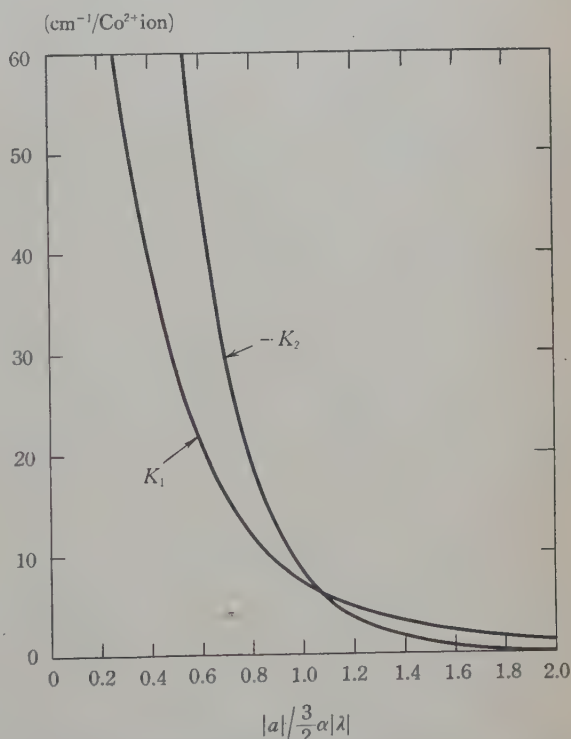


Fig. 3. Dependence of the calculated cubic anisotropy constant at the absolute zero of temperature on the strength of the crystalline field of low symmetry.

the value determined by Slonczewski in the case of cobalt substituted magnetite, namely $\alpha|\lambda|=132\text{ cm}^{-1}$. As seen from Fig. 3, K_1 is positive, K_2 is negative and the absolute values of K_1 and K_2 decrease rapidly as $|a/(3/2)\alpha\lambda|$ increases. $K_1 + (K_2/9)$, whose sign determines the easy axis of magnetization, is positive in the whole region of $|a/(3/2)\alpha\lambda|$ and therefore the easy direction of magnetization is always in $\langle 100 \rangle$. The temperature dependence of K_1 is shown in Fig. 4 in cases of $|a/(3/2)\alpha\lambda|=0.8, 1.0, 1.2$ and 1.6 .

Now, we compare the calculated anisotropy energy with the experimental one obtained by Shenker.⁵⁾ He has found that K_1 of a cobalt ferrite ($\text{Co}_{1.01}\text{Fe}_{2.00}\text{O}_{3.82}$) can be closely approximated by the following empirical formula in the temperature range from 20°K to 325°K:

$$K_1 = 19.6 \times 10^6 \exp(-1.90 \times 10^{-5} T^2) \text{ erg/cc.} \quad (3.4)$$

In his torque measurements, he applied the magnetic field near the $[100]$ direction in the (011) plane. In this case, the magnetization deviates slightly from the $[100]$ direction in the (011) plane. Denoting this angle by ψ , we can write (3.3) as

$$W_A = \text{const.} + K_1 \psi^2, \quad (3.5)$$

neglecting the higher order terms with respect to ψ . He has determined K_1 by measuring the dependence of the torque on the direction and strength of the applied magnetic field.

In order to compare more accurately the anisotropy energy obtained theoretically with the experimental results obtained by Shenker, we directly expand (3.2) in powers of ψ . If we denote the angles between the magnetization and $[111]$, $[\bar{1}\bar{1}\bar{1}]$, $[\bar{1}1\bar{1}]$, and $[1\bar{1}1]$ by $\theta_1, \theta_2, \theta_3$, and θ_4 respectively, the relation between θ_i in (3.2) and ψ is given by

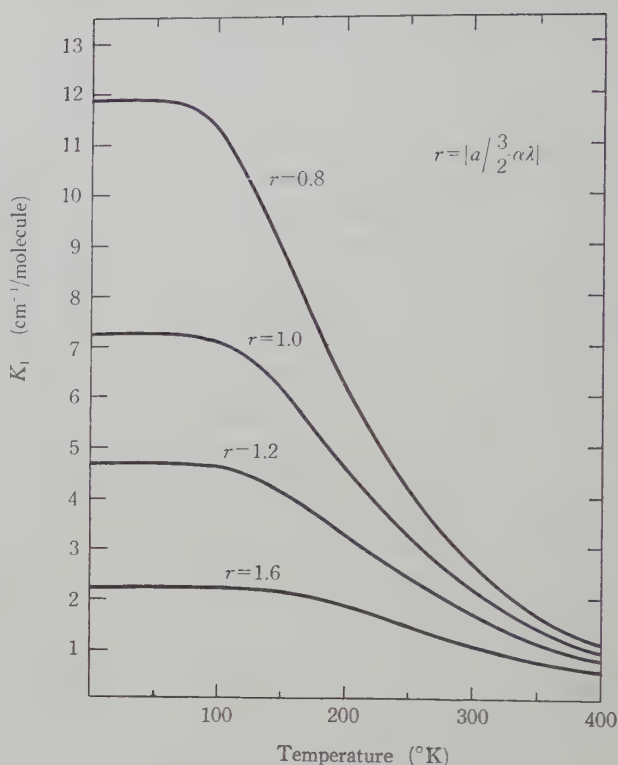


Fig. 4. Temperature dependence of the cubic anisotropy constant K_1 in cases of $|a/(3/2)\alpha\lambda|=0.8, 1.0, 1.2$ and 1.6 .

$$\begin{aligned}
\cos \theta_1 &= \cos \theta_2 = (1/3) \cos \phi, \\
\cos \theta_3 &= -(1/3) (\cos \phi - \sqrt{2} \sin \phi), \\
\cos \theta_4 &= -(1/3) (\cos \phi + \sqrt{2} \sin \phi).
\end{aligned} \tag{3.6}$$

Substituting (3.6) into (3.2) and expanding (3.2) in powers of ϕ , we obtain K_1 , which is the coefficient of ϕ^2 , as follows:

$$K_1 = \frac{9}{32} \cdot \frac{(\alpha\lambda)^4}{g^3} \cdot [f_0 + f_1], \tag{3.7}$$

$$f_0 = \tanh g\beta - g\beta \operatorname{sech}^2 g\beta, \tag{3.8}$$

$$\begin{aligned}
f_1 = & -\exp(-2\mu_B H_e \beta) \cdot \frac{\cos g'\beta}{\cos g\beta} \cdot \left[\tanh g\beta - \frac{1}{81} \cdot \frac{g^3}{g'^3} \tanh g'\beta \right. \\
& \left. - g\beta \left\{ \left(1 - \frac{1}{81} \cdot \frac{g^2}{g'^2} \right) - 2 \tanh g\beta \left(\tanh g\beta - \frac{1}{9} \cdot \frac{g}{g'} \tanh g'\beta \right) \right\} \right], \tag{3.9}
\end{aligned}$$

where

$$\begin{aligned}
\beta &= \frac{1}{kT}, \quad g = \sqrt{\frac{3}{4} (\alpha\lambda)^2 + |a|^2}, \\
\text{and} \quad g' &= \sqrt{\frac{1}{12} (\alpha\lambda)^2 + |a|^2}.
\end{aligned} \tag{3.10}$$

Now, if we consider the electrostatic interaction between the nearest neighbour cations in the B sites, the lowest energy configurations of Co^{2+} and Fe^{3+} in the B sites will be such that each tetrahedrons have two Co^{2+} and two Fe^{3+} ions.¹¹⁾ Paying attention to one Co^{2+} ion, we have its six nearest neighbour B sites forming two tetrahedrons jointed at this Co^{2+} ion. Under the restriction that one tetrahedron includes two Co^{2+} and two Fe^{3+} ions, we have two kinds of configuration A and B, as shown in Fig. 1, in one of which three Co^{2+} ions align in a straight line (Fig. 1. B) and in the other not in one line (Fig. 1. A). Then, K_1 of cobalt ferrite is given by the sum of the anisotropy constants of these two kinds of Co^{2+} ions,

$$K_1 = N_A K_1(a_A) + N_B K_1(a_B), \tag{3.11}$$

where a_A and a_B denote the a -values of the two kinds of Co^{2+} ions, and N_A and N_B their numbers. Since Shenker has measured K_1 almost continuously in the temperature range from 100°K to 300°K, a_A , a_B , N_A , N_B , and H_e can be determined so as to make the theoretical K_1 values fit the experimental ones in this temperature range.

Within the limit $1 > |a_A/a_B| > 1/2$, it was found difficult to make calculated values fit the experimental ones with any values of a_A , a_B , N_A and N_B . If we take $|a_A/a_B| = 1/3$, a good fit to the experimental curve could be obtained for $|a_A| = 140 \text{ cm}^{-1}$, $N_A/N_B = 0.67$, and $2\mu_B H_e = 540 \text{ cm}^{-1}$ (0°K) as shown in Fig. 5.1.

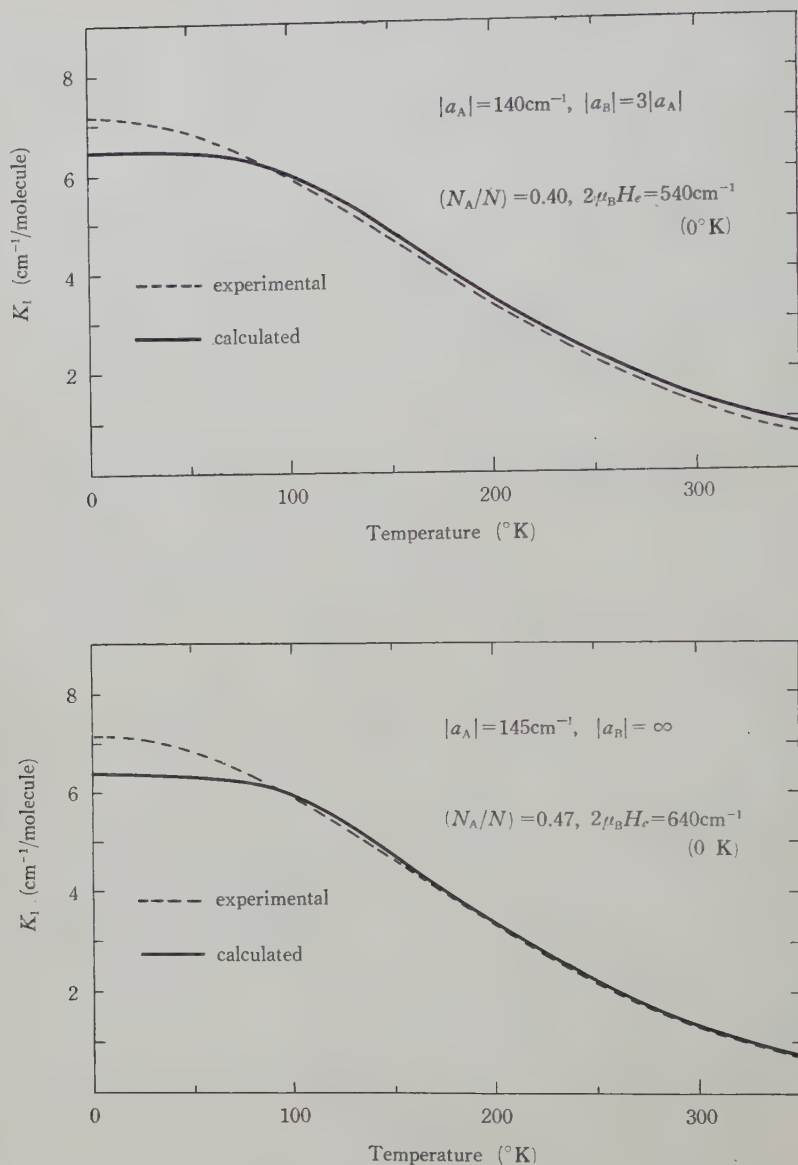


Fig. 5. Temperature dependence of the cubic anisotropy constant K_1 . Dotted curves show the experimental values obtained by Shenker. Solid curves show the calculated values with the use of the parameters obtained by fitting the calculated values to the experimental ones in the temperature range from 100°K to 300°K for $|a_A/a_B|=1/3$ (5.1) and $|a_A/a_B|=0$ (5.2).

If we take $|a_B|=\infty$, we obtain $|a_A|=145\text{ cm}^{-1}$, $N_A/N_B=0.89$ and $2\mu_B H_c=640\text{ cm}^{-1}$ (0°K), as shown in Fig. 5.2.

Next, we discuss the values of parameters determined above. As shown in

the Appendix, the calculated value of $|a_A/a_B|$ based on the point charge model is just $1/2$. If we use the Slater function, Hartree function, or Hartree-Fock function as the $3d$ -wave function used in calculating the averages of r^2 and r^4 (r is the distance between the d -electron and the nucleus), the calculated value of a_A becomes

$$\begin{aligned} |a_A| &= 123 \text{ cm}^{-1}, & \text{Slater function,} \\ |a_A| &= 75.5 \text{ cm}^{-1}, & \text{Hartree function,} \\ |a_A| &= 56.6 \text{ cm}^{-1}, & \text{Hartree-Fock function.*} \end{aligned} \quad (3.12)$$

Considering the approximation nature of the point charge model, the agreement in order of magnitude between the values of (3.12) and those obtained from the analysis of experimental data ($|a_A| = 140$ or 145 cm^{-1}) may be regarded as reasonable.

As the distance between the two Co^{2+} ions neighbouring the centre Co^{2+} ion in the B configuration is longer than that in the A configuration, the electrostatic energy of the B configuration should be lower than that of the A configuration. Denoting this energy difference by E , the ratio N_A/N_B may be roughly given by

$$N_A/N_B = 2 \exp\left\{-\frac{E}{kT_a}\right\}, \quad (3.13)$$

where T_a is the temperature of annealing of the sample. Shenker has found that the effect of the field cooling appears above 425°K . This means that Co^{2+} ions can diffuse above this temperature.^{7,8)} Therefore, since the sample was slowly cooled down in his experiment, we may take 425°K as T_a . Based on the point charge model, E is calculated to be $4374/\varepsilon$ degree, where ε is the reducing factor due to the electric polarization of the crystal and should be approximately equal to the dielectric constant. If we substitute the values of N_A/N_B determined above into (3.13), ε turns out to be 9.5 or 12.7 corresponding to $|a_A/a_B| = 1/3$ or $|a_B| = \infty$. Though the author could not find any measured values of ε for ferrites, these values may be reasonable as they are comparable with the known ε 's of other oxides.**

§ 4. Magnetic moment

Since the angular momentum of Co^{2+} ion is not completely quenched in the present case, both the spin and orbital moment contribute to the magnetic moment.

* This value is obtained with the use of the wave function interpolated between the Hartree-Fock wave function of neutral manganese atom and the corresponding hydrogen-like wave function (for infinite nuclear charge) by the method suggested by D. R. Hartree in *Rev. Mod. Phys.* **30** (1958), 63.

** FeO , $\varepsilon = 14.2$ (room temperature)

MgO , $\varepsilon = 9.65$ (25° C)

Cr_2O_3 , $\varepsilon = 12$ (room temperature)

American Institute of Physics Handbook, McGraw-Hill Co. Inc. (1957), 5-115

Using (2.13), we can express the spin magnetic moment at the absolute zero of temperature as

$$\mu_s = -2\mu_B \langle S_z \rangle = -\partial E / \partial H_s = 3\mu_B \left(1 - \frac{(\alpha\lambda \sin\theta)^2}{8(\mu_B H_s)^2} \right). \quad (4.1)$$

We shall next calculate the orbital magnetic moment. Denoting by E' the lowest eigenvalue of H' defined by the following equation:

$$H' = H + cL_z, \quad (4.2)$$

we obtain the orbital magnetic moment at the absolute zero of temperature as the following formula:

$$\mu_L = -\mu_B \langle L_z \rangle = -\mu_B \lim_{c \rightarrow 0} \partial E' / \partial c. \quad (4.3)$$

If we take one of $\langle 111 \rangle$ as z -axis and take the x - and y -axes perpendicular to it, as shown in Fig. 1, and denote the direction cosines of the magnetization with respect to these axes by l , m and n , the matrix of L_z will be given by

$$L_z = \begin{vmatrix} \varphi_1 & \varphi_{-1} & \varphi_0 \\ -\alpha n & 0 & -\frac{\alpha'}{\sqrt{2}}(l-im) \\ 0 & \alpha n & -\frac{\alpha'}{\sqrt{2}}(l+im) \\ -\frac{\alpha'}{\sqrt{2}}(l+im) & -\frac{\alpha'}{\sqrt{2}}(l-im) & 0 \end{vmatrix}. \quad (4.4)$$

In calculating the lowest eigenvalue E' of H' , we cannot neglect the effect of the off-diagonal elements between $\varphi_{\pm 1}$ and φ_0 in contrast with the case of the anisotropy energy. Therefore, we take into account the effect by the second order perturbation. Then, the effective Hamiltonian H' for the lowest doublet, specified by $\varphi_1 \chi_{M=-3/2}$ and $\varphi_{-1} \chi_{M=-3/2}$ can be expanded as

$$H' = H + cG_1 + c^2G_2 + \dots \quad (4.5)$$

The term important for the magnetic moment is G_1 . Using (2.3), (2.6), (2.8), and (4.4), we obtain G_1 as

$$G_1 = \begin{pmatrix} -\alpha n & 0 \\ 0 & \alpha n \end{pmatrix} - \frac{1}{J} \begin{pmatrix} p-q & r \\ r^* & p+q \end{pmatrix}, \quad (4.6)$$

where the second term comes from the second order perturbation via the upper φ_0 state, and p , q , and r are defined by

$$\begin{aligned} p &= -(3/2) \alpha'^2 \lambda (l^2 + m^2), \\ q &= (\alpha' / \sqrt{2}) [(l+im)b + (l-im)b^*], \\ r &= -(3/2) \alpha'^2 \lambda (l-im)^2. \end{aligned} \quad (4.7)$$

If we express the wave function of the ground state of H as

$$\phi_g = u\varphi_1\chi_{M=-3/2} + v\varphi_{-1}\chi_{M=-3/2}, \quad (4.8)$$

μ_L is given with the use of (4.3) by

$$\begin{aligned} \mu_L/\mu_B = & -\langle\phi_g|G_1|\phi_g\rangle = \alpha n(u^*u - v^*v) \\ & + (1/\mathcal{A})[p - q(u^*u - v^*v) + ru^*v + r^*uv^*]. \end{aligned} \quad (4.9)$$

If we average (4.9) over all the possible configurations of Co^{2+} and Fe^{3+} ions neighbouring the Co^{2+} ion under consideration, the number of these possible configurations being six in the case A and three in the case B, q and r in the second term vanish. Substituting the following relation:

$$u^*u - v^*v = \frac{\frac{3}{2}\alpha|\lambda|n}{\sqrt{(\frac{3}{2}\alpha\lambda n)^2 + |a|^2}}, \quad (4.10)$$

into (4.9), we obtain μ_L as

$$\mu_L/\mu_B = \frac{\frac{3}{2}\alpha^2|\lambda|n^2}{\sqrt{(\frac{3}{2}\alpha\lambda n)^2 + |a|^2}} + \frac{1}{\mathcal{A}} \cdot \frac{3}{2} \cdot \alpha'^2|\lambda|(1 - n^2). \quad (4.11)$$

The spin and orbital magnetic moments per Co^{2+} ion are obtained by averaging n^2 over four kinds of sites. If the magnetization lies along the easy direction [100], n_i^2 's for the four kinds of site are all equal to $1/3$. If $|\alpha\lambda|$ and $|\alpha'\lambda|$ are assumed to be 132 cm^{-1} and \mathcal{A} is taken as 1000 cm^{-1} , μ_S , μ_L , and the total magnetic moment $\mu = \mu_S + \mu_L$ are obtained from (4.1) and (4.11) in two cases of $|a_A/a_B| = 1/3$ and $|a_B| = \infty$ as follows:

	μ_S/μ_B	μ_L/μ_B	μ/μ_B
for $ a_A = 140 \text{ cm}^{-1}$ and $ a_B = 3 a_A $	2.94	0.56	3.50
for $ a_A = 145 \text{ cm}^{-1}$ and $ a_B = \infty$	2.96	0.45	3.41.

On the other hand, the measured values of μ/μ_B are $3.3^{(12,a)}$, $3.67^{(12,b)}$, $3.70^{(12,b)}$ and $3.94^{(12,c)}$ according to different authors.

§ 5. Dependence of the anisotropy energy on the concentration of cobaltous ions

In this section we shall consider the dependence of the anisotropy energy on the concentration of Co^{2+} ions in Fe-Co ferrite $\text{Co}_x\text{Fe}_{1-x}\text{O} \cdot \text{Fe}_2\text{O}_3$. As in the previous sections, we assume that each tetrahedron has two divalent cations and two trivalent cations, that the crystalline field of low symmetry at Co^{2+} ions is produced only by the nearest neighbour ions in the B sites, and that the electron transfer between Fe^{2+} and Fe^{3+} ions is so frequently that we can consider the averaged-out charge distribution.

Under these assumptions, the cubic anisotropy constant K (the coefficient of $\alpha_1^2\alpha_2^2 + \alpha_2^2\alpha_3^2 + \alpha_3^2\alpha_1^2$) is given by

$$K = c_0 K_0 + c_1 K_1 + c_{2A} K_{2A} + c_{2B} K_{2B} + K(\text{Fe}^{3+} \text{ in the A site}) \\ + K(\text{Fe}^{3+} \text{ in the B site}) + (1-x)K(\text{Fe}^{2+} \text{ in the B site}), \quad (5.1)$$

where c_n is the concentration of Co^{2+} ion which has n Co^{2+} ions as its nearest neighbours; n takes 0, 1, and 2. The suffixes A and B in c_{2A} and c_{2B} mean that Co^{2+} ions are in the A and B configurations. K_0 , K_1 , K_{2A} and K_{2B} mean the anisotropy constants of Co^{2+} ions corresponding to the nearest neighbour configurations. The last three terms in (5.1) are the anisotropy constants arising from Fe^{3+} and Fe^{2+} ions which can be neglected in cobalt ferrite but are the main part of the anisotropy constant in magnetite.¹⁾ Assuming the random distribution of Fe^{2+} and Co^{2+} ions under the restriction that each tetrahedron has two Fe^{3+} ions and two divalent ions, we can express c_0 , c_1 , c_{2A} and c_{2B} as the following functions of the concentration of Co^{2+} ions x :

$$c_0 = x(1-x)^2, \quad c_1 = 2x^2(1-x), \\ c_{2A} = c_A x^3, \quad c_{2B} = c_B x^3. \quad (5.2)$$

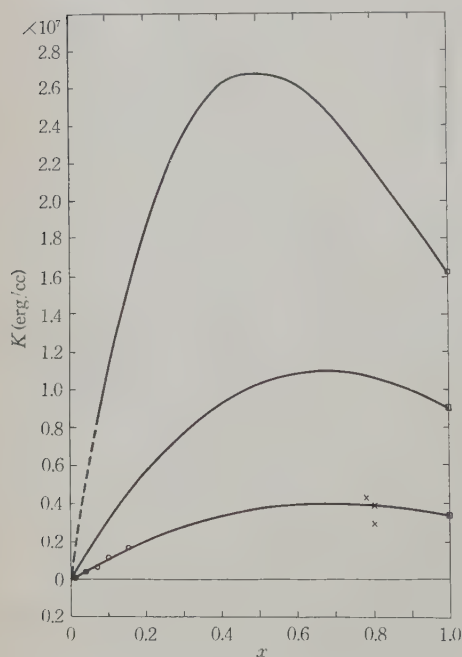


Fig. 6.1.

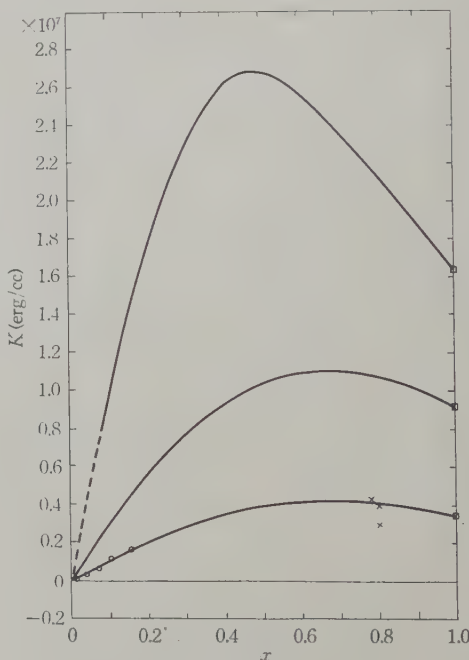


Fig. 6.2.

Dependence of the first order cubic anisotropy constant of the mixed ferrites $\text{Co}_x\text{Fe}_{1-x}\text{O} \cdot \text{Fe}_2\text{O}_3$ on the concentration of cobaltous ions. In these figures, the values of K_{2A} calculated by taking $|a_A| = 140 \text{ cm}^{-1}$ (6.1) or $|a_A| = 145 \text{ cm}^{-1}$ (6.2) are used in place of K_1 . (\circ are data from reference 7), \times are data from reference 4), and \square are data from reference 5)).

We assume that the ratio c_A/c_B is equal to $N_A/N_B=0.67$ or 0.89 (Section 3) in the case of pure cobalt ferrite, instead of the random value of 2. Substituting (5.2) into (5.1), K becomes

$$K=a_0+a_1x-a_2x^2+a_3x^3, \quad (5.3)$$

where

$$a_0=K_{\text{mag}},$$

$$a_1=K_0-K_{\text{Fe}^{2+}},$$

$$a_2=2(K_0-K_1),$$

and

$$a_3=K_0-2K_1+K_{\text{Co}},$$

where K_{mag} and K_{Co} are respectively the anisotropy constants of magnetite and cobalt ferrite. In (5.3), we use the experimental values for $K_0^{(7)}$ for isolated Co^{2+} ions, $K_{\text{mag}}^{(13)}$ and $K_{\text{Co}}^{(5)}$. As for K_1 , there is no experimental value, therefore, assuming the relation $K_1=K_{2A}$ which is exact in the point charge model as shown in the Appendix, we use the values of K_{2A} calculated in Section 3 in place of K_1 . Since $K_{\text{Fe}^{2+}}$ is very small compared with K_1 , $K_{\text{Fe}^{2+}}$ can be neglected in the expression of a_1 . K thus calculated by (5.3) is shown in Fig. 6 together with the experimental values. In Fig. 6.1 and 6.2, the values of K_{2A} calculated by taking $|a_A|=140\text{ cm}^{-1}$ or $|a_A|=145\text{ cm}^{-1}$ are used in place of K_1 as mentioned above.

§ 6. Discussion

(1) In this paper, we assumed that the low symmetry field for Co^{2+} ion mainly arises from the nearest neighbour ions in the B sites and introduced two parameters a_A and a_B in order to represent the strength of this part of the crystalline field. However, the actual values of a 's would fluctuate around these a_A and a_B values by the crystalline field from the second neighbour ions or remoter ions. The width of this fluctuation may, however, be narrow, because the parameter a is expressed by the sum of the third and fifth inverse powers of the distance between the Co^{2+} and other metallic ions, and the charge far distant from Co^{2+} ion are screened by the electric polarization of the crystal. Consequently, the present nearest neighbour approximation for treating the low symmetry field may be good.

(2) Since the chemical formula of Shenker's sample (with whose results we compared our theoretical calculation) is $\text{Co}_{1.01}\text{Fe}_{2.00}\text{O}_{3.62}$, it is expected that oxygen vacancies may exist in his sample. As Co^{2+} ions around oxygen vacancies are exposed to the crystalline field of low symmetry large enough to quench the orbital angular momentum, these Co^{2+} ions will make little contribution to the anisotropy energy. Accordingly, the effective number of Co^{2+} ions, namely N_A+N_B in (3.11), may be fewer than the total number of cobaltous ions, but we neglected this effect in this paper.

(3) For the discussion of the energy difference E between the A and B con-

figurations, only the electrostatic energy between point charges was considered. However, since the B configuration makes the larger crystalline field of low symmetry at the position of the centre Co^{2+} ion than the A configuration, the large energy splitting by the low symmetry field may further stabilize the B configuration. Therefore, it might have been better to add the difference of the energy splitting between these two configurations to the interaction energy between point charges. However, since the value of the dielectric constant is not known experimentally, our estimate of it might have included this effect.

(4) In Section 5, because we have no measured values of K_1 , we used the calculated values of K_{2A} in place of those of K_1 , as $K_1 = K_{2A}$ follows from the assumption of the point charge model and the assumption that the rate of the electron transfer between iron ions is so fast that the charges of Fe^{2+} and Fe^{3+} in the B sites are averaged out. However, the nearest neighbour Fe^{2+} would tend to be in line with $\text{Co}^{2+} - \text{Co}^{2+}$ due to electrostatic repulsion, and as the result of this tendency, a_1 in (A.6) may actually be a little larger than $|a_A|$ and so K_1 a little smaller than K_{2A} . If a_1 were equal to a_B , the maxima of the curves in Fig. 6 would practically disappear.

Acknowledgments

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Appendix

Calculation of the matrix elements of V'

We calculate the value of a and b in the matrix (2.8) for the Co^{2+} ion, the trigonal field of which has its principal axis in the $[111]$ direction.

We take x , y and z axes along the directions parallel respectively to $[-1-1\ 2]$, $[1\ -1\ 0]$, and $[1\ 1\ 1]$ referred to the cubic axes as shown in Fig. 1. The important parts of the crystalline field of low symmetry determining the orbital levels are represented by the following form:

$$\begin{aligned} V' = & r^2 (C_2^1 Y_2^1 + C_2^2 Y_2^2 + \text{comp. conj.}) \\ & + r^4 (C_4^1 Y_4^1 + C_4^2 Y_4^2 + C_4^4 Y_4^4 + \text{comp. conj.}), \end{aligned} \quad (\text{A.1})$$

$$Y_l^m = \sqrt{\frac{2l+1}{2} \cdot \frac{(l-m)!}{(l+m)!}} P_l^{|m|}(\cos \theta) e^{im\varphi},$$

where $P_l^{|m|}(\cos \theta)$ denote the associated Legendre function and (r, θ, φ) represents the polar coordinates of the d -electron, whose polar axis is the z -axis.

We approximate φ_m defined in (2.2) by the following form:

$$\begin{aligned}
 \varphi_1 &= \sqrt{5/6} \phi_{-2} - \sqrt{1/6} \phi_1, \\
 \varphi_{-1} &= -\sqrt{5/6} \phi_2 - \sqrt{1/6} \phi_{-1}, \\
 \varphi_0 &= (2/3) \phi_0 + (1/3) \sqrt{5/2} (\phi_3 - \phi_{-3}),
 \end{aligned}
 \tag{A.2}$$

where ϕ_m is the wave function of the F state of Co^{2+} ion and m represents the magnetic quantum number. These φ_m 's are the wave functions corresponding to the trigonal symmetry in the ground orbital triplet Γ_4 in the cubic field. Although the upper levels Γ_5 and Γ_2 mix with the ground orbital levels by the trigonal field, the change of the coefficients in (A.2) due to the mixing may be neglected in the case of large cubic field compared with the trigonal field. Also, 4P state mixes with the ground states by the cubic and trigonal field, but we neglect these effects. This approximation means to take both α and α' as $3/2$.

With the use of (A.1) and (A.2), a and b in (2.8) become

$$\begin{aligned}
 a &= \langle \varphi_1 | V | \varphi_{-1} \rangle = \langle r^2 \rangle [(1/35) \sqrt{5/3} C_2^2 + (1/7) \sqrt{5/6} C_2^{-1}] \\
 &\quad + \langle r^4 \rangle [-(\sqrt{5/21}) C_4^2 - (4/21) \sqrt{5} C_4^{-1} - (5/6) \sqrt{5/7} C_4^{-4}], \\
 b &= \langle \varphi_1 | V | \varphi_0 \rangle = -\langle \varphi_0 | V | \varphi_{-1} \rangle \\
 &= \langle r^2 \rangle [(1/10) \sqrt{5/6} C_2^1 + (1/14) \sqrt{5/3} C_2^{-2}] \\
 &\quad + \langle r^4 \rangle [\sqrt{5/6} C_4^1 + (1/6) \sqrt{5/7} C_4^4 - (5/42) \sqrt{5} C_4^{-2}].
 \end{aligned}
 \tag{A.3}$$

The coefficient C_i^m in (A.1) is calculated for the A and B configurations, neglecting the contributions from further ions than the nearest neighbours. Inserting these calculated values of C_i^m 's into (A.3), we obtain a and b as follows:

$$\begin{aligned}
 a_A &= (1 + e^{i(2/3)\pi}) F, \quad b_A = (1 + e^{-i(2/3)\pi}) G, \\
 a_B &= 2F, \quad b_B = 2G, \\
 F &= (3/70) (e^2 \langle r^2 \rangle / R^3) - (5/112) (e^2 \langle r^4 \rangle / R^5), \\
 G &= (3/140) (e^2 \langle r^2 \rangle / R^3) + (15/112) (e^2 \langle r^4 \rangle / R^5),
 \end{aligned}
 \tag{A.4}$$

where R represents the distance between the nearest neighbour ions in the B sites, and e the electronic charge.

According to (A.4), $|a_A/a_B|$ becomes $1/2$.

If we substitute the following values into (A.4):

$$\begin{aligned}
 R &= 2.95 \text{ \AA} \\
 \langle r^2 \rangle &= 0.741 \text{ \AA}^2 \text{ }^{(14)} && \text{Slater function,} \\
 \langle r^4 \rangle &= 0.882 \text{ \AA}^4, \\
 \langle r^2 \rangle &= 0.455 \text{ \AA}^2 \text{ }^{(15)} && \text{Hartree function,} \\
 \langle r^4 \rangle &= 0.53 \text{ \AA}^4,
 \end{aligned}
 \tag{A.5}$$

$$\begin{aligned}\langle r^2 \rangle &= 0.327 \text{ \AA}^2 \text{ }^{10)} \\ \langle r^4 \rangle &= 0.282 \text{ \AA}^4,\end{aligned}$$

Hartree-Fock function,

the values of $|a_A|$ become as shown in (3.12).

Similar calculations show that the parameter a for the Co^{2+} ion having only one Co^{2+} ion in its nearest neighbour sites is given by

$$a_1 = F. \quad (\text{A} \cdot 6)$$

Since a_1 is equal to $|a_A|$, K_1 is equal to K_{2A} , so far as the higher order contribution from the upper φ_0 state can be neglected.

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A Possible Symmetry in Sakata's Model for Bosons-Baryons System. II

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In the previous paper we have discussed a possible symmetry among the proton, neutron and Λ -particle in Sakata's model and obtained some physically interesting results in bosons-baryons system. This symmetry is equivalent to the invariance of the theory under transformations of the unitary group $U(3)$ of degree three. We shall study a mathematical structure of our work in more detail.

§ 1. Introduction

This paper is a continuation of the previous one,¹⁾ in which we have studied a possible symmetry in Sakata's model for the strongly interacting particles.

According to Sakata's idea,²⁾ proton, p , neutron, n , and Λ -particle, Λ , are the basic particles which compose other baryons and bosons in the sense of Fermi and Yang.³⁾ We cannot find any difference among these basic particles when the mass difference is neglected and the electromagnetic and weak interactions are switched off. Therefore we may expect a certain symmetry to exist among the basic particles and to be realized in their mutual interactions. This view, first advocated by Ogawa,⁴⁾ has recently been developed by the present authors¹⁾ and by Sawada and Yonezawa.⁵⁾ Their results are as follows: (a) There exists a π_0' -meson state which is Lorentz-pseudo-scalar and iso-scalar. (b) The spin of Ξ -particle may be $(3/2)^+$. (c) Several resonating states are anticipated in K - and π -nucleon scattering. (d) For boson and baryon systems a mass formula is found, which has a good correspondence to experimental data. Yamaguchi has also made an independent study from the same standpoint of view.⁶⁾

Our work is mathematically based on the representation of the unitary group $U(3)$ of degree three. This group and its representation do not seem so familiar to physicists and, moreover, there are some concrete mathematical problems worthy of special mention. In this paper, therefore, we shall expound a mathematical aspect of our work as plainly as possible.

In § 2 we clarify how the above symmetry is reduced to the invariance with respect to the unitary group $U(3)$. Under transformations of this group, the basic particles and anti-particles are contravariant and covariant vectors respectively, while the composite systems are mixed tensors. In § 3 we introduce the physical

quantities I_{\pm} , I_3 , N_B , S , Q and N_{ij} ($ij=13, 23, 31, 32$) associated with the group, and search for their commutation relations. There exist three quantities, N_B , M and M' , which are commutable with the above physical quantities. They are the analogues of I^2 in the case of isospin. We further obtain the transformation character of the basic particles and anti-particles under the application of the operators corresponding to the physical quantities.

The subsequent two sections (§ 4 and § 5) are devoted to the representation of $U(3)$. We first use the commutation relations to construct the basis vectors in the representation space. Each vector is a simultaneous eigenvector of isospin, charge, strangeness and baryon number. The representation thus obtained gives the usual values to isospin, but the eigenvalues of N_B , S and Q are not necessarily integers. This difficulty is inevitable when we take account of the commutation relations alone, and actually disappears in the representation of the group $U(3)$. An irreducible representation of the group $U(3)$ may be specified by three quantum numbers, n_B , s_0 and i_0 . In terms of them it is possible to express the values of M and M' and the degree of the representation.

In § 6 we deal with a composite system of the basic particles and anti-particles. It can be decomposed into its irreducible constituents by means of the "contraction operation" and Young's symmetrizers. We next assign the quantum numbers n_B , s_0 and i_0 to each constituent and express the basis vectors obtained in § 4 in terms of the basic particles and anti-particles.

In the concluding section we summarize the results which will be necessary in concrete applications.

§ 2. Symmetry in Sakata's model

In this paper we study a symmetry which will be realized in the mutual interactions among the basic particles. This symmetry involves the invariance under the exchange between p (or n) and Λ in addition to the ordinary charge independence and the conservation of baryon number. As is well known, the charge independence requires p and n systems to be invariant for the unimodular (special) unitary group $SU(2)$ of degree two. This group is to be replaced by the unimodular unitary group $SU(3)$ of degree three, if we admit the exchange between p (or n) and Λ . Further, in order to introduce the concept of baryon number, we are led to the unitary group $U(3)$ of degree three. Thus the above symmetry is expressible by the invariance with respect to this group. We shall describe this symmetry in more detail.

We first take up the basic particles: p , n and Λ . In view of the symmetry among these particles we represent their wave functions by vectors in a certain three-dimensional vector space (with complex numbers as scalars):

$$\begin{aligned} p: \chi_1^* &= (1, 0, 0), & n: \chi_2^* &= (0, 1, 0), \\ \Lambda: \chi_3^* &= (0, 0, 1), & \kappa &= 1, 2, 3, \end{aligned} \tag{2.1}$$

where the indices 1, 2 and 3 under the kernel letter χ indicate the vectors, while the superscript κ indicates the components of each vector. Then a generic vector χ^κ in the space can be expressed in the form $\chi^\kappa = c^i \chi_i^\kappa$ ($i=1, 2, 3$), the coefficients c 's being complex numbers. We next assume that when the mass difference between the basic particles is neglected and the electromagnetic and weak interactions are switched off, the theory must be invariant under an arbitrary unitary transformation:

$$\chi^{\kappa'} = A_{\kappa}^{\kappa'} \chi^\kappa, \quad \bar{A}_{\kappa'}^{\kappa} = A_{\kappa}^{\kappa'}, \quad (2.2)$$

where $\bar{A}_{\kappa'}^{\kappa}$ denotes the complex conjugate of $A_{\kappa}^{\kappa'}$ and the matrix $(A_{\kappa'}^{\kappa})$ is the inverse of $(A_{\kappa}^{\kappa'})$, i.e. $A_{\kappa'}^{\kappa'} A_{\kappa}^{\kappa'} = \delta_{\kappa}^{\kappa'}$, $A_{\kappa}^{\kappa'} A_{\kappa'}^{\kappa} = \delta_{\kappa'}^{\kappa}$. It seems necessary, in this connection, to add some comments concerning the notation. χ^κ are the components of a vector χ in the "old" coordinate system (κ), whereas $\chi^{\kappa'}$ are those of the same vector in the "new" system (κ'). It is the index κ and not the kernel letter χ that changes under the coordinate transformation (2.2).^{*} $A_{\kappa}^{\kappa'}$ has the two kinds of indices (i.e. with or without prime), which means that it refers to the two systems (κ) and (κ').

In our theory the anti-particles \bar{p} , \bar{n} and $\bar{\Lambda}$ are important as well as the basic particles. They will be called basic anti-particles for the sake of convenience. We assume that when the particles transform like (2.2), the anti-particles are subject to the transformation contragredient to (2.2). This involves that it is convenient to represent the wave functions of the anti-particles by χ_κ with a subscript:

$$\begin{aligned} \bar{p} : \chi_\kappa &= (1, 0, 0), & \bar{n} : \chi_\kappa &= (0, 1, 0), \\ \bar{\Lambda} : \chi_\kappa &= (0, 0, 1). \end{aligned} \quad (2.3)$$

Thus the transformation law of the basic anti-particles can be written as

$$\chi_{\kappa'} = A_{\kappa'}^{\kappa} \chi_\kappa. \quad (2.4)$$

If we regard a basic particle as a contravariant vector with respect to $U(3)$, a basic anti-particle must be taken as a covariant vector. On account of the unitarity of $(A_{\kappa'}^{\kappa})$ the coefficients of the transformation (2.4) are the complex conjugates of those of (2.2). The assumption (2.4) will be justified by the result at the end of the next section.

We next consider a system composed of k basic particles and l basic anti-particles. We denote its Salpeter-Bethe amplitude by

$$\begin{aligned} &\chi_{\lambda_1} \cdots \chi_{\lambda_l} \chi^{\kappa_1} \cdots \chi^{\kappa_k} \\ &= \langle \mathcal{Q} | T(\chi_{\lambda_1}(x_1) \cdots \chi_{\lambda_l}(x_l) \chi^{\kappa_1}(y_1) \cdots \chi^{\kappa_k}(y_k)) | B \rangle, \end{aligned} \quad (2.5)$$

where $|B\rangle$ is the eigenstate of the total Hamiltonian and $|\mathcal{Q}\rangle$ is the true vacuum.

^{*} Since (2.2) is considered as a coordinate transformation, the equalities in (2.1) are valid only in the original coordinate system.

On the left-hand side the order of χ_λ 's and that of $\chi^{\kappa'}$'s are significant, hence they must not be changed at random. As is obvious, (2.5) transforms like a mixed tensor of contravariant valence k and covariant valence l (valence k, l) under the transformation (2.2). It involves that the composite system under consideration gives a linear representation of the group $U(3)$. This representation may be decomposed into some irreducible representations. Then the states belonging to one of the irreducible representations will have a certain symmetry as a realization of the symmetry among the basic particles. That is, they will have the same spin and parity. This statement is still valid when the difference between A and nucleon is due to the adiabatic change.*

Thus we are interested in the problem of the representation of the group $U(3)$, which we shall subsequently discuss with special regard to its physical applications.

§ 3. Commutation relations

In this section we study the physical quantities associated with the unitary group $U(3)$ and the commutation relations between them.

An infinitesimal unitary transformation has the form

$$\chi^{\kappa'} = (\delta_{\kappa'}^{\kappa'} + \sqrt{-1} X_{\kappa'}^{\kappa'} \delta\alpha) \chi^{\kappa}, \quad (3.1)$$

where $\delta\alpha$ is an infinitesimal real parameter and $(X_{\kappa'}^{\kappa'})$ is an Hermitian matrix (κ' denotes the row and κ the column). $(X_{\kappa'}^{\kappa'})$ may be expressed linearly in terms of the matrices X_{ij} ($i, j=1, 2, 3$) defined by

$$(X_{ij})_{\kappa}^{\kappa'} \stackrel{\text{def}}{=} \frac{1}{2} [\delta_{i\kappa'} \delta_{j\kappa} (1 - \sqrt{-1}) + \delta_{j\kappa} \delta_{i\kappa'} (1 + \sqrt{-1})], \quad (3.2)$$

or

$$X_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_{12} = \frac{1}{2} \begin{pmatrix} 0 & 1 - \sqrt{-1} & 0 \\ 1 + \sqrt{-1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$X_{21} = \frac{1}{2} \begin{pmatrix} 0 & 1 + \sqrt{-1} & 0 \\ 1 - \sqrt{-1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and so on. By a direct calculation we can obtain the commutation relations

$$[X_{ij}, X_{kl}] \stackrel{\text{def}}{=} X_{ij} X_{kl} - X_{kl} X_{ij} \\ = \sqrt{-1} (\delta_{ik} X_{[jl]} - \delta_{lj} X_{[ki]} - \delta_{jk} X_{(il)} + \delta_{li} X_{(kj)}), \quad (3.3)$$

* According to Sawada and Yonezawa, some of the experimental evidences on the spins of particle states seem to be different from our expectation, but the configuration of the states derived from the symmetry theory still holds its significance (see ref. 5)). A further investigation will be needed on this point.

where () and [] for indices denote the ordinary processes of symmetrization and alternation respectively :

$$X_{(ij)} = \frac{1}{2} (X_{ij} + X_{ji}), \quad X_{[ij]} = \frac{1}{2} (X_{ij} - X_{ji}).$$

Now, when a physical theory is invariant under a (Lie) group of transformations, it is usual to make an infinitesimal operator correspond to some physical quantity. As an example we give the isospin operators $I_{\pm} = I_1 \pm \sqrt{3} I_2$ and I_3 in the case of the charge independence.* Similarly, in our theory which is invariant under $U(3)$, its infinitesimal operators can be identified with physical quantities in a certain way.

To this end we introduce the following matrices :

$$\begin{aligned} I_+ &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad I_- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad I_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ N_B &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \\ N_{13} &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad N_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad N_{31} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad N_{32} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \end{aligned} \quad (3.4)$$

If we write the basic particles (2.1) in the form of column vector and operate on them the matrices in (3.4), we have the left half of Table I. From this result we may interpret (3.4) as a particular representation of the following physical quantities: The first three quantities, I_{\pm} and I_3 , are nothing but the isospin operators. The next three, N_B , Q and S , are baryon number, charge and strangeness respectively. I_3 , N_B , Q and S are not linearly independent of each other but subject to the well-known relation

$$Q = I_3 + \frac{1}{2} (N_B + S). \quad (3.5)$$

The last four, N_{ij} ($ij=13, 23, 31, 32$), characteristic of our theory, are the operators interchanging the basic particles, i.e. transforming χ_j^* into χ_i^* .

It is easy to get the commutation relations between the physical quantities. For this we have only to form the commutators between the matrices in (3.4).

* There exists no intuitive physical quantity corresponding to I_1 or I_2 . But their linear combination I_+ (I_-) can be regarded as a physical quantity in the sense that it is the operator transforming n into p (p into n). We are using the term physical quantity in such a wide sense.

Table I. The effect of the infinitesimal operators on the basic particles and anti-particles

$I_+ p=0,$	$I_+ n=p,$	$I_+ A=0,$	$I_+ \bar{p}=-\bar{n},$	$I_+ \bar{n}=0,$	$I_+ \bar{A}=0,$
$I_- p=n,$	$I_- n=0,$	$I_- A=0,$	$I_- \bar{p}=0,$	$I_- \bar{n}=-\bar{p},$	$I_- \bar{A}=0,$
$I_3 p=\frac{1}{2}p,$	$I_3 n=-\frac{1}{2}n,$	$I_3 A=0,$	$I_3 \bar{p}=-\frac{1}{2}\bar{p},$	$I_3 \bar{n}=\frac{1}{2}\bar{n},$	$I_3 \bar{A}=0,$
$N_R p=p,$	$N_R n=n,$	$N_R A=A,$	$N_R \bar{p}=-\bar{p},$	$N_R \bar{n}=-\bar{n},$	$N_R \bar{A}=-\bar{A},$
$Qp=p,$	$Qn=0,$	$QA=0,$	$Q\bar{p}=-\bar{p},$	$Q\bar{n}=0,$	$Q\bar{A}=0,$
$Sp=0,$	$Sn=0,$	$SA=-A,$	$S\bar{p}=0,$	$S\bar{n}=0,$	$S\bar{A}=\bar{A},$
$N_{13} p=0,$	$N_{13} n=0,$	$N_{13} A=p,$	$N_{13} \bar{p}=-\bar{A},$	$N_{13} \bar{n}=0,$	$N_{13} \bar{A}=0,$
$N_{23} p=0,$	$N_{23} n=0,$	$N_{23} A=n,$	$N_{23} \bar{p}=0,$	$N_{23} \bar{n}=-\bar{A},$	$N_{23} \bar{A}=0,$
$N_{31} p=A,$	$N_{31} n=0,$	$N_{31} A=0,$	$N_{31} \bar{p}=0,$	$N_{31} \bar{n}=0,$	$N_{31} \bar{A}=-\bar{p},$
$N_{32} p=0,$	$N_{32} n=A,$	$N_{32} A=0,$	$N_{32} \bar{p}=0,$	$N_{32} \bar{n}=0,$	$N_{32} \bar{A}=-\bar{n}.$

When we apply these operators on the ordered products of $p, n, A, \bar{p}, \bar{n}$ and \bar{A} or their sum, the following rule must be used: If P and P' are any ordered products, we have

$$O_\alpha(aP+bP')=aO_\alpha(P)+bO_\alpha(P'), \quad a, b=\text{const.},$$

$$O_\alpha(PP')=O_\alpha(P)P'+PO_\alpha(P'), \quad O_\alpha: \text{the above operators.}$$

This rule cannot be applied to \bar{N}_{31} , which is not an infinitesimal operator of the group $U(3)$. For this operation the definition (4.7) should be remembered and the above rule must be applied to I_3, I_-, N_{31} and N_{32} separately.

Table II. Commutation relations

$[I_3, I_+]=I_+,$	$[I_3, I_-]=-I_-,$	$[I_+, I_-]=2I_3,$	$[I, S]=[I, N_R]=[S, N_R]=0,$
$[Q, I_+]=I_+,$	$[Q, I_-]=-I_-,$	$[Q, I_3]=[Q, I^2]=[I_3, I^2]=[Q, S]=[Q, N_R]=0,$	
$[N_R, N_{13}]=0,$	$[S, N_{13}]=N_{13},$	$[Q, N_{13}]=N_{13},$	$[I_3, N_{13}]=\frac{1}{2}N_{13},$
$[N_R, N_{23}]=0,$	$[S, N_{23}]=N_{23},$	$[Q, N_{23}]=0,$	$[I_3, N_{23}]=-\frac{1}{2}N_{23},$
$[N_R, N_{31}]=0,$	$[S, N_{31}]=-N_{31},$	$[Q, N_{31}]=-N_{31},$	$[I_3, N_{31}]=-\frac{1}{2}N_{31},$
$[N_R, N_{32}]=0,$	$[S, N_{32}]=-N_{32},$	$[Q, N_{32}]=0,$	$[I_3, N_{32}]=\frac{1}{2}N_{32},$
$[I_+, N_{13}]=0,$	$[I_-, N_{13}]=N_{23},$	$[N_{13}, N_{31}]=Q+S=\frac{1}{2}(N_R+3S)+I_3,$	
$[I_+, N_{23}]=N_{13},$	$[I_-, N_{23}]=0,$	$[N_{32}, N_{23}]=Q+S-2I_3=\frac{1}{2}(N_R+3S)-I_3,$	
$[I_+, N_{31}]=-N_{32},$	$[I_-, N_{31}]=0,$	$[N_{13}, N_{32}]=I_+,$	$[N_{23}, N_{31}]=I_-,$
$[I_+, N_{32}]=0,$	$[I_-, N_{32}]=-N_{31},$	$[N_{13}, N_{23}]=[N_{31}, N_{32}]=0.$	

We give the results in Table II, which should be valid for any representation of the physical quantities.

The above physical quantities may be considered as infinitesimal operators of $U(3)$. They can be expressed linearly in terms of X_{ij} 's as follows:

$$\begin{aligned}
 I_\pm &= X_{(12)} \pm \sqrt{-1} X_{[12]}, \quad I_3 = \frac{1}{2} (X_{11} - X_{22}), \\
 N_R &= \sum_{i=1}^3 X_{ii}, \quad Q = X_{11}, \quad S = -X_{33}, \\
 N_{ij} &= X_{(ij)} + \sqrt{-1} X_{[ij]}.
 \end{aligned} \tag{3.6}$$

Conversely, if we denote for convenience the operators I_\pm, I_3, \dots and N_{32} by O_α

($\alpha=1, 2, \dots, 10$) in the order in (3.4), a generic infinitesimal operator O may be expressed in the form

$$O = a^\alpha O_\alpha, \quad (3.7)$$

where the a 's are not uniquely determined on account of the linear relation (3.5). The coefficients of I_3, N_B, Q and S (i.e. a^3, a^4, a^5, a^6) can be chosen as real and those of I_+ and I_- (N_{13} and N_{31} , or N_{23} and N_{32}) are complex conjugate to each other: $a^1 = \bar{a}^2, a^7 = \bar{a}^9, a^8 = \bar{a}^{10}$. The commutation relations in Table II are satisfied not only by the physical quantities but also by the corresponding infinitesimal operators.*

We can easily see that the commutation relations in Table II have the following property:

$$[O_\alpha, O_\beta] = C_{\alpha\beta}^\gamma O_\gamma,$$

where the C 's are constants subject to the conditions

$$C_{\alpha\beta}^\gamma = -C_{\beta\alpha}^\gamma,$$

$$C_{\alpha\beta}^\delta C_{\delta\gamma}^\epsilon + C_{\beta\gamma}^\delta C_{\delta\alpha}^\epsilon + C_{\gamma\alpha}^\delta C_{\delta\beta}^\epsilon = 0, \quad (\text{Jacobi's identity}).$$

Commutation relations with this property play an important role in mathematics as well as in physics. That is, the infinitesimal operators of a group satisfy such commutation relations, which conversely determine the structure of the group to a great extent. This fact will be used in the next section to study the representation of $U(3)$. But it should be remarked that such a determination of the group structure is not complete. For example, the orthogonal group $O(n)$ and the proper orthogonal group $O(n)^+$ have the associated commutation relations in common. For a complete knowledge of the group structure, something is necessary in general besides the commutation relations. This "something" will be made clear in § 5 in the case of the representation of $U(3)$.**

We next mention the quantities which are expressible as polynomials of the infinitesimal operators of $U(3)$ and commutable with every operator. These quantities are the analogues of I^2 in the case of isospin, where I^2 is the "inner product" of isospin and is defined by

$$I^2 = \frac{1}{2} (I_+ I_- + I_- I_+) + (I_3)^2 = \sum_{i=1}^3 (I_i)^2. \quad (3.8)$$

From this one can easily see the importance of such quantities. It is obvious that

* We use the same symbols I_\pm, I_3, \dots and N_{32} to denote the infinitesimal operators of $U(3)$, or the associated physical quantities, or the representative matrices such as those in (3.4). This will not give rise to any confusion. The eigenvalues of the operators will be denoted in general by the corresponding small letters, i_3, q, s, n_B , etc.

** In mathematical terminology, the commutation relations determine only the local structure of the group, while the global structure refers to the topology of the group. The concepts "local" and "global" are explained elsewhere in some detail.⁷⁾ But such is not necessary to understand the following discussions.

N_B is one of required quantities, but there are two other independent ones, i.e.

$$\begin{aligned} 2M &= \sum_{ij} (X_{ij})^2, \\ 8M' &= \sum_{ijk} X_{ij} (\{X_{jk}, X_{ik}\} + \{X_{ik}, X_{kj}\} + \{X_{kj}, X_{ki}\} \\ &\quad - \{X_{jk}, X_{ki}\}), \quad \{X, X'\} = XX' + X'X. \end{aligned} \quad (3.9)$$

Making use of (3.3), we can prove that M and M' are commutable with every infinitesimal operator. The calculations being tedious, we are obliged to omit the proof.

In concluding this section we consider the basic anti-particles to complete the right half of Table I. When we carry out the transformation (3.1), the anti-particles are subject to the infinitesimal transformation whose coefficients are complex conjugate of those of (3.1), i.e.

$$\chi_{\kappa'} = (\partial_{\kappa'}' - \sqrt{-1} \bar{X}_{\kappa'}^* \partial \alpha) \chi_{\kappa} = (\partial_{\kappa'}^* - \sqrt{-1} \bar{X}_{\kappa'}^* \partial \alpha) \chi_{\kappa}. \quad (3.10)$$

If M_{α} ($\alpha=1, 2, \dots, 10$) denote the matrices in (3.4), 1, 2, \dots , 10 corresponding to the order there, a generic matrix $(X_{\kappa'}^*)$ in (3.1) can be written as (cf. (3.7))

$$(X_{\kappa'}^*) = a^{\alpha} M_{\alpha}.$$

Then the corresponding matrix $(-X_{\kappa'}^*)$ in (3.10) has the expression

$$(-X_{\kappa'}^*) = a^{\alpha} (-M_{\alpha}^T),$$

where M_{α}^T mean the transposed matrices of M_{α} . Further, we can easily see that $-M_{\alpha}^T$ satisfy the same commutation relations as M_{α} , i.e.

$$[-M_{\alpha}^T, -M_{\beta}^T] = ([M_{\beta}, M_{\alpha}])^T = C_{\beta\alpha}^{\gamma} M_{\gamma}^T = C_{\alpha\beta}^{\gamma} (-M_{\gamma}^T).$$

Thus the operators O_{α} are represented by the matrices $-M_{\alpha}^T$ in the case of the basic anti-particles. In view of this the right half of Table I can be obtained in the same manner as the left half. That is, we have only to write the anti-particles (2.3) in the form of column vector and then operate on them the matrices $-M_{\alpha}^T$.

§ 4. Construction of basis in the representation space

We shall deal with a matrix representation of the commutation relations obtained in last section (cf. Table II). The representative matrices of the physical quantities are assumed to be of finite degree and to have the complex elements. We are especially interested in the connection with isospin, charge, strangeness and baryon number. Our purpose is to construct a basis in the representation space so that each vector is a simultaneous eigenvector of I^2 , I_3 , Q , S and N_B .

We shall first give the following two facts, which are immediate consequences of the commutation relations.

(1) If V is an irreducible invariant subspace of the representation space, it is an eigenspace of N_B . Namely, any vector of V is an eigenvector of N_B cor-

responding to the same eigenvalue n_B . This comes from the commutability of N_B with all the infinitesimal operators: If $N_B v = n_B v$, $v \in V$, then $N_B (O_\alpha v) = O_\alpha (N_B v) = n_B (O_\alpha v)$.

(2) If v is a simultaneous eigenvector of N_B , S , Q and I_3 , so are $I_\pm v$ and $N_{ij} v$. The corresponding changes of the eigenvalues, Δn_B , Δs , Δq and Δi_3 , are given in the left half of Table III. These results can be easily obtained. For example, from $Sv = sv$ we have

$$S(N_{13}v) = (N_{13}S + N_{13})v = (s+1)(N_{13}v),$$

on account of the commutation relation. The right half of Table III describes the interchange of the basic particles and that of the basic anti-particles by means of I_\pm and N_{ij} . This fact has already been given in Table I in a concrete form.

Table III. Effect of the operators I_\pm and N_{ij}

Operator O_α	Change of the eigenvalues				Interchange of particles and anti-particles	
	Δn_B	Δs	Δq	Δi_3		
I_+	0	0	1	1	$n \rightarrow p$, $\bar{p} \rightarrow -\bar{n}$	
I_-	0	0	-1	-1	$p \rightarrow n$, $\bar{n} \rightarrow -\bar{p}$	
N_{13}	0	1	1	$\frac{1}{2}$	$\Lambda \rightarrow p$, $\bar{p} \rightarrow -\bar{\Lambda}$	
N_{23}	0	1	0	$-\frac{1}{2}$	$\Lambda \rightarrow n$, $\bar{n} \rightarrow -\bar{\Lambda}$	
N_{31}	0	-1	-1	$-\frac{1}{2}$	$p \rightarrow \Lambda$, $\bar{\Lambda} \rightarrow -\bar{p}$	
N_{32}	0	-1	0	$\frac{1}{2}$	$n \rightarrow \Lambda$, $\bar{\Lambda} \rightarrow -\bar{n}$	

If v is a simultaneous eigenvector of N_B , S , Q and I_3 with the eigenvalues n_B , s , q and i_3 , then $O_\alpha v$ is also an eigenvector corresponding to the eigenvalues $n_B + \Delta n_B$, $s + \Delta s$, $q + \Delta q$ and $i_3 + \Delta i_3$.

Now, let s_0 be an eigenvalue of S with the largest real part and let V_0 be the associated eigenspace. Since I_3 and S commute, the operator I_3 leaves the subspace V_0 invariant: If $v \in V_0$, i.e. $Sv = s_0 v$, then $S(I_3 v) = I_3 (Sv) = s_0 (I_3 v)$, $I_3 v \in V_0$. Accordingly, if we choose basis vectors in the representation space so that the first (say, n) vectors span V_0 , then I_3 takes the form

$$I_3 = \begin{pmatrix} I_3' & \times \\ \cdots & \cdots \\ \bigcirc & \times \end{pmatrix}$$

where I_3' is an $n \times n$ matrix and \bigcirc is a zero matrix. Next, let i_0 be an eigenvalue of I_3' with the largest real part. Although it is an eigenvalue of I_3 , its real part is not necessarily the largest of the real parts of the eigenvalues of I_3 . However, if v is a simultaneous eigenvector of S and I_3 with the eigenvalue s_0 of S , then the corresponding eigenvalue of I_3 is at most i_0 . We now denote by $v_{s_0 i_0}$ a simultaneous eigenvector of S and I_3 belonging to the eigenvalues s_0 and i_0 . Then (cf. Table III), $I_+ v_{s_0 i_0}$ is a simultaneous eigenvector of S and I_3 corresponding to the respective eigenvalues s_0 and $i_0 + 1$; $N_{13} v_{s_0 i_0}$ and $N_{23} v_{s_0 i_0}$ are eigenvectors of S with

the eigenvalue s_0+1 , hence we obtain from the definition of s_0 and i_0

$$I_+ v_{s_0 i_0} = N_{13} v_{s_0 i_0} = N_{23} v_{s_0 i_0} = 0. \quad (4.1)$$

In what follows we shall construct basis vectors of the irreducible invariant subspace involving $v_{s_0 i_0}$ in such a way that each vector is a simultaneous eigenvector of S , I^2 and I_3 . For this we start with the vector $v_{s_0 i_0}$.

Iterating the operation N_{32} on the vector $v_{s_0 i_0}$, we define the vectors

$$v_{s_0-r, i_0+r/2} \stackrel{\text{def}}{=} (N_{32})^r v_{s_0 i_0}, \quad (r=0, 1, \dots). \quad (4.2)$$

$v_{s_0-r, i_0+r/2}$ is a simultaneous eigenvector of S and I_3 with the respective eigenvalues s_0-r and $i_0+\frac{1}{2}r$, as is seen in Table III. From this we know that all the non-zero vectors in (4.2) are linearly independent of each other. Therefore, a zero vector must appear in the series (4.2), as we are considering a representation of finite degree. The vectors in (4.2) satisfy the relations

$$\begin{aligned} N_{23} v_{si} &= \alpha_{si} v_{s+1, i-1/2}, \quad \alpha_{si} = \text{const.}, \quad \alpha_{s_0 i_0} = 0, \\ (s, i) &= (s_0, i_0), \quad (s_0-1, i_0+1/2), \dots \end{aligned} \quad (4.3)$$

This equation is trivial for $(s, i) = (s_0, i_0)$ by virtue of (4.1). The equation for some $(s-1, i+\frac{1}{2})$ can be deduced from that for (s, i) :

$$\begin{aligned} N_{23} v_{s-1, i+1/2} &= N_{23} N_{32} v_{si} \\ &= [N_{32} N_{23} + (1/2)(N_B + 3S) - I_3] v_{si} = [\alpha_{si} + (1/2)(n_B + 3s) - i] v_{si}. \end{aligned}$$

Thus (4.3) can be proved by the inductive method. At the same time we obtain the recursion formula for α_{si}

$$\alpha_{s-1, i+1/2} = \alpha_{si} + (1/2)(n_B + 3s) - 1, \quad \alpha_{s_0 i_0} = 0,$$

whose solution gives at once

$$\alpha_{s_0-r, i_0+r/2} = (1/2) r [n_B + 3s_0 - 2i_0 - 2(r-1)].$$

Since a zero vector appears in the series (4.2), we may assume $v_{s'-1, i'+1/2} = 0$ and $v_{s't'} \neq 0$ for some s' and i' . Then we have from (4.3) that $\alpha_{s'-1, i'+1/2} = 0$, accordingly

$$\begin{aligned} r' &= (1/2)(n_B + 3s_0 - 2i_0) \geq 0, \\ s' &= (1/2)(-n_B - s_0 + 2i_0), \quad i' = (1/4)(n_B + 3s_0 + 2i_0), \end{aligned} \quad (4.4)$$

where r' is the value of r corresponding to $v_{s't'}$, i.e.

$$s' = s_0 - r', \quad i' = i_0 + (1/2)r'.$$

Next we repeatedly operate N_{31} on the last vector $v_{s't'}$ in the series (4.2):

$$\bar{v}_{s'-t, i'-t/2} \stackrel{\text{def}}{=} (N_{31})^t v_{s't'}. \quad (4.5)$$

These are the eigenvectors of S and I_3 belonging to the eigenvalues $s'-t$ and $i'-\frac{1}{2}t$ respectively. We make the same consideration for (4.5) as for (4.2) to obtain the following result:

$$t'' = 2i_0 \geq 0, \quad s'' = (1/2)(-n_B - s_0 - 2i_0), \quad i'' = (1/4)(n_B + 3s_0 - 2i_0), \quad (4.6)$$

where t'' , s'' and i'' are the values of t , s and i_3 corresponding to the last non-vanishing vector in the series (4.5).

For the future discussions it is convenient to introduce a matrix \bar{N}_{31} through

$$\bar{N}_{31} \stackrel{\text{def}}{=} 2N_{31}I_3 + N_{32}I_- = N_{31}(2I_3 + 1) + I_-N_{32} = 2(I_3 + 1)N_{31} + I_-N_{32}. \quad (4.7)$$

It satisfies the commutation relations

$$[N_{32}, \bar{N}_{31}] = 0, \quad [S, \bar{N}_{31}] = -\bar{N}_{31}, \quad [I_3, \bar{N}_{31}] = -(1/2)\bar{N}_{31}. \quad (4.8)$$

The last two relations indicate that \bar{N}_{31} is an operator decreasing the eigenvalues of S and I_3 by one and one half respectively: If v is a simultaneous eigenvector of S and I_3 with the eigenvalues s and i_3 , then $\bar{N}_{31}v$ is also an eigenvector corresponding to $s-1$ and $i_3-\frac{1}{2}$ respectively. In this point \bar{N}_{31} plays the same role as N_{31} , but the difference will be revealed soon.

Consider then the vectors defined by

$$v_{s_0-r-t, i_0+(r-t)/2} \stackrel{\text{def}}{=} (\bar{N}_{31})^t (N_{32})^r v_{s_0 i_0}, \\ r=0, 1, \dots, r', \quad t=0, 1, \dots, t'' \quad (=2i_0), \quad (4.9)$$

which are simultaneous eigenvectors of S and I_3 belonging to the eigenvalues s_0-r-t and $i_0+\frac{1}{2}(r-t)$ respectively. The vectors for $r=r'$ coincide with those in (4.5) to within scalar factors; it is trivial for $t=0$. Next, assume that it holds good for some t , and we have from the second expression for \bar{N}_{31} in (4.7) and the definition of r'

$$\bar{N}_{31}v = (2i_0 + r' - t + 1)N_{31}v + I_-(\bar{N}_{31})^t (N_{32})^{r'+1} v_{s_0 i_0} = (2i_0 + r' - t + 1)N_{31}v,$$

where v stands for $v_{s_0-r-t, i_0+(r-t)/2}$. Since we have $2i_0 + r' - t + 1 \neq 0$ from $t \leq 2i_0$, the above statement has been proved. On making use of this result we can show that each vector in (4.9) is not zero. That is, if $v_{s_0-r-t, i_0+(r-t)/2} = 0$ for some r and t , we operate $(\bar{N}_{31})^{t''-t} (N_{32})^{r'-r}$ on it to get $v_{s'' i''} = 0$. This is contradictory, as $v_{s'' i''}$ is equal to $\bar{v}_{s'' i''}$ to within a scalar factor.

Here we should notice a remarkable property of the vectors in (4.9), i.e.

$$I_+ v_{s_0-r-t, i_0+(r-t)/2} = 0. \quad (4.10)$$

Its proof is simple. In the case of $t=0$ we have from (4.1)

$$I_+ v_{s_0-r, i_0+r/2} = I_+ (N_{32})^r v_{s_0 i_0} = (N_{32})^r I_+ v_{s_0 i_0} = 0.$$

Then assuming $I_+ v_{s_0-r-t+1, i_0+(r-t+1)/2} = 0$, we obtain

$$I_+ v_{s_0-r-t, i_0+(r-t)/2} = I_+ \bar{N}_{31} v_{s_0-r-t+1, i_0+(r-t+1)/2} \\ = (2N_{31}I_3 + N_{32}I_- - 2N_{31}) I_+ v_{s_0-r-t+1, i_0+(r-t+1)/2} = 0.$$

(4.10) shows that each vector in (4.9) is the eigenvector of I_3 belonging to the largest eigenvalue in some irreducible representation of isospin. If we use N_{31}

instead of \bar{N}_{31} in (4.9), the resulting vectors do not satisfy (4.10).

If we operate I_- on one of the vectors in (4.9) repeatedly, we get all the eigenvectors of I_3 in the same representation of isospin, thus

$$\begin{aligned} \psi_{s_0-r-t, i_0+(r-t)/2, i_0+(r-t)/2-u} &\stackrel{\text{def}}{=} (I_-)^u (\bar{N}_{31})^t (N_{32})^r \psi_{s_0 i_0}, \\ r=0, 1, \dots, r' \quad (&= (1/2)(n_B + 3s_0 - 2i_0)), \quad t=0, 1, \dots, t'' \quad (=2i_0), \\ u=0, 1, \dots, u''' \quad (&=2i_0 + r - t). \end{aligned} \quad (4.11)$$

The three indices of the vectors, $s_0 - r - t$, $i_0 + \frac{1}{2}(r - t)$ and $i_0 + \frac{1}{2}(r - t) - u$, are the associated eigenvalues of S , I and I_3 respectively, where I is defined to be the positive root of

$$I(I+1) = \mathbf{I}^2. \quad (4.12)$$

It is obvious that these vectors are not zero. They are linearly independent of each other, because they have different sets of the eigenvalues of S , I and I_3 . In the Appendix we shall prove that they span the irreducible invariant subspace under consideration, namely, that the representation given by the vectors in (4.11) is irreducible.

For an irreducible representation the subspace spanned by the vectors (4.11) is coincident with the whole representation space, hence an arbitrary irreducible representation is equivalent to one of the representations constructed in this section.

§ 5. Properties of representation

In this section we shall give some properties of the representations. There exists a close connection between the representations of $U(3)$ and of the associated commutation relations. For instance, a representation of the group $U(3)$ reduces to that of the commutation relations; the former is reducible or irreducible when and only when the latter has the same property; and so on. Thus we may deduce the properties of the representations of $U(3)$ from the results in last section. However, one should notice that it is not a sufficient method for our purpose. We shall start with the observation of these circumstances.

It follows from (4.11) that $u''' = 2i_0 + r - t$ is a non-negative integer, hence $i = i_0 + \frac{1}{2}(r - t)$ is a non-negative half-integer and i_3 takes the values $i_0 + \frac{1}{2}(r - t) - u$, ($u = 0, 1, \dots, 2i_0 + r - t$). This coincides with the usual values for isospin. On the other hand, the eigenvalues of N_B , S and Q are ordinarily taken to be integers. This requirement is not necessarily fulfilled by the representations of the commutation relations. For example, the following matrices satisfy the commutation relations:

$$I_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad I_- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad I_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$N_B = \begin{pmatrix} 1-3z & 0 & 0 \\ 0 & 1-3z & 0 \\ 0 & 0 & 1-3z \end{pmatrix}, \quad S = \begin{pmatrix} z & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z-1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1-z & 0 & 0 \\ 0 & -z & 0 \\ 0 & 0 & -z \end{pmatrix}, \quad (5.1)$$

$$N_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad N_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad N_{31} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad N_{32} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

where z is an arbitrary complex number.* In the case $z=0$ (5.1) is reduced to (3.4). On the contrary, when $z \neq 0$, this representation is not equivalent to the representation (3.4), because N_B , S and Q have the different eigenvalues for (3.4) and (5.1).

We have pointed out in § 3 that the commutation relations cannot completely determine the group structure and that for the complete determination "something" is necessary besides the commutation relations. This "something" has appeared now. In fact, the non-integral values of n_B , s and q are forbidden in the representation of $U(3)$, which we are going to prove.

The infinitesimal operator S in (3.4) generates the one-dimensional subgroup G_1 of $U(3)$ defined by

$$\chi^1 = \chi^1, \quad \chi^{2'} = \chi^2, \quad \chi^{3'} = \exp(-\sqrt{-1}\alpha) \chi^3, \quad (5.2)$$

where α is a real parameter. If we denote the components χ^s and $\chi^{s'}$ in the matrix form χ and χ' respectively, (5.2) may be written as

$$\chi' = \exp(\sqrt{-1}\alpha S) \chi, \quad (5.3)$$

where S is the matrix given by (3.4). Next consider a continuous representation of $U(3)$. There corresponds to (5.3) the following transformation in the representation space:

$$v' = \exp(\sqrt{-1}\alpha S) v, \quad (5.4)$$

where we have employed the same symbol S to denote the representative matrix. We may regard this correspondence as a continuous representation of the subgroup G_1 . On the other hand, G_1 is isomorphic to the unitary group $U(1)$ of degree one and has the period 2π with respect to the parameter α . A continuous representation of such a group is fully reducible, and any irreducible representation is of degree one and has the form $\exp(\sqrt{-1}n\alpha)$, n being an arbitrary integer.⁸⁾ It follows then that by a suitable choice of basis vectors the matrix S in (5.4) can be reduced to a diagonal form with integral elements.

This proof can be applied to the cases of N_B and Q to show that their eigenvalues are also restricted to integers. Contrary to S , N_B and Q , the com-

* z cannot be removed even if we assume the matrices X_{ij} to be Hermitian.

mutation relations alone give the usual values to isospin, as was already observed. But we may make a consideration similar to the above with respect to I_3 . For this purpose we have merely to remark that in this case the one-dimensional group corresponding to the above G_1 has the period 4π with respect to the parameter, although it is also isomorphic to the unitary group $U(1)$.

Now, regardless of physical applications, many investigations have been made concerning the representations of the unitary group.^{(8), (9)} In the case of $U(3)$ their results are often stated in terms of the set of three integers f_i ($i=1, 2, 3$; $f_1 \geq f_2 \geq f_3$), which specifies an irreducible representation. To see the role of n_B , s_0 and i_0 more clearly, we shall seek for the relation between (n_B, s_0, i_0) and (f_1, f_2, f_3) .

Any unitary transformation can be reduced to the form

$$\chi' = \epsilon_\kappa \chi^\kappa, \quad (|\epsilon_\kappa| = 1), \quad (\text{not summed for } \kappa), \quad (5.5)$$

by a suitable choice of a basis. Then, in the irreducible representation of $U(3)$ specified by f_i , the corresponding representative matrix has the character

$$\chi(f_1, f_2, f_3) = \frac{\begin{vmatrix} \epsilon_1^{l_1} & \epsilon_1^{l_2} & \epsilon_1^{l_3} \\ \epsilon_2^{l_1} & \epsilon_2^{l_2} & \epsilon_2^{l_3} \\ \epsilon_3^{l_1} & \epsilon_3^{l_2} & \epsilon_3^{l_3} \end{vmatrix}}{\begin{vmatrix} \epsilon_1^2 & \epsilon_1 & 1 \\ \epsilon_2^2 & \epsilon_2 & 1 \\ \epsilon_3^2 & \epsilon_3 & 1 \end{vmatrix}}, \quad (5.6)$$

where

$$l_1 = f_1 + 2, \quad l_2 = f_2 + 1, \quad l_3 = f_3.$$

Each term of (5.6) has the form

$$\epsilon_1^{n_1} \epsilon_2^{n_2} \epsilon_3^{n_3}, \quad \left(\sum_{i=1}^3 n_i = \sum_{i=1}^3 f_i \right), \quad (5.7)$$

where n 's are integers. If we choose such a basis in the representation space that the representative matrix is diagonal, the quantities (5.7) become its diagonal elements. Each of them gives rise to the respective eigenvalues:

$$n_B = \sum_{i=1}^3 n_i, \quad s = -n_3, \quad i_3 = \frac{1}{2} (n_1 - n_2).$$

This comes from the facts that the transformations generated by N_B , S and I_3 in (3.4) are of the form (5.5) with

$$\begin{aligned} N_B: \quad \epsilon_1 = \epsilon_2 = \epsilon_3 = \exp(\sqrt{-1}\alpha), \quad S: \quad \epsilon_1 = \epsilon_2 = 0, \quad \epsilon_3 = \exp(-\sqrt{-1}\alpha), \\ I_3: \quad \epsilon_1 = \epsilon_2^{-1} = \exp[(1/2)\sqrt{-1}\alpha], \quad \epsilon_3 = 0, \quad (\alpha: \text{real parameter}), \end{aligned} \quad (5.8)$$

and that the corresponding representative matrices may be made diagonal at the same time.

On the other hand, the results in last section tell us the uniqueness of the set of eigenvalues (n_B, s_0, i_0) , hence the uniqueness of the corresponding diagonal element (5.7). Therefore, this element is not cancelled by other diagonal elements in

taking the spur of the matrix. In other words, the diagonal element corresponding to (n_B, s_0, i_0) should appear in the character (5.7). Thus we have from the definition of n_B, s_0 and i_0

$$n_B = \sum_{i=1}^3 n_i, \quad s_0 = \max(-n_3), \quad i_0 = \max_{n_3 = -s_0} \frac{1}{2} (n_1 - n_2), \quad (5.9)$$

where the first maximum is to be taken with respect to all the terms in (5.6), and the second to the terms with $n_3 = -s_0$. A close examination shows that $\min n_3 = f_3$ and the coefficient of $\epsilon_3^{f_3}$ is

$$(\epsilon_1^{f_1} \epsilon_2^{f_2} - \epsilon_1^{f_2} \epsilon_2^{f_1}) / \epsilon_1 \epsilon_2 (\epsilon_1 - \epsilon_2). \quad (5.10)$$

It follows then that $\min_{n_3 = -s_0} (n_1 - n_2) = f_1 - f_2$.

Summarizing the above we have the desired relation:

$$n_B = \sum_{i=1}^3 f_i, \quad s_0 = -f_3, \quad i_0 = (1/2) (f_1 - f_2), \quad (5.11)$$

or $f_1 = (1/2) (n_B + s_0) + i_0, \quad f_2 = (1/2) (n_B + s_0) - i_0, \quad f_3 = -s_0$.

For example, we have for the basic particles and anti-particles

$$\chi(f_1, f_2, f_3) = \sum_{\kappa=1}^3 \epsilon_\kappa, \quad (f_1, f_2, f_3) = (1, 0, 0), \quad (n_B, s_0, i_0) = (1, 0, 1/2);$$

$$\chi(f_1, f_2, f_3) = \sum_{\kappa=1}^3 \epsilon_\kappa^{-1}, \quad (f_1, f_2, f_3) = (0, 0, -1), \quad (n_B, s_0, i_0) = (-1, 1, 0).$$

In the next section we shall give a method of assigning (n_B, s_0, i_0) to irreducible composite systems without referring to f_i .

In view of this result we may use (n_B, s_0, i_0) instead of (f_1, f_2, f_3) to specify an irreducible representation of $U(3)$. In this case the following prescription is necessary to the values of n_B, s_0 and i_0 :

(i) n_B and s_0 are integers, and i_0 is a half-integer or an integer.

(ii) According as i_0 is an integer or a half-integer, so is $\frac{1}{2}(n_B + s_0)$.

(iii) $i_0 \geq 0, \quad n_B + 3s_0 - 2i_0 \geq 0. \quad (5.12)$

The condition (iii) is equivalent to $f_1 \geq f_2 \geq f_3$, or to $r' \geq 0, t' \geq 0 \mid (4.4), (4.6)$. The results in last section are valid for representations of the group $U(3)$ with the provisos (i), (ii) and (iii).

It seems to us that the set (n_B, s_0, i_0) is more convenient than (f_1, f_2, f_3) when we discuss physical problems. Thus we shall denote an irreducible representation of $U(3)$ by the symbol $\mathfrak{D}(n_B, s_0, i_0)$. Next, we state some properties of irreducible representations in terms of (n_B, s_0, i_0) .

Since M and M' defined by (3.9) are commutable with the quantities I_\pm, I_3, \dots and N_{32} , they have the same values for each vector in an irreducible invariant subspace. (The proof is the same as in the case of N_B . cf. (i) in § 4) Accordingly, they may be calculated through n_B, s_0 and i_0 . In fact, if we substitute (3.6) in (3.9) and use the commutation relations, we have

$$\begin{aligned}
2M &= Q^2 + S^2 + (N_B - Q + S)^2 + 2(Q + S) + 2(I_- I_+ + N_{31} N_{13} + N_{32} N_{23}), \\
8M' &= 4[Q^3 - S^3 + (N_B - Q + S)^3 + 3(Q^2 - S^2) + 3(Q - S) - 2N_B] \\
&\quad + 12[Q(I_- I_+ + N_{31} N_{13}) + (N_B - Q + S)(I_- I_+ + N_{32} N_{23}) \\
&\quad - S(N_{31} N_{13} + N_{32} N_{23}) + I_- I_+ - N_{32} N_{23}] + 6(\{N_{32}, I_-\} N_{13} + N_{31} \{N_{23}, I_+\}).
\end{aligned}$$

Operating these on $v_{s_0 i_0}$, we obtain on account of (4.1)

$$\begin{aligned}
m &= (1/2)[q_0^2 + s_0^2 + (n_B - q_0 + s_0)^2 + 2(q_0 + s_0)], \\
m' &= (1/2)[q_0^3 - s_0^3 + (n_B - q_0 + s_0)^3 + 3(q_0^2 - s_0^2) + 3(q_0 - s_0) - 2n_B],^*
\end{aligned} \tag{5.13}$$

where q_0 is charge corresponding to $v_{s_0 i_0}$ and is given by (cf. (3.5))

$$q_0 = i_0 + (1/2)(n_B + s_0).$$

(5.13) can be uniquely solved for s_0 and q_0 , because we have, by making use of (5.12),

$$\left| \frac{\partial m / \partial s_0}{\partial m' / \partial s_0} \frac{\partial m / \partial q_0}{\partial m' / \partial q_0} \right| = 3(2s_0 + n_B - q_0 + 1)(2q_0 - n_B - s_0 + 1)(q_0 + s_0 + 2)/2 \geq 3,$$

where s_0 , q_0 and n_B are taken to be independent of each other when calculating the derivatives. Thus we may use (n_B, m, m') instead of (n_B, s_0, i_0) to specify an irreducible representation of $U(3)$. (5.13) tells us that m and m' are integers or half-integers according as n_B is even or odd.

We can easily calculate the degree of an irreducible representation, which is equal to the dimension number of the subspace spanned by the vectors in (4.11). The calculation may be performed by taking account of the linear independence of those vectors. The result is

$$\begin{aligned}
d &= \sum_{r=0}^{r'} \sum_{t=0}^{t'} (2i_0 + r - t + 1) \\
&= (2i_0 + 1)(n_B + 3s_0 + 2i_0 + 4)(n_B + 3s_0 - 2i_0 + 2)/8.
\end{aligned} \tag{5.14}$$

From the discussion in last section we know the way an irreducible representation of $U(3)$ is decomposed into irreducible representations of isospin. We give the results in Table IV. There, $\mathfrak{D}(s, i)$ stands for an irreducible representation of isospin, s and i being the associated eigenvalues of S and I respectively. An arrow $\rightarrow [\checkmark]$ means that the vector belonging to the maximum eigenvalue of I_3 in $\mathfrak{D}(s, i)$ is transformed into that vector in $\mathfrak{D}(s-1, i+\frac{1}{2})$ [$\mathfrak{D}(s-1, i-\frac{1}{2})$] by the operator N_{32} [\bar{N}_{31}].

§ 6. Composite system

In this section we consider composite systems of the basic particles and anti-

* In Eq. (8) of the previous paper, the plus sign before s_0^3 should be replaced by the minus sign.

particles. One of the problems thereof is to decompose a system into its constituents irreducible and invariant under the unitary group $U(3)$. The states belonging to such a constituent give an irreducible representation of $U(3)$ and will have a certain symmetry as a realization of the symmetry among the basic particles. We shall solve the above problem in a concrete way, and then state a method of assigning (n_B, s_0, i_0) to each irreducible constituent and that of expressing the basis vectors (4.11) in terms of the basic particles and anti-particles.

As was mentioned in § 2, our problem is reduced to the problem of decomposing the tensor space V_l^k (the totality of mixed tensors of valence k, l). In this connection we first recall the decomposition of a tensor space with respect to the orthogonal group $O(n)$, where use is made of the trace operation by means of the fundamental tensor $\delta_{\lambda\kappa}$ and the application of Young's symmetrizers.^{8),9)} Under the orthogonal group the difference disappears between the contravariant and covariant indices, whereas we must distinguish between them with respect to the unitary group. Therefore, when we apply the above procedure of decomposition to our case, the fundamental tensor $\delta_{\lambda\kappa}$ in the case of $O(n)$ should be replaced by the Kronecker delta δ_λ^κ and Young's symmetrizers should be applied to superscripts and subscripts separately. Thus we may decompose the tensor space V_l^k through the following two steps:

(1) "Contraction operation", which corresponds to the trace operation in the case of $O(n)$. In the concrete,

$$T_{\lambda_1 \dots \lambda_l}^{\kappa_1 \dots \kappa_k} = T_{\lambda_1 \dots \lambda_l}^{(0) \kappa_1 \dots \kappa_k} + \sum_{(a,b)} \delta_{\lambda_b}^{\kappa_a} T_{\lambda_1 \dots \lambda_{l-1} \dots \lambda_l}^{\kappa_1 \dots \kappa_a \dots \kappa_b \dots \kappa_k} + \sum_{(ab,cd)} \delta_{\lambda_c}^{\kappa_a} \delta_{\lambda_d}^{\kappa_b} T_{\lambda_1 \dots \lambda_{l-2} \dots \lambda_{l-1} \dots \lambda_l}^{\kappa_1 \dots \kappa_a \dots \kappa_b \dots \kappa_c \dots \kappa_d \dots \kappa_k} + \dots \quad (6.1)$$

On the right-hand side, the indices (0), (a, b), (ab, cd), ..., designate the tensors T 's, and the symbol $\wedge (\vee)$ tells us to remove the index under (over) itself. Therefore, $T_{(a,b)}$ is a tensor of valence $k-1, l-1$; $T_{(ab,cd)}$ a tensor of valence $k-2, l-2$; and so on. *These tensors are reduced to zero if we contract them for any pair of a superscript and a subscript.* The summation $\sum_{ab, cd}$ is to be taken for $a, b=1, 2, \dots, k$ ($a < b$) and $c, d=1, 2, \dots, l$ independently.

Example I

$$\begin{aligned} T_\lambda^\kappa &= T_\lambda^{(0)\kappa} + \delta_\lambda^\kappa T, \quad T_\alpha^\alpha = 0; \\ T_{\lambda_1 \lambda_2}^{\kappa_1 \kappa_2} &= T_{\lambda_1 \lambda_2}^{(0) \kappa_1 \kappa_2} + \delta_{\lambda_1}^{\kappa_1} T_{\lambda_2}^{\kappa_2} + \delta_{\lambda_2}^{\kappa_2} T_{\lambda_1}^{\kappa_1} + \delta_{\lambda_1}^{\kappa_2} T_{\lambda_2}^{\kappa_1} + \delta_{\lambda_2}^{\kappa_1} T_{\lambda_1}^{\kappa_2} + \delta_{\lambda_1}^{\kappa_1} \delta_{\lambda_2}^{\kappa_2} T + \delta_{\lambda_2}^{\kappa_1} \delta_{\lambda_1}^{\kappa_2} T, \\ T_{\alpha\lambda}^{\alpha\kappa} &= T_{\lambda\alpha}^{\alpha\kappa} = T_{\alpha\lambda}^{(0) \alpha\kappa} = T_{\lambda\alpha}^{(0) \kappa\alpha} = T_{\alpha\lambda}^{(\alpha, b)} = T_{\lambda\alpha}^{(\alpha, b)} = 0, \quad (ab=11, 12, 21, 22); \\ T_\lambda^{\kappa_1 \kappa_2} &= T_\lambda^{(0) \kappa_1 \kappa_2} + \delta_\lambda^{\kappa_1} T^{\kappa_2} + \delta_\lambda^{\kappa_2} T^{\kappa_1}, \quad T_\alpha^{\alpha\kappa} = T_\alpha^{(0) \kappa\alpha} = 0; \\ T_{\lambda_1 \lambda_2}^{\kappa_1 \kappa_2 \kappa_3} &= T_{\lambda_1 \lambda_2}^{(0) \kappa_1 \kappa_2 \kappa_3} + \delta_{\lambda_1}^{\kappa_1} T_{\lambda_2}^{\kappa_2 \kappa_3} + \delta_{\lambda_2}^{\kappa_1} T_{\lambda_1}^{\kappa_2 \kappa_3} + \delta_{\lambda_1}^{\kappa_2} T_{\lambda_2}^{\kappa_1 \kappa_3} + \delta_{\lambda_2}^{\kappa_2} T_{\lambda_1}^{\kappa_1 \kappa_3} \\ &\quad + \delta_{\lambda_2}^{\kappa_2} T_{\lambda_1}^{\kappa_1 \kappa_3} + \delta_{\lambda_1}^{\kappa_3} T_{\lambda_2}^{\kappa_1 \kappa_2} + \delta_{\lambda_2}^{\kappa_3} T_{\lambda_1}^{\kappa_1 \kappa_2} + \delta_{\lambda_1}^{\kappa_1} \delta_{\lambda_2}^{\kappa_2} T_{\lambda_1}^{\kappa_3} \end{aligned}$$

$$\begin{aligned}
& + \delta_{\lambda_2}^{\kappa_1} \delta_{\lambda_1}^{\kappa_2} T^{\kappa_3} + \delta_{\lambda_1}^{\kappa_1} \delta_{\lambda_2}^{\kappa_3} T^{\kappa_2} + \delta_{\lambda_2}^{\kappa_1} \delta_{\lambda_1}^{\kappa_3} T^{\kappa_2} + \delta_{\lambda_1}^{\kappa_2} \delta_{\lambda_2}^{\kappa_3} T^{\kappa_1} + \delta_{\lambda_2}^{\kappa_2} \delta_{\lambda_1}^{\kappa_3} T^{\kappa_1}, \\
& T_{\alpha\lambda}^{(\alpha)\kappa_2\kappa_3} = T_{\lambda\alpha}^{(\alpha)\kappa_2\kappa_3} = T_{\alpha\lambda}^{(0)\kappa_1\alpha\kappa_3} = T_{\lambda\alpha}^{(0)\kappa_1\alpha\kappa_3} = T_{\alpha\lambda}^{(0)\kappa_1\kappa_2\alpha} = T_{\lambda\alpha}^{(0)\kappa_1\kappa_2\alpha} = T_{\alpha\alpha}^{(\alpha,b)} = T_{\alpha\alpha}^{(\alpha,b)} = 0, \\
& (ab=11, 12, 21, 22, 31, 32).
\end{aligned}$$

(2) Operation of Young's symmetrizers. After the process (1) is achieved, operate the symmetrizers on the tensors T 's on the right-hand side of (6.1) with respect to superscripts and subscripts separately. When the number of indices is two, this process is reduced to the ordinary symmetrization and alternation. For the indices more than two we refer to other books.^{8),9)} Since the tensors in question satisfy the property in italics stated in (1), there occurs some simplification. That is, for the tensor $T_{\lambda_1 \dots \lambda_l}^{\kappa_1 \dots \kappa_k}$ we have only to consider the following cases:

Case i)

for superscripts,

for subscripts,

$(l-l_1=0, 1, \dots, [l/2]);$

Case ii)

for superscripts,

$(k-k_1=0, 1, \dots, [k/2]);$

for subscripts,

where $[x]$ denotes the "integral part of x ", the largest integer which does not exceed x . We can easily prove that if we operate on T the symmetrizers corresponding to the other cases, the resulting tensors vanish identically. (See Example II.) The same simplification holds good for the tensors $T_{\lambda_1 \lambda_2}^{(\alpha,b)}$, $T_{\lambda_1 \lambda_2}^{(ab,cd)}$, \dots , in which cases we have merely to replace k and l by $k-1$ and $l-1$; $k-2$ and $l-2$; \dots respectively.

Example II

$$T_{\lambda_1 \lambda_2}^{\kappa_1 \kappa_2} = T_{(\lambda_1 \lambda_2)}^{\kappa_1 \kappa_2} + T_{[\lambda_1 \lambda_2]}^{\kappa_1 \kappa_2} + T_{\lambda_1 \lambda_2}^{[\kappa_1 \kappa_2]};$$

$T_{[\lambda_1 \lambda_2]}^{[\kappa_1 \kappa_2]}$ is identically zero on account of $T_{[\alpha \lambda]}^{[\alpha \kappa]}=0$.

$$T_{\lambda}^{\kappa_1 \kappa_2} = T_{\lambda}^{(\kappa_1 \kappa_2)} + T_{\lambda}^{[\kappa_1 \kappa_2]};$$

$$T_{\lambda_1 \lambda_2}^{\kappa_1 \kappa_2 \kappa_3} = T_{(\lambda_1 \lambda_2)}^{(\kappa_1 \kappa_2 \kappa_3)} + T_{[\lambda_1 \lambda_2]}^{(\kappa_1 \kappa_2 \kappa_3)} + {}'T_{(\lambda_1 \lambda_2)}^{\kappa_1 \kappa_2 \kappa_3} + {}''T_{(\lambda_1 \lambda_2)}^{\kappa_1 \kappa_2 \kappa_3},$$

where $'T$ and $''T$ are characterized by

$$'T_{\lambda_1 \lambda_2}^{\kappa_1 \kappa_2 \kappa_3} + {}'T_{\lambda_1 \lambda_2}^{\kappa_2 \kappa_1 \kappa_3} = {}'T_{\lambda_1 \lambda_2}^{\kappa_1 \kappa_2 \kappa_3} + {}'T_{\lambda_1 \lambda_2}^{\kappa_2 \kappa_1 \kappa_3} + {}'T_{\lambda_1 \lambda_2}^{\kappa_3 \kappa_1 \kappa_2} = 0,$$

$$''T_{\lambda_1 \lambda_2}^{\kappa_1 \kappa_2 \kappa_3} + {}''T_{\lambda_1 \lambda_2}^{\kappa_1 \kappa_2 \kappa_1} = {}''T_{\lambda_1 \lambda_2}^{\kappa_1 \kappa_2 \kappa_3} + {}''T_{\lambda_1 \lambda_2}^{\kappa_2 \kappa_1 \kappa_3} + {}''T_{\lambda_1 \lambda_2}^{\kappa_3 \kappa_1 \kappa_2} = 0;$$

$T_{\lambda_1 \lambda_2}^{[\kappa_1 \kappa_2 \kappa_3]}$, $'T_{[\lambda_1 \lambda_2]}^{\kappa_1 \kappa_2 \kappa_3}$ and $''T_{[\lambda_1 \lambda_2]}^{\kappa_1 \kappa_2 \kappa_3}$ are identically zero because of $T_{[\alpha \lambda]}^{[\alpha \kappa \kappa]}=0$, etc.

We notice that it is possible to decompose tensor spaces in a different form, e.g.

$$T_{\lambda}^{\kappa_1 \kappa_2} = T_{\lambda}^{(\kappa_1 \kappa_2)} + T_{\lambda}^{[\kappa_1 \kappa_2]}, \quad (\text{process (2)})$$

$$T_{\lambda}^{(\kappa_1 \kappa_2)} = \bar{T}_{\lambda}^{(\kappa_1 \kappa_2)} + \bar{T}_{\lambda}^{(1)}, \quad \bar{T}_{\alpha}^{(\alpha \kappa)} = 0, \quad (\text{process (1)})$$

$$T_{\lambda}^{[\kappa_1 \kappa_2]} = \bar{T}_{\lambda}^{[\kappa_1 \kappa_2]} + \bar{T}_{\lambda}^{(2)}, \quad \bar{T}_{\alpha}^{[\alpha \kappa]} = 0.$$

Here the tensors $\bar{T}_{\lambda}^{(\kappa_1 \kappa_2)}$ and $\bar{T}_{\lambda}^{[\kappa_1 \kappa_2]}$ are nothing but those resulting from the above method of decomposition. The other two are of the form

$$\bar{T}_{\lambda}^{(1)} = (1/2) (\delta_{\lambda}^{\kappa_1} \bar{T}^{\kappa_2} + \delta_{\lambda}^{\kappa_2} \bar{T}^{\kappa_1}), \quad \bar{T}_{\lambda}^{(2)} = (1/2) (\delta_{\lambda}^{\kappa_1} \bar{T}^{\kappa_2} - \delta_{\lambda}^{\kappa_2} \bar{T}^{\kappa_1}),$$

and are irreducible as well as $\delta_{\lambda}^{\kappa_1} \bar{T}^{\kappa_2}$ and $\delta_{\lambda}^{\kappa_2} \bar{T}^{\kappa_1}$. This shows that the decomposition of tensors into irreducible constituents is not necessarily unique. But the irreducible constituents obtained from $\bar{T}_{\lambda_1 \dots \lambda_l}^{\kappa_1 \dots \kappa_k}$ in (6.1) are uniquely determined. This fact seems to be of interest, because only these (unique) irreducible constituents have been employed to obtain a physically significant mass formula for baryons and bosons.⁵⁾

Now we consider an irreducible invariant subspace thus obtained, and denote by $T_{(\lambda)}^{(\kappa)}$ a non-vanishing component $T_{\lambda_1 \dots \lambda_l}^{\kappa_1 \dots \kappa_k}$ of a generic tensor belonging to this subspace. $T_{(\lambda)}^{(\kappa)}$'s are not linearly independent but subject to some linear relations. Such relations are expressible either as outer and inner products with $\delta_{\lambda}^{\kappa}$ (process (1)), or as symmetrizations and alternations with respect to superscripts and subscripts (process (2)). As examples of the former type, we give the tensor $T_{\lambda_1 \dots \lambda_l}^{\kappa_1 \dots \kappa_k} = \delta_{\lambda_1}^{\kappa_1} T_{\lambda_2 \dots \lambda_l}^{\kappa_2 \dots \kappa_k}$ for the outer product and the relation $\delta_{\lambda_1}^{\kappa_1} T_{\lambda_1 \dots \lambda_l}^{\kappa_1 \dots \kappa_k} = 0$ for the inner product, i.e.

$$\begin{aligned} T_{1\lambda_2 \dots \lambda_l}^{\kappa_1 \dots \kappa_k} &= T_{2\lambda_2 \dots \lambda_l}^{\kappa_2 \dots \kappa_k} = T_{3\lambda_2 \dots \lambda_l}^{\kappa_3 \dots \kappa_k} \\ T_{1\lambda_2 \dots \lambda_l}^{\kappa_1 \dots \kappa_k} &+ T_{2\lambda_2 \dots \lambda_l}^{\kappa_2 \dots \kappa_k} + T_{3\lambda_2 \dots \lambda_l}^{\kappa_3 \dots \kappa_k} = 0. \end{aligned} \quad (6.2)$$

Here we introduce $N_{[(\kappa), (\lambda)]}$, the number of indices 1 of the component $T_{(\lambda)}^{(\kappa)}$. This is defined as the remainder when we subtract the number of 1 of the subscripts λ 's in $T_{(\lambda)}^{(\kappa)}$ from that of the superscripts κ 's. In the same way we define

the number of indices 2 (3) of the component $T_{(\lambda)}^{(\kappa)}$, which we designate by $N_2[(\kappa), (\lambda)]$ ($N_3[(\kappa), (\lambda)]$). All the terms in a relation of the form (6.2) possess $N_i[(\kappa), (\lambda)]$ in common ($i=1, 2, 3$); that is, $N_i[(\kappa), (\lambda)]$ are the same for the components $T_{(\lambda)}^{(\kappa)}$'s linearly depending through the process (1). This fact is also valid with respect to the process (2). Thus, if we take up any component $T_{(\lambda_0)}^{(\kappa_0)}$, then the components linearly depending on it have the same values of $N_i[(\kappa), (\lambda)]$, i.e. $N_i[(\kappa_0), (\lambda_0)]$.

Next, consider the tensor with the components

$$\begin{aligned} T_{(\lambda)}^{(\kappa)} &= 0 \text{ when } T_{(\lambda)}^{(\kappa)} \text{ is linearly independent of } T_{(\lambda_0)}^{(\kappa_0)}, \\ T_{(\lambda)}^{(\kappa)} &\neq 0 \text{ when } T_{(\lambda)}^{(\kappa)} \text{ is linearly dependent on } T_{(\lambda_0)}^{(\kappa_0)}, \end{aligned} \quad (6.3)$$

and we may regard $N_i[(\kappa_0), (\lambda_0)]$ as associated with this tensor. By the same method we may introduce similar tensors starting with the non-vanishing components other than those in the second line of (6.3). These tensors are linearly independent of each other and form a basis in the subspace in question. They are simultaneous eigenvectors of N_B , S , I_3 and Q and the corresponding eigenvalues are respectively given by

$$\begin{aligned} n_B &= k - l = \sum_{i=1}^3 N_i[(\kappa_0), (\lambda_0)], \quad s = -N_3[(\kappa_0), (\lambda_0)], \\ 2i_3 &= N_1[(\kappa_0), (\lambda_0)] - N_2[(\kappa_0), (\lambda_0)], \quad q = N_1[(\kappa_0), (\lambda_0)], \end{aligned} \quad (6.4)$$

for, under the transformation (5.2) generated by S , the above tensor is multiplied by the scalar factor $\exp(-\sqrt{-1} \alpha N_3[(\kappa_0), (\lambda_0)])$, which implies that the tensor is an eigenvector of S with the eigenvalue $-N_3[(\kappa_0), (\lambda_0)]$. This reasoning can also be applied to N_B , I_3 and Q .

From (6.4) we have, by remembering the definition of s_0 and i_0 in § 4,

$$\begin{aligned} s_0 &= \max(-N_3[(\kappa), (\lambda)]), \\ i_0 &= (1/2) \max_{N_3 = -s_0} (N_1[(\kappa), (\lambda)] - N_2[(\kappa), (\lambda)]). \end{aligned} \quad (6.5)$$

The first maximum is to be taken with respect to all non-vanishing components $T_{(\lambda)}^{(\kappa)}$'s, and the second to all such components $T_{(\lambda)}^{(\kappa)}$'s as $N_3[(\kappa), (\lambda)] = -s_0$. Thus we have determined the quantum numbers s_0 and i_0 associated with the irreducible invariant subspace, or with the corresponding irreducible representation. The values (s_0, i_0) are actually taken by a certain tensor, which is nothing but $v_{s_0 i_0}$ introduced in § 4. It can be obtained by putting to zero all the components corresponding to $(s, i) \neq (s_0, i_0)$. The uniqueness of $v_{s_0 i_0}$ is obvious from the results in § 4.

It is obvious from the above consideration that we may take the following step after the processes (1) and (2):

(3) A practical method of obtaining (n_B, s_0, i_0) and $v_{s_0 i_0}$ for the irreducible representations resulting from $T_{(0)}$.

Case i) For a fixed l_1 (see (2)), make the diagrams

1	2		k
---	---	--	-----

for superscripts,

$1'$	$2'$			l_1'
$(l_1+1)'$	$(l_1+2)'$			

for subscripts,

where $(1', 2', \dots, l')$ is such a permutation of $(1, 2, \dots, l)$ that the number increases from left to right in each row and from top to bottom in each column. A non-vanishing component of $v_{s_0 s_0}$ is $T_{\lambda_1 \dots \lambda_l}^{x_1 \dots x_l}$ with

$$\kappa_a = 1, \quad a = 1, 2, \dots, k,$$

$$\lambda_a=3, \quad b=1', 2', \dots, l_1'; \quad \lambda_b=2, \quad b=(l_1+1)', \dots, l',$$

and the other non-vanishing components are obtained from this by performing alternations of the subscripts λ 's belonging to the same columns. n_B , s_0 and i_0 are given by

$$n_R = k - l, \quad s_0 = l_1, \quad i_0 = (1/2) (k + l - l_1).$$

The representations corresponding to different $(1', 2', \dots, l')$ are equivalent to each other and their number is⁸⁾

$$\frac{l! (2l_1 - l + 1)}{(l_1 + 1)! (l - l_1)!}.$$

Case ii) We can treat this case in the same way as i). For a fixed k_1 the diagrams are

$1'$	$2'$			k_1'
$(k_1+1)'$	$(k_1+2)'$			

for superscripts,

1	2		l
---	---	--	-----

for subscripts.

The indices of a non-vanishing component of $v_{s_0 t_0}$ are given by

$$\kappa_a=1, \quad a=1', 2', \dots, k_1'; \quad \kappa_a=2, \quad a=(k_1+1)', \dots, k',$$

$$\lambda_b=3, \quad b=1, 2, \dots, l.$$

n_B , s_0 , i_0 and the number of equivalent representations are

$$n_B = k - l, s_0 = l, i_0 = (1/2)(2k_1 - k), \frac{k! (2k_1 - k + 1)}{(k_1 + 1)! (k - k_1)!}$$

respectively.

We next describe a method of expressing the tensors given by (6.3) in terms of $p, n, A, \bar{p}, \bar{n}$ and \bar{A} . The basic particles χ_i^α or anti-particles $\bar{\chi}_i^\alpha$ form a basis in the vector space V_0^1 or V_1^0 respectively. Therefore, any tensor $T_{\lambda_1 \dots \lambda_l}^{\kappa_1 \dots \kappa_k}$ is expressible in the form

$$T_{\lambda_1 \dots \lambda_l}^{\kappa_1 \dots \kappa_k} = \sum_{j_1, \dots, j_l} C_{j_1 \dots j_l}^{i_1 \dots i_k} \chi_{\lambda_1}^{j_1} \dots \chi_{\lambda_l}^{j_l} \chi_{i_1}^{\kappa_1} \dots \chi_{i_k}^{\kappa_k}. \quad (6.6)$$

In this expression, replace χ_i^{κ} 's and χ_{λ}^j 's by p, n, \dots and \bar{A} [(2.1), (2.3)] without changing their order, and we get a sum of ordered products of k particles and l anti-particles. For example,

$$\partial_{\lambda}^{\kappa} = \chi_{\lambda}^{\kappa} \chi^{\kappa} = \bar{p}p + \bar{n}n + \bar{A}A. \quad (6.7)$$

Hereafter we denote such a sum of ordered products by $P_l^k(p, n, A, \bar{p}, \bar{n}, \bar{A})$. We remark that one must not change the order of the products at random. Particular attention should be paid to the expression of tensors involving $\partial_{\lambda}^{\kappa}$, such as the second and the following terms in (6.1). For instance, $\partial_{\lambda_b}^{\kappa_a} T_{\lambda_1}^{\kappa_1} \dots \hat{\chi}_{\lambda_b}^{\kappa_b} \dots \chi_{\lambda_l}^{\kappa_l}$ may be ex-

pressed as the product of (6.7) and the expression $P_{l-1}^{k-1}(p, n, \dots, \bar{A})$ of $T_{\lambda_1}^{\kappa_1} \dots \hat{\chi}_{\lambda_b}^{\kappa_b} \dots \chi_{\lambda_l}^{\kappa_l}$.

In this case, p, n and A in the expression $P_1^1(p, n, \dots, \bar{A})$ of $\partial_{\lambda_b}^{\kappa_a}$ are to be written at the a -th place of the particles, and \bar{p}, \bar{n} and \bar{A} at the b -th place of the anti-particles. Thus it is now easy to express $v_{s_0 i_0}$ in terms of p, n, \dots and \bar{A} :

$$v_{s_0 i_0} = P_l^k(p, n, \dots, \bar{A}). \quad (6.8)$$

In Table V we give the non-vanishing components of $v_{s_0 i_0}$, the expression (6.8), the values of (n_B, s_0, i_0) , m and m' ; and the degree d of the representations.

For the expression of tensors other than $v_{s_0 i_0}$, the above procedure is also useful but the following is simpler and more convenient for the concrete applications: Let O_a be the infinitesimal operators of $U(3)$ (see § 3). Their effect on the basic particles and anti-particles is given in Table I. As for the ordered products of p, n, \dots and \bar{A} or their sum the following rule is valid: If P and P' are any ordered products, we have

$$\begin{aligned} O_a(aP + bP') &= aO_a(P) + bO_a(P'), \quad a, b = \text{const.}, \\ O_a(PP') &= O_a(P)P' + PO_a(P'). \end{aligned} \quad (6.9)$$

This can be easily seen by performing the infinitesimal transformation corresponding to O_a on $aP + bP'$ and PP' . The rule (6.9) cannot be applied to \bar{N}_{31} , which is not an infinitesimal operator of the group $U(3)$. For this operation the definition (4.7) should be remembered and the above rule must be applied to I_3, I_-, N_{31} and N_{32} separately. Thus, operating N_{32}, \bar{N}_{31} and I_- on (6.8) repeatedly, we have all the tensors (4.11) in the form $P_l^k(p, n, \dots, \bar{A})$. These are nothing but the tensors given by (6.3), which we see from the corresponding values of i, i_3 and s .

Finally, we touch on a property of the transformation

$$P_l^k(p, n, A, \bar{p}, \bar{n}, \bar{A}) \rightarrow P_l^k(\bar{p}, \bar{n}, \bar{A}, p, n, A).$$

Under this transformation, basis vectors of an irreducible representation become those of another irreducible representation. The corresponding changes of n_B, s_0 ,

q_0 , m and m' are easily obtained by means of (3), i.e.

$$n_B \rightarrow -n_B, \quad s_0 \rightarrow q_0, \quad q_0 \rightarrow s_0, \quad m \rightarrow m, \quad m' \rightarrow -m'.$$

§ 7. Summary for applications

We shall summarize the above results for the convenience of concrete applications.

(1) Physical quantities associated with the unitary group $U(3)$ are

I (isospin), N_B (baryon number),

S (strangeness), Q (charge),

N_{ij} (characteristic of our theory; $ij=13, 23, 31, 32$).

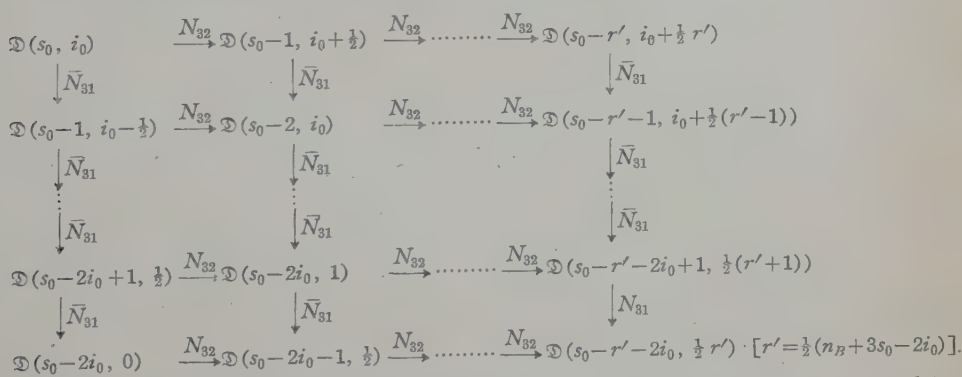
They are not linearly independent but subject to the well-known relation (3.5). I_{\pm} and N_{ij} are the operators transforming the basic particles into each other (cf. Tables I and III). The commutation relations between the quantities are given in Table II.

(2) An irreducible representation may be specified by the following three numbers:

n_B (baryon number), s_0 (the largest eigenvalue of S),

i_0 (the largest isospin of the states with $S=s_0$).

Table IV. Decomposition of $\mathfrak{D}(n_B, s_0, i_0)$ into irreducible representations of isospin



This table tells how the irreducible representation of $U(3)$, $\mathfrak{D}(n_B, s_0, i_0)$, is decomposed into irreducible representations of isospin. $\mathfrak{D}(s, i)$ stands for an irreducible representation of isospin, s and i being the associated eigenvalues of S and I respectively. An arrow $\rightarrow [\downarrow]$ means that the vector corresponding to the maximum eigenvalue of I_3 in $\mathfrak{D}(s, i)$ is transformed into that vector in $\mathfrak{D}(s-1, i+\frac{1}{2})$ [$\mathfrak{D}(s-1, i-\frac{1}{2})$] by the operator N_{32} [\bar{N}_{31}].

These must satisfy the conditions (i), (ii) and (iii) in § 5. We denote this representation by $\mathfrak{D}(n_B, s_0, i_0)$. Table IV tells us how $\mathfrak{D}(n_B, s_0, i_0)$ is decomposed into irreducible representations of isospin. The degree of $\mathfrak{D}(n_B, s_0, i_0)$ is given by (5.14). There exist three quantities N_B , M and M' that commute with all quan-

tities given in (1). M and M' are defined by (3.9) and their eigenvalues may be calculated by means of (5.13). It is possible to specify an irreducible representation by (n_B, m, m') in place of (n_B, s_0, i_0) .

(3) The decomposition of the composite system of k basic particles and l anti-particles is reducible to the decomposition of the tensor space V_l^k (the totality of tensors of valence k, l). The latter can be performed through two steps: the "contraction operation" and Young's symmetrization. They have been explained in detail in (1) and (2) of § 6. It is easy to assign (n_B, s_0, i_0) to an irreducible constituent thus obtained and to find the eigenvector $v_{s_0 i_0}$ with the eigenvalues $S=s_0$ and $I_3=i_0$. A method is given in (3) of § 6. (cf. Table V). The basis vectors (4.11) can be constructed from this $v_{s_0 i_0}$ according to Table I. In this case Table IV will be helpful. The concrete expressions of the vectors have already been published.^{1), 5)}

Table V. The non-vanishing components and the expression of $v_{s_0 i_0}$, the values of (n_B, s_0, i_0) , m and m' , and the degree d of the representations

		$v_{s_0 i_0}$	n_B, s_0, i_0	m	m'	d
T^e	T^1	\bar{p}	$1, 0, \frac{1}{2}$	$3/2$	$5/2$	3
T_λ	T_3	\bar{A}	$-1, 1, 0$	$3/2$	$-5/2$	3
$T_\lambda^{(0)}$	T_3^1	$\bar{A}\bar{p}$	$0, 1, \frac{1}{2}$	3	0	8
$T_{(\lambda_1 \lambda_2)}^{(0)}$	T_{33}^{11}	$\bar{A}\bar{A}\bar{p}p$	$0, 2, 1$	8	0	27
$T_{[\lambda_1 \lambda_2]}^{(0)}$	$T_{23}^{11} = -T_{32}^{11}$	$[\bar{n}, \bar{A}]p\bar{p}/\sqrt{2}$	$0, 1, 3/2$	6	9	10
$T_{[\lambda_1 \lambda_2]}^{(0)}$	$T_{33}^{12} = -T_{33}^{21}$	$\bar{A}\bar{A}[\bar{p}, n]/\sqrt{2}$	$0, 2, 0$	6	-9	10
$T_\lambda^{(0)}$	T_3^{11}	$\bar{A}p\bar{p}$	$1, 1, 1$	$11/2$	$17/2$	15
$T_\lambda^{(0)}$	$T_3^{12} = -T_3^{21}$	$\bar{A}[\bar{p}, n]/\sqrt{2}$	$1, 1, 0$	$7/2$	$-\frac{1}{2}$	6
$T_{(\lambda_1 \lambda_2)}^{(0)}$	T_{33}^{111}	$\bar{A}\bar{A}\bar{p}p\bar{p}p$	$1, 2, 3/2$	$23/2$	$35/2$	42
$T_{[\lambda_1 \lambda_2]}^{(0)}$	$T_{32}^{111} = -T_{32}^{111}$	$[\bar{n}, \bar{A}]p\bar{p}p/\sqrt{2}$	$1, 1, 2$	$19/2$	$53/2$	15
$T_{(\lambda_1 \lambda_2)}^{(0)}$	$T_{33}^{121} = -T_{33}^{211}$	$\bar{A}\bar{A}[\bar{p}, n]p/\sqrt{2}$	$1, 2, \frac{1}{2}$	$17/2$	$-\frac{1}{2}$	24
$T_{(\lambda_1 \lambda_2)}^{(0)}$	$T_{33}^{112} = -T_{33}^{211}$	$\bar{A}\bar{A}(p\bar{p}n - n\bar{p}p)/\sqrt{2}$	$1, 2, \frac{1}{2}$	$17/2$	$-\frac{1}{2}$	24

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Appendix

Proof that the vectors (4.11) span an irreducible invariant subspace

In § 4 we constructed the linearly independent vectors (4.11) in the representation space. In this Appendix we shall show that they form a basis in an irreducible invariant subspace.

For the sake of simplicity, we denote the vectors as follows:

$$\begin{aligned} v_{utr} &\stackrel{\text{def}}{=} (I_-)^u (\bar{N}_{31})^t (N_{32})^r v_{s_0 t_0}, \\ r &= 0, 1, \dots, r' \quad (= (1/2)(n_B + 3s_0 - 2i_0)), \\ t &= 0, 1, \dots, t'' \quad (= 2i_0), \\ u &= 0, 1, \dots, u''' \quad (= 2i_0 + r - t). \end{aligned} \quad (\text{A} \cdot 1)$$

Since we have

$$N_{32} v_{otr} = \bar{N}_{31} v_{ot'r} = I_- v_{u't'r} = 0,$$

it is possible to extend the domain of the variables r , t and u in (A.1) such that

$$v_{utr} = 0, \quad \begin{cases} r > r', \quad t \leq t'', \quad u \geq 0; \\ r \leq r', \quad t > t'', \quad u \geq 0; \\ r \leq r', \quad t \leq t'', \quad u > u'''. \end{cases} \quad (\text{A} \cdot 2)$$

As (A.1) is obviously closed under the operations of N_B , S , I_3 and I_{\pm} , we have merely to examine the effect of the operation of N_{31} on (A.1).

If we operate on v_{otr} the third expression of \bar{N}_{31} in (4.7), we have

$$v_{o,\ell+1,r} = \rho N_{31} v_{otr} + I_- v_{o,\ell,r+1}, \quad \rho = 2i_0 + r - t + 1 \neq 0.$$

Further operating $(I_-)^u$ on this and taking account of (A.2), we get

$$N_{31} v_{utr} = \rho^{-1} (v_{u,\ell+1,r} - v_{u+1,\ell,r+1}) = L(v_{utr}), \quad (\text{A} \cdot 3)$$

where $L(v_{utr})$ stands for a linear combination of the vectors in (A.1).

Next we have

$$[N_{32}, (I_-)^u] = u N_{31} (I_-)^{u-1}.$$

If we operate this on v_{otr} , we obtain by virtue of (A.3)

$$N_{32} v_{utr} = v_{u,\ell,r+1} + u N_{31} v_{u-1,\ell,r} = L(v_{utr}). \quad (\text{A} \cdot 4)$$

As for N_{23} , we first consider the case $r=u=0$, in which

$$N_{23} v_{ot_0} = L(v_{utr}) \quad (\text{A} \cdot 5)$$

holds good. This is trivial for $t=0$, because $N_{23}v_{000}=0$; assuming it for some t , we have

$$\begin{aligned} N_{23}v_{0,t+1,0} &= N_{23}\bar{N}_{31}v_{0t0} = \{(\bar{N}_{31}+N_{31})N_{23} + (Q+S+2)I_-\}v_{0t0} \\ &= \{(2N_{31}I_3 + N_{32}I_-) + N_{31}\}L(v_{utr}) + (Q+S+2)v_{1t0} = L(v_{utr}), \end{aligned}$$

where (A.3) and (A.4) have been used in the last equality. In the case $u=0$, the assumption

$$N_{23}v_{0tr} = L(v_{utr})$$

leads to

$$N_{23}v_{0,t,r+1} = N_{23}N_{32}v_{0tr} = (N_{32}N_{23} + Q + S - 2I_3)v_{0tr} = L(v_{utr}).$$

Further, from

$$N_{23}v_{utr} = L(v_{utr}), \quad (\text{A.6})$$

it follows that

$$N_{23}v_{u-1,t,r} = I_- N_{23}v_{utr} = L(v_{utr}).$$

Thus by the inductive method (A.6) can be proved for all u , t and r .

Lastly we consider N_{13} . From

$$N_{13}v_{00r} = L(v_{utr}), \quad (\text{A.7})$$

we get

$$N_{13}v_{0,0,r+1} = (N_{32}N_{13} + I_+)v_{00r} = L(v_{utr}),$$

hence (A.7) is valid in general. (For $r=0$, $N_{13}v_{000}=0$.) Next, the assumption

$$N_{13}v_{0tr} = L(v_{utr})$$

results in

$$N_{13}v_{0,t+1,r} = \{(\bar{N}_{31} - N_{13})N_{13} - N_{32}N_{13} + 2(Q+S)I_3 + I_+I_-\}v_{0tr} = L(v_{utr}),$$

and from

$$N_{13}v_{utr} = L(v_{utr}) \quad (\text{A.8})$$

it follows that

$$N_{13}v_{u+1,t,r} = (I_- N_{13} - N_{23})v_{utr} = L(v_{utr}).$$

Thus we have (A.8) in general.

It follows from the above that the vectors (A.1) generate an invariant subspace of the representation space. The irreducibility of the subspace is known from the construction of (A.1) in § 4 and the first equation of (A.3).

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Bound States in Four-Nucleon Coupling

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An interaction Lagrangian of general four-nucleon coupling is assumed and equations for nucleon-antinucleon and two-nucleon systems in three types of chain approximation are derived from the interaction and solved exactly. Then the mathematical structure of the solutions is studied in various cases, and the regions where the value of the coupling constant must lie in order to give the bound states are obtained in the case of pseudoscalar coupling. Some interesting features are pointed out.

§ 1. Introduction and summary

Recently, composite particle models have been presented and studied by many authors and the strong and weak points of the models have gradually become clear. Aims of the models are to derive elementary particles, which have been found by experiment, from as fewer particles as possible and to seize a multiplicity of interactions between elementary particles in a unified way. Because of difficulty in mathematical treatment of the models, studies on this problem are mainly devoted to make clear the qualitative characters of the interactions from the standpoint of the models. In regard to Sakata's model,¹⁾ especially, energetic investigations were performed by the Nagoya group. The model seems to have no qualitative difficulty except for Σ^+ decay ($\rightarrow n + e^+ + \nu$).²⁾ As the experiment of Σ^+ may, however, be unreliable, the validity of the model seems to be sure enough.

On the other hand, there is an interesting work of Maki³⁾ on the mathematical treatment of Sakata's model which assumes a strong four-fermi coupling on baryons (i.e. N and \bar{N}) and constructs π - and K -mesons as bound states of $(N\bar{N})$ and $(\bar{N}N)$, respectively, through this interaction. His results are very satisfactory and it seems that all the physically interesting points are worked out in his article. From the mathematical standpoint, however, his interaction type (pseudoscalar coupling) is not the only one which satisfies the necessary conditions. Therefore it is not meaningless to study other possibilities. The necessary conditions which we call here are as follows:

- i) A nucleon and an antinucleon make a π -meson as a bound state and do not make any other bound state through the interaction.
- ii) Two nucleons make no bound state with small extension.
- iii) Usual π - N coupling constant is deduced from the theory.

Beside the above conditions, we may require that a nucleon and an antinucleon make another bound state called a $\pi^{0'}$ -meson which is charge singlet and pseudoscalar. The last requirement is, however, not of serious meaning unless we take account of Λ -particle because a $\pi^{0'}$ -meson might contain a Λ -particle⁴⁾ and even its existence might not be sure.

We start with an interaction Lagrangian of general four-nucleon coupling. Though the interaction

$$-L' = \sum_A \{g_A^S : (\bar{\psi} O^A \psi) (\bar{\psi} O^A \psi) : + g_A^T \sum_{i=1}^3 : (\bar{\psi} \tau_i O^A \psi) (\bar{\psi} \tau_i O^A \psi) : \} \quad (1.1)$$

seems to be the most general one which satisfies charge independence, g_A^S 's and g_A^T 's are related to each other through Fierz's identity.⁵⁾ Therefore the most general four-nucleon coupling results in the following forms:*

$$-L' = \sum_A g_A^S : (\bar{\psi} O^A \psi) (\bar{\psi} O^A \psi) :, \quad (1.2a)$$

$$-L' = \sum_A g_A^T \sum_{i=1}^3 : (\bar{\psi} \tau_i O^A \psi) (\bar{\psi} \tau_i O^A \psi) :, \quad (1.2b)$$

$$\text{or} \quad -L' = \sum_A g_A^K \sum_{L=1}^4 : (\bar{\psi} \tau_L O^A \psi) (\bar{\psi} \tau_L O^A \psi) :, \quad (1.2c)$$

where O^A is a matrix operator, "A" standing for scalar (S), vector (V), tensor (T), axial vector (A) or pseudoscalar (P). We see, therefore, that there are five independent coupling constants. Adding the cutoff parameter, there are six independent parameters. What we must do is to determine the values of these parameters, but the above conditions are too lax for determining them. They are to be determined in future by adding other conditions (for instance, we require that K - and $\pi^{0'}$ -mesons are composed of nucleons and Λ -particles through the N - N and N - Λ couplings of the same type, whose coupling constants are related to each other by I - O - O symmetry⁴⁾).

As the model contains many parameters in this manner, the parameters cannot be determined at present. It has therefore no serious meaning to determine the values of coupling constants by taking up only a special type. What we must do at present might be to examine the properties and possibilities of the model in various coupling types so as to be of use in the future time when the conditions enough to determine the parameters appear altogether.

Among many ways to study the mathematical structure of the model, it is desirable to have such a way that brings the characteristics of the model to light well. Though we can derive the two-body equation from the general interaction (1.2) and can naturally solve it, the problem will become unintelligible in this case because of the complexity. Therefore we assume the following interaction Lagrangian:

* The author is indebted to Dr. Z. Maki for pointing out this form.

$$-L' = g_A^S : (\bar{\psi} O^A \psi) (\bar{\psi} O^A \psi) : + g_A^T \sum_{i=1}^3 : (\bar{\psi} \tau_i O^A \psi) (\bar{\psi} \tau_i O^A \psi) :, \quad (1.3)$$

where summation over A is not taken. As g_A^S and g_A^T are independent of each other in this case, there are three adjustable parameters. The Lagrangian (1.3) corresponds to the special case of (1.1) where all other coupling constants than g_A^S and g_A^T are equal to zero. Three cases in (1.3), i.e. $g_A^T=0$, $g_A^S=0$ and $g_A^S=g_A^T$ correspond to the special cases of (1.2a) (1.2b) and (1.2c) respectively.

From the above interaction (1.3) we derive the equations for the nucleon-antinucleon system and two-nucleon system in general chain approximation. Then we see that there are two types of chain approximation for the nucleon-antinucleon system. We call here the two types "the first and second approximations" respectively, and the mixture of them "the general one". We solve the equations exactly and calculate the coupling constants of pseudoscalar type ($O^A=\gamma_5$) as an example, for two cases where the total energy of the system (i.e. the mass of bound state) is equal to a two-nucleon mass and to a vanishing mass. The calculated coupling constants give the boundary values of the region where the value of the coupling constant must lie in order to give the bound state.

We studied the mathematical structure of the model in such a way as mentioned above and obtained the following results.

(1) Any type of coupling satisfies the above conditions provided the coupling constants are suitably chosen, even if unwilling bound states are existent for the smaller coupling constant. For pseudoscalar type, the value of the coupling constant is the smallest and there is no other bound state below. This situation, however, is not necessary for the above condition.

(2) In the first chain approximation, the allowed bound state is the only one which has the same assignment as the coupling type, that is, if the coupling is of charge triplet pseudoscalar type, for example, only the charge triplet pseudoscalar state is allowed.

(3) In the second chain approximation, various states are allowed for the only one coupling type. This fact is a matter of course because this case is reduced to the first chain approximation with many coupling types by making use of Fierz's formulae.⁵⁾ We may notice further the following facts, though they are self-evident: If $g_A^T=0$, charge singlet and triplet states are degenerate, and if $g_A^S=0$, there is a relation $g_{A1}^T = -3g_{A0}^T$ where g_{A0}^T and g_{A1}^T are the coupling constants which are determined for charge singlet and triplet states, respectively, to have the same binding energy. If $g_A^S=g_A^T$, no charge triplet state is allowed.

(4) In the general chain approximation, the first and second ones compete with each other.

(5) The equation for two-nucleon system is reduced to the equation for nucleon-antinucleon system in the second chain approximation by suitable replacement. The coupling constants in this case are obtainable from those for nucleon-antinucleon system in the second one. The similar argument as paragraph (3) holds in this

case.

(6) There is a fact which is contrary to our ordinary sense. That is, the larger the absolute value of the coupling constant grows, the smaller the binding energy becomes for some bound states. This fact might be regarded as an effect of the relativity.

(7) When the coupling constant exceeds the boundary value of the region which we mentioned above, the binding energy of the system overcomes the masses of two particles and the square of the total energy becomes negative, i.e. $E^2 < 0$. This might be regarded as a "ghost" in a sense. The ghost disappears if we require that the wave function vanishes in the distance.

§ 2. Formulation

We rewrite our interaction Lagrangian (1.3) as

$$-L' = (g_A^L/4) : (\bar{\psi} \tau_L O^A \psi) (\bar{\psi} \tau_L O^A \psi) :, \quad (2.1)$$

where ψ is nucleon operator, $\tau_L (L=0, 1, 2, 3)$ isotopic spin operator of nucleon, $\tau_0=1$, $g_A^0 \equiv g_A^S$, $g_A^1 = g_A^2 = g_A^3 = g_A^T$, O^A γ -matrices and A denotes scalar (S), vector (V), tensor (T), pseudovector (A) or pseudoscalar (P). Then we have

$$(\gamma_\mu \partial_\mu + \kappa) \psi(x) = - (g_A^L/2) \tau_L O^A : \psi(x) (\bar{\psi}(x) \tau_L O^A \psi(x)) :, \quad (2.2)$$

or

$$\psi(x) = \psi^0(x) - i(g_A^L/2) \int S_F(x-x') \tau_L O^A : \psi(x') (\bar{\psi}(x') \tau_L O^A \psi(x')) : (dx'), \quad (2.3)$$

where κ is the nucleon mass, and ψ^0 and $S_F(x)$ are the operator and propagator of the free nucleon and satisfy the following equations respectively,

$$(\gamma_\mu \partial_\mu + \kappa) \psi^0 = 0 \quad (2.4)$$

$$(\gamma_\mu \partial_\mu + \kappa) S_F(x) = -i\delta(x), \quad S_F(x) = \frac{i}{(2\pi)^4} \int \frac{i\gamma_\mu p_\mu - \kappa}{p^2 + \kappa^2 - i\epsilon} e^{ipx} (dp). \quad (2.5)$$

i) Equation for nucleon-antinucleon system in a bound state

We define the wave function of a nucleon-antinucleon system as

$$\phi(1, 2) \equiv \langle 0 | T[\psi(x_1) \bar{\psi}(x_2)] | N\bar{N} \rangle, \quad (2.6)$$

and operating $\bar{\psi}(x)$ on both sides of Eq. (2.3) from right and placing it between $\langle 0 |$ and $|N\bar{N}\rangle$ after taking the T -product, we have

$$\begin{aligned} \phi(1, 2) &= -i(g_A^L/2) \int S_F(x_1-x_1') \tau_L O^A \\ &\quad \times \langle 0 | T[: \psi(x_1') (\bar{\psi}(x_1') \tau_L O^A \psi(x_1')) : \bar{\psi}(x_2)] | N\bar{N} \rangle (dx_1'). \end{aligned} \quad (2.7)$$

As we are dealing with bound state, it is assumed here that free part of the wave function does not exist. Using Wick's theorem, the T -product of the integrand

in Eq. (2.7) is rewritten as

$$\begin{aligned} & \langle 0 | T [: \phi(x_1') (\bar{\psi}(x_1') \tau_L O^A \phi(x_1')) : \bar{\psi}(x_2)] | N \bar{N} \rangle \\ &= \langle 0 | N [\phi(x_1') (\bar{\psi}(x_1') \tau_L O^A \phi(x_1')) \bar{\psi}(x_2)] | N \bar{N} \rangle \\ & \quad - S_F'(x_1' - x_2) \text{Tr}(\tau_L O^A \phi(1', 1')) + \phi(1', 1') \tau_L O^A S_F'(x_1' - x_2). \end{aligned} \quad (2.8)$$

If we take the Tamm-Dancoff approximation, the first term of Eq. (2.8) is neglected and $S_F'(x)$ becomes $S_F(x)$. Then we have

$$\begin{aligned} \phi(1, 2) &= i(g_A^L/2) \int S_F(x_1 - x_1') \tau_L O^A S_F(x_1' - x_2) \text{Tr}(\tau_L O^A \phi(1', 1')) (dx_1') \\ & \quad - i(g_A^L/2) \int S_F(x_1 - x_1') \tau_L O^A \phi(1', 1') \tau_L O^A S_F(x_1' - x_2) (dx_1'). \end{aligned} \quad (2.9)$$

As the second and third terms of Eq. (2.9) have quite different character, we take them up separately in two equations:

$$\begin{aligned} \phi(1, 2) &= i(g_A^L/2) \int S_F(x_1 - x_1') \tau_L O^A S_F(x_1' - x_2) \\ & \quad \times \text{Tr}(\tau_L O^A \phi(1', 1')) (dx_1'), \end{aligned} \quad (2.10)$$

$$\begin{aligned} \phi(1, 2) &= -i(g_A^L/2) \int S_F(x_1 - x_1') \tau_L O^A \phi(1', 1') \\ & \quad \times \tau_L O^A S_F(x_1' - x_2) (dx_1'). \end{aligned} \quad (2.11)$$



Fig. 1. Feynman diagram of nucleon-antinucleon system in chain approximation.
(a) first type
(b) second type



Fig. 2. Feynman diagram of two-nucleon system in chain approximation.

Equation (2.10) is the same formula as that has been obtained by Maki if we put $g_A^S = g_A^T$. We shall call Eqs. (2.10), (2.11) and (2.9) "the first, second and general types of chain approximation". The name "chain approximation" is in accordance with Maki's. In order to understand the equations intuitively, we show them in Feynman diagrams (Fig. 1). Eqs. (2.10) and (2.11) correspond to (a) and (b) of Fig. 1 respectively and Eq. (2.9) is the mixture of (a) and (b).

ii) Equation for 2 nucleons in bound state

The equation for the 2-nucleon system is obtained in the same manner as that for the nucleon-antinucleon system. Defining the wave function of 2-nucleon system as

$$\Psi(1, 2) \equiv \langle 0 | T[\phi(x_1)\phi(x_2)] | 2N \rangle, \quad (2.12)$$

and considering

$$\begin{aligned} & \langle 0 | T[: \phi(x_1') (\bar{\psi}(x_1') \tau_L O^A \phi(x_1')) : \phi(x_2)] | 2N \rangle \\ &= \langle 0 | N[\psi(x_1') (\bar{\psi}(x_1') \tau_L O^A \psi(x_1')) \phi(x_2)] | 2N \rangle \\ & \quad + S_F(x_2 - x_1') \tau_L O^A \Psi(1', 1'), \end{aligned} \quad (2.13)$$

we see

$$\Psi(1, 2) = -i(g_A^L/2) \int S_F^{(1)}(x_1 - x_1') S_F^{(2)}(x_2 - x_1') (\tau_L O^A)^{(1)} (\tau_L O^A)^{(2)} \Psi(1', 1') (dx_1'). \quad (2.14)$$

Here we take the Tamm-Dancoff approximation as in last paragraph i). It must be noticed that there exists only one equation in this case and it is different from the case of $N\bar{N}$ system as being convinced from the Feynman diagram (Fig. 2).

Hereafter in § 3 and § 4 we shall solve these equations, (2.10) (2.11) (2.9) and (2.14), in order. Since all these equations, however, involve a divergent integral, we make use of the cutoff method in order to avoid this divergence. That is, we use cutoff propagator $S_F^c(x)$ as a substitute for usual $S_F(x)$.

$$\begin{aligned} S_F^c(x) &= i(2\pi)^{-4} \int (i\gamma_\mu p_\mu - \kappa) \{ [p^2 + \kappa^2 - i\varepsilon]^{-1} - [p^2 + \lambda^2 - i\varepsilon]^{-1} \} e^{ipx} (dp) \\ &= \frac{i}{(2\pi)^4} \int \frac{i\gamma_\mu p_\mu - \kappa}{p^2 + \kappa^2 - i\varepsilon} \cdot \frac{\lambda^2 - \kappa^2}{p^2 + \lambda^2 - i\varepsilon} e^{ipx} (dp). \end{aligned} \quad (2.15)$$

§ 3. Solution for nucleon-antinucleon system in bound state

i) First type of chain approximation

The equation for nucleon-antinucleon system in the first type of chain approximation (2.10) have already been solved by Maki when $g_A^S = g_A^V$ and $O^A = \gamma_5$, but it is necessary to solve it briefly for comparison with solutions of the other equations (2.9) (2.11) and (2.14) and for comprehension of this approximation. First of all, we separate the coordinate of centre of mass $X = (x_1 + x_2)/2$ from Eq. (2.10) and reduce it to the equation which contains the relative coordinate $x = x_1 - x_2$ only. That is, if we put

$$\phi(1, 2) = \phi(x) \exp(iPX), \quad (3.1)$$

then we have the equation for $\phi(x)$,

$$\begin{aligned} \phi(x) = & -(i/2) (2\pi)^{-4} (\lambda^2 - \kappa^2)^2 g_A^L F_x \left[\{i\gamma(p+P/2) - \kappa\} \tau_L O^A \right. \\ & \left. \times \{i\gamma(p-P/2) - \kappa\} \text{Tr}(\tau_L O^A \phi(0)) \right], \end{aligned} \quad (3.2)$$

where

$$\begin{aligned} F_x[u(p)] = & \int (dp) \exp(ipx) u(p) [(p+P/2)^2 + \kappa^2 - i\varepsilon]^{-1} [(p-P/2)^2 + \kappa^2 - i\varepsilon]^{-1} \\ & \times [(p+P/2)^2 + \lambda^2 - i\varepsilon]^{-1} [(p-P/2)^2 + \lambda^2 - i\varepsilon]^{-1} \end{aligned} \quad (3.3)$$

and $P=p_1-p_2$ and $p=(p_1+p_2)/2$ are total and relative energy momentum. We take hereafter the centre of mass system $P=(\mathbf{0}, iE)$. As $\phi(x)$ generally has 64 independent components, we describe them in the following form,

$$\phi(x) = (\phi_L^S(x) + \gamma_\mu \phi_{\mu L}^V(x) + \sigma_i \phi_{iL}^T(x) + \tilde{\sigma}_i \tilde{\phi}_{iL}^T + \gamma_5 \gamma_\mu \phi_{\mu L}^A(x) + \gamma_5 \phi_L^P(x)) \tau_L, \quad (3.4)$$

where $i(=1, 2, 3)$, $L(=0, 1, 2, 3)$ and $\mu(=1, 2, 3, 4)$ are dummy indices. We shall denote these 64 $\phi_L^S, \phi_{\mu L}^V, \dots$, as ϕ^a *en bloc* and corresponding matrices $1\tau_L, \gamma_\mu \tau_L, \dots$ as Γ^a .

Now $\text{Tr}(\tau_L O^A \Gamma^a \phi^a)$ is non-vanishing only if $\Gamma^a = \pm \tau_L O^A$; for instance, when $O^A = \gamma_5$,

$$\text{Tr}(\tau_L \gamma_5 \phi(0)) = 8\phi_L^P(0). \quad (3.5)$$

Therefore,

$$\begin{aligned} \phi(x) = & -i4(2\pi)^{-4} (\lambda^2 - \kappa^2)^2 g_L^L \tau_L \phi_L^P(0) F_x [\{i\gamma(p+P/2) - \kappa\} \gamma_5 \\ & \times \{i\gamma(p-P/2) - \kappa\}]. \end{aligned} \quad (3.6)$$

Considering

$$\begin{aligned} \{i\gamma(p+P/2) - \kappa\} \gamma_5 \{i\gamma(p-P/2) - \kappa\} = & \gamma_5 \{\mathbf{p}^2 + (p_4^2 - P_4^2/4) + \kappa^2\} \\ & + i\boldsymbol{\sigma} \cdot \mathbf{p} P_4 - i\gamma_5 \gamma_4 P_4 \kappa, \end{aligned} \quad (3.7)$$

we see

$$\phi_L^S = \phi_{\mu L}^V = \tilde{\phi}_{iL}^T = \phi_{iL}^A = 0. \quad (3.8)$$

Consequently the solution of Eq. (3.2) is expressed as

$$\phi(x) = (\boldsymbol{\sigma} \cdot \boldsymbol{\phi}_L^T(x) + \gamma_5 \gamma_4 \phi_{4L}^A(x) + \gamma_5 \phi_L^P(x)) \tau_L, \quad (3.9)$$

where $\boldsymbol{\phi}_L^T, \phi_{4L}^A$ and ϕ_L^P are given by the following equations ($\epsilon = E/2$),

$$\begin{aligned}
\phi_L^P(x) &= -i4(2\pi)^{-4}(\lambda^2 - \kappa^2)^2 g_L^P \phi_L^P(0) F_x[\mathbf{p}^2 - p_0^2 + \epsilon^2 + \kappa^2], \\
\phi_L^T(x) &= \quad \quad \quad F_x[-2\mathbf{p}\epsilon], \\
\phi_{4L}^A(x) &= \quad \quad \quad F_x[2\epsilon\kappa].
\end{aligned} \tag{3.10}$$

The solution (3.9) represents charge triplet, pseudo-scalar and spin 0 state (briefly hereafter $T, -, 0$) when $g_A^S=0$ which corresponds to π -meson, and ($S, -, 0$) when $g_A^T=0$. If $g_A^S=g_A^T$ according to Maki, both states are possible, or in other words, they are degenerate.

In order to obtain the eigenvalue of coupling constant, we set $x=0$ in the first equation of (3.10) and define new constant as follows,

$$G_A^L \equiv (\lambda^2 - \kappa^2)^2 (2^6 \cdot 3\pi^2)^{-1} g_A^L, \tag{3.11}$$

$$F_0[\mathbf{p}^2] \equiv (i\pi^2/4) A_1, \quad F_0[1] \equiv (i\pi^2/6) A_2, \quad F_0[p_0^2] \equiv (i\pi^2/12) (A_1 - 2A_2). \tag{3.12}$$

Then we have

$$8G_A^L \{A_1 + A_2 + (\epsilon^2 + \kappa^2) A_2\} = 1. \tag{3.13}$$

This is the equation that gives the relation between energy and coupling constant.

ii) Second type of chain approximation

In the same manner as in the case of the first type, we see that wave function $\phi(x)$ for relative coordinate satisfies

$$\phi(x) = (i/2) (2\pi)^{-4} (\lambda^2 - \kappa^2)^2 g_A^L F_x \{i\gamma(p + P/2) - \kappa\} \tau_L O^A \phi(0) \tau_L O^A \{i\gamma(p - P/2) - \kappa\}. \tag{3.14}$$

Substituting the equation (3.4) and

$$g_A^L \tau_L O^A \phi \tau_L O^A = g_A^S \phi' + g_A^T \phi'', \tag{3.15}$$

$$\begin{aligned}
O^A \phi O^A &\equiv \phi' = (\phi_L'^S + \gamma_\mu \phi_{\mu L}'^V + \dots) \tau_L, \\
\tau_L O^A \phi \tau_L O^A &\equiv \phi'' = (\phi_L''^S + \gamma_\mu \phi_{\mu L}''^V + \dots) \tau_L \\
&= 3(\phi_L'^S + \gamma_\mu \phi_{\mu 0}'^V + \dots) - (\phi_L'^S + \gamma_\mu \phi_{\mu 1}'^V + \dots) \tau_3,
\end{aligned} \tag{3.16}$$

into Eq. (3.15) and comparing factors of Γ^a 's in the both sides, we have the following equations ($\kappa=1$),

$$\begin{aligned}
\phi_L^S(x) &= (i/2) (2\pi)^{-4} (\lambda^2 - 1)^2 F_x [(-\mathbf{p}^2 + p_0^2 - \epsilon^2 + 1) (g_A^S \phi_L'^S(0) + g_A^T \phi_L''^S(0)) \\
&\quad - 2i\mathbf{p} \cdot (g_A^S \phi_L'^V(0) + g_A^T \phi_L''^V(0)) + 2p_0 (g_A^S \phi_{4L}'^V(0) + g_A^T \phi_{4L}''^V(0)) \\
&\quad + 2\epsilon\mathbf{p} \cdot (g_A^S \tilde{\phi}_L'^T(0) + g_A^T \tilde{\phi}_L''^T(0))],
\end{aligned}$$

$$\begin{aligned}
\phi_L^V(x) &= \quad \quad \quad F_x [(\mathbf{p}^2 - p_0^2 + \epsilon^2 + 1) (g_A^S \phi_L'^V(0) + g_A^T \phi_L''^V(0)) \\
&\quad - 2\mathbf{p}\mathbf{p} \cdot (g_A^S \phi_L'^V(0) + g_A^T \phi_L''^V(0)) - 2i\mathbf{p} (g_A^S \phi_L'^S(0) + g_A^T \phi_L''^S(0)) -
\end{aligned}$$

$$\begin{aligned}
& -2ip_0\mathbf{p}(g_A^S\phi_{4L}^{\prime\prime V}(0)+g_A^T\phi_{4L}^{\prime\prime V}(0))+2i\epsilon(g_A^S\tilde{\phi}_L^{\prime T}(0)+g_A^T\tilde{\phi}_L^{\prime\prime T}(0)) \\
& +2i\epsilon\mathbf{p}\times(g_A^S\phi_L^{\prime A}(0)+g_A^T\phi_L^{\prime\prime A}(0))], \\
\phi_{4L}^V(x)= & \quad,, \quad F_x[(\mathbf{p}^2+p_0^2-\epsilon^2+1)(g_A^S\phi_{4L}^{\prime V}(0)+g_A^T\phi_{4L}^{\prime\prime V}(0)) \\
& +2p_0(g_A^S\phi_L^{\prime S}(0)+g_A^T\phi_L^{\prime\prime S}(0))-2ip_0\mathbf{p}\cdot(g_A^S\phi_L^{\prime V}(0)+g_A^T\phi_L^{\prime\prime V}(0))], \\
\phi_L^T(x)= & \quad,, \quad F_x[(\mathbf{p}^2+p_0^2-\epsilon^2+1)(g_A^S\phi_L^{\prime T}(0)+g_A^T\phi_L^{\prime\prime T}(0)) \\
& -2\mathbf{p}\mathbf{p}\cdot(g_A^S\phi_L^{\prime T}(0)+g_A^T\phi_L^{\prime\prime T}(0))-2ip_0\mathbf{p}\times(g_A^S\tilde{\phi}_L^{\prime T}(0)+g_A^T\tilde{\phi}_L^{\prime\prime T}(0)) \\
& -2ip_0(g_A^S\phi_L^{\prime A}(0)+g_A^T\phi_L^{\prime\prime A}(0))+2\mathbf{p}(g_A^S\phi_{4L}^{\prime A}(0)+g_A^T\phi_{4L}^{\prime\prime A}(0)) \\
& -2\epsilon\mathbf{p}(g_A^S\phi_L^{\prime P}(0)+g_A^T\phi_L^{\prime\prime P}(0))], \\
\tilde{\phi}_L^T(x)= & \quad,, \quad F_x[(-\mathbf{p}^2-p_0^2+\epsilon^2+1)(g_A^S\tilde{\phi}_L^{\prime T}(0)+g_A^T\tilde{\phi}_L^{\prime\prime T}(0)) \\
& +2\mathbf{p}\mathbf{p}\cdot(g_A^S\tilde{\phi}_L^{\prime T}(0)+g_A^T\tilde{\phi}_L^{\prime\prime T}(0))-2\epsilon\mathbf{p}(g_A^S\phi_L^{\prime S}(0)+g_A^T\phi_L^{\prime\prime S}(0)) \\
& -2i\epsilon(g_A^S\phi_L^{\prime V}(0)+g_A^T\phi_L^{\prime\prime V}(0))+2ip_0\mathbf{p}\times(g_A^S\phi_L^{\prime T}(0)+g_A^T\phi_L^{\prime\prime T}(0)) \\
& +2\mathbf{p}\times(g_A^S\phi_L^{\prime A}(0)+g_A^T\phi_L^{\prime\prime A}(0))], \\
\phi_L^A(x)= & \quad,, \quad F_x[(-\mathbf{p}^2+p_0^2-\epsilon^2+1)(g_A^S\phi_L^{\prime A}(0)+g_A^T\phi_L^{\prime\prime A}(0)) \\
& +2\mathbf{p}\mathbf{p}\cdot(g_A^S\phi_L^{\prime A}(0)+g_A^T\phi_L^{\prime\prime A}(0))-2i\epsilon\mathbf{p}\times(g_A^S\phi_L^{\prime V}(0)+g_A^T\phi_L^{\prime\prime V}(0)) \\
& +2ip_0(g_A^S\phi_L^{\prime T}(0)+g_A^T\phi_L^{\prime\prime T}(0))+2\mathbf{p}\times(g_A^S\tilde{\phi}_L^{\prime T}(0)+g_A^T\tilde{\phi}_L^{\prime\prime T}(0)) \\
& +2ip_0\mathbf{p}(g_A^S\phi_{4L}^{\prime A}(0)+g_A^T\phi_{4L}^{\prime\prime A}(0))], \\
\phi_{4L}^A(x)= & \quad,, \quad F_x[(-\mathbf{p}^2-p_0^2+\epsilon^2+1)(g_A^S\phi_{4L}^{\prime A}(0)+g_A^T\phi_{4L}^{\prime\prime A}(0)) \\
& -2\mathbf{p}\cdot(g_A^S\phi_L^{\prime T}(0)+g_A^T\phi_L^{\prime\prime T}(0))+2ip_0\mathbf{p}\cdot(g_A^S\phi_L^{\prime A}(0)+g_A^T\phi_L^{\prime\prime A}(0)) \\
& -2\epsilon(g_A^S\phi_L^{\prime P}(0)+g_A^T\phi_L^{\prime\prime P}(0))], \\
\phi_L^P(x)= & \quad,, \quad F_x[(\mathbf{p}^2-p_0^2+\epsilon^2+1)(g_A^S\phi_L^{\prime P}(0)+g_A^T\phi_L^{\prime\prime P}(0)) \\
& +2\epsilon\mathbf{p}\cdot(g_A^S\phi_L^{\prime T}(0)+g_A^T\phi_L^{\prime\prime T}(0))-2\epsilon(g_A^S\phi_{4L}^{\prime A}(0)+g_A^T\phi_{4L}^{\prime\prime A}(0))].
\end{aligned} \tag{3.17}$$

In order to obtain eigenvalue equations for the coupling constant, we set $x=0$ in Eqs. (3.17) and rewrite them, using Eqs. (3.11) and (3.12).

$$\left. \begin{aligned}
& \{-A_1-A_3+(1-\epsilon^2)A_2\}(G_A^S\phi_L^{\prime S}+G_A^T\phi_L^{\prime\prime S})+\phi_L^S=0, \\
& \{A_3+(1+\epsilon^2)A_2\}(G_A^S\phi_L^{\prime V}+G_A^T\phi_L^{\prime\prime V})+\phi_L^V \\
& \quad +2i\epsilon A_2(G_A^S\tilde{\phi}_L^{\prime T}+G_A^T\tilde{\phi}_L^{\prime\prime T})=0, \\
& \{-A_1+A_3+(1+\epsilon^2)A_2\}(G_A^S\tilde{\phi}_L^{\prime T}+G_A^T\tilde{\phi}_L^{\prime\prime T})+\tilde{\phi}_L^T \\
& \quad -2i\epsilon A_2(G_A^S\phi_L^{\prime V}+G_A^T\phi_L^{\prime\prime V})=0, \\
& \{2A_1-A_3+(1-\epsilon^2)A_2\}(G_A^S\phi_{4L}^{\prime V}+G_A^T\phi_{4L}^{\prime\prime V})+\phi_{4L}^V=0, \\
& \{A_1-A_3+(1-\epsilon^2)A_2\}(G_A^S\phi_L^{\prime T}+G_A^T\phi_L^{\prime\prime T})+\phi_L^T=0,
\end{aligned} \right\} \tag{3.18}$$

$$\left. \begin{aligned}
 \{-A_3 + (1 - \epsilon^2) A_2\} (G_A^S \phi_L'^A + G_A^T \phi_L''^A) + \phi_L^A &= 0, \\
 \{-2A_1 + A_3 + (1 + \epsilon^2) A_2\} (G_A^S \phi_{4L}'^A + G_A^T \phi_{4L}''^A) + \phi_{4L}^A \\
 - 2\epsilon A_2 (G_A^S \phi_L'^P + G_A^T \phi_L''^P) &= 0, \\
 \{A_1 + A_3 + (1 + \epsilon^2) A_2\} (G_A^S \phi_L'^P + G_A^T \phi_L''^P) + \phi_L^P \\
 - 2\epsilon A_2 (G_A^S \phi_{4L}'^A + G_A^T \phi_{4L}''^A) &= 0.
 \end{aligned} \right\}$$

Among Eqs. (3.18) the second and third equations and the seventh and eighth equations are simultaneous equations.

It must be noticed here that various states (which are represented by $[\Gamma^a]$ or ϕ^a hereafter) are possible in this case and different coupling constants are determined for each state respectively. This is a remarkable difference between the first and second types of chain approximation. If $g_A^T = 0$, two states which are designated by the same spin and parity but different charge state (charge singlet $L=0$ and triplet $L=1, 2, 3$) are degenerate, or conversely they have an equal coupling constant (g_A^S) for a given energy. If $g_A^S = 0$, there is a relation between the coupling constants which are determined from charge singlet state g_{A0}^T and triplet state g_{Ai}^T respectively so as to give an equal energy. That is, $g_{Ai}^T = -3g_{A0}^T$. When we take Maki's interaction ($g_A^S = g_A^T \equiv g_A$), we see

$$g_A^T \tau_L O^A \phi \tau_L O^A = 4g_A (\phi_0^S + \gamma_\mu \phi_{\mu 0}^T + \cdots + \phi_0^P), \quad (3.19)$$

therefore charge triplet states do not exist as the solutions of the equation at all. If we put $O^A = \gamma_5$ (pseudoscalar coupling) for example,

$$\phi' = (\phi_0^S - \gamma_\mu \phi_{\mu L}^T + \sigma_i \phi_{iL}^T + \tilde{\sigma}_i \tilde{\phi}_{iL}^T - \gamma_5 \gamma_\mu \phi_{\mu L}^A + \gamma_5 \phi_L^T) \tau_L. \quad (3.20)$$

iii) General type of chain approximation

We have the solutions for the first and second chain approximations in paragraphs i) and ii). Utilizing these solutions, we can very easily obtain the solution of Eq. (2.9) for the general case. If we take $O^A = \gamma_5$, we have equations which correspond to Eqs. (3.13) and (3.18):

$$\left. \begin{aligned}
 [1 - \{-2A_1 + A_3 + (1 + \epsilon^2) A_2\} (G_A^S + 3G_A^T)] \phi_{40}^A - 6\epsilon A_2 (3G_A^S + G_A^T) \phi_0^P &= 0, \\
 [1 + \{A_1 + A_3 + (1 + \epsilon^2) A_2\} (-7G_A^S + 3G_A^T)] \phi_0^P + 2\epsilon A_2 (G_A^S + 3G_A^T) \phi_{40}^A &= 0, \\
 [1 - \{-2A_1 + A_3 + (1 + \epsilon^2) A_2\} (G_A^S - G_A^T)] \phi_{4i}^A - 2\epsilon A_2 (G_A^S + 7G_A^T) \phi_i^P &= 0, \\
 [1 + \{A_1 + A_3 + (1 + \epsilon^2) A_2\} (G_A^S - 9G_A^T)] \phi_i^P + 2\epsilon A_2 (G_A^S - G_A^T) \phi_{4i}^A &= 0.
 \end{aligned} \right\} \quad (3.21)$$

The equations for other states than ϕ_L^A and ϕ_{4L}^A are quite the same as Eqs. (3.18). If $g_A^S = 0$, the first two equations of (3.21) are the same as the last two of (3.18) for charge singlet state, and if $g_A^T = 0$, the last two of (3.21) are also the same as those for charge triplet states. Further, when $g_A^S = g_A^T$, the last of (3.21) is equivalent to Eq. (3.13).

§ 4. Solution for two-nucleon system in bound state

As in the case of a nucleon-antinucleon system, the wave function for two nucleons Ψ also has 64 components which are conveniently expressed in 4×4 matrix in regard to spinor suffices and 2×2 matrix in regard to isospin suffices.⁽⁶⁾ If we use this expression the equation (2.14) is reduced to

$$\Psi(x) = -(i/2) (2\pi)^{-4} (\lambda^2 - \kappa^2)^2 g_A^L F_i [\{i\gamma(p + P/2) - \kappa\} \tau_L \bar{O}^A \Psi(0) \tau_L \bar{O}^A \\ \times \{-i\gamma(p - P/2) - \kappa\}], \quad (4.1)$$

where total and relative energy-momenta are written as $P = p_1 + p_2$ and $p = (p_1 - p_2)/2$ which are slightly different from those for the nucleon-antinucleon system, and $\bar{O}^A \equiv A O^A T A^{-1}$, A being defined by $A \gamma_\nu^T A^{-1} = \gamma_\nu$ ($\nu = 1, 2, 3, 4, 5$). That is

$$\bar{1} = 1, \quad \bar{\gamma}_\mu = \gamma_\mu, \quad \bar{\sigma}_i = -\sigma_i, \quad \bar{\tilde{\sigma}}_i = -\tilde{\sigma}_i, \quad \bar{\gamma}_5 \bar{\gamma}_\mu = -\gamma_5 \gamma_\mu, \quad \bar{\gamma}_5 = \gamma_5. \quad (4.2)$$

Now, if we operate γ_5 on both sides of Eq. (4.1) from the right and substitute

$$\Phi(x) \equiv \Psi(x) \gamma_5 \\ = (\Psi_L^P(x) - \gamma_\mu \Psi_{\mu L}^A(x) - \sigma_i \tilde{\Psi}_{iL}^T(x) - \tilde{\sigma}_i \Psi_{iL}^T(x) - \gamma_5 \gamma_\mu \Psi_{\mu L}^V(x) + \gamma_5 \Psi_L^S(x)) \tau_L, \quad (4.3)$$

$$\Phi'(x) \equiv \Psi'(x) \gamma_5, \quad \Phi''(x) \equiv \Psi''(x) \gamma_5, \quad g_A'^L \equiv -g_A^L, \quad (4.4)$$

$$-g_A^L \tau_L O^A \Psi(x) \tau_L \bar{O}^A \gamma_5 = -g_A^S \Psi'(x) \gamma_5 - g_A^T \Psi''(x) \gamma_5 \\ = g_A'^S \Phi'(x) + g_A'^T \Phi''(x), \quad (4.5)$$

where

$$\Psi'(x) \equiv O^A \Psi(x) \bar{O}^A, \quad \Psi''(x) \equiv \tau_i O^A \Psi(x) \tau_i \bar{O}^A, \quad (4.6)$$

then we see Φ , Φ' and Φ'' satisfy Eqs. (3.17) and (3.18) formally, so the equation for two nucleon system has characters of extreme resemblance to that for nucleon-antinucleon system in the second approximation. Therefore, similar argument related under Eq. (3.18) holds also in this case. It must, however, be noticed that anti-symmetry with respect to two nucleons is required in this case and therefore a half of states are physically meaningless because of the violation of the Fermi statistics. From the definitions (4.3) and (4.4), we see that a coupling constant for a state $[\Gamma^a]$ of two-nucleon system $g(\Gamma^a)_{2N}$ is obtainable from that for a state $[\Gamma^a \gamma_5]$ of the nucleon-antinucleon system in the second approximation $g(\Gamma^a \gamma_5)_{N\bar{N}}$, i.e. $g(\Gamma^a)_{2N} = -g(\Gamma^a \gamma_5)_{N\bar{N}}$.

§ 5. Calculation of eigenvalue for coupling constant and its interpretation

As a result of sections 3 and 4 we see that, in order to obtain the eigenvalue of coupling constant, it is necessary to calculate A_1 , A_2 and A_3 which are defined by Eqs. (3.12) and (3.13). Rewriting $F_0[u(p)]$ with the use of Feynman's formula,

we have

$$F_0[u(p)] = (3/16) \int_{-1}^1 dx \int_{-1}^1 dy \int_{-1}^1 dz (1+z)(1-z) \int (dp) u(p) \\ \times \left[\mathbf{p}^2 - p_0^2 + \{\epsilon(x+y) + \epsilon z(y-x)\} p_0 - \epsilon^2 + (\lambda^2 + \kappa^2)/2 + (\lambda^2 - \kappa^2)z/2 \right]^{-4}. \quad (5.1)$$

Substituting $u(p) = \mathbf{p}^2$, 1 and p_0^2 and integrating over p , we obtain

$$A_1 = (3/16) \int_{-1}^1 dx \int_{-1}^1 dy \int_{-1}^1 dz (1+z)(1-z) [(\epsilon^2/4) \{(1-z)x + (1+z)y\}^2 \\ - \epsilon^2 + (\lambda^2 + \kappa^2)/2 + (\lambda^2 - \kappa^2)z/2]^{-1}, \quad (5.2)$$

$$A_2 = \quad , \quad [\quad , \quad]^{-2}, \quad (5.3)$$

$$A_3 = \quad , \quad [\quad , \quad]^{-2} \\ \times \{-\epsilon^2 + (\lambda^2 + \kappa^2)/2 + (\lambda^2 - \kappa^2)z/2\}. \quad (5.4)$$

Now, if $(N\bar{N})$ and (NN) are in a bound state, energy E lies in region $0 \leq E \leq 2\kappa$, and coupling constant should, therefore, be in some corresponding region. Then we calculate the coupling constants for $E=0$ and $E=2\kappa$ so as to obtain this region. The integrations in Eqs. (5.2), (5.3) and (5.4) become easy in these cases and the results are as follows:

for $E=0$,

$$A_1 = A_3 = \frac{3}{\kappa^2(\alpha-1)^2} \left\{ \alpha + 1 - \frac{2\alpha}{\alpha-1} \log \alpha \right\}, \quad (5.5)$$

$$A_2 = \frac{6}{\kappa^4(\alpha-1)^2} \left\{ -2 + \frac{\alpha+1}{\alpha-1} \log \alpha \right\}, \quad (5.6)$$

and for $E=2\kappa$

$$A_1 = \frac{1}{16\kappa^2} \left[-8 + \frac{128}{\sqrt{\alpha-1}} \tan^{-1} \frac{1}{\sqrt{\alpha-1}} + \frac{\alpha^2 - 14\alpha + 37}{\alpha-1} \log \alpha \right. \\ \left. - \frac{(\sqrt{\alpha-9})^3}{\sqrt{\alpha-1}} \log \left\{ \frac{\alpha+3-\sqrt{(\alpha-1)(\alpha-9)}}{\alpha+3+\sqrt{(\alpha-1)(\alpha-9)}} \cdot \frac{\alpha-5+\sqrt{(\alpha-1)(\alpha-9)}}{\alpha-5-\sqrt{(\alpha-1)(\alpha-9)}} \right\} \right], \quad (5.7)$$

$$A_2 = \frac{3}{2\kappa^4(\alpha-1)} \left[\log \alpha - \frac{8}{\sqrt{\alpha-1}} \tan^{-1} \frac{1}{\sqrt{\alpha-1}} - \sqrt{\frac{\alpha-9}{\alpha-1}} \log \left\{ \quad , \quad \right\} \right], \quad (5.8)$$

$$A_3 = \frac{1}{16\kappa^2} \left[8 + \frac{64}{\sqrt{\alpha-1}} \tan^{-1} \frac{1}{\sqrt{\alpha-1}} - \frac{\alpha^2 - 2\alpha - 23}{\alpha-1} \log \alpha \right. \\ \left. + \frac{(\alpha+3)\sqrt{\alpha-9}}{\sqrt{\alpha-1}} \log \left\{ \quad , \quad \right\} \right], \quad (5.9)$$

where we put $\alpha = \lambda^2/\kappa^2$ for simplicity. If we take $\alpha=10$ and $\kappa=1$, the values of

A_1, A_2 and A_3 are

$$\text{for } E=0, \quad A_1=A_3=0.218, \quad A_2=0.06 \quad (5.10)$$

$$\text{and for } E=2; \quad A_1=0.288, \quad A_2=0.190, \quad A_3=0.265. \quad (5.11)$$

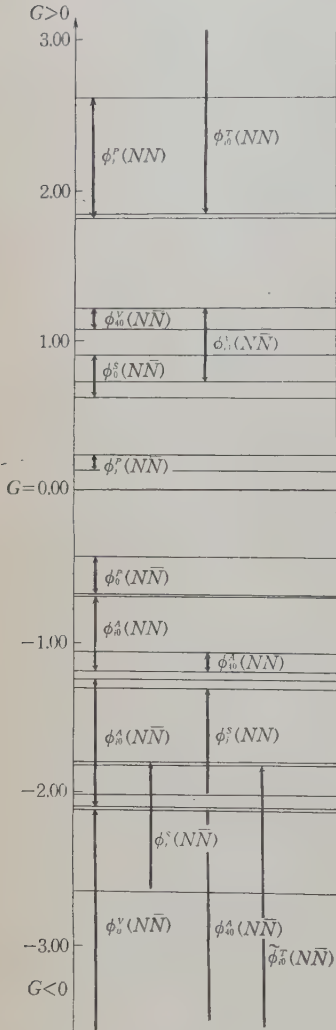


Fig. 3. Existence regions of bound states.

Substituting these values and $O^A=\gamma_5$ in Eqs. (3.13), (3.18) and (3.21), we have eigenvalues for coupling constants as shown in three cases of Table I which correspond to the cases $g_A^T=0$, $g_A^S=0$, and $g_A^T=g_A^S=g_A$ respectively.

As mentioned before, the coupling constant has different eigenvalue for each state which we describe as $[I^a]$ or ϕ^a . We see in the tables that two states are yet possible which have the same parity, spin and charge state. As we see from Eqs. (3.18) and (3.21) two sets of states (ϕ_{4L}^A , ϕ_L^T) and (ϕ_{4L}^r , ϕ_{4L}^s) for $(N\bar{N})$ are combined by the simultaneous equations and therefore give two coupling constants respectively. But in the case of $E=0$, these equations become independent of each other and give different coupling constants which correspond to each state. Then we write the value of the coupling constants for $E=2\kappa$ also in the corresponding positions of the table. The symbol "×" means that the corresponding states have no existence as the solutions of the equations, and in the case of (NN) , especially except the charge triplet states in the case III, these states are symmetric with respect to two nucleons and are, therefore, physically meaningless because of the violation of the Fermi statistics. In the case of $(N\bar{N})$, however, every state is possible, because the antisymmetry with respect to two particles is not required in this case on account of discrepancy of the two particles.

The first thing which draws our attention might be that the states which have spin larger than 2 do not exist as solutions of the equations. This is, however, a matter of course because we take the chain approximation. Next, it seems to be plausible to us that a larger absolute value of coupling constant is necessary to give a bound state of smaller energy (or larger binding energy), but there are many exceptions which are contrary to our ordinary sense. This fact may represent an effect of the relativity, since the coupling constant $|G|$ in question is con-

siderably large as expected.

Now let us consider to make coupling constant G increase gradually from zero. We take the case II ($g=g^T, g^S=0$) as a typical example and illustrate it by a diagram (Fig. 3) so as to make easy to understand the table. When $G=0$, all particles move quite freely. If G grows larger in the region $0 < G < 0.116$, every particle begins to interact with each other but composes no real bound state. Once G arrives at 0.116, there appears the first bound state $(T, -, 0)$ of $(N\bar{N})$ with mass 2κ (or zero binding energy) which corresponds to π -meson. The larger G grows further, the smaller the mass of the bound state becomes. When G gets to 0.226, the binding energy overcomes the mass of the two particles 2κ , and the energy (i.e. mass) of the bound state vanishes at all. If we make G larger, the bound state disappears. In other words, it becomes a composite particle of imaginary mass which is regarded as a ghost in a certain sense. This is an interesting fact and is always possible not only in the case $(N\bar{N})$ but also in (NN) when G is large enough for binding energy to overcome the mass of two particles. According to the study by S. Tanaka²⁾ the wave function of such a particle satisfies $(p^2 - m^2)\psi = 0$ instead of $(p^2 + m^2)\psi = 0$, and propagate with super-light velocity. If such a particle were possible as an intermediate state, a phenomena should appear in nucleon-nucleon scattering which violates the causality.

Beyond a region where is no bound state, there appears the next bound state of $(N\bar{N})$ at $G=0.60$, i.e. $\phi_0^S(S, +, 0)$. When G grows further to 0.71, another bound state $\phi_{10}^T(N\bar{N}) (S, -, 1)$ becomes possible before the former bound state disappears. Therefore the two bound states coexist for a while. Repeating such circumstance, G grows larger and larger and when G arrives at 1.81 there appears the first bound state of two nucleons, $\phi_j^T(NN) (T, -, 0)$. Successively the next bound state, $\phi_{10}^T(NN) (S, +, 1)$, is formed at $G=1.83$ which corresponds to the deuteron with very small range. We do not hope the existence of such bound states. The similar situation holds also in the case $G < 0$ but it is somewhat different from the case $G > 0$ that the bound state of (NN) appears for a considerably small coupling constant $|G|$.

It is of interest that bound states of $(N\bar{N})$ and (NN) appear and disappear in such a manner as mentioned above according to the increase of coupling constant. What we must do might, however, be to determine the most suitable type and constant of coupling from the circumstances. This is however difficult for the following reasons: We take a special type of interaction, i.e. pseudoscalar coupling ($O^4 = \gamma_5$) only and a special approximation, i.e. chain approximation, and do not consider the interaction between N and Λ . But we may assert at least that pseudo-scalar coupling is not always necessary to compose the π -meson as a bound state of $(N\bar{N})$ and it is always possible, even though we take any type of coupling, only if we choose a suitable coupling constant. The value of coupling constant giving π -meson is smallest if we take pseudoscalar coupling. Further, it is possible for this theory to give the usual π -meson-nucleon coupling constant

($f=0.08$), if we take a suitable cutoff parameter λ^2 , because meson-nucleon coupling constant is dependent only upon the range of meson and independent of the value of our coupling constant g .³⁾

In conclusion we notice that the original coupling constant g (erg cm³) is obtainable by multiplying G by 3.2×10^{-43} .

§ 6. Possible extension

i) Nucleon-nucleon scattering

It seems necessary to calculate nucleon-nucleon scattering in order to investigate the effects which the four fermi coupling have upon nucleon-nucleon interaction. According to Nishijima,⁷⁾ the S -matrix is written in this case as

$$S_{ab} = \delta_{ab} - i(g_A^L/4) \int \bar{\Psi}_b^0(1, 1) (\tau_L O^A)^{(1)} (\tau_L O^A)^{(2)} \Psi_a(1, 1) (dx_1), \quad (6.1)$$

where $\Psi_b^0(1, 1)$ denotes the free part of wave function. If we put $\Psi_a(1, 2) = \exp(iP_a(x_1 + x_2)/2) \phi_a(x_1 - x_2)$, then S -matrix (6.1) is rewritten as

$$S_{ab} = \delta_{ab} - i(g_A^L/4) (2\pi)^4 \delta^4(P_a - P_b) \bar{\phi}_b^0(0) (\tau_L O^A)^{(1)} (\tau_L O^A)^{(2)} \phi_a(0). \quad (6.2)$$

At first sight, we see that only the waves which have values at $x=0$ are scattered. This is a considerably different character from what we usually encounter. Actually the states which are influenced by this scattering are only the states that appear in Table I. States $J \geq 2$ are not scattered so far as we take chain approximation.

ii) Applications to K , Σ and Ξ

According to Sakata's model, K -meson is a bound state of $(A\bar{N})$ or $(\bar{A}N)$. As the two particles are different from each other, only one type of chain approximation is possible according as the coupling is of form $(\bar{A}N)(\bar{N}A)$ or $(\bar{A}A)(\bar{N}N)$. As the discussion on K -meson is quite similar to that on π -meson, we do not mention further here.

Σ and Ξ are composed of $(A\bar{N}\bar{N})$ and $(A\bar{A}\bar{N})$ respectively and the equation for these particles are easily formulated as in section 2. If we define the wave function of Σ by

$$\Psi_\Sigma(1, 2, 3) \equiv \langle 0 | T[\phi_N(1) \phi_A(2) \bar{\phi}_N(3)] | \Sigma \rangle, \quad (6.3)$$

neglect NA interaction and consider only the first type of chain approximation for $N\bar{N}$ and $A\bar{N}$, then $\Psi_\Sigma(1, 2, 2)$ satisfies

$$\begin{aligned} \Psi_\Sigma(1, 2, 3) = & i(g_A^L/2) \int S_F^{(1)}(1-1') (\tau_L O^A)^{(1,3)} S_F^{(3)}(1'-3) \\ & \times \text{Tr}^{(13)}[(\tau_L O_A)^{(1,3)} \Psi_\Sigma(1', 2, 1')] (dx_1') \\ & + i(g_A^L/2) \int S_F^{(2)}(2-2') (\tau_L O^A)^{(2,3)} S_F^{(3)}(2'-3) \text{Tr}^{(2,3)}[(\tau_L O_A)^{(2,3)} \Psi_\Sigma(1, 2', 2')] (dx_1'). \end{aligned} \quad (6.4)$$

Table. Eigen values of coupling Constant G

State(ϕ^2)		ϕ^S_0	ϕ^V_{40}	ϕ^T_{10}	$\phi^{A_{40}}$	ϕ^{P_0}	ϕ^S_{40}	ϕ^V_{10}	$\tilde{\phi}^T_{10}$	ϕ^S_j	ϕ^V_{4j}	ϕ^T_{1j}	$\phi^{A_{4j}}$	ϕ^{P_j}	$\phi^{A_{4j}}$	ϕ^V_{1j}	$\tilde{\phi}^T_{1j}$
Charge state(S: Singlet T: Triplet)		S	S	S	S	S	S	S	S	T	T	T	T	T	T	T	T
Parity(+ : even - : odd)		+	+	+	+	-	-	-	-	+	+	+	+	-	-	-	-
Spin		0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
(NN)	1st	$E=2\kappa$ $E=0$	\times	\times	\times	0.134 0.252	\times	\times	\times	\times	\times	\times	\times	\times	\times	\times	\times
	2nd	$E=2\kappa$ $E=0$	1.81 2.66	3.22 3.60	-43.5 -16.7	-3.78 -6.33	-1.32 -2.02	2.13 3.60	-5.48 -16.7	1.81 2.66	3.22 3.60	-43.5 -16.7	-3.78 -6.33	-1.32 -2.02	-9.55 -6.33	2.13 3.60	-5.48 -16.7
	Gen-eral	$E=2\kappa$ $E=0$	do.	do.	do.	do.	0.157 0.289	do.	do.	do.	do.	do.	do.	do.	do.	do.	do.
	(NN)	$E=2\kappa$ $E=0$	\times	\times	5.48 16.7	-2.13 -3.60	\times	-3.22 -3.60	\times	43.5 16.7	1.32 2.02	9.55 6.33	\times	\times	-1.81 -2.66	\times	3.78 6.33
(NN)	1st	$E=2\kappa$ $E=0$	\times	\times	\times	\times	\times	\times	\times	\times	\times	\times	\times	0.134 0.252	\times	\times	\times
	2nd	$E=2\kappa$ $E=0$	0.60 0.89	1.07 1.20	-14.5 -5.57	-1.26 -2.11	-0.44 -0.69	0.71 1.20	-1.83 -5.57	-1.81 -2.66	-3.22 -3.60	43.5 16.7	3.78 6.33	1.32 2.02	9.55 6.33	-2.13 -3.60	-5.48 -16.7
	Gen-eral	$E=2\kappa$ $E=0$	do.	do.	do.	do.	do.	do.	do.	do.	do.	do.	do.	0.116 0.226	-5.36 6.33	do.	do.
	(NN)	$E=2\kappa$ $E=0$	\times	\times	1.83 5.57	-0.71 -1.20	\times	-1.07 -1.20	\times	14.5 5.57	-1.32 -2.02	-9.55 -6.33	\times	\times	1.81 2.66	-3.78 -6.33	\times
(NN)	1st	$E=2\kappa$ $E=0$	\times	\times	\times	0.134 0.252	\times	\times	\times	\times	\times	\times	\times	0.134 0.252	\times	\times	\times
	2nd	$E=2\kappa$ $E=0$	0.452 0.665	0.805 0.900	-10.9 -4.18	-0.945 -1.58	-0.330 -0.505	-2.39 -1.58	0.533 0.900	\times	\times	\times	\times	\times	\times	\times	\times
	Gen-eral	$E=2\kappa$ $E=0$	do.	do.	do.	do.	\times	\times	do.	\times	\times	\times	\times	0.134 0.252	\times	\times	\times
	(NN)	$E=2\kappa$ $E=0$	\times	\times	1.37 4.18	-0.533 -0.900	\times	-0.805 0.900	\times	10.9 4.18	\times	\times	\times	\times	\times	\times	\times
Case I $G=G_S, G^T=0$																	
Case II $G=G^T, G^S=0$																	
Case III $G=G^S=G^T$																	



Fig. 4. Feynman diagram. of $\Sigma = (ANN)$

where $g_A^{\prime L}$ is the coupling constant of $(\bar{\psi}_A \tau_L O^A \psi_N) (\bar{\psi}_N \tau_L O^A \psi_A)$ and $T_r^{(i,j)}$ means to take the trace about spinor suffices of two particles i and j . It is quite trivial to write the equation in which all types of interaction are taken into account. If we show Eq. (6.4) by the Feynman diagram, it becomes as Fig. 4, for instance. The equation is very interesting if we can solve it by some excellent prescription and obtain the mass of Σ , but it is very hard because the equation essentially contains the difficulty of the three-body problem.

The author is indebted to many people for valuable discussions and he wishes to express his gratitude to all of them though he does not write their names here.

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Interactions Induced by High Energy Neutrinos

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High energy neutrino-electron and neutrino-nucleon collisions are discussed. In particular the "elastic" processes $\bar{\nu} + p \rightarrow n + e^+$ and $\nu + n \rightarrow p + e^-$ are investigated in detail. The cross section of $\nu + n \rightarrow p + e^-$ is estimated to be $\sim 0.8 \times 10^{-38}$ cm² for neutrino energy in lab system > 1 Bev under the reasonable assumptions.

§ 1. Introduction

A high energy neutrino flux obtained by the multi-Gev accelerator (e.g., the proton synchrotron at CERN) may be used for investigating reactions induced by neutrinos. If such a plan turns out to be feasible, there will open a new branch of the particle physics, of which no direct experimental information is available up to the present. On the other hand, considerable amount of knowledge have been accumulated on the weak processes at "low energy" regions. Noticeably, the antineutrino-reactions, e. g.

$$\bar{\nu} + p \rightarrow n + e^+,$$

$$\bar{\nu} + d \rightarrow n + n + e^+,$$

were successfully investigated¹⁾ by making use of an intense beam of "pile"-neutrinos; whereas the reaction²⁾

$$\bar{\nu} + \text{Cl}^{37} \rightarrow \text{A}^{37} + e^-$$

has not been detected. These facts, failure in detecting any double β -decay processes and all evidences for weak processes, support the conservation law of lepton number.

Extremely energetic processes induced by weak interactions, or more definitely leptonic processes at very high energies, must be one of the most interesting problems in particle physics: they are quite sensitive to mechanisms and/or structures of interactions involved. Speculations³⁾ on these phenomena have been published from time to time, after the Fermi theory of nuclear β -decays was established. However, even at the present days, it remains to be entirely unclear what we could ever expect about very high energy leptonic processes. In the present paper, we cannot, of course, state anything new about such cases. Rather we shall be reasona-

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bly modest and conservative; in particular one of the purposes of the present paper is to estimate the "elastic" cross-sections of $\bar{\nu} + p \rightarrow n + e^+$ and $\nu + n \rightarrow p + e^-$ at multi-Gev regions, for which we can hope to, say, extrapolate the present information obtained at low energy (sub-Gev) regions.

First of all it would be instructive to list the threshold energies for various reactions initiated by neutrinos (target particles are assumed to be at rest) (Table I).

Table I. Threshold Energies

Reactions		Threshold energy in Gev
$\bar{\nu} + p \rightarrow$	$n + e^+$	0.0018
	$n + \mu^+$	0.11
	$n + e^+ + \pi^0$	0.15
	$\mathcal{N}^0 + e^+$	0.36 (\mathcal{N} =the 33 isobar, its mass being assumed to be 1.23 Gev)
	$\Lambda + e^+ + K^0$	0.92
	$\Lambda + e^+$	0.19
	$\Lambda + \mu^+$	0.33
	$\Sigma^0 + e^+$	0.29
	$\Sigma^0 + \mu^+$	0.43
$\bar{\nu} + e^- \rightarrow$	$\mu^- + \bar{\nu}$	11
	$\pi^- + \pi^0$	74
	$K^- + \pi^0$	400
	$K^- + K^0$	960

From the threshold values, we expect to observe neither $\bar{\nu}e^-$ nor νe^- reactions at the present stage of large accelerators. The cosmic-ray neutrinos seem to be the only source for such a purpose.

We shall next mention obvious remarks about the conservation laws of lepton number. Conventionally, the neutrino, the electron e^- and the negative muon μ^- are defined as leptons, and the total number of leptons minus that of antileptons is believed to be conserved in leptonic processes. For the time being, let us assume there exist two kinds of neutrinos,* ν_μ and ν_e , the ν_μ is always accompanied with the muon and the ν_e with the electron, and moreover the number of μ plus ν_μ and the number of e and ν_e are separately conserved in all processes. For example,

$$\left. \begin{array}{l} \pi^+ \\ K^+ \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \mu^+ + \nu_\mu \\ e^+ + \nu_e \end{array} \right.$$

$$n \rightarrow p + e^- + \bar{\nu}_e$$

* Such a possibility must have been known to many people. See for example, B. Pontecorvo, J. Expl. Theoret. Phys. (U.S.S.R.) **37** (1959), 1751.

$$K^+ \rightarrow \begin{cases} \pi^0 + e^+ + \nu_e \\ \pi^0 + \mu^+ + \nu_\mu \end{cases}$$

$$\mu^- + p \rightarrow n + \nu_\mu, \quad \mu^+ \rightarrow \bar{\nu}_\mu + e^+ + \nu_e, \text{ etc.,}$$

are allowed. Such a postulate is sufficient to make the unwanted processes forbidden:

$$\mu^- + p \not\rightarrow p + e^-$$

$$\mu^- \not\rightarrow e^- + \gamma$$

$$\mu^\pm \not\rightarrow e^\pm + e^+ + e^-.$$

The neutrinos available by multi-Gev accelerators must be almost ν_μ and $\bar{\nu}_\mu$ since the major source of "neutrinos" are π - μ and $K_{\mu 2}$ decays. Then, the artificially produced "neutrinos" (ν_μ , $\bar{\nu}_\mu$) can induce the following reactions:*

$$\begin{aligned} \bar{\nu}_\mu + p &\rightarrow n + \mu^+ \\ \nu_\mu + n &\rightarrow p + \mu^-, \end{aligned} \tag{1}$$

but they cannot induce the following reaction:

$$\begin{aligned} \bar{\nu}_\mu + p &\rightarrow n + e^+ \\ \nu_\mu + n &\rightarrow p + e^-. \end{aligned} \tag{2}$$

Experimental check of such a possibility must be relatively easy once neutrino reactions become detectable. In what follows we shall no more consider the possibility of two kinds of neutrinos; we assume hereafter only one kind of "two-component" neutrinos. We also accept the (universal) V-A type (lepton number conserving, time reversal invariant) weak Fermi-interactions currently fashionable (unless the contrary is stated explicitly)⁴⁾.

For the sake of completeness, we shall begin our discussions with neutrino-electron interactions (Section 2). We shall add there some speculation on leptonic interactions. Next we shall discuss the "elastic processes"

$$\begin{aligned} \bar{\nu} + p &\rightarrow n + e^+ \\ \nu + n &\rightarrow p + e^- \end{aligned} \tag{3}$$

in Section 3. In this section, however, we shall be extremely conservative. If we take the conserved (strangeness conserving) weak vector current, the vector part (and the weak magnetism) of the relevant nuclear matrix elements can be evaluated

* Under this assumption, the reactions

$$\begin{aligned} \bar{\nu}_\mu + e^- &\rightarrow e^- + \bar{\nu}_\mu \\ \nu_\mu + e^- &\rightarrow \begin{cases} e^- + \nu_\mu \\ \mu^- + \nu_e \end{cases} \end{aligned}$$

are allowed, while $\bar{\nu}_\mu + e^- \rightarrow \mu^- + \nu_e$, etc., are forbidden.

with the aid of the electromagnetic form factors of the physical nucleons. We have no way of estimating the axial vector parts reliably.

Above the pion thresholds there are no reasons why the inelastic processes, e. g.

$$\bar{\nu} + p \rightarrow e^+ + n + (\text{several pions}) + \dots,$$

should not have large cross-sections as compared with the "elastic" processes (3). Such a possibility shall be briefly considered in Section 4.

§ 2. Neutrino-electron interactions

Discussions given in this section are mostly of academic nature, because all reactions induced by the neutrino-electron collision seem to have too small cross-sections to be measurable (or energetically forbidden even for the largest accelerators which will be available in the immediate future). We shall discuss this interaction and give some remarks on it simply for the sake of completeness and comparison with neutrino-nucleon case.

Existence of the μ -decay, $\mu \rightarrow e + \nu + \bar{\nu}$, tells us the reactions

$$\begin{aligned} \nu + e^- &\rightarrow \mu^- + \nu \\ \bar{\nu} + e^- &\rightarrow \mu^- + \bar{\nu} \end{aligned} \quad (4)$$

must occur. The conventional μ -decay interaction

$$\frac{G}{\sqrt{2}} (\bar{\nu} \gamma_\alpha (1 + \gamma_5) \mu) (\bar{e} \gamma_\alpha (1 + \gamma_5) \nu) + \text{h. c.} \quad (5)$$

predicts the following cross-sections for high energy neutrinos:*

$$\begin{aligned} \sigma(\nu + e^- \rightarrow \mu^- + \nu) &= 3\sigma(\bar{\nu} + e^- \rightarrow \mu^- + \bar{\nu}) \approx \frac{4}{\pi} G^2 p^2 \\ &= 0.57 \times 10^{-37} \text{ cm}^2 \left(\frac{p}{M} \right)^2 \text{ for } p \gg m_\mu \end{aligned} \quad (6)$$

where p is the neutrino momentum in the CM system, M is the proton mass, and the muon mass m_μ has been neglected.**

Estimation of the cross-sections for

$$\left. \begin{aligned} \nu + e^- &\rightarrow e^- + \nu \\ \bar{\nu} + e^- &\rightarrow e^- + \bar{\nu} \end{aligned} \right\}$$

depends on a model and the corresponding specific interactions we happen to choose. It is commonly believed that leptons have no strong interactions among themselves

* We assume that the target electron is at rest. In heavy material, the inner atomic electrons have considerable internal motion, hence the cross-sections would increase for such electrons as compared with the case of free electrons at rest.

** More accurate evaluation tells us $\sigma(\nu + e^- \rightarrow \mu^- + \nu) = 1.0 \times 10^{-40} \text{ cm}^2$ for $\nu_{lab} = 20 M$, where ν_{lab} is the incident neutrino energy in the lab system.

and with strongly interacting particles other than the electromagnetic coupling.

There are no positive evidences against this point of view. Nevertheless experimental information about neutrino-electron interactions is in fact extremely meagre. We have only two conditions: (i) Firstly, we know from pile-neutrino experiment⁵⁾

$$\sigma(\bar{\nu} + e^- \rightarrow e^- + \bar{\nu}) < 6 \times 10^{-43} \text{ cm}^2 \quad (7)$$

or the anomalous magnetic moment of $\bar{\nu}$ must be smaller than 1.4×10^{-9} Bohr magneton.* (ii) The second piece of information comes from the great success of the quantum electrodynamics (QED) applied to electrons (particularly the Lamb shifts of hydrogen-like atoms, and the QED correction to the magnetic moment of the electron). An additional electron-neutrino interactions, if they ever exist, should not disturb the excellent fit between experiments and QED-predictions.

One of the appealing and simplest possibilities of guessing the cross-sections for (6) would be given by a universal generalization,

$$\frac{G}{\sqrt{2}} \left[(\bar{\nu} \gamma_\alpha (1 + \gamma_5) \mu) + (\bar{\nu} \gamma_\alpha (1 + \gamma_5) e) \right] \left[(\bar{\mu} \gamma_\alpha (1 + \gamma_5) \nu) + (\bar{e} \gamma_\alpha (1 + \gamma_5) \nu) \right], \quad (8)$$

of the μ -decay interaction (5). If the weak interactions are intermediated by a charged boson,⁷⁾ (8) would be quite a reasonable choice. Then we find again

$$\sigma(\nu + e^- \rightarrow e^- + \nu) = 3\sigma(\bar{\nu} + e^- \rightarrow e^- + \bar{\nu}) = \frac{4}{\pi} G^2 p^2, \quad \text{for } p \gg m_e \quad (9)$$

where we have ignored the electron mass m_e . For example, $\sigma(\nu + e^- \rightarrow e^- + \nu)$ turns out to be $1.6 \times 10^{-40} \text{ cm}^2$ at $\nu_{lab} = 10M$ (where ν_{lab} is the *lab* energy of the incident neutrino). The angular distributions in the *CM* system are as follows: Isotropic for $\nu + e^- \rightarrow e^- + \nu$, $(1 + \cos \theta)^2$ for $\bar{\nu} + e^- \rightarrow e^- + \bar{\nu}$, where θ is the scattering angle of the antineutrino.

Pomeranchuk⁸⁾ has asserted that the *total* cross-section for a *particle-particle* collision must be equal to that for an *antiparticle-particle* collision at high energy limit. If this assertion remains to be true for all interactions including weak interactions (a reasonable guess, if his assertion happens to be true for the strong interactions), one can expect the equality of total cross-sections

$$\sigma_{tot}(\nu + e^-) = \sigma_{tot}(\bar{\nu} + e^-) \text{ at extremely high energies.} \quad (10)$$

The "elastic" cross-sections do *not* satisfy** such an equality (see (6) and

* If the neutrino ν obeys two-component theory,⁶⁾ ν and $\bar{\nu}$ cannot have any anomalous magnetic moment. However, they can have *in principle* an "electric" structure.

** This is, of course, not surprising, because (6) and (9) should not be trusted at very high energies. For example, if some sorts of structures are associated with the leptonic interactions, we must expect that the validity of simple formulae like (6) and (9) is necessarily limited to low momentum transfer region. Even if we suppose that leptonic structures are unimportant until exceedingly high energies, the unitarity condition will tell us their validity; if numerical values of cross sections predicted from (6) and (9) are of the order of the wave mechanical limit $\sim \pi/p^2$, these formulae (6) and (9) will no longer be valid. We can define the critical energy by (6) or (9) $\sim \pi/p^2$. Such a critical energy (in the center of mass system) is of the order $G^{-1/2} \approx 300 \text{ Gev}$, and above this energy the reactions of (i.e. the higher order effects of) "weak" interactions must be duly taken into account.

(9)). However, one can naturally expect that there are many inelastic processes, e. g.

$$\left. \begin{aligned} \nu + e^- &\rightarrow e^- + \nu \\ \bar{\nu} + e^- &\rightarrow e^- + \bar{\nu} \end{aligned} \right\} + (\text{many electron pairs or photons, etc.})$$

available at high energies, which might make the equality (10) valid at high energy limit.

Finally we must remark the possible deviation of cross-sections from simple formulae (6) and (9). If the weak leptonic interactions have some non-locality,⁹⁾ in particular if a possible intermediary boson⁷⁾ (a catalyzer of weak processes) exist, we can expect quite substantial deviation from (6) and (9). A possible effect of an intermediary boson has recently been noticed by Glashow,¹⁰⁾ who has given the cross-section:

$$\sigma(\bar{\nu} + e^- \rightarrow e^- + \bar{\nu}) = \frac{4}{3\pi} G^2 p^2 \left\{ \frac{m_b^4}{(m_b^2 - 4p^2)^2 + (m_b \tau)^2} \right\}$$

where m_b and τ is the mass and mean lifetime of the intermediate boson. However, any deviations should occur at extremely high neutrino energies (provided the target electron is at rest)

$$2\nu_{lab} \doteq 4p^2/m_e \sim m_b^2/m_e.$$

Since m_b should not be smaller than the K -meson mass m_K , m_b^2/m_e must be larger than $1000 \times m_K \approx 500$ Gev.

Let us add a (drastic) speculation about possible existence of an extra neutrino-electron interaction.¹¹⁾ We assume that apparent lack of any strong interactions between leptons are merely limited within the "low energy" regions (more precisely, small values of 4-momentum transfer q) (see (i) and (ii) mentioned above). We introduce here a characteristic momentum κ ; and we shall imagine that our extra neutrino-electron interaction is effectively very "weak" for $q^2 \ll \kappa^2$ but is not necessarily "weak" for $q^2 \gtrsim \kappa^2$. A phenomenological description of such a possibility would be given by, say, a non-local e - ν interaction density:

$$\frac{1}{\sqrt{2}} g \int (\bar{\nu}(x) \gamma_\alpha (1 + \gamma_5) e(x)) F(x-y) (\bar{e}(y) \gamma_\alpha (1 + \gamma_5) \nu(y)) d^4 y, \quad (11)$$

where $F(x-y)$ is a scalar function of $(x-y)$, and whose Fourier transform is arbitrarily assumed to be*

$$F(q) = \frac{q^2}{\kappa^2 + q^2}.$$

* At low energy, $|p| \ll \kappa$, this choice is equivalent to an interaction of Konopinski-Uhlenbeck type:

$$\frac{1}{\kappa^2} \frac{g}{\sqrt{2}} [\partial_\beta (\bar{\nu} \gamma_\alpha (1 + \gamma_5) e)] [\partial_\beta (\bar{e} \gamma_\alpha (1 + \gamma_5) \nu)].$$

Then we shall find

$$\sigma(\bar{\nu} + e^- \rightarrow e^- + \bar{\nu}) = \frac{4}{3\pi} g^2 p^2 \left(\frac{4p^2}{\kappa^2 - 4p^2} \right)^2 (p < \kappa) \quad (11')$$

where p is again the CM momentum of the neutrino and m_e has been ignored. If we assume $\kappa \sim M$ and use $\sigma(\bar{\nu} + e^- \rightarrow e^- + \bar{\nu}) \sim 6 \times 10^{-43} \text{ cm}^2$ at $\nu_{lab} \simeq 2m_e$ (see Eq. (7)) in order to fix the coupling constant g , we find very large neutrino cross-sections for large ν_{lab} ; for example $\sigma(\bar{\nu} + e^- \rightarrow e^- + \bar{\nu}) \sim 10^{-34} \text{ cm}^2$ at $\nu_{lab} \simeq 10 M$. Furthermore, once we admit some non-weak interactions between leptons, we may try to assign the strangeness to all leptons:¹¹⁾ e.g. strangeness zero for electrons and neutrinos (as well as positrons and antineutrinos), and strangeness ± 1 for the μ^\pm meson, and assume the extra non-weak interactions among leptons being not only lepton number conserving but also "strangeness conserving". Then we conclude

$$\left. \begin{array}{l} \sigma(\nu + e^- \rightarrow \nu + e^-) \\ \sigma(\bar{\nu} + e^- \rightarrow \bar{\nu} + e^-) \end{array} \right\} \gg \left\{ \begin{array}{l} \sigma(\nu + e^- \rightarrow \mu^- + \nu) \\ \sigma(\bar{\nu} + e^- \rightarrow \mu^- + \bar{\nu}) \end{array} \right. \quad (12)$$

at very high energies, since the latter processes, $\nu + e^- \rightarrow \mu + \nu$, etc., are caused by the weak interactions (compare (11) with (6)).

These possibilities have been mentioned in order to warn a credulity about non-existence of non-weak leptonic interactions (other than electromagnetic coupling). It would be needless to say that all numerical statements, Eq. (11) and others, should by no means be taken seriously. Though not repeated below, similar statements can be made on possible existence of new interactions between leptons and strongly interacting particles.* In the subsequent two sections, we come back to and keep to hold a conservative and conventional point of view about leptonic processes.

§ 3. "Elastic" neutrino-nucleon processes**

We shall treat in this section the well-defined problem; the "elastic" neutrino-nucleon processes***

$$\left. \begin{array}{l} \bar{\nu} + p \rightarrow n + e^+ \\ \nu + n \rightarrow p + e^- \end{array} \right\} \quad (13)$$

These processes must exist according to the well-established weak interactions for nuclear β -decays. Other elastic processes,

* A possible non-weak interaction between muons and other particles shall be discussed fully in a forthcoming paper by A. Peterman and Y. Yamaguchi. See also J. Schwinger, ref. 7).

** Prof. G. Cocconi has kindly informed me that essentially the same analysis as presented in this section has been performed also by T. D. Lee and by N. Cabibbo and R. Gatto (to be published).

*** For high energy neutrinos, atomic nuclei may be replaced to a good approximation by an assembly of free nucleons.

$$\begin{aligned}\nu + p &\rightarrow n + e^+ \\ \bar{\nu} + n &\rightarrow p + e^-, \end{aligned}$$

are forbidden from the conservation law of lepton number. The μ^- -capture interactions predict the existence of two additional "elastic" processes:

$$\left. \begin{aligned} \bar{\nu} + p &\rightarrow n + \mu^+ \\ \nu + n &\rightarrow p + \mu^- \end{aligned} \right\} \quad (14)$$

For definiteness, we assume the universal weak interaction (CP invariant):⁴⁾

$$\frac{G}{\sqrt{2}} (J_\alpha^V + J_\alpha^A) \left[(\bar{e}\gamma_\alpha(1+\gamma_5)\nu) + (\bar{\mu}\gamma_\alpha(1+\gamma_5)\nu) \right] + \text{h. c.} \quad (15)$$

being responsible to (13) and (14), where J_α^V and J_α^A are the (strangeness conserving) weak vector and axial vector currents, respectively, and these currents are assumed to obey the transformation laws¹²⁾

$$\begin{aligned} J_\alpha^V &\rightarrow -J_\alpha^V \\ J_\alpha^A &\rightarrow +J_\alpha^A \end{aligned}$$

under the combined operation of charge conjugation and charge symmetry. J_α^V contains terms $(\bar{p}\gamma_\alpha n)$ and so on, whereas J_α^A must contain at least the term $(\bar{p}\gamma_\alpha\gamma_5 n)$. Their explicit forms depend on a special choice of the "model". For example, if the divergenceless weak vector current $(\partial_\alpha J_\alpha^V = 0)$ ⁴⁾ is postulated (as far as the charge independence holds), the weak vector current is completely fixed:

$$\begin{aligned} J_\alpha^V &= (\bar{p}\gamma_\alpha n) + \sqrt{2} \{ (\bar{\Sigma}^+ \gamma_\alpha \Sigma^0) - (\bar{\Sigma}^0 \gamma_\alpha \Sigma^-) \} + (\bar{\Xi}^0 \gamma_\alpha \Xi^-) \\ &+ \sqrt{2} \{ (\pi^+) * \partial_\alpha \pi^0 - \pi^0 \partial_\alpha \pi^- \} + \{ (K^+) * \partial_\alpha K^0 - K^0 \partial_\alpha K^- \}. \end{aligned} \quad (16)$$

The special choice of our weak interaction (15) tells us that the elastic scatterings of the neutrinos

$$\begin{aligned} \nu + p &\rightarrow p + \nu \\ \bar{\nu} + p &\rightarrow p + \bar{\nu}, \quad \text{etc.}, \end{aligned}$$

are the second order effects in the weak constant G , and have very small cross-sections.

Under the assumptions of weak interaction (15), the matrix element for $\nu + n \rightarrow p + e^-$ in the first order in the weak interaction must have the general form¹³⁾

$$\begin{aligned} \frac{G_V}{\sqrt{2}} \bar{u}_p \left[F_1^V(q^2) \gamma_\alpha + i \frac{F_2^V(q^2)}{2M} (p-n)_\beta \sigma_{\alpha\beta} \right. \\ \left. + i F_A(q^2) \gamma_\alpha \gamma_5 + i b F_P(q^2) (p-n)_\alpha \gamma_5 \right] u_n \times \bar{u}_e \gamma_\alpha (1+\gamma_5) u_\nu \end{aligned} \quad (17)$$

(the same formula holds for $\nu + n \rightarrow p + \mu^-$ if we replace the Dirac spinor \bar{u}_e by \bar{u}_μ), where we have used the charge symmetry for strong interactions, M =nucleon mass, 4-momenta of the neutrino, the proton, the final electron and the neutron are written by the particle symbols ν , p , e , n and q is the momentum transfer

$$q = e - \nu = p - n.$$

Form factors F_1, F_2, F_A, F_P are functions of q^2 , the invariant momentum transfer squared, and they are normalized to one at $q^2=0$:

$$F_1(0) = F_2(0) = F_A(0) = F_P(0) = 1.$$

G_V and $G_A \equiv -\lambda G_V$ ($\lambda \approx 1.25$) are the renormalized vector and axial vector constants, respectively, and the second and the fourth terms in the curl bracket of Eq. (17) represent the so-called weak magnetism and the induced pseudoscalar effect. If we take the conserved weak vector current J_α^V , we find no renormalization for the vector constant: $G_V = G$, μ equal to the difference of the anomalous magnetic moment of the proton and that of the neutron in the units of nuclear magneton:

$$\mu = 3.71 \quad (18)$$

and the two form factors F_1 and F_2 must be equal to the electromagnetic form factors of nucleons measured by Hofstadter et al.¹⁴⁾

$$F_1(q^2) \approx F_2(q^2) \approx \frac{1}{\left(1 + \frac{r^2 q^2}{12}\right)^2} \equiv F(q^2) \quad (19)$$

$$r \approx 0.8 \times 10^{-13} \text{ cm}$$

which hold at least for small q^2 . Magnitude of the pseudoscalar constant has been estimated by Wolfenstein¹⁵⁾ and Goldberger-Treiman¹⁸⁾

$$|G_V b| \approx 10 \times |G_V| / m_\mu \quad (20)$$

where m_μ is the muon mass. For the electron case (13), the pseudoscalar term contributes very little to the reaction rates and can be neglected. We know little about F_A and F_P (however, see ref. 13)). Time reversal invariance of strong and weak interactions guarantees G_V, λ, μ and b being real; more accurately, all form factors are in general real below the pion threshold (e.g. $\bar{\nu} + p \rightarrow n + e^+ + \pi^0$) but they are complex above the pion threshold.

Using the general form of the matrix element (17), we can easily calculate the cross-section* for the elastic processes

$$\nu + n \rightarrow p + e^-, \text{ and}$$

$$\bar{\nu} + p \rightarrow n + e^+ :$$

* The cross-section in the lab system is given in the Appendix.

$$\begin{aligned}
& \left. \begin{aligned} & \sigma(\nu + n \rightarrow p + e^-) \\ & \sigma(\bar{\nu} + p \rightarrow n + e^+) \end{aligned} \right\} \\
&= \frac{G_V^2}{\pi} \int \frac{d\Omega_e}{4\pi} \left(\frac{pE}{E+p} \right)^2 \left[|F_1(q^2)|^2 \left(1 + \frac{p}{E} \right)^2 (1 + \cos\theta) \right. \\
&\quad + |F_1(q^2) + \mu F_2(q^2)|^2 \left(\frac{p}{E} \right)^2 (1 - \cos\theta)^2 \\
&\quad + |\mu F_2(q^2)|^2 \frac{p^2}{2M^2} \left(1 + \frac{p}{E} \right)^2 (1 - \cos^2\theta) \\
&\quad + |\lambda F_A(q^2)|^2 \left\{ \left(1 + \frac{p}{E} \right)^2 (1 + \cos\theta) + 2(1 - \cos\theta) - \left(\frac{p}{E} \right)^2 (1 - \cos^2\theta) \right\} \\
&\quad \pm 2\operatorname{Re}\{(F_1(q^2) + \mu F_2(q^2))^* \lambda F_A(q^2)\} \\
&\quad \times \left\{ 2 \frac{p}{E} (1 - \cos\theta) + \left(\frac{p}{E} \right)^2 (1 - \cos^2\theta) \right\} \\
&\quad + |m_e b F_P(q^2)|^2 \frac{p^2}{E^2} \left(\frac{1 + \cos^2\theta}{2} - \cos\theta \right) \\
&\quad \left. - \operatorname{Re}(\lambda F_A^*(q^2) m_e b F_P(q^2)) \frac{M m_e}{E^2} (1 - \cos\theta) \right] \quad (21)
\end{aligned}$$

where p and E are the momentum and energy of the nucleon in the CM system, θ is the angle in the CM system between the incident ν and produced e , the integrand represents the angular distribution of the produced electron. The electron mass m_e has been ignored except for $(m_e b)$; hence $q^2 = 2p^2(1 - \cos\theta)$. If $m_e b$ is replaced by $m_\mu b$, (21) gives the cross-sections for

$$\left. \begin{aligned} & \nu + n \rightarrow p + \mu^- \\ & \bar{\nu} + p \rightarrow n + \mu^+ \end{aligned} \right\} \quad (\text{for } p \gg m_\mu).$$

We quote here two limiting cases of the elastic cross-sections (21).

(I) The non-relativistic limit, $p \ll E \approx M$, all F 's = 1. In such a case, weak magnetism (and pseudoscalar term) can be ignored.

$$\sigma(\bar{\nu} + p \rightarrow n + e^+) = \sigma(\nu + n \rightarrow p + e^-) = \frac{1}{\pi} G_V^2 (1 + 3\lambda^2) p^2, \quad \left(\lambda = -\frac{G_A}{G_V} \right) \quad (22)$$

This is a well-known result.*

(II) Extreme relativistic limit, $p \gg M$. There the values of the cross-sections depend critically on a choice of form factors as well as μ . If we choose the conserved weak vector current and further assume

* Notice that in (22) we put $m_e = 0$, proton mass M_p = neutron mass M_n ; otherwise we find

$$\begin{aligned}
& \sigma(\bar{\nu} + p \rightarrow n + e^+) = \frac{1}{\pi} G_V^2 (1 + 3\lambda^2) [p - (M_n - M_p)] [(p - M_n + M_p)^2 - m_e^2]^{1/2} \\
&= 2.292(1 \pm 2.6\%) \times 10^{-44} \text{ cm}^2 \left(\frac{p}{m_e} - 2.530 \right) \left[\left(\frac{p}{m_e} - 2.530 \right)^2 - 1 \right]^{1/2} \quad (22')
\end{aligned}$$

where p is the neutrino momentum.

$$\left. \begin{aligned} F_1(q^2) = F_2(q^2) &= 1 \left/ \left(1 + \frac{r^2}{12} q^2 \right)^2 \right. \equiv F(q^2) \\ F_P(q^2) &\approx F_A(q^2) = F(q^2), \end{aligned} \right\} \quad (23)$$

then we find the limiting value of cross-sections

$$\begin{aligned} \sigma(\nu + n \rightarrow p + e^-) &= \sigma(\bar{\nu} + p \rightarrow n + e^+) = \frac{2}{\pi} G_V^2 M^2 \frac{1}{r^2 M^2} \left[1 + \lambda^2 + \frac{1}{2} \left(\frac{3}{r^2 M^2} \right) \mu^2 \right] \\ &\approx \begin{cases} 0.75 \times 10^{-38} \text{ cm}^2 & (\text{for } \lambda = 1.25, \mu = 3.71) \\ 0.64 \times 10^{-38} \text{ cm}^2 & (\text{for } \lambda = 1.00, \mu = 3.71) \end{cases}. \end{aligned} \quad (24)$$

Notice that this limiting cross-section is constant. This is because the form factors restrict the scattering angle θ within very small forward cone $\theta \lesssim 1/rp$. In the extremely relativistic limit, contributions from all interference terms and the pseudo-scalar term are of the order of (or even less than) M^2/p^2 . Thus (24) can be applied also to $\nu + n \rightarrow p + \mu^-$ and $\bar{\nu} + p \rightarrow n + \mu^+$.

Between these two limiting cases, the cross-sections increase rapidly (roughly as the square of the CM momentum p , i.e. linear in the *lab* energy ν_{lab} of the neutrino). Under the assumption (23), $\sigma(\nu + n \rightarrow p + e^-)$ reaches essentially to the high energy limit for $\nu_{lab} \gtrsim M$.

The estimation of the cross-section at intermediate energy range can be tried on the basis of a simplified assumption (23).

We find:

$$\begin{aligned} \sigma(\nu + n \rightarrow p + e^-) &= \frac{1}{\pi} G_V^2 \left(\frac{pE}{E+p} \right)^2 \left[\left(1 + \frac{p}{E} \right)^2 (f_0 + f_1) \right. \\ &\quad + (1 + \mu)^2 \left(\frac{p}{E} \right)^2 (f_0 - 2f_1 + f_2) + \mu^2 \frac{p^2}{2M^2} \left(1 + \frac{p}{E} \right)^2 (f_0 - f_2) \\ &\quad + \lambda^2 \left\{ \left(1 + \frac{p}{E} \right)^2 (f_0 + f_1) + 2(f_0 - f_1) - \left(\frac{p}{E} \right)^2 (f_0 - f_2) \right\} \\ &\quad \left. \pm 2\lambda(1 + \mu) \left\{ 2 \frac{p}{E} (f_0 - f_1) + \left(\frac{p}{E} \right)^2 (f_0 - f_2) \right\} \right]. \end{aligned} \quad (25)$$

We have neglected the small contribution from the induced pseudoscalar term. $f_l (l=0, 1, 2)$ is defined by*

* For any reasonable choice of the form factor $F(q^2)$ (assumed to be monotonic decreasing), we verify the following limiting values for $f_l (l=0, 1, 2)$:

$$\begin{aligned} \left. \begin{aligned} f_0 &\rightarrow 1 \\ f_1 &\rightarrow 0 \\ f_2 &\rightarrow 1/3 \end{aligned} \right\} \quad \text{for } p \rightarrow 0 \\ \lim_{p^2 \rightarrow \infty} p^2 f_0 &= \lim_{p^2 \rightarrow \infty} p^2 f_1 = \lim_{p^2 \rightarrow \infty} p^2 f_2 \\ \lim_{p^2 \rightarrow \infty} p^4 (f_0 - f_1) &= 1/2 \lim_{p^2 \rightarrow \infty} p^4 (f_0 - f_2) \end{aligned}$$

$$f_i(p^2) = \frac{1}{4\pi} \int d\Omega_\theta F^2(q^2) (\cos\theta)^i, \quad q^2 = 2p^2(1 - \cos\theta),$$

more explicitly :

$$\begin{aligned} f_0 &= \frac{1}{3a} \left\{ 1 - \frac{1}{(1+a)^3} \right\} \\ f_1 &= \frac{1}{3a^2} \left\{ -1 + a + \frac{2}{(1+a)^2} - \frac{1}{(1+a)^3} \right\} \\ f_2 &= \frac{1}{3a^3} \left\{ -6a + (a+2)^2 - \frac{7}{1+a} + \frac{4}{(1+a)^2} - \frac{1}{(1+a)^3} \right\} \end{aligned} \quad (26)$$

and

$$\left. \begin{aligned} f_0 &\rightarrow \frac{1}{3a} \\ f_1 &\rightarrow \left(1 - \frac{1}{a}\right) \frac{1}{3a} \\ f_2 &\rightarrow \left(1 - \frac{2}{a}\right) \frac{1}{3a} \end{aligned} \right\} \quad \text{for } a \rightarrow \infty$$

where

$$a = \frac{1}{3} r^2 p^2 \approx 5 \left(\frac{p}{M} \right)^2.$$

It would be evident from Eq. (25) how the formula should be modified if different form factors F_1 , F_2 and F_A are taken.

The difference between the cross-sections for $\bar{\nu} + p$ and $\nu + n$ comes from the interference between J^V and J^A contributions to the matrix element; both in non-relativistic and extreme relativistic limits such a difference does vanish.

The cross-sections up to $\nu_{lab} = 10 M$ have been calculated from a simplified expression (25) and have been tabulated in Table II.

We must emphasize here the important contribution to the cross-section coming from the weak magnetism. Roughly one half of the total cross-section $\sigma(\nu + n \rightarrow p + e^-)$ shown in Table II is in fact due to the weak magnetism. Or, expressing in a converse manner, we may regard $\sigma(\nu + n \rightarrow p + e^-)$ as one of the sensible tests for the starting hypothesis: the conserved weak vector current.

For the sake of comparison, we quote here the result for $\lambda=1$, $\mu=0$ and "no" form factors $F_1(q^2) = F_A(q^2) = 1$:

$$\frac{\sigma_0(\nu + n)}{\sigma_0(\bar{\nu} + p)} = \frac{4G_V^2}{\pi} \left(\frac{p E}{E + p} \right)^2 \times \left\{ \frac{\left(1 + \frac{p}{E}\right)^2}{\left(1 + \frac{p^2}{3E^2}\right)} \right\}. \quad (27)$$

We find from this expression

Table II. Elastic cross-section in 10^{-38} cm^2 .

The following table is based on Eq.(25). Parameters are chosen as follows:

$$G_V = (1.01) \times 10^{-5} / M^2, \quad \mu = 3.71$$

$$F_1(q^2) = F_2(q^2) = F_A(q^2) = \frac{1}{\left(1 + \frac{r^2}{12} q^2\right)^2}$$

$$a = \frac{r^2}{3} p^2 = 5 \left(\frac{p}{M}\right)^2 \quad (r \approx 0.8 \times 10^{-13} \text{ cm})$$

$$1/M = 2.10_3 \times 10^{-14} \text{ cm (nucleon Compton wavelength)}.$$

We have tried two different values of the axial vector constant: $\lambda \equiv -G_A/G_V = 1.25$ and 1.00 ($\lambda = 1.25 \pm 0.04$ has been reported in the "Proceedings of 1958 Annual International Conference on High Energy Physics at CERN", edited by B. Ferretti, 1958, Geneva (p. 241).).

$\lambda \equiv -G_A/G_V$	$\lambda = 1.25$		$\lambda = 1.00$	
process	$\nu + n \rightarrow p + e^-$	$\bar{\nu} + p \rightarrow n + e^+$	$\nu + n \rightarrow p + e^-$	$\bar{\nu} + p \rightarrow n + e^+$
ν_{lab}/M				
0.5	0.701	0.190	0.565	0.157
1	0.886	0.331	0.711	0.296
2	0.836	0.479	0.699	0.415
3	0.810	0.551	0.682	0.475
4	0.794	0.592	0.671	0.509
5	0.785	0.620	0.665	0.533
6	0.779	0.639	0.661	0.549
7	0.774	0.653	0.657	0.561
8	0.771	0.664	0.656	0.571
9	0.769	0.673	0.655	0.578
10	0.767	0.681	0.653	0.585

$$\sigma_0(\nu + n) = \begin{cases} 1.9 \times 10^{-38} \text{ cm}^2 \\ 2.7 \times 10^{-37} \text{ cm}^2 \end{cases} \quad \text{for } \nu_{lab}/M = \begin{cases} 1 \\ 10. \end{cases}$$

Another reason for quoting this formula will be described in the subsequent section.

Essentially the same cross-sections as (25) can be expected for the cross-sections for $\bar{\nu} + p \rightarrow n + \mu^+$ and $\nu + n \rightarrow p + \mu^-$ apart from the pseudoscalar term. However, the pseudoscalar term contributes still substantially to the cross-sections for ν_{lab} below and around M . For example, we find

$$\left. \begin{aligned} \sigma(\nu + n \rightarrow p + \mu^-) &\approx 0.9 \times 10^{-38} \text{ cm}^2 \\ \sigma(\bar{\nu} + p \rightarrow n + \mu^+) &\approx 0.6 \times 10^{-38} \text{ cm}^2 \end{aligned} \right\}$$

$$\text{at } \nu_{lab} = 3M \text{ (and } \lambda = 1.25)$$

if we grant the estimation (20) of the pseudoscalar constant and $F_1 = F_2 = F_A = F_p = \text{Eq. (23)}$ (other parameters are same as Table II). The pseudoscalar contribution decreases with increasing $\nu_{lab} (\gg M)$.

It would be desirable to discuss the sensitivity of total cross-sections to form factors assumed. For this purpose we have tried two other possibilities:

$$F_1(q^2) = F_2(q^2) = F_A(q^2) = F_V(q^2) = \begin{cases} \left(1 + \frac{r^2}{6} q^2\right)^{-1} & \text{(I)} \\ \exp\left[-\frac{r^2}{6} q^2\right] & \text{(II)} \end{cases}$$

We have, of course, to choose all form factors so as to reproduce the same result for small q^2 :

$$F(q^2) = 1 - \frac{1}{6} r^2 q^2 + \dots$$

The first choice (I) gives larger cross-sections (about two times) than those listed in Table II. However, it should be noticed that (I) has already been excluded by Hofstadter et al.¹⁴⁾ for the electromagnetic structure of physical nucleons, and hence (I) cannot be accepted for our weak (vector) form factor provided we take the conserved weak vector current. On the other hand, the second choice (II) leads to essentially the same results as tabulated in Table II. This is not surprising, because both form factors (23) and (II) decrease rapidly with increasing q^2 and values of the cross-sections are essentially determined by the integral on the scattering angle θ within the narrow forward cone ($\theta \lesssim 1/rp$ if $rp \gg 1$) common to both cases.

Next, we have to discuss $F_A(q^2)$. As stated above, we know very little about it. However, we may expect in any sensible model of strong interactions

$$r_A \lesssim (\text{Hofstadter radius}) = r \quad (28)$$

where r_A is defined again by*

$$F_A(q^2) = 1 - \frac{1}{6} r_A^2 q^2 + \dots \text{ for small } q^2$$

(in this connection see ref. 13). From this we may conclude that

$$F_A(q^2) \gtrsim F(q^2) \text{ given in (23).}$$

If our reasoning is true (which seems to be quite reasonable), we can assert that (provided the weak interaction is given by the point Fermi interaction (15) and the conserved weak vector current is assumed) the results given in Table II and the

* If we take such an axial vector current J_{α}^A as suggested by R. P. Feynman (unpublished)

$$\partial_{\alpha} J_{\alpha}^A = -\frac{2Mi}{f_{\pi}} \varphi_{\pi} \quad (a)$$

where f_{π} is the pion-nucleon coupling constant ($f_{\pi}^2/4\pi \simeq 15$) and φ_{π} is the pion field operator (the same isovector component as J^A), then the difference of the pion lifetime predicted from (a) and the experimental lifetime can be used to guess the value of r_A ; the result is not incompatible with (28).

high energy limit (24) give the *lower bounds** of the "elastic" cross-sections $\sigma(\bar{\nu} + p \rightarrow n + e^+)$ and $\sigma(\nu + n \rightarrow p + e^-)$. (Evidently the *total* cross-sections for (anti) neutrino-nucleon collisions must be larger than the *partial* cross-sections (25).)

Up to now we did not take into account a possible non-local effect^(7,9) associated with the original weak interaction (15). If such an effect does exist, all the form factors F_1, F_2, F_A and F_P must be regarded as resultant non-localities due to the weak interaction itself and the contribution from strong interactions. As a typical example, we take the intermediate charged boson with the mass m_B . Then F_1 and F_2 are equal to the Hofstadter form factors

$$F(q^2) = \left(1 + \frac{1}{12} r^2 q^2\right)^{-2}$$

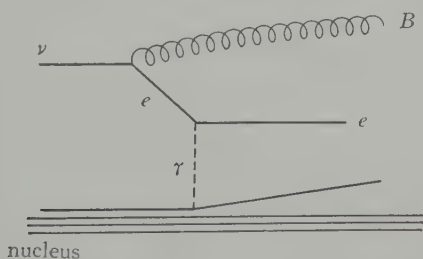
multiplied by $m_B^2/(m_B^2 + q^2)$. As long as m_B is rather large, say, $m_B > 3/r$, the non-locality due to intermediary bosons, $m_B^2/(m_B^2 + q^2)$, is not important. Therefore, provided any non-locality inherent to the weak interaction is *small*—it is not unreasonable—we feel that the results given in Table II are fairly close to the truth.**

We have many other two-body processes induced by neutrinos and consistent with the law of lepton number conservation when we take into account hyperons:

$$\bar{\nu} + p \rightarrow \begin{cases} \Lambda^0 + e^+, & \Lambda^0 + \mu^+ \\ \Sigma^0 + e^+, & \Sigma^0 + \mu^+ \end{cases} \quad (29)$$

* This is quite true, if $F(q^2)$ is not monotonically decreasing. As was emphasized by R. G. Sachs, $F_1(q^2), F_2(q^2)$ may not be smoothly decreasing.

** On the other hand if the intermediary boson B have a relatively small mass (say, $\sim m_K$), then the neutrino-nucleon collisions can produce this boson B, and the B-production cross-section σ_B will be linear in the Fermi constant G_F . In order to balance the total energy and total momentum, one virtual photon must be involved in the process:



where the nucleus may or may not stay in the original ground state. Depending on these two possibilities we find the proportionalities for σ_B (per proton):

$$\sigma_B \propto \begin{cases} G_F \times \frac{Z}{137} \\ \text{or } G_F \times \frac{1}{137} \end{cases}$$

where Z is the number of protons in the target nucleus. In either case σ_B seems to be large compared with the elastic cross-sections (25), well above the B-threshold ($\nu_{iab} \geq M$).

$$\bar{\nu} + n \rightarrow \Sigma^- + e^+, \Sigma^- + \mu^+ \quad (30)$$

$$\nu + n \rightarrow \Sigma^+ + e^-, \Sigma^+ + \mu^- \quad (31)$$

$$\left. \begin{aligned} \bar{\nu} + p &\rightarrow \Xi^0 + e^+, \Xi^0 + \mu^+ \\ \bar{\nu} + n &\rightarrow \Xi^- + e^+, \Xi^- + \mu^+ \end{aligned} \right\} \quad (32)$$

These are induced by the weak interaction responsible to leptonic decays of strange particles:

$$\frac{G'}{\sqrt{2}} J_\alpha^{SNC} \left[(\bar{e} \gamma_\alpha (1 + \gamma_5) \nu) + (\bar{\mu} \gamma_\alpha (1 + \gamma_5) \nu) \right] + \text{h. c.} \quad (33)$$

where J^{SNC} is a strangeness-non-conserving weak current.¹⁶⁾ If J^{SNC} induces only the strangeness change $\Delta S = \pm 1$, (32) must be extremely slow processes (whose reaction rates are proportional to $(G')^4$). If the J^{SNC} is such that only processes with $\Delta S/\Delta Q = +1$ are allowed,⁴⁾ then (31) is also forbidden in the lowest order in the weak interaction (33). Experiments seem to show that the renormalized $(G')^2$ is one order of magnitude smaller than G_F^2 at least for $\Lambda \rightarrow p + e^- + \bar{\nu}$.¹⁷⁾ We may therefore expect naïvely

$$\sigma(\bar{\nu} + p \rightarrow \Lambda^0 + e^+) / \sigma(\bar{\nu} + p \rightarrow n + e^+) \lesssim 10^{-1} \quad (34)$$

for high neutrino energies ($\sqrt{2}m_e \nu_{lab} \gg \Lambda$ - p mass difference), unless the relevant "form factors" for these two-body processes differ considerably. To derive any better guess of the cross-sections for (29)–(32), apparently we need more experimental data on leptonic decays of hyperons.

So much about the "elastic" processes, we are going to discuss the inelastic processes in the subsequent section.

§ 4. Reactions induced by energetic neutrinos

In this section we give a brief discussion on other reactions than two-body processes described in the preceding section. First of all, we have the pion production; there are six single pion production modes:

$$\left. \begin{aligned} \bar{\nu} + p &\rightarrow \begin{cases} n + \pi^0 + e^+ \\ p + \pi^- + e^+ \end{cases} \\ \bar{\nu} + n &\rightarrow n + \pi^- + e^+ \\ \nu + p &\rightarrow p + \pi^+ + e^- \\ \nu + n &\rightarrow \begin{cases} p + \pi^0 + e^- \\ n + \pi^+ + e^- \end{cases} \end{aligned} \right\} \quad (35)$$

and as far as the available energy allows, we can expect many modes of multiple production of strongly interacting particles. All these reactions are induced by

the weak interactions (15) or (33) in combination with strong interactions.

Number n of open channels increases so rapidly with increasing neutrino energy. The cross-section per each open channel will be, on the average, of the order of $\sigma(\nu+n \rightarrow p+e^-)$ given in (25),* and thus the total cross-sections $\sigma_{tot}(\nu+N)$ and $\sigma_{tot}(\bar{\nu}+N)$ (we denote the nucleon by N)

$$\sigma_{tot}(\nu+N) \simeq \sigma_{tot}(\bar{\nu}+N) \sim \sigma(\nu+n \rightarrow p+e^-) \times n$$

will increase as $p^2 \propto \nu_{lab}$ (p : CM momentum to the incident neutrino): or more optimistically σ_{tot} is of the order of $\sigma_0(\nu+n)$ given in (27), the simplest prediction from perturbation theory (without renormalization complication):

$$\sigma_{tot}(\text{neutrino-nucleon}) \sim \sigma_0(\nu+n) \quad (36)$$

for large ν_{lab} . This conclusion can also be shown to be very plausible by applying the so-called closure approximation¹⁸⁾ (with an appropriate modification) to estimate σ_{tot} (neutrino-nucleon).

We can, in principle, make a more definite statement on the contribution $\sigma_{\nu}^{(V)}$ to the neutrino cross-section due to J_{α}^V based on the assumption of divergenceless J_{α}^V . J_{α}^V and the isovector part j_{α}^V of the electric current ($ej_{\alpha} = e(j_{\alpha}^S + j_{\alpha}^V)$) are the components of one isovector. Therefore, as long as the charge independence is valid for the system consisting of strongly interacting particles, the neutrino reaction $\sigma_{\nu \rightarrow e}^{(V)}$ (or $\sigma_{\nu \rightarrow \mu}^{(V)}$)

$$\nu + n \rightarrow F + e^- \quad (37)$$

$$\text{or } \nu + n \rightarrow F + \mu^- \quad (37')$$

due to J_{α}^V are intimately connected with the electron (or muon) induced reaction and $\sigma_e^{(V)}$ (or $\sigma_{\mu}^{(V)}$)

$$e + p \rightarrow F + e, \quad (38)$$

$$\text{or } \mu + p \rightarrow F + \mu, \quad (38')$$

due to the isovector part j_{α}^V , where F denotes a final state of strongly interacting particles and F must have the (total) nucleon number 1, and total electric charge 1 (in unit of the protonic charge). Under the assumption of keeping only J_{α}^V and j_{α}^V , the cross-sections of

$$\bar{\nu} + p \rightarrow F + e^+$$

$$\bar{\nu} + p \rightarrow F + \mu^+$$

$$e + n \rightarrow F' + e$$

$$\mu + n \rightarrow F' + \mu$$

* We can estimate the cross-section $\sigma(\nu+N \rightarrow N+\pi+e)$ for low energy pion production by using the Chew-Low static model for pion-nucleon interactions. Such an estimation shows that $\sigma(\nu+N \rightarrow N+\pi+e)$ is of the order of $\sigma(\nu+n \rightarrow p+e^-)$.

are trivially related to those of (37), (38) by "isospin rotation" (F and F' are identical states except for orientation of isospin). More precisely, the relationship¹⁹⁾ between "differential" cross-sections of these processes (37) and (38) is given by* (m_e is set equal to 0)

$$\frac{d\sigma_{\nu \rightarrow e}^{(F)}(q^2, E_F; F)}{d\sigma_e^{(F)}(q^2, E_F; F)} = 2 \left(\frac{G_V}{e^2} q^2 \right)^2 = 2.4 \left(\frac{q^2}{M^2} \right)^2 \times 10^{-8} \quad (39)$$

where q is the 4-momentum transfer (the neutrino momentum minus the electron momentum for (37), and the initial electron momentum minus the final electron momentum for (38)), E_F is the total energy of the strongly interacting system F , $e^2/4\pi = 1/137$ is the fine structure constant. It should be noticed that in (39) we are comparing two cross-sections for the *same momentum transfer* squared q^2 and the same *final energy* E_F of the system F . It is evident that

$$\frac{\sum_F d\sigma_{\nu \rightarrow e}^{(F)}(q^2, E; F)}{\sum_F d\sigma_e^{(F)}(q^2, E; F)} \equiv \frac{d\sigma_{\nu \rightarrow e}^{(V)}(q^2, E)}{d\sigma_e^{(V)}(q^2, E)}$$

is again equal to the right-hand side of (39). If the muon is energetic it is evident that we can replace $\sigma_e^{(V)}$ by $\sigma_\mu^{(V)}$ in Eq. (39). For the total contribution

$$\sigma_\nu^{(V)} = \sigma_{\nu \rightarrow e}^{(V)} + \sigma_{\nu \rightarrow \mu}^{(V)},$$

we find

$$\sigma_\nu^{(V)} \simeq 5 \times 10^{-8} \langle (q^2/M^2)^2 \rangle_{av} \times \sigma_e^{(V)}. \quad (40)$$

Evidently, experimental information about the electron- and muon-induced reactions can be used to estimate $d\sigma_\nu^{(V)}(q^2, E)$ (or even the total contribution $\sigma_\nu^{(V)}$). Unfortunately, available data on electro-pion production is limited to low energy regions²⁰⁾ and are not sufficient enough to guess $\sigma_\nu^{(V)}$ at multi-Gev neutrino energies. Some information on nuclear interactions due to energetic muons is available from the cosmic-ray experiments underground.²¹⁾ Difficulties of using such data lie in the following points:

- (a) We must separate the cross-section into partial contributions $\sigma_\mu^{(V)}$ and $\sigma_\mu^{(S)}$ due to j_α^V and j_α^S (and possibly the interference term);
- (b) nuclear interactions of muons are intermediated by the virtual photon, and resulting reaction rates favours to small momentum transfer (q^2).

Particularly, (b) makes difficult to guess the average quantities $\langle (q^2/M^2)^2 \rangle_{av}$ which is so vital when one tries to apply the formula (40). Under the precaution of these comments, we have tried preliminary analysis to estimate neutrino cross-

* If a charged intermediate boson with mass m_R exists, $2(G_V/e^2)^2(q^2)^2$ must be replaced by $2(G_V/e^2)^2 \cdot (q^2)^2 \cdot (m_R^2/(m_R^2 + q^2))^2$.

section $\sigma_\nu^{(r)}$. The result appears to show that the meson production of low multiplicity (say $E_F \lesssim M$) due to energetic neutrinos (say, $\nu_{lab} \gtrsim 10M$) is of the order of 10^{-38} cm^2 .

Conclusion

We know that the neutrino is electrically neutral, spin 1/2, and has the rest mass less than several 100 ev.²²⁾ Neutrinos appear to interact with other particles only through "weak" interactions. The last statement is confirmed as long as neutrino-reactions involve only "small" momentum transfer. The leptons, in particular the neutrinos, are so curious particles that any kind of experiments on leptonic processes at high energies are extremely interesting, even though experimental results are negative (for example, check the process $\nu + e^- \rightarrow e^- + \nu$).

We have estimated the cross-sections of two-body neutrino processes $\bar{\nu} + p \rightarrow n + e^+$, and $\nu + n \rightarrow p + e^-$ based on the weak Fermi interaction, and found $\sigma(\nu + n \rightarrow p + e^-) \gtrsim 0.8 \times 10^{-38} \text{ cm}^2$ for the neutrino energy (lab system) $\nu_{lab} \gtrsim M$. Total cross-section σ_ν for high energy neutrino must be substantially larger than the elastic cross-section $\sigma(\nu + n \rightarrow p + e^-)$ quoted above; therefore detection of neutrino interactions with $\nu_{lab}/M = 1 \sim 10$ does not seem to be quite hopeless.

Acknowledgements

The author wishes to thank Dr. J. Prentki for his extremely valuable discussions. He is also indebted to Dr. S. L. Glashow for his discussions and to Mr. W. Klein for some numerical computations.

It should be mentioned that the experiment of seeing neutrino reactions is now under consideration at CERN. In this connection the author has enjoyed discussing the matter with Professors G. Bernardini and G. Cocconi.

Appendix

The differential cross-section for elastic processes in the lab system is given by

$$\begin{aligned} \frac{d\sigma}{d\omega} (\nu + n \rightarrow p + e^-) &= \frac{G_F^2}{2\pi^2} \nu^2 \frac{\left(\cos \frac{\theta}{2}\right)^2}{\left[1 + \frac{2\nu}{M} \left(\sin \frac{\theta}{2}\right)^2\right]^3} \\ &\times \left[|F_1(q^2)|^2 + \frac{q^2}{4M^2} \left\{ 2|F_1(q^2) + \mu F_2(q^2)|^2 \left(\tan \frac{\theta}{2}\right)^2 + \mu^2 |F_2(q^2)|^2 \right\} \right. \\ &\quad \left. + \lambda^2 |F_A(q^2)|^2 \left\{ 1 + 2 \left(\tan \frac{\theta}{2}\right)^2 + \frac{q^2}{2M^2} \left(\tan \frac{\theta}{2}\right)^2 \right\} \right. \\ &\quad \left. \pm 2 \text{Re} \left\{ (F_1(q^2) + \mu F_2(q^2)) * \lambda F_A(q^2) \right\} \times \left\{ 2 \frac{\nu}{M} - \frac{q^2}{2M^2} \right\} \left(\tan \frac{\theta}{2}\right)^2 \right] \end{aligned}$$

where

ν = incident neutrino energy in lab system ;

θ = angle between the neutrino and the electron ;

$$q^2 = (\text{momentum transfer})^2 = \frac{\left(2\nu \sin \frac{\theta}{2}\right)^2}{1 + \frac{2\nu}{M} \left(\sin \frac{\theta}{2}\right)^2}.$$

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Solar Modulation of Primary Cosmic Rays*

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It is proposed that the solar modulation of cosmic rays is accounted for in terms of a combined model, for which disordered as well as ordered magnetic fields are responsible. These magnetic fields are considered to be of solar origin. The diffusion region of disordered magnetic fields is assumed to extend widely over the solar system and takes part in the modulation of the low energy part of primary cosmic rays. Special account is given for the interpretation of the characteristic shape of the low energy spectra common to all heavy primary cosmic rays recently observed.²⁾ The eleven-year variations are ascribed to the superposition effect of ordered magnetic fields distributed near the earth orbit. The diurnal variations are shown to be a direct consequence of the present model.

Our arguments are based upon the assumption of two kinds of solar streams: the continuously ejected streams and the more intense ones produced by solar eruptions. A quantitative estimate of the physical parameters to describe the present model is worked out.

§ 1. Introduction

The relations between the solar ejection of highly ionized gases and the various types of time variations of cosmic rays have been disclosed by many authors, though no theory is yet complete enough. If we lay aside the unusual cosmic-ray production associated with solar flare events, we can retain the essential constancy and isotropy of primary cosmic rays, since cosmic rays are regarded to be substantially of galactic origin.³⁾ The idea that the solar corpuscular streams cause a variation in the intensity of primary cosmic rays was first suggested by Alfvén.⁴⁾ Alfvén's model in its original form seems at present to be of limited applicability. Since then many hypothetical mechanisms of hydrodynamic nature have been proposed to explain the observed time variations of cosmic rays; some of them have given at least positive results, to which we shall refer here.

Morrison⁵⁾ has suggested the statistical modulation of primary cosmic rays by interplanetary magnetic fields of solar origin, in an effort to give a consistent interpretation of several phenomena, such as the cosmic-ray storms (Forbush decreases), the eleven-year variations and the 27-day recurrent phenomena. The cosmic-ray storms are ascribed to a screening effect of cosmic rays by the intense magnetic clouds which are ejected from the equatorial region of the sun and correlate strongly

* This work was reported partly in the paper titled under "Interpretation of the energy spectrum of heavy primary cosmic rays" by S. Hayakawa, M. Koshiba and Y. T. ¹⁾ at the Moscow Conference on Cosmic Rays.

with solar activity. It has also been proposed that a diffusion region of random magnetic fields, extending widely over a bulky portion of the solar system, is responsible for the eleven-year variations as well as for the low energy cutoff of the primary spectrum. In order to sustain the assumed diffusion region, the continual emission of corpuscular streams should be needed even from high heliographic latitudes. Several difficulties have been argued concerning Morrison's model, for example, by a critical work of Parker.⁶⁾ Here we remark one of the difficulties that both the low energy spectral shape and eleven-year variations cannot be explained by the same mechanism, since the latter indicate the modulation of cosmic rays with energies as high as 30 Gev, while the former is concerned mainly with the energy region less than 1 Gev.

The continual emission of corpuscular streams is thought to be a serious difficulty with which such a diffusion model as Morrison's one meets. Parker has expressed a positive view on this point.⁷⁾ As is generally believed, the solar corona is found to be not in hydrostatic equilibrium, but to keep its stability by steadily ejecting corpuscular streams. Obviously, Parker's theory is not an established one but is open to question. It would, nevertheless, be permissible to regard the continual emission of solar streams as a working hypothesis, since there is no astrophysical evidence against it. One may accept it as reasonable to use cosmic rays as test particles travelling through interplanetary magnetic fields.

Referring to the diurnal variations of cosmic rays, a revised version of Alfvén's original idea has been proposed by Nagashima⁸⁾ and by Dorman,^{9),10)} independently. They have assumed the presence of an outward steady flow of solar streams, which will cause energy changes and consequently intensity changes of cosmic rays incident upon the earth.

The characteristic shape of the low energy spectra of heavy primaries observed recently by Aizu et al.²⁾ makes it possible to investigate the solar modulation mechanism more quantitatively. There are naturally many possibilities of interpreting the implications in this result.^{1),11)} We here resort only to the solar modulation mechanism.

In this paper we shall develop a combined model by which we mean the modulation due to both disordered and ordered magnetic fields. In Section 1, the observed spectra of heavy primaries are accounted for in terms of the diffusion in disordered magnetic fields, extending far out of the earth orbit. In Section 2, the presence of a decelerating region of ordered magnetic fields near the earth orbit is inferred from the eleven-year variations. In Section 3, the other types of time variations are discussed referring to the interplanetary magnetic fields suggested by the present model.

§ 2. Modulation due to disordered magnetic fields

Let us consider such corpuscular streams of highly ionized gases that are steadily emitted from the solar corona, as was proposed by Parker.⁷⁾ The solar streams thus

produced are supposed to have uniform magnetic fields near the sun.¹²⁾ As the distance from the sun increases, hydrodynamical instability in them will grow. When the solar streams pass through the earth orbit, they will become waved or kinked and their magnetic fields are entangled at the same time. Therefore the formation of a region of disordered magnetic fields beyond the earth orbit may well be possible. It is important to distinguish between the role of rather regular magnetic fields near the earth and that of random ones far beyond it. We call attention only to the latter for the present.

The diffusion region is assumed to lie outside the earth orbit at distances from the sun, r_i to r_s ($r_s > r_i > r_e$, r_e = the sun-earth distance). The magnetic fields inside the diffusion region are regarded to consist of magnetic clouds of the average dimension $l(r)$, of the strength $B(r)$, and with the average distance between clouds, $\lambda_0(r)$, all depending upon the distance from the sun, r . For simplicity these magnetic clouds are assumed to be distributed symmetrically about the sun and to move steadily outwards with constant velocity V . Such a region of magnetic clouds will cause a reduction of intensity of low energy cosmic rays. Such simplifications will not lose essential points in comparison with Morrison's model with a diffusion region of ellipsoidal shape. The assumption of constant outward velocity does not seem unreasonable because a deceleration of the magnetic clouds due to collisions with interplanetary gases is less effective.

We define the modulation factor $b(R)$ as

$$j(R) dR = b(R) j_0(R) dR \quad (1)$$

where $j_0(R) dR$ is the differential rigidity spectrum of cosmic rays in the interstellar space and $j(R) dR$ is that modulated by the diffusion region. (We express the cosmic-ray spectrum in rigidity form hereafter.) Since we here deal with the equilibrium distribution of cosmic-ray particles, the net flux of cosmic-ray particles inside the diffusion region is zero, that is,

$$Vf(r, R) - \frac{\lambda(r, R)}{3} \frac{\partial}{\partial r} [vf(r, R)] = 0$$

where $f(r, R)$ is the differential rigidity spectrum at r , v the velocity of a cosmic-ray particle with the rigidity R and $\lambda(r, R)$ its effective mean free path. With the boundary conditions $f(r_s, R) = j_0(R)$ and $f(r \leq r_i, R) = j(R)$, we find

$$j(R) = j_0(R) \exp \left(-\frac{3}{v} \int_{r_i}^{r_s} \frac{V}{\lambda(r, R)} dr \right),$$

that is,

$$b(R, v) = \exp \left[-\frac{3V}{v} \int_{r_i}^{r_s} \frac{dr}{\lambda(r, R)} \right]. \quad (2)$$

It should be noted that $b(R) \equiv b(R, v)$ is dependent not only on R but also on v . $\lambda(r, R)$, the effective mean free path can be expressed as

$$\lambda(r, R) = \lambda_0(r) \left\{ 1 + (R/R_c(r))^2 \right\}, \quad (3)$$

with

$$R_c(r) \cong B(r)l(r). \quad (4)$$

The r -dependences of $\lambda_0(r)$, $l(r)$ and $B(r)$ enable us to perform the integral in (2). We make further assumptions: The configuration of the magnetic clouds is stable and stationary, and thus both the magnetic flux of a cloud, $\pi l^2(r)B(r)$, and the number of clouds are conserved which results in

$$\frac{\partial}{\partial t} N(r) + \nabla \cdot [N(r) \mathbf{V}] = \frac{V}{r^2} \frac{\partial}{\partial r} [r^2 N(r)] = 0,$$

where $N(r)$ is the density of the clouds, so that there holds $N(r) \propto r^{-2}$. Then $\lambda_0(r) = 1/\pi l^2(r)N(r)$ and $\lambda_0(r) \cong l(r)$ lead us to $\lambda_0(r) \propto r^{2/3}$. The conservation of the magnetic flux gives us $B(r) \propto r^{-4/3}$. Summarizing the above arguments, we have

$$\lambda_0(r) \cong l(r) \propto r^{2/3}, \quad B(r) \propto r^{-4/3} \text{ and } R_c(r) \propto r^{-2/3}. \quad (5)$$

Now we are able to give an explicit form of the modulation factor, neglecting the ratio of r_i/r_s compared with unity, as

$$b(R, R_0) = \exp \left[-9 \left(\frac{r_s}{r_i} \right)^{1/3} \frac{V r_i}{c \lambda_0(r_i)} \cdot \frac{(R^2 + R_0^2)^{1/2}}{R} \right] \text{ for } R \ll R_c, \quad (6a)$$

$$b(R, R_0) = \exp \left[-3 \frac{V r_i}{c \lambda_0(r_i)} \cdot \frac{(R^2 + R_0^2)^{1/2} R_c^2(r_i)}{R^3} \right] \text{ for } R \gg R_c. \quad (6b)$$

Here we have taken into account

$$v = cR / (R^2 + R_0^2)^{1/2} \quad \text{with} \quad R_0 \equiv AMc^2 / Ze, \quad (7)$$

where A and Z are respectively the mass number and the atomic number of a cosmic-ray particle and Mc^2 the rest energy of a nucleon. We have rewritten $b(R, v)$ as $b(R, R_0)$. Interpolating (6a) and (6b), we obtain

$$b(R, R_0) = \exp \left[-K \frac{\bar{R}_c^2 (R^2 + R_0^2)^{1/2}}{R(R^2 + \bar{R}_c^2)} \right], \quad (8)$$

with

$$K = 9Vr_s / c\lambda_0(r_s) \quad \text{and} \quad \bar{R}_c = R_c(r_i) / \sqrt{3} (r_s/r_i)^{1/6}. \quad (9)$$

It should be noted that the modulation factor in this form is simply expressed in terms of two parameters, K and \bar{R}_c , depending upon the characteristic quantities of the diffusion region. It is plotted in Fig. 1 with suitable sets of parameters. According to (8), the modulated spectrum independent of particle species is

expected only for $R \gg R_0$, far above 2 Gv. Below R_0 the rigidity spectra of protons and heavy nuclei are certainly different. The difference between the maximum position of the proton spectrum, R_{mp} , and that of the heavy primary spectrum, R_{mh} , is reasonably small, when \bar{R}_0 is taken as smaller than 1 Gv. Such a small value of \bar{R}_0 results in a large value of K .

In a quiet period of solar activity, 1954, the observed α -spectrum^{13),14)} showed a peak at

$$R_{mh}('54) \cong 1.6 \text{ Gv}, \quad (10)$$

while the proton spectrum deduced from the latitude effect¹⁵⁾ has a peak at $R_{mp} \cong 1.4 \text{ Gv}$, slightly different from R_{mh} in (10). As to the proton spectrum, there exist uncertainties. It has recently been reported that the proton component has the same rigidity spectrum as heavier nuclei.¹⁶⁾ If strictly so, the diffusion model will fail due to the R_0 -dependence. But the estimation of R_{mp} by this experiment is not free from ambiguities. As will be shown later, the predicted spectrum of the proton component is only slightly different from that of heavier nuclei, and then

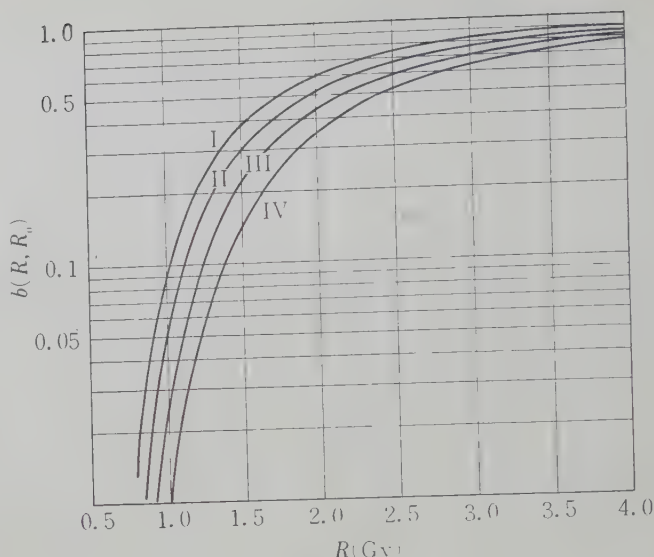


Fig. 1-a.

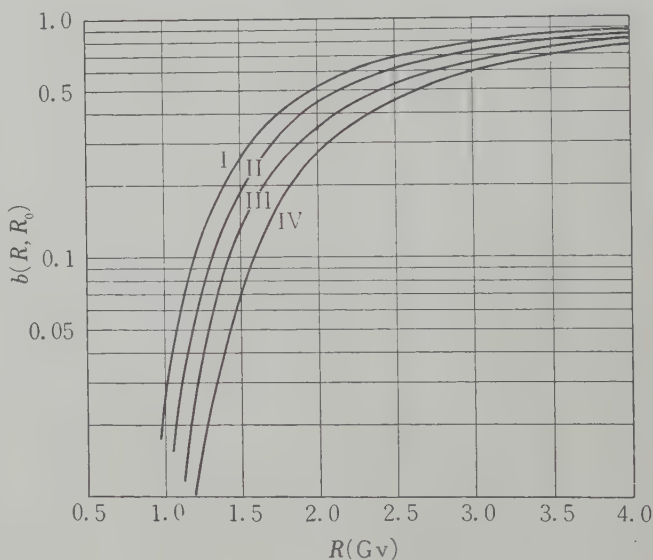


Fig. 1-b.

The modulation factor of the diffusion in disordered magnetic fields $b(R, R_0)$ defined in Eq. (8), is plotted against the rigidity of a cosmic ray particle R . $b(R, R_0)$ has two parameters, K and \bar{R}_0 , which are, as defined in (9), dependent upon the characteristic quantities of the diffusion region. The values of $b(R, R_0)$ are calculated for the four sets of assignment of these parameters. Fixing $K=20$, we take $\bar{R}_0=0.31, 0.35, 0.40$ and 0.45 for the case I, II, III and IV respectively. Fig. 1-a shows $b(R, R_0)$ of protons for respective cases and 1-b that of heavy nuclei.

we could rather say that this experiment would support the present analysis. As the sun becomes active, the maximum position may shift to

$$R_{mh}('57) \cong 2.2 \text{ Gv} \quad (11)$$

as was given in reference 2) for all heavy nuclei and by the Minnesota groups¹⁴⁾ for α -particles, both observed in 1957. The integral intensity of heavy primaries is reduced by factor 2.

If the spectrum in the interstellar space is expressed as

$$j_0(R) dR = \text{const. } R^{-\gamma} dR \quad \text{with } \gamma = 2.5, \quad (12)$$

then the values of R_{mp} and R_{mh} allow us to choose the values of the parameters defined in (9). The characteristic quantities of the diffusion region in the solar quiet period are summarized in Table I, in which the value of V is inferred from retardation of the geomagnetic disturbances as compared to solar events, and the inner radius of the diffusion region is taken plausibly as $r_i \cong 3r_e$, as will be discussed in the next section. In the active period of solar activity, the values in Table I change; perhaps the strength of the magnetic fields increases. We change the values of the parameters, so that the shift of R_{mh} and the reduction in the intensity level fit the observations.* The modulated rigidity spectra are calculated and are shown in Fig. 2-a and 2-b. The results are in good agreement with the observations.** The energy output from the sun to sustain the steady outward flow of such magnetic clouds is estimated to be $\sim 10^{29}$ erg sec⁻¹, if the gas density of the clouds is of the order of 10^2 cm⁻³ at the earth orbit. This and the values in Table I are not inconsistent with those estimated by Parker.⁷⁾

Table I. Possible sets of the values of the quantities to describe the diffusion region lying at distances from the sun, r_i to r_s
 V : the outward velocity of magnetic clouds,
 l : their typical dimension,
 λ_0 : the average distance between them,
 B : the strength of magnetic fields.

r_i	$3r_e$	$3r_e$
r_s	$20r_e$	$30r_e$
V	10^8 cm/sec	10^8 cm/sec
$\lambda_0(r_i) \cong l(r_i)$	1.3×10^{11} cm	1.5×10^{11} cm
$\lambda_0(r_s) \cong l(r_s)$	4.5×10^{11} cm	6.8×10^{11} cm
$B(r_i)$	1.9×10^{-5} gauss	1.7×10^{-5} gauss
$B(r_s)$	1.5×10^{-6} gauss	0.8×10^{-6} gauss

(r_e = the sun-earth distance $\cong 1.5 \times 10^{13}$ cm)

* For $R \gg R_0$, a simple relation exists between the shifts of the maximum position of the spectrum and the changes in the intensity level, as was pointed out by J. Nishimura.

** At the Moscow Conference on Cosmic Rays in 1959, C. J. Waddington reported the eleven-year variation of the α -particle spectrum during the last solar cycle. His results fit quite well with our theory.

The spectra of heavy primaries we here referred to were measured by means of a balloon-flight on September 11, 1957. At that time a solar eruption took place and a solar radio outburst was observed.¹⁷⁾ Therefore one might raise a question that the shift of R_{mh} and the reduction in the intensity level might be attributed to such streams that are emitted sporadically from the equatorial region of the sun and beared with more intensive magnetic fields than above. However, we believe that the influence of the denser streams will be reflected upon a change in the intensity of rather high energy particles and is less effective for the reduction of the overall intensity of low energy particles, since they occupy a small portion of the diffusion region.* It may be remarked that the diffusion region varies with

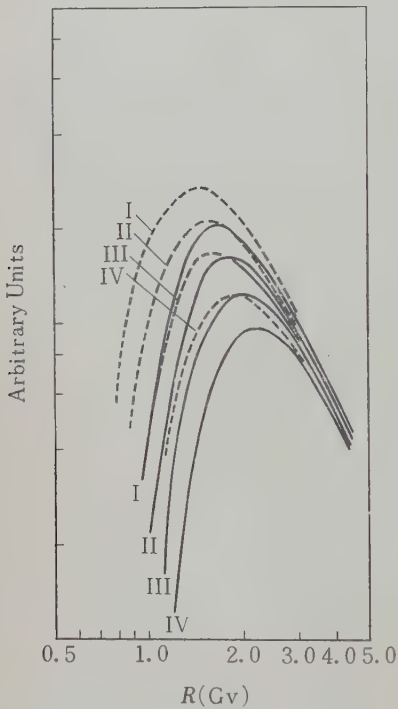


Fig. 2-a. The rigidity spectra modulated by the diffusion region are plotted. We take the primary spectrum as $R^{-2.5} dR$ and calculate the modulated rigidity spectrum $b(R, R_0) R^{-2.5}$ for the four cases. The solid lines correspond to heavy nuclei and the dotted lines to protons.

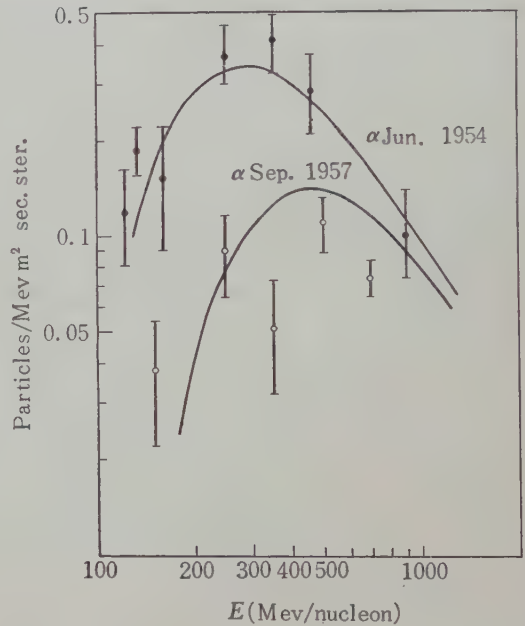


Fig. 2-b. The modulated energy spectra of heavy nuclei for the cases I and IV are compared with the observations of α -spectra in 1954 and 1957.

* The further progress of this experiment finds that the intensity of heavy primaries shows a considerable decrease up to 20 Gev and, in detail, the heavier component decreases more.¹⁸⁾ The mere fact that the changes in intensity are remarkable even in the high energy region may be attributed to such an effect of ordered magnetic fields as will be discussed in the next section. The intensity differences among the different species of heavy nuclei seem to be of confusion, but may be caused by unusual absorption of high energy particles in the atmosphere due to magnetic disturbance, as was pointed out by S. Hayakawa.

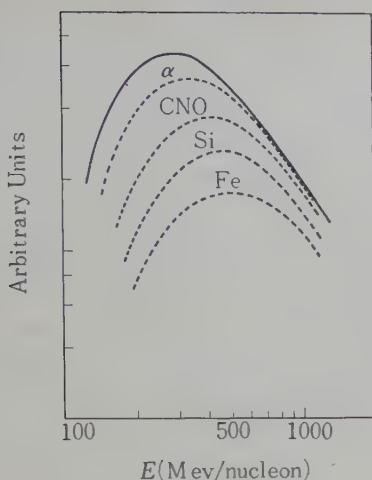


Fig. 3-a.

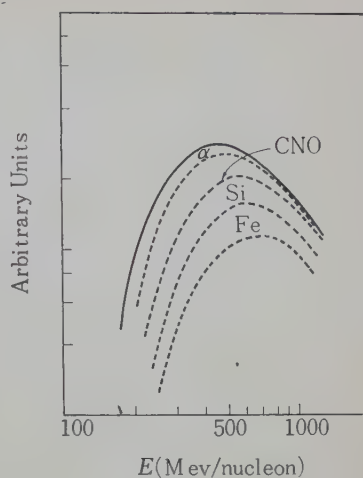


Fig. 3-b.

The ionization energy loss of primary cosmic rays in the interstellar matter, say 3 gcm^{-2} , is corrected for the results of Figs. 2-a and b. The modulated energy spectra of heavy nuclei, corrected for the ionization loss are shown by the dotted lines, while the spectra not corrected for it by the solid lines. Fig. 3-a corresponds to the case I and Fig. 3-b to the case IV.

solar cycle, though it connects loosely with sunspots themselves.

Finally we remark the effect of the ionization loss of heavy primaries in the interstellar space. Starting from the spectrum (12) at the sources of cosmic rays, the ionization loss in the interstellar space with, say, the thickness of 3 gcm^{-2} causes different shapes of the spectra for different nuclei. As is shown in Fig. 3, the ionization effect would be observable in a solar quiet period. In a solar active period, however, the solar modulation is so effective that the effect of the ionization loss could not be observed because of the limitation in statistics of the available experiments.

§ 3. Role of ordered magnetic fields

An effect of ordered magnetic fields we wish to remark is a depression in intensity of high energy cosmic rays as far as $15 \sim 30 \text{ GeV}$. The eleven-year variations may indicate the presence of such a region of ordered magnetic fields as will be discussed in the following. We again assume the solar streams, which are steadily emitted from the sun and are supposed to be distributed nearly spherically. Now we are concerned with the stable configuration of them before the onset of hydrodynamical instability. Let us consider a stream emitted from the solar region of the latitude θ with the initial velocity $V \cong 10^8 \text{ cm/sec}$. When it is at the distance r from the sun, its velocity will have two components, the radial component and the azimuthal one, respectively,

$$V_r \cong V \text{ and } V_\phi \cong \omega r \sin \theta, \quad (13)$$

where ω is the angular velocity of the rotation of the sun, $2.7 \times 10^{-6} \text{ sec}^{-1}$. (Here we take the frame of reference fixed in the solar system as in the preceding section.) We see $V_\phi \ll V_r$ inside the earth orbit and V_ϕ will become comparable with V_r beyond it, but the increase of V_ϕ is to be limited by instability of the stream. One may expect that it is stable so far as $V_\phi \lesssim V_r$, or to the distance $r_i \cong (2-3)r_e$. If the magnetic field in the stream is uniform and have a component B_\perp normal to the direction of the stream motion, the electric field will be induced, seen in the fixed frame of reference, which has a radial component

$$(E)_r = -\frac{1}{c} [\mathbf{V} \times \mathbf{B}_\perp]_r \quad \text{i. e.} \quad E_r = \mp \frac{V_\phi}{c} B_\perp. \quad (14)$$

The plus sign, for example, in the case of $\theta=0$, relates to B_\perp upwards in the ecliptic plane and the minus sign to downwards (when we stand on the northern hemisphere).

The magnetic fields of the streams considered here connect possibly to the general solar magnetic field. If the solar magnetic field has the same direction as the terrestrial one, the radial component of induced electric fields will have the outward direction, corresponding to the plus sign in (14). The solar streams will then form a decelerating region against impinging cosmic rays, which lays inside the diffusion region discussed in the preceding section.

Let us follow a cosmic-ray particle which is assumed to be confined in the plane inclined by angle θ to the ecliptic plane and to pass straightly in reversed radial direction through the decelerating region. The resultant change in the rigidity of the particle is found from (13) and (14) as

$$\Delta R = - \sum E_r d(r) = - \sum \frac{\omega r}{c} \sin \theta B_\perp(r) d(r), \quad (15)$$

where $d(r)$ is the width of the streams. The summation is taken over the whole streams traversed by the particle, sufficiently from r_i to r_e , since E_r is very small for $r < r_e$ because of $V_\phi \ll V_r$.

$d(r)$ and $B_\perp(r)$ certainly vary with the distance from the sun, r , as

$$B_\perp(r) = B_\perp(r_e) (r_e/r)^k \quad \text{and} \quad d(r) = d(r_e) (r/r_e)^h. \quad (16)$$

If we resort to the relation between magnetic field and gas density, $B \propto \rho^{2/3}$ for a free isotropic expansion,¹⁹⁾ we find roughly

$$B_\perp(r) \propto r^{-4/3} \quad \text{and} \quad d(r) \propto r^{2/3}, \quad (h=2/3 \text{ and } k=4/3, \text{ Case a}). \quad (17a)$$

Or the component of the magnetic field normal to the stream motion is presumed to vary in a different manner from the parallel one, and then

$$B_\perp(r) \propto r^{-1} \quad \text{and} \quad d(r) \propto r, \quad (h=k=1, \text{ Case b}). \quad (17b)$$

Taking into account the possible passages of a particle through the given

region, the net result of rigidity change is the deceleration so far as a particle passes through the region from the outer side and has a rigidity larger than $R_e \cong B_\perp(r_e) \times d(r_e)$. Obviously, low-rigidity particles are only scattered by the streams and suffered from no rigidity change. The net decrease of rigidity can be calculated from (15) and (17), in the case of b, as

$$\Delta \bar{R} \cong - (1/\pi) (\omega r_e/c) \cdot B_\perp(r_e) r_i (1 - r_e/r_i) \text{ for } R > R_e \cong B_\perp(r_e) d(r_e). \quad (18)$$

There enters only a factor of the order of unity in case a.

According to the Liouville theorem, the deceleration of cosmic-ray particles due to the steady magnetic field results in the decrease of their intensity. As in (1) we define the reduction factor $b_0(R)$, the ratio of the intensity of particles with the rigidity R inside the earth orbit, $r \leq r_e$ to that of $r \geq r_i$. For the latter we can assume the spectrum in (12), since we are interested in the rigidity region larger than several Gv, in which the effect of the diffusion region is negligible. Then we obtain

$$\begin{aligned} b_0(R) &= 1 \quad \text{for } R < R_e, \\ b_0(R) &= 1 - (\gamma + 2) (\Delta \bar{R}/R) \quad \text{for } R > R_e. \end{aligned} \quad (19)$$

Namely the fractional change of intensity is given by

$$\delta j(R)/j(R) = (\gamma + 2) (\Delta \bar{R}/R) \quad \text{for } R > R_e. \quad (20)$$

The main properties of the eleven-year variations to be compared with the above model are of gross structure.^{20),21)} Smoothing out all of the irregular fluctuations of primary cosmic rays, there exists a pronounced trend, apparently associated with solar cycle. The amplitude of variations is about four percent. The intensity correlates inversely with the sunspot number. The change of the primary spectrum shows a considerable effect up to 30 Gv. The variations are also mainly isotropic and world-wide. We accept the observational information as

$$\left[\frac{j(R)}{j(R)} \right]_{\text{sunspot max.}} - \left[\frac{j(R)}{j(R)} \right]_{\text{sunspot min.}} \cong -4\% \text{ for } R \cong 15-20 \text{ Gv}. \quad (21)$$

These properties are entirely consistent with the above model, though available information is not sufficient to determine the rigidity dependence of intensity change. Comparing (21) with (18) and (20), we find reasonable values of the quantities of the ordered magnetic field: In the active period of solar activity, they are

$$r_i \cong 3r_e, \quad B_\perp(r_e) \cong 5 \times 10^{-5} \text{ gauss} \quad \text{and} \quad d(r_e) \cong 5 \times 10^{11} \text{ cm}, \quad (22)$$

and thus the resultant fractional change of intensity is given by

$$\left[\frac{\delta j(R)}{j(R)} \right]_{\text{sunspot max.}} = -\frac{0.9}{R} \quad \text{for } R > R_e = 7 \text{ Gv}. \quad (23)$$

At the time of solar activity minimum, perhaps $B_\perp(r_e)$ decreases to 2×10^{-5} gauss, $r_i \cong 2r_e$, and $d(r_e) \cong 10^{12}$ cm. The expected amplitude with these values is reduced

to the one-fifth of the above value.* Here $d(r_e)$ is chosen so as to give the same value of R_e . The broadening of the width of the streams and the decrease of r_i are considered to arise from that the streams are subject to the earlier onset of hydrodynamical instability due to decreasing magnetic fields than in the epoch of greater solar activity. These values of the quantities of ordered magnetic fields thus found are consistent with the values in Table I.

The superposition of ordered as well as disordered magnetic fields will have the combined modulation factor as shown in Fig. 4. It is needless to say that our model is too idealistic; the transient region, particularly, is neglected, in which the regular magnetic fields turn out to be entangled. It will modify the results mentioned above and a true form of a combined modulation factor would be such as shown by the dotted line in Fig. 4.

§ 4. Other related phenomena

A direct consequence of our model is the diurnal variations; the solar streams with uniform magnetic fields will cause the quiescent diurnal variations (S_q) as well as the ones associated with magnetic disturbances (DS). These effects have been suggested by Nagashima⁸⁾ and Dorman.⁹⁾ According to the extensive analysis of Dorman and Feinberg,^{9),10)} the real change in the primary spectrum for the S_q is found to be

$$\left[\frac{\Delta j(R)}{j(R)} \right]_{S_q} = 0 \quad \text{for } R < R_d \quad \text{and} \quad \left[\frac{\Delta j(R)}{j(R)} \right]_{S_q} \cong \pm \frac{a}{R} \quad \text{for } R > R_d, \quad (24)$$

where R_d is taken as 7 Gv and the amplitude is about 0.15 for cosmic rays vertically incident upon the equatorial region of the earth. The phase of maximum is effectively at about 90° to the left of the earth-sun direction for an observer standing on the northern hemisphere. The phenomena would be accounted for in terms of

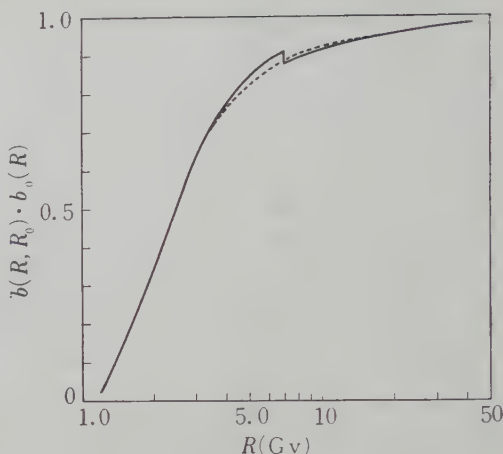


Fig. 4. The combined modulation factor due to the superposition of disordered and ordered magnetic fields is plotted. The solid line shows the product of $b(R, R_0)$ and $b_0(R)$, which are given in (8) and (19), respectively. The dotted line denotes an expected form of the "real" combined modulation factor.

* If the reversal of the direction of the general solar magnetic field occurs periodically, then the region of ordered magnetic fields in a certain period would, contrary to the above case, accelerate cosmic-ray particles, resulting an increase in their intensity. Thus the eleven-year variations might be interpreted as fluctuation around the ground level of the intensity of primary cosmic rays. We would reserve this possibility, and do not go into a further discussion on it.

the rigidity changes due to the solar streams. The magnetic fields must have the component upward on the ecliptic plane in order to induce the electric field parallel to the direction of the earth revolution, that is, to explain the phase of maximum. If a cosmic-ray particle passes through ten or more such streams, the observed amplitude will be accounted for. R_d is quite the same as R_e defined in (18). These results strongly support our arguments in the previous section.

In order to explain the DS, in which the intensity variation of low energy particles is enhanced and the time of maximum is displaced to earlier hours, Nagashima²²⁾ has proposed a superposition of such streams that have more intensive and more turbulent magnetic fields, and reflect particles of rigidities smaller than R_d .

If the steadily emitted streams have appreciable irregularities near the earth in the period of minimum of solar activity, the time of maximum of the S_q will shift also to earlier hours. At this time the magnetic fields are supposed to be weak, so that the amplitude will decrease. These effects will qualitatively account for the observed trend of change of the S_q with solar cycle.¹⁰⁾

The above considerations reveal the relationship between the eleven-year variations and the diurnal ones. We should like to emphasize the persistent nature of these phenomena, which is the very characteristic of the steady configuration of solar streams. On the other hand, the phenomena like the cosmic-ray storms accompanied with solar eruptions show only transient decreases, being rather irregular in space and time.

§ 5. Concluding remarks

We have developed the model of the interplanetary magnetic fields mainly from the cosmic-ray aspect and have left it for future problems to base our model on reasonable astrophysical and hydrodynamical arguments. We believe that the assumption of the continual emission of solar streams, upon which the present model is based, may not be far from reality. Many questions will arise on the detailed nature of the steady configuration of such streams presented here. Particularly, the onset of hydrodynamical instability is assumed to take place at the distance of a few times the sun-earth distance and the regular magnetic fields distributed inside it to have the unique direction. Future investigation must be made to confirm these assumptions, and also many variations on our model will be possible.

The modulation of low energy cosmic rays by the diffusion in disordered magnetic fields shows not only rigidity-dependence, but also velocity-dependence, and there may result in a slight difference between the spectrum of the proton component and that of heavier nuclei. The reliable observation of the proton spectrum will be important in this respect. It is predicted that both the eleven-year variations and the diurnal ones have the same energy-dependence, and the further analysis of the eleven-year variations is to be expected. The observation of a change of the

rigidity spectrum of primary cosmic rays with solar cycle is also waited for, since the continuously ejected streams are believed to change slowly with solar cycle. To make clear the electromagnetic conditions of another kind of solar streams sharply connected to sunspot activity, it is desirable to observe, if possible, the difference between the cosmic-ray spectra associated with and without solar eruptions in each period of solar activity.

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Integral Representations of Bethe-Salpeter Amplitudes

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An integral representation is shown for the matrix element between the vacuum state and a one-particle state of the T-product of two field operators (two-body Bethe-Salpeter amplitudes). It is a generalization of the representation introduced by Wick. The derivation makes use of the Lorentz invariance, microcausality, asymptotic conditions, spectral conditions plus the stability condition.

§ 1. Introduction

Recently, in connection with dispersion relations, our knowledge about the analytic properties of scattering amplitudes, all the external particles being on their mass shells, has undergone appreciable progress. On the other hand, Jost and Lehmann¹⁾ found an integral representation of the matrix elements of causal commutators for the case of two fields of equal mass, which was extended to the case of unequal masses by Dyson.²⁾ It gives the analytic properties of various amplitudes when external particles are not necessarily on their mass shells. The reason why we consider the knowledge off the mass shell as important comes from the hope that it may bring us information on the structure of elementary particles. This is indeed the case with the nucleon form factors connected with high energy electron-nucleon collisions.

Vertex functions are the simplest sources of such information, if we exclude the familiar propagation functions. They are classified into (three-point) Green's functions, (two-body) Bethe-Salpeter amplitudes and vertex functions in the narrow sense, according to the numbers of particles on the physical mass shell. In this paper we shall concentrate our attention to two-body B-S amplitudes. Already in 1954 Wick³⁾ and Cutkosky⁴⁾ introduced an integral representation of B-S amplitudes for the special case of a bound system of two scalar particles exchanging massless "photons". It is, however, based essentially on the ladder approximation of the B-S equation and the vanishing mass of the exchanged "photon". In order to remove these restrictions, we start from the definition of B-S amplitudes instead of starting from B-S equations. We derive an integral representation from the stability condition (3), in addition to the basic assumptions usually accepted in the axiomatic formulation of quantized field theories.

First we express B-S amplitudes in terms of the vacuum expectation values of double retarded commutators. Then, with the use of the Dyson representation

for double commutators⁵⁾ we can obtain an integral representation for B-S amplitudes. A restriction on the integration interval is found by comparing it with the one for the case of causal commutators. In § 3 analytic properties of the B-S amplitudes are studied. They are analytic functions of two complex variables with singular region very different from that of scattering amplitudes, which are conjectured to be representable in the form of double dispersion relations.

As a consequence of generality in our assumptions, we are able to study, for example, the B-S equation in the ladder-approximation for a bound system of two particles exchanging "mesons" with non-vanishing mass. It will be discussed in a subsequent paper to be published before long.

§ 2. Integral representations of B-S amplitudes

The two-body Bethe-Salpeter amplitude for a one-particle state $|p\rangle$ with a momentum p is defined as the matrix element, between the state $|p\rangle$ and the vacuum state $|0\rangle$, of the time ordered product of two field operators ϕ_1 and ϕ_2 ,

$$f(x, p) = e^{-ip \cdot X} \langle 0 | T \phi_1(x_1) \phi_2(x_2) | p \rangle = \langle 0 | T \phi_1\left(\frac{x}{2}\right) \phi_2\left(-\frac{x}{2}\right) | p \rangle, \quad (1)$$

where $x = x_1 - x_2$, $X = (x_1 + x_2)/2$ and $p \cdot X = \mathbf{p} \cdot \mathbf{X} - p_0 \cdot X_0$. Our discussion is based on the four assumptions accepted in the general framework of conventional quantized field theories: the Lorentz invariance (including the existence of the vacuum state), microcausality, asymptotic conditions and spectral conditions. The last assumption means that

$$\langle 0 | \phi_1 | p_a \rangle \cdot \langle p_a | \phi_2 | p \rangle = 0, \quad \langle 0 | \phi_2 | p_b \rangle \cdot \langle p_b | \phi_1 | p \rangle = 0, \quad (2)$$

for p_a and p_b satisfying $-p_a^2 < m_1^2$ and $-p_b^2 < m_2^2$, respectively. Besides these, the stability condition,

$$m \equiv \sqrt{-p^2} < m_1 + m_2, \quad (3)$$

plays an important role in the derivation of an integral representation of B-S amplitudes.

For simplicity we consider a model in which the particle in question has zero spin and both ϕ_1 and ϕ_2 are scalar fields. The B-S amplitude is simply related to the corresponding retarded amplitude,

$$f_R(x, p) = \langle 0 | R \phi_1\left(\frac{x}{2}\right) \phi_2\left(-\frac{x}{2}\right) | p \rangle, \quad (4)$$

which is rewritten, in terms of the incoming field, as the vacuum expectation value,

$$\begin{aligned} f_R(x, p) &= \langle 0 | R \left[\phi_1\left(\frac{x}{2}\right) \phi_2\left(-\frac{x}{2}\right) \right] \cdot \phi_{in}^\dagger(p) | 0 \rangle \\ &= \left\langle \left[R \left[\phi_1\left(\frac{x}{2}\right) \phi_2\left(-\frac{x}{2}\right) \right], \phi_{in}^\dagger(p) \right] \right\rangle_0. \end{aligned} \quad (5)$$

As a consequence of the asymptotic condition,* it follows that

$$f_R(x, p) = \frac{1}{(2\pi)^{3/2}} \int d^4 z e^{ip \cdot z} (-\square_z + m^2) r(x, z; p) \quad (6)$$

in an analogous way to Lehmann, Symanzik and Zimmermann,⁶⁾ where

$$r(x, z; p) = \theta(x) \theta(-p \cdot y) \left\langle \left[\left[\phi_1\left(\frac{x}{2}\right), \phi_2\left(-\frac{x}{2}\right) \right], \phi^\dagger(z) \right] \right\rangle_0, \quad (7)$$

with $y = -x/2 - z$. Then, with the use of Dyson's representation for the vacuum expectation value of a double commutator,

$$\left\langle \left[\left[\phi_1\left(\frac{x}{2}\right), \phi_2\left(-\frac{x}{2}\right) \right], \phi(z) \right] \right\rangle_0 = \int_0^1 d\lambda \int_0^\infty ds \int_0^\infty dt \Phi(\lambda, s, t) \mathcal{A}(x; s) \mathcal{A}(y + \lambda x; t), \quad (8)$$

where $\mathcal{A}(x; s)$ is the usual invariant commutator function for a free field with mass \sqrt{s} , we can express the retarded amplitude as

$$f_R(x, p) = \int_0^1 d\lambda \int_0^\infty ds \int_0^\infty dt \Phi(\lambda, s, t) \int d^4 z e^{ip \cdot z} (-\square_z + m^2) A(x, z; p). \quad (9)$$

Here we have omitted an irrelevant constant, and $A(x, z; p)$ is given by

$$\begin{aligned} A(x, z; p) &= \theta(x) \theta(-p \cdot y) \mathcal{A}(x; s) \mathcal{A}(y + \lambda x; t) \\ &= \theta(-p \cdot y) \mathcal{A}_R(x; s) \mathcal{A}_R(y + \lambda x; t) \\ &= \frac{-1}{2\pi i} \int_{-\infty}^\infty d\tau \frac{e^{i\tau p \cdot y}}{\tau + i\epsilon} \int \frac{d^4 k}{(2\pi)^4} \frac{e^{ik \cdot z}}{\bar{k}^2 + s} \int \frac{d^4 l}{(2\pi)^4} \frac{e^{il \cdot (y + \lambda x)}}{\bar{l}^2 + t}, \end{aligned} \quad (10)$$

where $\bar{k} = k + i\epsilon p$, with a positive infinitesimal number ϵ .

Taking the Fourier transform of (9), we obtain

$$\begin{aligned} f_R(q, p) &= \int d^4 x e^{-iq \cdot x} f(x, p) \\ &= \int_0^1 d\lambda \int_0^\infty ds \int_0^\infty dt \Phi(\lambda, s, t) \lim_{\substack{-p^2 \rightarrow m^2 \\ p_0 > 0}} (p^2 + m^2) A(q, p). \end{aligned} \quad (11)$$

$A(q, p)$ is the Fourier transform of $A(x, z; p)$ and is calculated by replacing the τ -integration path by a large semi-circle in the upper-half of the complex τ plane:

$$A(q, p) = \int d^4 x \int d^4 z e^{-iq \cdot x} e^{ip \cdot z} A(x, z; p)$$

* If the particle is not elementary, we have to make use of the arguments for composite particles by Nishijima⁷⁾ and Zimmermann.⁸⁾

$$\begin{aligned}
&= \frac{-1}{2\pi i} \int_{-\infty}^{\infty} d\tau \frac{1}{\{\bar{q} + (\frac{1}{2} - \lambda)p + \lambda\tau p\}^2 + s} \cdot \frac{1}{\tau + i\epsilon} \cdot \frac{1}{(\bar{p} - \tau p)^2 + t} \\
&= \frac{1}{\{\bar{q} + (\frac{1}{2} - \lambda)p + \lambda(1 + \sqrt{t/p^2 - p^2})p\}^2 + s} \cdot \frac{1}{(\sqrt{t/p^2 - p^2} + \sqrt{t})2\sqrt{t}} \\
&+ \frac{1}{\{\bar{q} + (\frac{1}{2} - \lambda)p + \lambda(1 - \sqrt{t/p^2 - p^2})p\}^2 + s} \cdot \frac{1}{(-\sqrt{t/p^2 - p^2} + \sqrt{t})2\sqrt{t}}. \quad (12)
\end{aligned}$$

The spectral function $\phi(\lambda, s, t)$ has the following form, as is assured by the stability condition (3) :

$$\phi(\lambda, s, t) = \delta(t - m^2) \phi_0(\lambda, s) + \theta(t - (m_1 + m_2)^2) \phi_1(\lambda, s, t). \quad (13)$$

We then get from (11), (12) and (13)

$$f_R(q, p) = \int_0^1 d\lambda \int_0^\infty ds \frac{\phi_0(\lambda, s)}{\{\bar{q} + (\frac{1}{2} - \lambda)p\}^2 + s}. \quad (14)$$

By a linear transformation of λ to $\zeta = 2\lambda - 1$, (14) can be written in the form

$$f_R(q, p) = \int_{-1}^1 d\zeta \int_0^\infty ds \frac{\phi(\zeta, s)}{\left(\bar{q} - \zeta \frac{p}{2}\right)^2 + s}. \quad (15)$$

In order to find a restriction on the carrier, outside of which the spectral function $\phi(\zeta, s)$ vanishes, we compare the integral representation (15) with the one for the matrix element of the corresponding causal commutator,

$$f_R(q, p) = \int d^4u \int ds \frac{\varphi(u, s)}{(\bar{q} - u)^2 + s}. \quad (16)$$

$\varphi(u, s)$ vanishes outside the region where the vectors $p, 2 \pm u$ both lie in the forward light cone and

$$\sqrt{s} \geq \text{Max} \left\{ 0; m_1 - \sqrt{-\left(\frac{p}{2} + u\right)^2}; m_2 - \sqrt{-\left(\frac{p}{2} - u\right)^2} \right\}. \quad (17)$$

By comparison we have a restriction on the carrier of $\phi(\zeta, s)$:

$$\sqrt{s} \geq \text{Max} \left\{ m_1 - \frac{1+\zeta}{2} m; m_2 - \frac{1-\zeta}{2} m \right\}. \quad (18)$$

As an integral representation for the B-S amplitude we finally obtain

$$f(q, p) = \int_{-1}^1 d\zeta \int_0^\infty ds \frac{\phi(\zeta, s)}{\left(q - \zeta \frac{p}{2}\right)^2 + s - i\epsilon}, \quad (19)$$

with the restriction (18) on the carrier of $\Phi(\zeta, s)$ (Fig. 1).

§ 3. Analytic properties of B-S amplitudes

In the center of mass system $f(q, p)$ is a function of q^2 and q_0 , and we can easily see from (19) and (18) that $f(q, p)$ is, for fixed q^2 , regular in the complex q_0 plane with two cuts from $\sqrt{m_1^2 + q^2} - \frac{m}{2}$ to $+\infty$ and from $-\infty$ to $\frac{m}{2} - \sqrt{m_2^2 + q^2}$. The gap between the two cuts is due to the stability condition (3).³⁾

In order to see the analytic properties of $f(q, p)$ studied in § 2, we write (19) in the form

$$f(q^2, q \cdot p) = \int_{-1}^1 d\zeta \int_0^\infty dt \frac{\tilde{\Phi}(\zeta, t)}{q^2 - \zeta q \cdot p + t - \left(\frac{m}{2}\right)^2 - i\epsilon} \quad (20)$$

after linear transformation of s to $t = s + (1 - \zeta^2) \cdot (m/2)^2$. The restriction on the carrier of $\tilde{\Phi}(\zeta, t)$ is given by the inequality (21) (see Fig. 2):

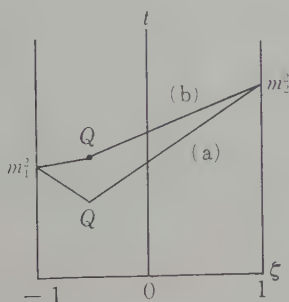


Fig. 2. The lower bound for the carrier of $\tilde{\Phi}(\zeta, t)$ is shown. The coordinate of the point Q is given by $Q(\zeta, t) = Q\left(-\frac{m_2 - m_1}{m}, \left(m_1 - \frac{m}{2}\right) \cdot \left(m_2 - \frac{m}{2}\right) + \left(\frac{m}{2}\right)^2\right)$. (a) and (b) correspond to the cases of $m < 2m_1$ and of $m > 2m_1$, respectively.

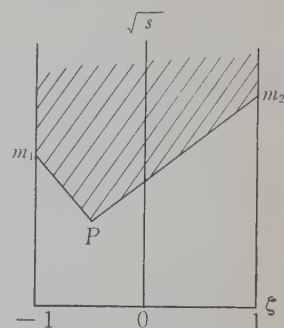


Fig. 1. The spectral function $\Phi(\zeta, s)$ vanishes outside the shaded region. $m_1 \leq m_2$ is assumed without the loss of generality. The coordinate of the point P is given by $P(\zeta, \sqrt{s}) = P\left(-\frac{m_2 - m_1}{m}, (m_1 + m_2 - m)/2\right)$. Here one must notice that m_2 is smaller than $m + m_1$ by its definition.

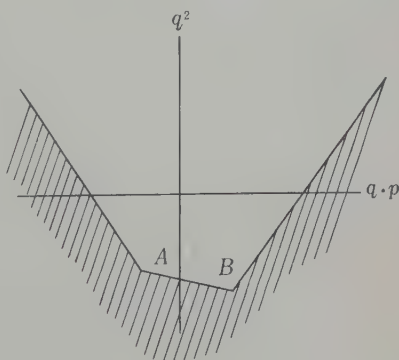


Fig. 3. $f(q^2, q \cdot p)$ is regular outside the shaded region for the case of $m < 2m_1$. The coordinates of the points A and B are given by $A(q \cdot p, q^2) = A\left(-m\left(m_1 - \frac{m}{2}\right), -\left(m_1 - \frac{m}{2}\right)^2\right)$ and $B(q \cdot p, q^2) = B\left(m\left(m_2 - \frac{m}{2}\right), -\left(m_2 - \frac{m}{2}\right)^2\right)$. For $m > 2m_1$ we have only to shift the q^2 -axis to the left of the point A .

$$t \geq \text{Max} \left\{ \left(m_1 - \frac{m}{2} \right)^2 + \left(\frac{m}{2} \right)^2 - m \left(m_1 - \frac{m}{2} \right) \zeta ; \right. \\ \left. \left(m_2 - \frac{m}{2} \right)^2 + \left(\frac{m}{2} \right)^2 + m \left(m_2 - \frac{m}{2} \right) \zeta \right\}. \quad (21)$$

As an analytic function of two complex variables, $f(q^2, q \cdot p)$ has singular region very different from that of scattering amplitudes. Fig. 3 shows the necessary region of regular $f(q^2, q \cdot p)$ in the plane of real q^2 and $q \cdot p$.

Recently Deser, Gilbert and Sudarshan⁹ have published similar work. However, their purpose and the method of derivation seem to be somewhat different from ours.

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Effect of Lattice-Electron Interaction on the Landau Diamagnetism

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A theory of the Landau diamagnetism is presented, in which the interaction of electrons with lattice vibrations connected with virtual exchange of phonons is taken into account. Calculation is made up to the second order perturbation; a suitable canonical transformation can eliminate the first order term of lattice-electron interaction. The result gives only a small correction to the diamagnetism.

§ 1. Introduction

The simplest theory of metals is based on the assumption that the electrons are free, and in quantum theory it was first shown based on this model by Landau¹⁾ that there is a non-zero diamagnetic effect due to the helical paths of the electrons in a magnetic field. We should expect some additional effect on this original work if we extend the free electron model to include the interaction of electrons with the lattice vibrations. It is the principal object of the present paper to determine in a quantitative way the influence of this interaction on the Landau diamagnetism.

The calculation for current density of the electron-lattice system is carried out by employing the perturbation formalism adopted by Schafroth²⁾ which has such an advantage as that the expressions obtained are everywhere regular, safe from vanishing energy denominators.

We restrict ourselves to weak magnetic fields so that the induced current density $\mathbf{i}(\mathbf{x})$ may be expressed as linear in a field with vector potential $\mathbf{A}(\mathbf{x})$, and in terms of the Fourier transforms, it becomes

$$i_{\mu}(\mathbf{q}) = \sum_{\nu=1}^3 K_{\mu\nu}(\mathbf{q}) A_{\nu}(\mathbf{q}). \quad (1.1)$$

The translational and the gauge invariance requires

$$K_{\mu\nu}(\mathbf{q}) = (q_{\mu} q_{\nu} - \delta_{\mu\nu} q^2) K(q^2). \quad (1.2)$$

The Meissner-Ochsenfeld effect is given when $K(q^2)$ has a first-order pole at the origin; that is,

$$K(q^2) = K_0/q^2 + K_r(q^2), \quad (1.3)$$

where K_0 is a positive constant, and $K_r(q^2)$ is a regular function of q , to which the normal diamagnetism corresponds. The purpose of this paper is to decide quantitatively how a perturbation treatment of the current density, when one takes into

account the electron-lattice interaction in the Hamiltonian, exhibits a correction to the Landau diamagnetism, after verifying that K_0 surely vanishes as Schafroth indicated.

§ 2. Model

We consider an aggregate of electrons whose density is n_{el} in the vibrational field and a weak static magnetic field. We shall use the formalism of second quantization³⁾ defined by

$$\left. \begin{aligned} \psi(\mathbf{x}) &= \sum_{k\sigma} b_{k\sigma} \exp(i\mathbf{k} \cdot \mathbf{x}) \\ \psi^*(\mathbf{x}) &= \sum_{k\sigma} b_{k\sigma}^* \exp(-i\mathbf{k} \cdot \mathbf{x}) \end{aligned} \right\}. \quad (2.1)$$

Here $b_{k\sigma}^*$ and $b_{k\sigma}$ are respectively the creation and destruction operators for electrons, and possess the commutation properties

$$\left. \begin{aligned} [b_{k\sigma}, b_{k'\sigma'}^*]_+ &= \delta_{kk'} \delta_{\sigma\sigma'} \\ [b_{k\sigma}, b_{k'\sigma'}]_+ &= [b_{k\sigma}^*, b_{k'\sigma'}^*]_+ = 0 \\ b_{k\sigma}^* b_{k\sigma} &= n_{k\sigma} \end{aligned} \right\}, \quad (2.2)$$

and σ denotes the electron spin. The Hamiltonian for our system may be written as

$$H = H_0 + H_1 + H', \quad (2.3)$$

where

$$H_0 = \sum_{k\sigma} (\hbar^2 k^2 / 2m) b_{k\sigma}^* b_{k\sigma} + \sum_{w < w_D} \hbar w s (a_w^* a_w + 1/2), \quad (2.3a)$$

$$H_1 = i \sum_{\substack{w k \\ \sigma}} D_w (a_w b_{k+w\sigma}^* b_{k\sigma} - a_w^* b_{k-w\sigma}^* b_{k\sigma}), \quad (2.3b)$$

$$H' = (e\hbar/mc) \sum_{\substack{k q \\ \sigma}} (\mathbf{k} \cdot \mathbf{A}(\mathbf{q})) b_{k-q/2\sigma}^* b_{k+q/2\sigma}. \quad (2.3c)$$

a_w^* and a_w are the creation and destruction operators for phonons, possessing the relations:

$$\begin{aligned} [a_w, a_{w'}^*] &= \delta_{ww'} \\ [a_w, a_{w'}] &= [a_w^*, a_{w'}^*] = 0 \\ a_w^* a_w &= N_w. \end{aligned} \quad (2.4)$$

Here we take H_0 to describe a system of free electrons and a phonon gas in unit volume, and H_1 and H' represent respectively the lattice-electron interaction and the interaction between the electrons and a magnetic field. The maximum value of w is denoted by w_D satisfying the relation in Debye's approximation

$$2w_D^3 / 3(2\pi)^2 = n, \quad (2.5)$$

where n is the number of atoms per unit volume. D_w is an interaction constant

of the lattice-electron system given by

$$D_w = (2F\zeta \hbar \omega_s / 3n)^{1/2}, \quad (2.6)$$

in Frohlich's notation.⁴⁾ Besides, $A(\mathbf{q})$ is the Fourier transform of $A(\mathbf{x})$:

$$A(\mathbf{x}) = \sum_{\mathbf{q}} A(\mathbf{q}) \exp(-i\mathbf{q} \cdot \mathbf{x}), \quad (2.7)$$

therefore,

$$A(\mathbf{q}) = \int A(\mathbf{x}) \exp(i\mathbf{q} \cdot \mathbf{x}) d\mathbf{x}. \quad (2.7a)$$

Our current density operator is given by

$$\mathbf{j}(\mathbf{x}) = \mathbf{j}_0(\mathbf{x}) + \mathbf{j}_1(\mathbf{x}), \quad (2.8)$$

where

$$\mathbf{j}_0(\mathbf{x}) = -e\hbar/m \cdot \sum_{\substack{kq \\ \sigma}} k \exp(-i\mathbf{q} \cdot \mathbf{x}) b_{k+q/2\sigma}^* b_{k-q/2\sigma}, \quad (2.8a)$$

$$\mathbf{j}_1(\mathbf{x}) = -e^2/mc \cdot \sum_{\substack{k k' \\ \sigma}} A(\mathbf{x}) \exp(i(\mathbf{k}-\mathbf{k}') \cdot \mathbf{x}) b_{k'\sigma}^* b_{k\sigma}. \quad (2.8b)$$

Then the average current density is represented by

$$\mathbf{i}(\mathbf{x}) = \text{Spur}(\mathbf{j}(\mathbf{x}) \exp(\beta(\Phi - H))). \quad (2.9)$$

Here $\beta = 1/\kappa T$ (κ is the Boltzmann constant) and Φ is the free energy of the system. We shall hereafter calculate the above average up to the second order terms in the lattice-electron interaction and the terms linear in the magnetic field.

§ 3. Discussion of current density

We now consider a canonical transformation similar to that of Bohm and Pines in their plasma theory⁵⁾ to eliminate the lattice-electron interaction H_1 . When our generating function is given by

$$S = \sum_{k, w, \sigma} D_w (a_w b_{k+w\sigma}^* b_{k\sigma} + a_w^* b_{k\sigma}^* b_{k+w\sigma}) \frac{1}{\hbar \omega_s + \varepsilon(\mathbf{k}) - \varepsilon(\mathbf{k} + \mathbf{w})}, \quad (3.1)$$

we can easily verify

$$H_1 + i[H_0, S] = 0, \quad (3.2)$$

where $\varepsilon(\mathbf{k}) = \hbar^2 k^2 / 2m$. Our Hamiltonian may be written as

$$H_{new} = e^{-iS} H e^{iS} = H_0 + H' + H_1^{(1)} + H'^{(1)} + H'^{(2)} + \dots, \quad (3.3)$$

where we put $H_1^{(1)} = i/2 \cdot [H_1, S]$, $H'^{(1)} = i[H', S]$, and $H'^{(2)} = -1/2 \cdot [[H', S], S]$.

Our new current density operator is likewise expressed as

$$\mathbf{j}_{new}(\mathbf{x}) = e^{-iS} \mathbf{j}(\mathbf{x}) e^{iS} = \mathbf{j}_0(\mathbf{x}) + \mathbf{j}_1(\mathbf{x}) + \mathbf{j}_0^{(1)}(\mathbf{x}) + \mathbf{j}_0^{(2)}(\mathbf{x}) + \mathbf{j}_1^{(2)}(\mathbf{x}) + \dots \quad (3.4)$$

In this expression we can leave the last term $\mathbf{j}_1^{(2)}(\mathbf{x})$ out of consideration, because it is easily seen that

$$\sum_{\nu} \langle \mathbf{j}_1^{(2)}(\mathbf{x}) \rangle_{\nu\nu} \exp(\beta(\Phi_0 - E_{\nu})) = 0. \quad (3.4a)$$

The free energy Φ is determined by

$$\text{Spur}(\exp(\beta(\Phi_0 - H_{new}))) = 1. \quad (3.5)$$

We wish to find the free energy up to the second order term, so we put

$$\Phi = \Phi_0 + \Phi_2, \quad (3.6)$$

where Φ_0 is the zeroth-order free energy and Φ_2 is the second-order perturbation by the lattice-electron interaction. Φ_2 is determined by Buckingham and Schafroth⁽⁶⁾ in their calculation of the specific heat of metals as follows:

$$\begin{aligned} \Phi_2 = -2 \sum_{k,w} D_w^2 \frac{1}{\hbar\omega_s + \varepsilon(\mathbf{k}) - \varepsilon(\mathbf{k} + \mathbf{w})} & (f(\mathbf{k} + \mathbf{w})(1 - f(\mathbf{k})) \\ & + F(\mathbf{w})(f(\mathbf{k} + \mathbf{w}) - f(\mathbf{k}))), \end{aligned} \quad (3.7)$$

where

$$\begin{aligned} f(\mathbf{k}) &= \{1 + \exp(\beta(\varepsilon(\mathbf{k}) - \zeta))\}^{-1}, \\ F(\mathbf{w}) &= (\exp(\beta\hbar\omega_s) - 1)^{-1}. \end{aligned} \quad (3.7a)$$

As it is easily seen that the second term goes to zero with vanishing temperature, we may write

$$\Phi_2 = -2 \sum_{k,w} D_w^2 \frac{f(\mathbf{k} + \mathbf{w})(1 - f(\mathbf{k}))}{\hbar\omega_s + \varepsilon(\mathbf{k}) - \varepsilon(\mathbf{k} + \mathbf{w})}. \quad (3.8)$$

The average current density is represented by H_{new} and j_{new} as follows:

$$\begin{aligned} \mathbf{i}(\mathbf{x}) = \text{Spur} \{ (\mathbf{j}_0(\mathbf{x}) + \mathbf{j}_1(\mathbf{x}) + \mathbf{j}_0^{(1)}(\mathbf{x}) + \mathbf{j}_0^{(2)}(\mathbf{x})) \exp(\beta(\Phi_0 + \Phi_2 \\ - H_0 - H' - H_1^{(1)} - H'^{(1)} - H'^{(2)})) \}. \end{aligned} \quad (3.9)$$

Then we get the expression for the average current density determined by the terms linear in the magnetic field and quadratic in the lattice-electron interaction:

$$\mathbf{i}^{(2)}(\mathbf{x}) = \mathbf{i}^{(1)}(\mathbf{x}) + \mathbf{i}^{(II)}(\mathbf{x}) + \mathbf{i}^{(III)}(\mathbf{x}) + \mathbf{i}^{(IV)}(\mathbf{x}) + \mathbf{i}^{(V)}(\mathbf{x}). \quad (3.10)$$

Here

$$\mathbf{i}^{(I)}(\mathbf{x}) = \sum_{\nu} \langle \mathbf{j}_1(\mathbf{x}) \rangle_{\nu\nu} (\langle H_1^{(1)} \rangle_{\nu\nu} - \Phi_2) F_1(E_{\nu}, E_{\nu}), \quad (3.10a)$$

$$\begin{aligned} \mathbf{i}^{(II)}(\mathbf{x}) = \sum_{\nu, \nu'} \langle \mathbf{j}_0(\mathbf{x}) \rangle_{\nu\nu'} \{ (\langle H' \rangle_{\nu'\nu} \langle H_1^{(1)} \rangle_{\nu\nu} - \langle H' \rangle_{\nu'\nu} \Phi_2) F_2(E_{\nu}, E_{\nu}, E_{\nu'}) \\ + (\langle H_1^{(1)} \rangle_{\nu'\nu} \langle H' \rangle_{\nu'\nu} - \Phi_2 \langle H' \rangle_{\nu'\nu}) F_2(E_{\nu'}, E_{\nu'}, E_{\nu}) \}, \end{aligned} \quad (3.10b)$$

$$\begin{aligned} \mathbf{i}^{(III)}(\mathbf{x}) = \sum_{\substack{\nu, \nu', \nu'' \\ \nu \neq \nu'}} (\langle \mathbf{j}_0(\mathbf{x}) \rangle_{\nu''\nu'} \langle H' \rangle_{\nu'\nu''} \langle H_1^{(1)} \rangle_{\nu''\nu} \\ + \langle H' \rangle_{\nu\nu'} \langle \mathbf{j}_0(\mathbf{x}) \rangle_{\nu'\nu''} \langle H_1^{(1)} \rangle_{\nu''\nu}) F_2(E_{\nu}, E_{\nu'}, E_{\nu''}), \end{aligned} \quad (3.10c)$$

$$\mathbf{i}^{(\text{IV})}(\mathbf{x}) = \sum_{\nu\nu'} \langle \mathbf{j}_0^{(2)}(\mathbf{x}) \rangle_{\nu\nu'} \langle H' \rangle_{\nu'\nu} + \langle \mathbf{j}_0(\mathbf{x}) \rangle_{\nu\nu'} \langle H'^{(2)} \rangle_{\nu'\nu} F_1(E_\nu, E_{\nu'}), \quad (3.10d)$$

$$\mathbf{i}^{(\text{V})}(\mathbf{x}) = \sum_{\nu\nu'} \langle \mathbf{j}_0^{(1)}(\mathbf{x}) \rangle_{\nu\nu'} \langle H'^{(1)} \rangle_{\nu'\nu} F_1(E_\nu, E_{\nu'}), \quad (3.10e)$$

where ν, ν' , etc., are the eigenstates of H_0 and the corresponding eigenvalues $E_\nu, E_{\nu'}$, etc. When we put

$$F_0(E) = \exp(\beta(\Phi_0 - E)), \quad (3.11)$$

$F_1(E_\nu, E_{\nu'})$ and $F_2(E_\nu, E_{\nu'}, E_{\nu''})$ are, according to Schafroth, the difference quotients of $F_0(E)$ given by

$$F_1(E_\nu, E_{\nu'}) = \frac{F_0(E_\nu) - F_0(E_{\nu'})}{E_\nu - E_{\nu'}}, \quad (3.12)$$

$$F_2(E_\nu, E_{\nu'}, E_{\nu''}) = \frac{F_0(E_\nu)}{(E_\nu - E_{\nu'})(E_\nu - E_{\nu''})} + \text{cycl.} \quad (3.13)$$

Each term of the average current density is noteworthy in that it has the following common expression

$$\mathbf{i}_\mu^{(N)}(\mathbf{x}) = \sum_{\mathbf{q}} K_{\mu\nu}^{(N)}(q^2) A_\nu(\mathbf{q}) \exp(-i\mathbf{q} \cdot \mathbf{x}), \quad (3.14)$$

where N denotes I, II, III, IV , and V .

The kernel $K_{\mu\nu}^{(N)}(q^2)$ may be expanded up to the second order terms of q :

$$K_{\mu\nu}^{(N)}(q^2) = K_{\mu\nu}^{(N0)} + K_{\mu\nu}^{(N2)}. \quad (3.15)$$

Here $K_{\mu\nu}^{(N0)}$ and $K_{\mu\nu}^{(N2)}$ have the following relation:

$$\sum_{N=I}^V K_{\mu\nu}^{(N0)} = K_{\mu\nu}^{(0)} = \frac{(q_\mu q_\nu - \delta_{\mu\nu} q^2)}{q^2} K_0, \quad (3.15a)$$

$$\sum_{N=I}^V K_{\mu\nu}^{(N2)} = K_{\mu\nu}^{(2)} = (q_\mu q_\nu - \delta_{\mu\nu} q^2) K_2. \quad (3.15b)$$

It is easily seen that K_0 surely vanishes, but here it is worth noticing that $K_{\mu\nu}^{(0)}$, the zeroth-order term of $K_{\mu\nu}^{(I)}(q^2)$, does not vanish but remains. Therefore we find that there is no singularity in our kernel, and consequently no Meissner effect at least within our approximation.

Now we shall proceed to determine the coefficient K_2 which gives a correction to the Landau diamagnetism.

§ 4. Landau diamagnetism

As $K_{\mu\nu}^{(0)}$ vanishes, we may rewrite (1.2) in our approximation into the form

$$\mathbf{K}_{\mu\nu}(\mathbf{q}) = (q_\mu q_\nu - \delta_{\mu\nu} q^2) K_2; \quad (4.1)$$

on the other hand, the magnetic field \mathbf{H} is given by $\mathbf{H} = \text{rot } \mathbf{A}(\mathbf{x})$ and the magnetization \mathbf{M} is related to the current density by

$$\mathbf{i}(\mathbf{x}) = c \operatorname{rot} \mathbf{M}, \quad (4.2)$$

and so we have the equation

$$\mathbf{M} = -1/c \cdot K_2 \mathbf{H}, \quad (4.3)$$

hence the diamagnetic susceptibility is obtained by the relation $\chi = -K_2/c$. The calculation is carried out by changing our sum over \mathbf{k} and \mathbf{w} to an integral with the restriction $k < k_0$ and $w < w_D$.

Then we find that the diamagnetic susceptibility including our effect is given by

$$\begin{aligned} \chi = \chi_L \bigg[& 1 + (\pi/8)^{2/3} F \frac{\hbar s w_D}{\zeta_0} \left\{ w_D/k_0 - 4k_0^2/3w_D (ms/\hbar + k_0) \right. \\ & - 2(k_0 - w_D)^2/3w_D (ms/\hbar + k_0 - w_D/2) \\ & - 2(k_0 + w_D)^2/3w_D (ms/\hbar + k_0 + w_D/2) + \log \left| \frac{ms/\hbar + k_0 - w_D}{ms/\hbar + k_0 + w_D} \right| \\ & + 2w_D^{-1} (ms/\hbar + 2k_0/3 - 2m^2 s^2/\hbar^2 k_0) \\ & \left. \times \log \left| \frac{(ms/\hbar - k_0)^2}{(ms/\hbar + k_0 - w_D/2)(ms/\hbar + k_0 + w_D/2)} \right| \right] \bigg], \quad (4.4) \end{aligned}$$

where

$$\chi_L = -n\mu_0^2/2\zeta_0.$$

We find that the correction is extremely small, and for Na amounts to 0.06 percent. Recently May⁷⁾ has investigated the influence of the periodic field on the Landau diamagnetism and got a correction which agrees with our result. In this paper we discuss the effect of the electron-lattice interaction alone separated from the Coulomb interaction. Wentzel⁸⁾ investigated a dense electron gas coupled with longitudinal sound waves, using the equivalent Hamiltonian to that of the pair theory and concluded that the effect on the Landau diamagnetism was unaffected, which must have been due to the incompleteness of the Hamiltonian.

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S-Wave K Meson-Nucleon Interaction

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An approach to s -wave K - N interaction is made along the same line with our previous study for π - N scattering. The characteristic property of K - N interaction in nucleon core is expressed in terms of the change of the value of coupling constant, and it is suggested that the introduction of indefinite metric may be necessary in some sense to describe the phenomena in nucleon core correctly. Moreover, an attempt to obtain the forward scattering amplitude for K - N scattering is made on the basis of the above consideration about characteristic property of K - N interaction. It is shown that our results are not inconsistent with the experimental ones. Finally some discussion about the coupling constant is made in comparison with the results obtained by many authors.

§ 1. Introduction

In order to see the characteristic property of nucleon core, it is necessary to examine the following two kinds of phenomena, (i) the phenomena in which the nucleon core does not undergo any essential change in the course of collision, (ii) the phenomena in which the nucleon core undergoes some remarkable change in the course of collision. We have previously examined the problems of s -wave pion-nucleon scattering as an approach to the process belonging to the former and have seen that the current pion field theory should be modified within the nucleon core although it can give a satisfactory explanation with respect to the phenomena in the neighborhood of pion cloud. Let us now concentrate our consideration to the process belonging to the latter. In this paper we intend to study the problems in the case of K - N scattering with the purpose of finding a clue to the correct dynamics in nucleon core. It may be expected that some new result can be derived from our study of this process because here the core in the intermediate state is changed into the hyperon core. Our approach to K - N interaction in nucleon core is made along the same line with the previous paper.¹⁾

When the forward scattering amplitudes for K^+p and K^-p scattering at the energy ω are denoted by $D_+(\omega) + iA_+(\omega)$ and $D_-(\omega) + iA_-(\omega)$ respectively, we pay our attention to $D^{(1)}(m_K) = (1/2)[D_-(m_K) + D_+(m_K)]$ and $D^{(2)}(m_K) = (1/2)[D_-(m_K) - D_+(m_K)]$, where m_K is the K -meson mass. If both M_A (or M_S) = M and $m_K = \mu$ were fulfilled in the case of pseudoscalar KNA (or $KN\Sigma$) interaction, as will be shown in § 2, the contributions from the bound state to K - N scattering should have the same forms with those in π - N scattering, so our discussion about $D^{(1)}(m_K)$

and $D^{(2)}(m_K)$ is developed by tracing the correspondence between these two processes throughout this paper, where M_A (or M_Σ), M and μ are the masses of A -particle (or Σ -particle), nucleon and pion, respectively.

If M_A (or M_Σ) were equal to M , we should of course obtain the following conclusions. (i) The correct dynamics should have such a property as that the effective coupling constant g_A (or g_Σ) is reduced to $(m_K/2M)g_A=f_A$ (or $(m_K/2M)g_\Sigma=f_\Sigma$) in appearance when the dynamics is applied to the problems within the nucleon core.* (ii) The p - v -coupling theory is promising to describe the behavior of K - N interaction in nucleon core. In the practical case, however, the above conclusions are obliged to suffer a remarkable change owing to the relation $(M_A^2 - M^2 - m_K^2) > 0$ (or $M_\Sigma^2 - M^2 - m_K^2 > 0$), that is, in nucleon core the value of g_A^2 (or g_Σ^2) is reduced to a negative one in appearance.* It may be necessary in some sense to introduce the indefinite metric in describing the phenomena in nucleon core.

On the basis of the results obtained in § 2, some attempt is made in § 3 in order to derive a significant part from the expression for $D^{(1)}(\omega)$ in dispersion relation in which a divergent one may be contained, and it is shown that the value of $D^{(1)}(m_K)$ obtained by our method is consistent with the experimental results. But it is necessary to have much experimental data in order to make sure of the validity of our method. In § 4 we deal with the problems for $D^{(1)}(m_K)$ and $D^{(2)}(m_K)$ in the case where both $KN\Delta$ and $KN\Sigma$ interactions are of scalar type. In § 5 some discussion about the coupling constant is made in comparison with the results obtained by many authors.^{2)~5)}

§ 2. S-wave K - N interaction

As was shown by Igi²⁾ and Matthews-Salam,³⁾ the dispersion relations for K - N scattering⁶⁾ are as follows:

$$D^{(1)}(\omega) = (1/2) [D_-(\omega) + D_+(\omega)] = F_1(\omega) + G_1(\omega), \quad (1)$$

$$D^{(2)}(\omega) = (1/2) [D_-(\omega) - D_+(\omega)] = F_2(\omega) + G_2(\omega), \quad (2)$$

$$F_1(\omega) = (1/2) (X' + X), \quad F_2(\omega) = (1/2) (X' - X), \quad (3)$$

$$X = \frac{1}{\pi} \int_0^{\omega_{A\pi}} \frac{A_-(\omega')}{\omega' + \omega} d\omega', \quad X' = \frac{1}{\pi} \int_0^{\omega_{A\pi}} \frac{A_-(\omega')}{\omega' - \omega} d\omega', \quad (4)$$

$$G_1(\omega) = \frac{1}{\pi} \int_{\omega_{A\pi}}^{m_K} \frac{\omega' A_-(\omega')}{\omega'^2 - \omega^2} d\omega' + \frac{1}{4\pi^2} \int_{m_K}^{\infty} \frac{\omega' k' [\sigma_-(\omega') + \sigma_+(\omega')]}{\omega'^2 - \omega^2} d\omega', \quad (5)$$

$$G_2(\omega) = \frac{\omega}{\pi} \int_{\omega_{A\pi}}^{m_K} \frac{A_-(\omega')}{\omega'^2 - \omega^2} d\omega' + \frac{\omega}{4\pi^2} \int_{m_K}^{\infty} \frac{k' [\sigma_-(\omega') - \sigma_+(\omega')]}{\omega'^2 - \omega^2} d\omega', \quad (6)$$

* The so-called cutoff theory with cutoff energy $\omega_{max} = M$ may be regarded as the theory in which the coupling constant in nucleon core is assumed to be equal to zero.

and

$$\omega_{A\pi} = \{(M_A + \mu)^2 - M^2 - m_K^2\} / 2M = 0.404 m_K, \quad (7)$$

where k is the wave number of K -meson and $\sigma_{\pm}(\omega)$ are the total cross sections for $K^{\pm}p$ scattering respectively. If the K -interaction Lagrangian contains $\phi_K^2(x)$ or $\phi_K^4(x)$ terms explicitly, an additional term C must be involved on the right-hand side of Eq. (1). But we assume for a moment that $\phi_K^2(x)$ or $\phi_K^4(x)$ terms are not explicitly contained in the Lagrangian. Some discussion about this problem will be made later.

At present we have not enough experimental data for $K^{\pm}p$ scattering to see the properties both of $G_1(\omega)$ and of $G_2(\omega)$ with confidence. If there is such a tendency as $\sigma_{-}(\omega') \cong \sigma_{+}(\omega')$ at high energy, then the value of $D^{(2)}(\omega)$ turns out to be finite, while Eq. (1) loses its meaning because $G_1(\omega)$ turns out to be a divergent quantity. Therefore Eq. (1) should be rewritten as follows:

$$D^{(1)}(\omega) = (1/2) [D_{-}(\omega) + D_{+}(\omega)] = F_1(\omega) + G_{1r}(\omega), \quad (1)'$$

where $G_{1r}(\omega)$ is the significant physical quantity which can be derived from $G_1(\omega)$ by a suitable method. Some attempt to obtain this $G_{1r}(\omega)$ is made in § 3.

In this section we pay our attention to the $F_1(m_K)$ and $F_2(m_K)$ which represent the contributions from the bound states to KN scattering in the limiting case as $k=0$. As was mentioned in the previous paper,¹⁾ we may regard these $F_1(m_K)$ and $F_2(m_K)$ as *the perturbation results in the true field theory*. We here deal with the case of pseudoscalar interaction. As was shown in reference 2 or 3,

$$X = \frac{-f_A^2}{\omega + \omega_A} + \frac{-f_{\Sigma}^2}{\omega + \omega_{\Sigma}}, \quad X' = \frac{f_A^2}{\omega - \omega_A} + \frac{f_{\Sigma}^2}{\omega - \omega_{\Sigma}}, \quad (8)$$

$$\omega_A = (M_A^2 - M^2 - m_K^2) / 2M \cong 0.107 m_K, \quad (9)$$

$$\omega_{\Sigma} = (M_{\Sigma}^2 - M^2 - m_K^2) / 2M \cong 0.268 m_K, \quad (10)$$

where f_A and f_{Σ} are the renormalized coupling constants in KNA and $KN\Sigma$ interactions respectively. Let us discuss only the first terms in the expressions of X and X' for simplicity, since the second terms in them respectively have the same forms as those of the first terms. Then

$$F_{1A}(\omega) = f_A^2 \frac{\omega_A}{\omega^2 - \omega_A^2} = f_A^2 \frac{(M_A^2 - M^2 - m_K^2) / 2M}{\omega^2 - \{(M_A^2 - M^2 - m_K^2) / 2M\}^2}, \quad (11)$$

$$F_{2A}(\omega) = f_A^2 \frac{\omega}{\omega^2 - \omega_A^2} = f_A^2 \frac{\omega}{\omega^2 - \{(M_A^2 - M^2 - m_K^2) / 2M\}^2}, \quad (12)$$

where the suffix A (or Σ) stands for the effect due to KNA (or $KN\Sigma$) interaction. In order to see the correspondence between KN scattering and πN scattering, we try to adjust the factors which are derived from the consideration

about isotopic spin. For this purpose we set as follows:

$$f_A^2 = 2f_A'^2, \quad f_\pi^2 = 2f_\pi'^2. \quad (13)$$

If we neglect the strangeness of particles and set $M_A = M$, then the situation for the bound state in K^+p scattering is similar to that in π^+p scattering and, as will be shown below, the expressions for $F_{1A}(\omega)$ and $F_{2A}(\omega)$ have the same forms with those in πp scattering except for the difference in the mass between m_K and μ (μ is pion mass). If $M_A = M$,

$$F_{1A}(\omega) \longrightarrow f_A'^2 \frac{-m_K^2/M}{\omega^2 - (m_K^2/2M)^2}, \quad (11)'$$

$$F_{2A}(\omega) \longrightarrow 2f_A'^2 \frac{\omega}{\omega^2 - (m_K^2/2M)^2}. \quad (12)'$$

In πN scattering, the contributions from the bound state to πp scattering are

$$F_1(\omega) = 2f^2 \frac{(\mu^2/2M)^2 - \mu^2}{\omega^2 - (\mu^2/2M)^2} \cdot \frac{1}{2M} \cong f^2 \frac{-\mu^2/M}{\omega^2 - (\mu^2/2M)^2}, \quad (11)''$$

$$F_2(\omega) = -2f^2 \frac{(\mu^2/2M)^2 - \mu^2}{\omega^2 - (\mu^2/2M)^2} \cdot \frac{\omega}{\mu^2} \cong 2f^2 \frac{\omega}{\omega^2 - (\mu^2/2M)^2}. \quad (12)''$$

The f in Eqs. (11)'' and (12)'' is equal to $(\mu/2M)g$, where g is the renormalized pseudoscalar coupling constant in pion-nucleon system. Then it may be said that $F_{1A}(m_K)$ and $F_{2A}(m_K)$ will mainly represent the contributions from the nucleon core (inner region $\sim 1/M$) and those from K -meson cloud (outer region $\sim 1/m_K$) respectively. This can also be seen through the following expressions,

$$F_{1A}(m_K) \cong -f_A'^2/M, \quad F_{2A}(m_K) \cong 2f_A'^2/m_K. \quad (14)$$

On the other hand, the practical calculation based on the Lorentz-covariant perturbation gives the following results. (Hereafter the effect due to $KN\pi$ -interaction is not written down because it can be expressed by the same form as that due to KNA -interaction.)

$$D_A^{(1)}(m_K) = \frac{-g_A^2(M_A + M)}{(1 + m_K/M) \{(M_A + M)^2 - m_K^2\}}, \quad (15)$$

$$D_A^{(2)}(m_K) = \frac{g_A^2 m_K}{(1 + m_K/M) \{(M_A + M)^2 - m_K^2\}}, \quad (16)$$

where g_A is not the rationalized coupling constant, but the unrationalized one. If $M_A = M$,

$$D_A^{(1)}(m_K) \cong -g_A'^2/M, \quad D_A^{(2)}(m_K) \cong g_A'^2 m_K/2M^2 = 2f_A'^2/m_K, \quad (17)$$

where $g_A'^2 = 2g_A'^2$, $f_A' = (m_K/2M)g_A'$. Comparing (17) with (14), we obtain the similar conclusions with those in the case of πp scattering. The phenomena in the neighborhood of K -meson cloud can be explained by the current field theory,

while those in nucleon core cannot be explained. The correct dynamics should have such a property as that the effective coupling constant g_A is reduced to f_A in appearance when the dynamics is applied to the problems within the nucleon core. When $p\nu$ -coupling theory is adopted, we obtain similar ones with the expressions in Eqs. (15) and (16) except for the difference in coupling constant between f_A^2 and g_A^2 . Thus it may be said that $p\nu$ -coupling theory is promising to describe the behavior of *K*-*N* interaction in nucleon core.

The above conclusions, however, must be modified by the fact that M_A is not equal to M . Since $(M_A^2 - M^2 - m_K^2)/2M \cong 0.107 m_K > 0$, the $F_{1A}(m_K)$ in Eq. (11) and $F_{2A}(m_K)$ in Eq. (12) are

$$F_{1A}(m_K) \cong f_A^2 \frac{(M_A^2 - M^2 - m_K^2)/2M}{m_K^2} > 0, \quad (18)$$

$$F_{2A}(m_K) \cong f_A^2 \frac{1}{m_K} = 2f_A^2/m_K. \quad (19)$$

$F_{2A}(m_K)$ in Eq. (19) is almost identical with $F_{2A}(m_K)$ in Eq. (14) (or $D_A^{(2)}(m_K)$ in Eq. (17)). This may be due to the following situation. Since the $F_{2A}(m_K)$ represents the contribution from *K*-meson cloud, it is not affected by the change of nucleon core apart from the question of strangeness. On the other hand, $F_{1A}(m_K)$ ought to be remarkably affected by the change of nucleon core because it represents the contribution from the nucleon core. The sign of $F_{1A}(m_K)$ in Eq. (18) is positive, while that of $F_{1A}(m_K)$ in Eq. (14) (or that of $D_A^{(1)}(m_K)$ in Eq. (17)) is negative. Thus we may say that the value of g_A^2 (or g_π^2) should be reduced to a negative one in appearance when the correct dynamics is applied to the phenomena in nucleon core. It may be necessary in some sense to introduce the indefinite metric in describing the phenomena in nucleon core.*

§ 3. On the estimation of $D^{(1)}(\omega)$

Since the effects due to the internal structure of nucleon core play an important role in the $D^{(1)}(\omega)$, it is needless to say that our treatment for *K*-*N* scattering in which the core in the intermediate state is changed into a hyperon core is more difficult than that for π -*N* scattering. In this section, however, let us try to obtain the $G_{1\nu}(\omega)$ along the same line with the method in the previous paper¹⁾ as much as possible.

If the practical effect brought by the application of the correct dynamics to the phenomena in the region of nucleon core may be expressed in terms of the change of the value of coupling constant, our result may be expressed as follows.

* This conclusion is not inconsistent with our result in the case of pion-nucleon scattering, because the f^2 may be interpreted as the value of coupling constant averaged over the spread of nucleon core.

(i) In such phenomena as that the nucleon core does not undergo any essential change in the course of collision, the value of coupling constant in nucleon core ought to be reduced in appearance to $f_A = (m_K/2M)g_A$ in spite of the fact that its value in the neighborhood of K -meson cloud is g_A . (ii) In addition to this, in such phenomena as that the nucleon core (mass M) is changed into the hyperon core (mass M_A) in the course of collision, the value of coupling constant in nucleon core ought to be approximately modified as follows* (apart from the question of strangeness),

$$f_A^2 \simeq -f_A^2 \frac{(M_A^2 - M^2 - m_K^2)}{m_K^2 - \{(M_A^2 - M^2 - m_K^2)/2M\}^2}. \quad (20)**$$

(Of course Eq. (20) is reduced to $f_A^2 \simeq f_A^2$ when $M_A = M$.)

The $D^{(1)}(\omega)$ in Eq. (1)' can generally be expressed by the following form,

$$\begin{aligned} D^{(1)}(\omega) = & F_1(\omega) + \frac{1}{\pi} \int_{m_A\pi}^{m_K} \left(\frac{1}{\omega'^2 - \omega^2} - \frac{1}{\omega'^2 - m_K^2} \right) \omega' A_-(\omega') d\omega' \\ & + \frac{1}{4\pi^2} \int_{m_K}^{\infty} \left(\frac{1}{\omega'^2 - \omega^2} - \frac{1}{\omega'^2 - m_K^2} \right) \omega' k' [\sigma_-(\omega') + \sigma_+(\omega')] d\omega' \\ & + \alpha(m_K, M, M_A, M_S, f_A^2, f_S^2). \end{aligned} \quad (21)$$

One of the important problems is to obtain the unknown $\alpha(m_K, M, M_A, M_S, f_A^2, f_S^2)$. Now we assume that the $\alpha(m_K, M, M_A, M_S, f_A^2, f_S^2)$ can approximately be expressed by the sum of the following three terms. (i) The effect due to $KN\Lambda$ -interaction $\alpha_A(m_K, M, M_A, f_A^2)$, (ii) the effect due to $KN\Sigma$ -interaction $\alpha_S(m_K, M, M_S, f_S^2)$ and (iii) the effect due to absorption B .

$$\begin{aligned} \alpha(m_K, M, M_A, M_S, f_A^2, f_S^2) = & \alpha_A(m_K, M, M_A, f_A^2) \\ & + \alpha_S(m_K, M, M_S, f_S^2) + B. \end{aligned} \quad (22)$$

Then

$$D^{(1)}(m_K) = D_A^{(1)}(m_K) + D_S^{(1)}(m_K) + B, \quad (23)$$

$$D_A^{(1)}(m_K) = F_{1A}(m_K) + \alpha_A(m_K, M, M_A, f_A^2), \quad (24)$$

$$D_S^{(1)}(m_K) = F_{1S}(m_K) + \alpha_S(m_K, M, M_S, f_S^2). \quad (25)$$

Since $D_S^{(1)}(m_K)$ ought to have a similar form to that of $D_A^{(1)}(m_K)$, we now try to determine the form of $D_A^{(1)}(m_K)$. In the case where the nucleon core does not undergo any essential change in the course of the collision, the form of $D_A^{(1)}(m_K)$

* Compare the $F_{1A}(m_K)$ in Eq. (11) with the $F_1(\mu)$ in Eq. (11)'. But note that the difference in the factor between these two quantities is derived from the consideration about isotopic spin.

** The description of the other K -meson reactions will be influenced by the existence of such an effect as this. Some discussion about this problem will be made in near future.

should be reduced to that of $D^{(1)}(\mu)$ in the case of π - N scattering except for the difference in the mass between m_K and μ . From the relation between Eq. (11)' and Eq. (11)'', we may determine the form of $\alpha_A(m_K, M, M_A=M, f_A^2) \equiv \alpha_A(m_K, M, f_A^2)$ in the case of $M_A=M$, that is,

$$\alpha_A(m_K, M, f_A^2) = 0 \quad (26)$$

or

$$\alpha_A(m_K, M, f_A^2) = -f_A^2/2M, \quad (27)$$

corresponding respectively to Method (I) or Method (II) in the previous paper,¹⁾ where the factor 1, 2 in the right-hand side of Eq. (27) is the coefficient derived from the consideration about isotopic spin. Making use of the relation (20), we get

$$\alpha_A(m_K, M, M_A, f_A^2) = 0 \quad (26)'$$

or

$$\alpha_A(m_K, M, M_A, f_A^2) = \frac{f_A^2}{2} \frac{(M_A^2 - M^2 - m_K^2)/M}{m_K^2 - \{(M_A^2 - M^2 - m_K^2)/2M\}^2}. \quad (27)'$$

Thus

$$D^{(1)}(m_K) = \xi \left[f_A^2 \frac{(M_A^2 - M^2 - m_K^2)/2M}{m_K^2 - \{(M_A^2 - M^2 - m_K^2)/2M\}^2} + f_s^2 \frac{(M_s^2 - M^2 - m_K^2)/2M}{m_K^2 - \{(M_s^2 - M^2 - m_K^2)/2M\}^2} \right] + B, \quad (28)$$

where $\xi=1$ or 2 corresponding to Method (I) or (II) respectively.

In order to estimate the magnitude of B in Eq. (28), it is convenient to describe the s -wave scattering processes of low energy K^-p collisions in terms of the complex scattering length $A=a+ib$ which has been employed by many authors,^{7)~9)} that is,

$$k \cot \delta = 1/A, \quad (29)$$

where δ is the complex phase shift ($\alpha+i\beta$). For simplicity we now omit the isotopic dependence.

From Eq. (29) it follows that

$$\sin \alpha \cosh \beta = (kb) \sin \alpha \sinh \beta + (ka) \cos \alpha \cosh \beta, \quad (30)$$

$$\cos \alpha \sinh \beta = (kb) \cos \alpha \cosh \beta - (ka) \sin \alpha \sinh \beta. \quad (31)$$

In the limit $k \rightarrow 0$, both $\sin \alpha$ and $\sinh \beta$ must tend to zero, since both a and b are finite quantities (cf. Eqs. (30) and (31)). Strictly speaking, in the limit as $k \rightarrow 0$,

$$\sin \alpha \cong \pm (ka), \quad \beta \cong (kb), \quad (32)$$

where the double signs (\pm) correspond to the cases of $\cos \alpha = \pm 1$ respectively. Then the R -matrix for s -wave scattering turns out to be

$$R = e^{2i\delta} - 1 = e^{-2\beta + 2i\alpha} - 1 \Rightarrow e^{-2kb} [1 + 2i(ka)] - 1. \quad (33)$$

The real part of forward scattering amplitude $D(\omega)_{k \rightarrow 0}$ can be expressed as follows:

$$D(m_K) = \lim_{k \rightarrow 0} \frac{\text{Im } R}{2ik} = \lim_{k \rightarrow 0} \frac{2ika}{2ik} e^{-2kb} = a. \quad (34)$$

So far as the $D(m_K)$ alone is concerned, the effect due to absorption reaction is not so important.* Thus we may neglect the B -term in Eq. (28) which represents the absorption effect on K - p scattering in the limiting case of $k=0$.

According to the result obtained by Igi,²⁾ it seems that the value of coupling constant f_A^2 (or f_π^2) is of the order of $1/4$.** If this value of coupling constant is adopted, then we obtain the following result by means of the relation (28):

$$D^{(1)}(m_K) = 0.1/m_K \quad (\text{for } \xi=1) \quad (35)$$

or

$$D^{(1)}(m_K) = 0.2/m_K \quad (\text{for } \xi=2). \quad (36)$$

Igi²⁾ has analyzed the experimental data on low energy K^\pm - p scattering and has obtained the results:

$$D_+(m_K) = (-1.33/m_K), \quad D_-(m_K) = 2.26/m_K. \quad (37)$$

From these values it follows that

$$D^{(1)}(m_K) = 0.47/m_K. \quad (38)$$

Although there is enough ground for controversy with regard to our method, it may be said that Method (II) is more suitable than Method (I). If $f_A^2 = f_\pi^2 = 1/2$, the calculation based on Method (II) predicts the result $D^{(1)}(m_K) = (0.4/m_K)$ which agrees fairly well with the result mentioned in Eq. (38). But it may be premature to discuss the value in detail, since the statistics of experimental data on K^\pm - p scattering are too poor to supply detailed quantitative information on K - N interaction.

§ 4. On scalar K - N interaction

In the case where both $KN\Lambda$ - and $KN\Sigma$ -interactions are of scalar type, the expressions for $F_1(\omega)$ and $F_2(\omega)$ can be obtained by the substitution of $f_A^2 \rightarrow -g_A^2(s)$ in Eqs. (11) and (12).*** ($g_A(s)$ is the renormalized scalar coupling constant.****) When $M_A = M$, we obtain

* On the contrary, the imaginary part of forward scattering amplitude $A(m_K)$ can be expressed mostly in terms of the absorption effect.

** This value corresponds nearly to g_A^2 (or g_π^2) = 4 in pseudoscalar coupling constant. Naturally, this value may not be a very reliable one.

*** See reference 2).

**** Our $g_A^2(s)$ corresponds to the $g_A^2[(M_A + M)^2 - m_K^2]/4M^2$ in reference 3).

$$F_{1A}(m_K) = g_A'^2(s)/M, \quad F_{2A}(m_K) = -2g_A'^2(s)/m_K, \quad (14)'$$

where $g_A'^2(s) = 2g_A'^2(s)$.

The perturbation results corresponding to Eqs. (15) and (16) are

$$D_A^{(1)}(m_K) = \frac{-g_A'^2(s)(M_A - M)}{(1 + m_K/M) \{m_K^2 - (M_A - M)^2\}}, \quad (15)'$$

$$D_A^{(2)}(m_K) = \frac{-g_A'^2(s)m_K}{(1 + m_K/M) \{m_K^2 - (M_A - M)^2\}}. \quad (16)'$$

When $M_A = M$,

$$D_A^{(1)}(m_K) \cong 0, \quad D_A^{(2)}(m_K) = -2g_A'^2(s)/m_K. \quad (17)'$$

In the case of pseudoscalar coupling theory, we may regard a suppression of contributions from virtual baryon-antibaryon pairs as one of the reasons why the coupling constant g_A is reduced to f_A in nucleon core. But it may not be expected that, in the case of scalar coupling theory, there exists such an effect as this. Therefore the fact that $D_A^{(1)}(m_K)$ in Eq. (17)' differs from $F_{1A}(m_K)$ in Eq. (14)' should be interpreted in terms of the modification of the current field theory in nucleon core.

According to the analysis which has been made by Igi,²⁾ the $D_-(m_K)$ in Eq. (37) may change its sign in the case of scalar-type interaction, and for the value of $D_+(m_K)$ it seems that $D_+(m_K) = (-0.75/m_K)$ may also be allowed. Then the value of $D^{(1)}(m_K)$ in Eq. (38) is replaced by

$$D^{(1)}(m_K) \cong (-1.5/m_K) \quad \text{or} \quad (-1.8/m_K). \quad (38)'$$

In the case of scalar-type interaction there is no correspondence between the treatment for K - N scattering and that for π - N scattering in the strict sense of the word, so $D^{(1)}(\omega)$ in this case may not be estimated by the same way as in § 3, but if an attempt to estimate the value of $D^{(1)}(m_K)$ is made under the assumption that $D^{(1)}(m_K)$ may be obtained by the substitution of $f_A'^2$ (or $f_\pi'^2$) $\rightarrow -g_A'^2(s)$ (or $-g_\pi'^2(s)$) in Eq. (28),

$$D^{(1)}(m_K) = \hat{\epsilon} \left[-g_A'^2(s) \frac{(M_A^2 - M^2 - m_K^2)/2M}{m_K^2 - \{(M_A^2 - M^2 - m_K^2)/2M\}^2} - g_\pi'^2(s) \frac{(M_\pi^2 - M^2 - m_K^2)/2M}{m_K^2 - \{(M_\pi^2 - M^2 - m_K^2)/2M\}^2} \right]. \quad (28)'$$

When $g_A'^2(s) = g_\pi'^2(s) = 2.5$, we have

$$D^{(1)}(m_K) = -1.0/m_K \quad (\text{for } \hat{\epsilon} = 1) \quad (35)'$$

or

$$D^{(1)}(m_K) = -2.0/m_K \quad (\text{for } \hat{\epsilon} = 2). \quad (36)'$$

It may also be said in this case that our results do not contradict with the values mentioned in Eq. (38)'.

§ 5. Discussions

The f_A in Eq. (11) or (12) is of course equal to $(m_K/2M)g_A$ in the limiting case $M_A=M$ (for example $f_A=m_K g_A/(M_A+M)$).^{*} Therefore the values of $f_A^2=f_z^2=1/4$ nearly correspond to those of $g_A^2=g_z^2=4$. But these values given by Igi or Matthews-Salam may suffer a considerable change with the increasing accuracy of experimental data. According to the perturbation result it seems that $g_A^2 \cong 1$ (or $g_z^2 \cong 1$). Indeed the cross sections for $K^\pm p$ scattering may be explained in terms of perturbation theory if only the value of g_A^2 (or g_z^2) is of the order of unity, but there has been no experimental evidence for validity of the current field theory within nucleon core. As is seen through Eqs. (15) and (16), the perturbation calculation gives the following characteristic results.

$$|D_A^{(1)}(m_K)/D_A^{(2)}(m_K)| \cong |D_z^{(1)}(m_K)/D_z^{(2)}(m_K)| \cong 4.5, \quad (39)$$

$$D_A^{(1)}(m_K) \cong -g_A^2/2M, \quad D_z^{(1)}(m_K) \cong -g_z^2/2M \quad (40)$$

Since the contributions from nucleon core to the scattering cross section are predominantly larger than those from K -meson cloud (cf. Eqs. (39) and (40)), it may be said that the determination of coupling constant in perturbation theory has been made by comparing the experimental data with the contributions from nucleon core which have probably been estimated incorrectly by means of perturbation calculation. The perturbation calculation will predict the result $\sigma_-(m_K) \cong \sigma_+(m_K)$ because $|D^{(1)}(m_K)| \gg |D^{(2)}(m_K)|$. This is obviously inconsistent with the experimental results although one of the reasons of the inconsistency is attributed to the neglect of an important effect due to the absorption reaction of K -meson. Thus we may say that there will be no reliable foundation for the conclusion $g_A^2 \cong 1$ (or $g_z^2 \cong 1$). Now we should like to emphasize that the determination of coupling constant in the current field theory must be performed on the basis of the phenomena in the neighborhood of K -meson cloud. A detailed discussion about the coupling constant will be made in near future by taking into consideration the other K -meson reactions too (for example, K -meson production by γ -ray).

In § 3 we have tried to estimate the value of $D^{(1)}(m_K)$ along the same line with the method in the previous paper.¹⁾ As the treatment for $K^\pm p$ scattering is more complicated than that for $\pi^\pm p$ scattering in such a meaning that two kinds of coupling constants (f_A and f_z) are involved in the former apart from the question of the effect due to absorption, we are obliged to introduce the assumption of (22). There is naturally enough ground for controversy with regard to this assumption. Therefore it may be difficult to examine the propriety of the method employed in the case of πN scattering through our study of $K N$ scattering. If the value of coupling constant f_A^2 (or f_z^2) is of the order of $1/2$, there will be a considerable difference between the result obtained by Method (I) and that obtained

^{*} The f_A^2 corresponds to the $g_A^2[m_K^2 - (M_A - M)^2]/4M^2$ in reference 3).

by Method (II) as was shown in § 3, although the experimental results on π - N scattering can be satisfactorily explained by Method (I) as well as by Method (II) in the previous paper.¹⁾ Thus it will be possible to make a discrimination between them when the reliable experimental data for K^\pm - p scattering will be reported in detail.

Up to now we have taken into account only the KNA - and $KN\Sigma$ -interactions. When the K -interaction Lagrangian contains $\phi_K^2(x)$ or $\phi_K^4(x)$ terms explicitly, an additional term C must be involved on the right-hand side of Eq. (1). If we are forced to reinterpret our subtraction method under the assumption that interaction Lagrangian contains $\phi_K^2(x)$ or $\phi_K^4(x)$ -terms, it may be said that the additional C term ought to be included in our expression $G_{1r}(\omega) - G_1(\omega)$.

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A Unified Model for Elementary Particles

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By extending the Sakata model, a unified model for elementary particles is proposed, the basic particles in the Sakata model are assumed to be constructed of the lepton and B^+ , which is regarded as a new kind of matter. The full symmetry among three basic particles and the symmetrical property of the weak interactions which was recently pointed out by Gamba, Marshak and Okubo come automatically of this model. The nature of B^+ and the new mechanics which accounts for the binding of B^+ to the lepton will be the central problem to be studied in the future.

§ 1. Introduction

A few years ago, one of the present authors (S. S.)¹⁾ proposed the composite model for elementary particles, in which all particles belonging to the boson-baryon family were considered to be constructed of protons, neutrons, Λ -particles and their anti-particles. On the basis of this model, many works²⁾ have so far been carried out; especially, postulating the full symmetry among the three basic particles,^{3), 4)} Ikeda, Ogawa and Ohnuki,⁵⁾ and Sawada and Yonezawa^{6), 7)} have recently obtained very interesting results which are in good agreement with experimental data.

In spite of these successes, the Sakata model has been concerned only with the strongly interacting particles, and nothing has been done with the lepton. Moreover, even in the case of strong interactions, the charge independence or the full symmetry (which we shall call the I.O.O. symmetry) among the three basic particles was introduced in a formal way, and it has not yet been clear why the strong interaction should have such invariance properties. In order to describe both baryon and lepton in a unified way, and to find out a profound meaning hidden behind the charge independence or the I.O.O. symmetry, we must go further from the Sakata model.

For this purpose let us turn our attention to the weak interaction. The theory of weak interaction has now come to a very definite point after the great success of the current-current interaction theory proposed by Feynman and Gell-Mann.⁸⁾ Furthermore, it has recently become clear that the Feynman-Gell-Mann current derived from the Sakata model is quite sufficient to account for the experimental facts concerning the weak processes.^{9), 10)} Especially, it should be noted that the terms which obeyed the conditions $\Delta S/\Delta Q = -1$ and $|\Delta S/\Delta Q| > 1$ automatically dropped out of the current. Thus, by adopting the Feynman and Gell-Mann theory and

taking the standpoint of the Sakata model, we may write down the interaction Hamiltonian for the weak interaction as follows:

$$H_{weak} = \mathcal{J}_\mu \cdot \mathcal{J}_\mu^+, \quad (1.1)$$

where

$$\mathcal{J}_\mu = f_1(\bar{\nu}\gamma_\mu(1+\gamma_5)e^-) + f_2(\bar{\nu}\gamma_\mu(1+\gamma_5)\mu^-) + f_3(\bar{P}\gamma_\mu(1+\gamma_5)N) + f_4(\bar{P}\gamma_\mu(1+\gamma_5)\Lambda) \quad (1.2)$$

and

$$f_1 \simeq f_2 \simeq f_3, \quad (f_4/f_1)^2 \simeq \frac{1}{10} \sim \frac{1}{100}^{11)} \quad (1.3)$$

As we see from (1.1), the weak interactions have the universal character independent of the family concept of the elementary particles. Not only the coupling constants are of the same order of magnitude, but also the forms have the ($A-V$) type in common. Moreover, if we disregard the smallness of f_4 in comparison with other f 's, the interaction Hamiltonian (1.1) is invariant under the simultaneous transformation

$$P \leftrightarrow \nu, \quad N \leftrightarrow e^-, \quad \Lambda \leftrightarrow \mu^-. \quad (1.4)$$

This fact was first pointed out by Gamba, Marshak and Okubo¹²⁾ at the Kiev Conference, so we shall call it the Kiev symmetry. The existence of the Kiev symmetry tells us that, apart from the strong and electromagnetic interactions, proton, neutron and Λ -particle have the same properties as those of neutrino, electron and muon respectively. Therefore, if we imagine that P , N and Λ are the compound system of positively charged matter B^+ with ν , e^- and μ^- respectively, all weak interactions will be reduced to those of leptons, and the strong interaction will be regarded as the properties of B^+ .

In the next section, we shall discuss this model in detail, and show how the main features of the weak and strong interactions can be derived from it.

§ 2. Qualitative aspects of the model

We have proposed a unified model for elementary particles in which the three basic baryons are supposed to be composed of B^+ together with ν , e^- and μ^- respectively. So we may denote

$$P \equiv \langle B^+ \nu \rangle^*$$

$$N \equiv \langle B^+ e^- \rangle$$

$$\Lambda \equiv \langle B^+ \mu^- \rangle.$$

* It should be remarked that we must adopt the four-component theory of the neutrino to get the real proton:

At present, however, the nature of B^+ as well as the mechanism by which B^+ is bound to the lepton are entirely unknown. Thus it is open to question whether B^+ behaves like the ordinary matter or not. For example, we may imagine B^+ having a charge-like nature, as was suggested by Taketani.¹³⁾ Even if B^+ has a particle-like nature, it would be quite doubtful whether it obeys the quantum law or not. These points will be discussed in the next section. In any case, we should now be satisfied with qualitative considerations.

Let us first consider the properties of B^+ . Irrespective of its real nature, B^+ should have the following properties:

- (i) B^+ is bound to only one lepton, and not to any anti-lepton.
- (ii) Separation of a baryon into B^+ and lepton is very hard or may be impossible.
- (iii) All of the particle-like natures of the baryon (spin, statistics, and so on) come solely from those of the lepton. The role of B^+ in the baryon is to make it massive and active.
- (iv) B -matter must be conserved for all processes. This will account for the conservation of baryon number in the usual theory.

Next we shall discuss the weak interaction. In our model, it is assumed that the lepton is the only source of weak interactions. Therefore, adopting the current-current interaction theory proposed by Feynman and Gell-Mann,⁸⁾ we shall start from the primary interaction of the form

$$H_{weak} = j_\mu \cdot j_\mu^+, \quad (2.1)$$

where j_μ is the Feynman and Gell-Mann current (F.G.-current) of the lepton:

$$j_\mu = f \left\{ (\bar{e}\gamma_\mu(1+\gamma_5)\nu) + (\bar{\mu}\gamma_\mu(1+\gamma_5)\nu) \right\}. \quad (2.2)$$

Now, if B^+ is bound to the lepton to form the baryon, the F.G.-current of the lepton will take the following form:

$$J_\mu = \langle j_\mu \rangle_B = f \left\{ (\bar{N}\gamma_\mu(1+\gamma_5)P) + (\bar{A}\gamma_\mu(1+\gamma_5)P) \right\}. \quad (2.3)$$

Therefore, in order to take the baryon into consideration, we shall have to use, instead of (2.1) and (2.2), the effective Hamiltonian

$$H_{weak}^{eff} = \mathcal{J}_\mu \cdot \mathcal{J}_\mu^+ \quad (2.4)$$

with the effective current

$$\mathcal{J}_\mu = j_\mu + J_\mu. \quad (2.5)$$

(2.4) and (2.5) are obviously identical with (1.1) and (1.2) respectively.

Here it should be noted that we have assumed the conservation of B^+ in each of the F.G.-current. If we require only the conservation of B^+ through the whole weak process, the effective Hamiltonian would in general take the form

$$H_{weak}^{eff} = J_\mu \cdot J_\mu^{*+} + J_\mu' \cdot J_\mu'^{*+} + J_\mu'' \cdot J_\mu''^{*+} \quad (2.6)$$

with

$$J_\mu' = f \left\{ (\bar{N} \gamma_\mu (1 + \gamma_5) \nu) + (\bar{A} \gamma_\mu (1 + \gamma_5) \nu) \right\} \quad (2.7)$$

and

$$J_\mu'' = f \left\{ (\bar{e} \gamma_\mu (1 + \gamma_5) P) + (\bar{\mu} \gamma_\mu (1 + \gamma_5) P) \right\} \quad (2.8)$$

(2.6) is, however, quite unfavourable, because the unwanted processes such as

$$K^+ \rightarrow \pi^+ + \nu + \bar{\nu}$$

or

$$\mu^- + P \rightarrow e^- + P$$

would occur. This is the reason why we assume that B^+ is conserved not only through the whole weak process, but also in each of the F.G.-current.*

As was mentioned in § 1, the coupling constant of the Fermi interaction including $A(f_4$ is (1.4)) seems to be slightly smaller than those of the other,¹¹⁾ hence the effective Hamiltonian (2.4) should be regarded as an approximate one. We may, however, expect that such differences among the coupling constants will be explained, for instance, by taking into account the unknown mechanism of binding B^+ to the muon, which is not considered here.

Now we shall discuss the strong interaction. In our model, it is assumed that the strength of the strong interaction between the basic baryons should be determined by the nature of B -matter, and be independent or almost independent of the *kind* of lepton to which B -matter is bound. Moreover, the lepton in the baryon cannot change its *kind* by the strong interaction, because such a transition happens only when the lepton suffers a weak interaction. It is then conceivable to suppose that, in virtue of the property (ii) of B^+ , the *currents* defined as

$$I_P \equiv \langle \bar{\nu} O \nu \rangle_B = (\bar{P} O P),$$

$$I_N \equiv \langle \bar{e} O e \rangle_B = (\bar{N} O N),$$

$$I_A \equiv \langle \bar{\mu} O \mu \rangle_B = (\bar{A} O A),$$

where O denotes the Dirac matrix, for each basic baryon plays an essential role to provide the strong interaction, although nothing is known about its profound meaning.** From these assumptions, one can easily conclude that the effective Hamiltonian for the strong interaction should have a full symmetrical form. Accordingly if we put

* If the weak interactions were mediated by a heavy charged boson introduced by Feynman and Gell-Mann,⁸⁾ this assumption would come out automatically.

** It should be mentioned that an example of such a theory was once proposed by Y. Fujii.¹⁴⁾

$$\chi = \begin{matrix} P \\ N \\ A \end{matrix}, \quad (2.9)$$

the effective Hamiltonian will be expressed by a function of $(\bar{\chi}O\chi)$. If we take, for example, the four-fermion interaction as a primary one, it should have the following form:

$$H_{strong}^{eff} = \sum_A g_A (\bar{\chi}O^A\chi) (\bar{\chi}O^A\chi). \quad (2.10)$$

We here notice that the Hamiltonian for the strong interaction satisfies not only the charge independence, but also the I.O.O. symmetry. These properties have hitherto been explained on the basis of the hypothetical charge space, whereas in our model such symmetries are an immediate consequence derived from the simple properties of B -matter. It should be stressed that it is no more necessary to assume the existence of any formal space whose physical meaning is quite obscure.

Thus we see that the main features of both strong and weak interactions can be ascribed to the three leptons and B -matter.

Finally we shall remark a possibility of the process

$$\mu^- + N \rightleftharpoons A + e^-,$$

which represents an exchange of free and bound leptons. This is supposed to be caused through the following steps:

$$\mu^- + N \rightleftharpoons \mu^- + \langle B^+ e^- \rangle \rightleftharpoons \langle B^+ \mu^- \rangle + e^- \rightleftharpoons A + e^-,$$

that is, by a collision of the neutron (A -particle) and the muon (electron), the former is dissociated into the B^+ and the electron (muon), and thereafter the B^+ is bound to the muon (electron) leaving the electron (muon) free. Since, as was mentioned previously, such separation of the baryon into B^+ and lepton is thought to be very hard (even in the virtual process), this reaction would be a very high energy phenomena if it occurs. In spite of the difficulty of its observation, we are interested in it, for it will give us some information about the binding of B^+ to the lepton.

§ 3. Further outlook

In the preceding sections, we have described a new model of elementary particles which, unifying all elementary particles into the three leptons (ν , e^- and μ^-) and a new kind of matter B^+ , could account for the existence of three basic baryons (P , N and A) strongly interacting in charge-independent or I.O.O. symmetric ways. Further, our model was shown to afford a conceivable interpretation of the Kiev symmetry of weak interactions.

To make further progress, however, it would be necessary to clarify many

problems being left unsolved in the present discussions. Among these the following may be of fundamental importance:

1. To investigate the physical meaning lying behind the differences among the basic baryons. This may be closely related to a problem of finding the structures of ν , e^- and μ^- .¹⁵⁾

2. To find a physical ground for the space-time structure of interactions which has been assumed more or less formally or phenomenologically. In other words, it will be necessary to search for the reason of parity-conserving character of the strong interactions on the one hand, and of parity-violating $A-V$ form of the weak interactions on the other. Especially the former problem would be settled in connection with discovering the true nature of B -matter.

3. To discover a dynamical law which governs the compound system of lepton and B^+ . In order to search for the new physical principles which reveal themselves in the sub-quantum-mechanical region of nature, and thus to enable us to develop the present model in more concrete form, it would be instructive to draw our attention to the various possibilities of constructing the lepton- B^+ system. The possible forms of construction may be classified in accordance with our intuitive pictures into the following categories:

a) *The hydrogen-atom-like model.* In this model, both B^+ and lepton are regarded as point-like particles. We then treat B^+ as a kind of charged heavy bosons with spin 0 and the baryons are considered to be composite systems of leptons and B^+ "particles", although the applicability of present-day quantum field theory to such systems might be more or less doubtful.

b) *The core type model.* One of the constituents (the lepton or B^+) is assumed to be the point-like core of the other, whereas the latter is imagined to have an intrinsic extension in space-time.

c) *The two-fluid model.* Both B -matter and lepton are nonlocally distributed in a small region of space, and combined dynamically by each other as if they were two different kinds of fluid.

d) *The vessel model.* The lepton is imagined to play a role as the vessel or the bag in which B -matter can be filled up. An empty vessel is then nothing but the lepton itself, whereas a filled one corresponds to the baryon. In this model, B -matter would not be able to be observed in any way except in the case where B -matter exists as a filling material of the lepton-vessels. In this respect, B may have a property similar to that of the charge as was mentioned in § 2. Along this line of thought Matumoto and one of the present authors (M. N.) developed a model¹⁶⁾ which could well interpret, e.g. the mass formula of elementary particles proposed by Matumoto.²⁾ Besides these models, we may picture many other models for our compound system. The models a)–d) are merely the ones which we take as the typical representatives being expected to be of some use in future development of our model.

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On the Structure of the Elementary Particles

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Recently a unified model¹⁾ for the elementary particles (called "the Nagoya Model") has been proposed in which the basic baryons in the Sakata model are assumed to be composed of the leptons and a new kind of matter, B . Through the investigation of the mass formula for the compound particles in the Sakata model, it is shown that the B -matter has a charge-like character similar to the electric charge ($\varepsilon = -\bar{\varepsilon}$). On the basis of this character of the B -matter, i) the mass formula (the bond character and the mass-bond relation) is reproduced, ii) the physical ground is given to Taketani's idea,²⁾ and his idea is developed. A dynamical law of the basic baryons within the compound particle is discussed. Remarks on new boson-baryon family and photon are given.

§ 1. Introduction

Through the investigations of the elementary particles and their mutual interactions, it has been found that the elementary particles are classified into the lepton (ν, e, μ) and the boson-baryon families and have a kind of universal interaction, e.g. the weak interaction, and that the boson-baryon family has another kind of universal interaction, e.g., the strong interaction.

For the strong interaction, the well-known rule of Nakano-Nishijima and Gell-Mann³⁾ has been established phenomenologically. The more profound and realistic way of understanding of this rule has been proposed by Sakata.⁴⁾ According to Sakata's idea, only proton, p , neutron, n , Λ -particle, Λ , and their anti-particles, $(\bar{p}, \bar{n}, \bar{\Lambda})$, are assumed to be the basic particles which compose other baryons and bosons. Through the analysis of the strong interactions based on the Sakata model, it has been pointed out that the full symmetry (I.O.O.-symmetry)⁵⁾ among p, n and Λ exists when the nucleon- Λ mass difference is neglected. This idea has been further developed and brought into fruitful results.⁶⁾

On the other hand, for the weak interaction, it is clarified that if ν, e, μ are assumed to be the basic particles as well as p, n, Λ , the Gamba, Marshak and Okubo symmetry⁷⁾ for the exchange $(\nu, e, \mu) \rightleftharpoons (p, n, \Lambda)$ is approximately valid.⁸⁾

Based on these symmetries, the Nagoya model¹⁾ has been proposed in which the basic baryons p, n, Λ in the Sakata model are assumed to be composed of the leptons ν, e, μ and a new kind of matter, B .

According to the Nagoya model, the characteristics of the interaction among B -matters are supposed to be reflected in the characteristics of the strong interaction in the Sakata model.

Based on the Sakata model, the semi-empirical mass formula⁹⁾ for the compound particles has been proposed and fairly well succeeded.* Here, we would consider that the characteristics of the interaction among B -matters may be strongly reflected in the mass formula.

The semi-empirical mass formula for the compound particles in the Sakata model are as follows:

$$\begin{aligned} M = & m_B(n_B + n_{\bar{B}}) - V_B(n_{B\bar{B}} - n_{BB} - n_{\bar{B}\bar{B}}) \\ & + \Delta m(n_A + n_{\bar{A}}) + \Delta V(n_{N\bar{A}} + n_{NA} - n_{NA} - n_{\bar{N}\bar{A}}) \\ & + 2\Delta V(n_{A\bar{A}} - n_{AA} - n_{\bar{A}\bar{A}}), \end{aligned} \quad (1.1)$$

where $m_B \approx 1840 m_e$,

$V_B \approx 3400 m_e$,

$\Delta m = \Delta V \approx 300 \sim 400 m_e$,

n_B, n_A : the number of basic baryon, A -particle in the compound particles,

$n_{B\bar{B}}, n_{NA}$, etc.: the number of the bond $B\bar{B}'$, $N\bar{A}$, etc., in the compound particles.

The main part of the mass formula is reduced to

$$M = m_B(n_B + n_{\bar{B}}) - V_B(n_{B\bar{B}} - n_{BB} - n_{\bar{B}\bar{B}}), \quad (1.2)$$

by neglect of $\Delta m, \Delta V$.

Here, the bond character in this mass formula can be shown as just reproducible** if we assume as follows.

- (i) B -matter distributes in ν, e, μ which are something like the sphere of radius l_0 .
- (ii) There exists the very short range ($\ll l_0$) interaction among B -matters. The interaction energy of $B\bar{B}$ is equal but of the opposite sign to that of $B\bar{B}$ and $\bar{B}\bar{B}$.

Namely, the interaction energies of $B\bar{B}$, $B\bar{B}$, $\bar{B}\bar{B}$ are given by

$$\begin{aligned} CC & \iint \rho(\mathbf{r}) v(\mathbf{r} - \mathbf{r}') \bar{\rho}(\mathbf{r}') d\mathbf{r} d\mathbf{r}', \\ CC & \iint \rho(\mathbf{r}) v(\mathbf{r} - \mathbf{r}') \rho(\mathbf{r}') d\mathbf{r} d\mathbf{r}', \\ \bar{C}\bar{C} & \iint \bar{\rho}(\mathbf{r}) v(\mathbf{r} - \mathbf{r}') \bar{\rho}(\mathbf{r}') d\mathbf{r} d\mathbf{r}', \\ C & = -\bar{C}, \end{aligned} \quad (1.3)$$

* The main consequences obtained by the mass formula are, e.g. the existence of the resonance of $K\bar{N}$ scattering (incident energy ≈ 300 Mev) and new bosons ($\approx 1700 m_e$) with strangeness ± 2 . These are supported by the recent experimental analysis.¹⁰⁾

** The authors thank Drs. Z. Maki and S. Sawada for the discussions on this point.

where $\rho(\mathbf{r})$, $\bar{\rho}(\mathbf{r})$ are the densities of the B - and anti- B -matter and $v(\mathbf{r}-\mathbf{r}')$ is a very short range ($\ll l_0$) interaction.

- (iii) The energy of the ground state of the compound particle is given by the classical minimum energy.

According to this analysis, we are tempted to suppose that the B -matter has a charge-like character* ($C=-\bar{C}$) similar to the electric charge ($\varepsilon=-\bar{\varepsilon}$).

Recently, an attempt (the Taketani-Katayama model)¹¹⁾ has been made to understand the mutual differences among the leptons ν , e , μ due to the differences among the distributions of the electric charge within the leptons.

The fact that the B -matter has the charge-like character ($C=-\bar{C}$) leads us to Taketani's idea²⁾ and gives the physical ground to his idea. Taketani attempted to unify the Nagoya model (lepton B -matter baryon) and the Taketani-Katayama model in the following manner. (Fig. 1)

- (1) e and μ are constructed by loading the ε -charge to ν .
- (2) The basic baryons (p , n , Λ) are constructed by loading the B -charge to the leptons (ν , e , μ) respectively.

In this paper, we shall introduce the assumptions about the structure of the basic particles and of the interaction (§ 2), derive the mass formula of the compound particles (§ 3), and examine the conditions required for the law of dynamics inside the region of the compound particles (§ 4).

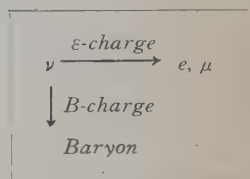


Fig. 1.

§ 2. Basic assumptions about the model

Our basic assumptions on the structure of the elementary particles are as follows.

- (1) ν is the basic matter which can load** the ε -charge and the B -charge. There exists its anti-matter $\bar{\nu}$. ν has the spin 1/2 and the mass nearly equal to zero.

At present, we consider ν as a rigid sphere with radius l_0 .

- (2) $\bar{\varepsilon}$ (anti- ε -charge) $= -\varepsilon$, $\varepsilon = \varepsilon \int \rho_\varepsilon(\mathbf{r}) d\mathbf{r}$, (electric charge-),

$$\bar{B}(\text{anti-}B\text{-charge}) = -B, \quad B = B \int \rho_B(\mathbf{r}) d\mathbf{r}, \quad (\text{electric charge+}). \quad (2.1)$$

There exists a very short range ($\ll l_0$) interaction between the charges. The interaction energies are given by

$$BB' \iint \rho_B(\mathbf{r}) v(\mathbf{r}-\mathbf{r}') \rho_{B'}(\mathbf{r}') d\mathbf{r} d\mathbf{r}', \quad (2.2)$$

and

$$\varepsilon\varepsilon' \iint \rho_\varepsilon(\mathbf{r}) v(\mathbf{r}-\mathbf{r}') \rho_{\varepsilon'}(\mathbf{r}') d\mathbf{r} d\mathbf{r}'.$$

* The B -matter is conserved in the interaction as in the case for the ε -charge.

** The fundamental origin of the charge and the mechanism of the loading are yet unknown in the present stage.

The ε - and the B -charge are conserved in the interaction respectively.

(3) ν can load only the ε - and the B -charge but neither the $\bar{\varepsilon}$ - nor the \bar{B} -charge.

$\bar{\nu}$ can load only the $\bar{\varepsilon}$ - and the \bar{B} -charge but neither the ε - and nor the B -charge.

(4) e and μ are constructed by loading the ε -charge to ν . The basic baryons (p, n, A) are constructed by loading the B -charge to the leptons (ν, e, μ) respectively.

There exists the freedom of the different charge distribution and this appears as the e - μ difference.

§ 3. The masses and the mutual interaction of the basic particles and the mass formula for the compound particles

[1] The mass of the basic particle

The masses are produced by the interactions between the charges as

$$m_B = \frac{1}{2} B^2 \iint \rho_B(\mathbf{r}) v(\mathbf{r}-\mathbf{r}') \rho_B(\mathbf{r}') d\mathbf{r} d\mathbf{r}', \quad (3.1)$$

$$m_\varepsilon = \frac{1}{2} \varepsilon^2 \iint \rho_\varepsilon(\mathbf{r}) v(\mathbf{r}-\mathbf{r}') \rho_\varepsilon(\mathbf{r}') d\mathbf{r} d\mathbf{r}'.$$

If we assume that $\rho_B(\mathbf{r})$ and $\rho_\varepsilon(\mathbf{r})$ are constants and

$$v(\mathbf{r}-\mathbf{r}') = \alpha l_0^2 \delta(\mathbf{r}-\mathbf{r}') \quad (3.2)$$

(α is a numerical factor of order 1),

then we obtain

$$m_B = \frac{3}{8\pi} \frac{\alpha B^2}{l_0}, \quad (3.3)$$

$$m_\varepsilon = \frac{3}{8\pi} \frac{\alpha \varepsilon^2}{l_0}. \quad (3.4)$$

From these, it holds

$$\varepsilon^2/B^2 = m_\varepsilon/m_B. \quad (3.5)$$

In this case, if we equate ε to the electric charge, e , and m_ε to the electron mass, m_e , then the length, l_0 , will be about $0.1 y^*$ from Eq. (3.4).** Therefore, from this result of our model it may be expected that the quantum electrodynamics loses its validity near the range of this order.

* $1y$ (1 yukawa) denotes a unit of length ($=10^{-13}\text{cm}$).

** If we assume $v(\mathbf{r}-\mathbf{r}') = 1/|\mathbf{r}-\mathbf{r}'|$, l_0 will be of the order of the classical electron radius ($\sim 2.8y$), and this conflicts with the present knowledge obtained from the experiments about the applicability of quantum electrodynamics.¹²⁾ The very short range, of $v(\mathbf{r}-\mathbf{r}')$ is also required from this point. We consider that the interaction between the parts of the unit electric charge is not necessarily due to the exchange of γ (photon).

[2] *The interaction between the basic baryons ($p, n, \Lambda, \bar{p}, \bar{n}, \bar{\Lambda}$)*

The interaction energy between the baryon 1 and the baryon 2 is given as follows,*

$$V_B^{12}(\mathbf{R}_1 - \mathbf{R}_2) = B^1 B^2 \iint \rho_{B^1}(\mathbf{r}_1) v(\mathbf{r}_1 - \mathbf{r}_2) \rho_{B^2}(\mathbf{r}_2) d\mathbf{r}_1 d\mathbf{r}_2, \quad (3.6)$$

where $\mathbf{R}_1, \mathbf{R}_2$ are the center-of-mass coordinates of the baryon 1 and 2 respectively. It should be remarked that there exists the following relation,

$$V_B^{12}(0) = 2m_B \epsilon^{12},$$

$$\epsilon^{12} = \begin{cases} +1 & (1, 2 \text{ are } BB \text{ or } \bar{B}\bar{B}), \\ -1 & (1, 2 \text{ are } B\bar{B}), \end{cases} \quad (3.7)$$

namely $V_B^{12}(\mathbf{R}_1 - \mathbf{R}_2)$ is the interaction with the depth not larger than $2m_B$ and the range of the order of l_0 .

The total energy H of the system of n basic baryons is then given by

$$H = \sum_{i=1}^n \sqrt{p_i^2 + m_i^2} + \sum_{i < j}^n V_B^{ij}(\mathbf{R}_i - \mathbf{R}_j). \quad (3.8)$$

If we assume that $\rho_B(\mathbf{r})$ is constant, then we obtain from Eq. (3.2)

$$V_B^{12}(\mathbf{R}_1 - \mathbf{R}_2) = V_B^{12}(0) \left[\frac{1}{16} \left(\frac{|\mathbf{R}_1 - \mathbf{R}_2|}{l_0} \right)^3 - \frac{3}{4} \frac{|\mathbf{R}_1 - \mathbf{R}_2|}{l_0} + 1 \right]$$

$$\text{for } |\mathbf{R}_1 - \mathbf{R}_2| < 2l_0,$$

$$= 0 \quad \text{for } |\mathbf{R}_1 - \mathbf{R}_2| > 2l_0. \quad (3.9)$$

[3] *Mass formula for the compound particles*

The masses of the compound particles are determined by the minimum eigenvalue of H . The classical minimum energy, E_{min} , for H is given by

$$E_{min} = \sum_{i=1}^n m_B + \sum_{i < j}^n V_B^{ij}(0), \quad (3.10)$$

and from Eq. (3.7), we obtain

$$E_{min} = \sum_{i=1}^n m_B + \sum_{i < j}^n 2m_B \epsilon^{ij}. \quad (3.11)$$

Equations (3.10), (3.11) just reproduce Eq. (1.2), namely the main part of the original mass formula and the empirical relation between m_B and V_B such as,

$$V_B \simeq 2m_B. \quad (3.12)$$

* We assume that in the interacting time $\rho_B(\mathbf{r})$ has the same distribution as in the free state. The various effects due to the deformation and the polarization, etc., are considered to be added further to Eq. (3.6) as corrections.

§ 4. The law of dynamics of the basic baryons within the compound particle

[1] Based on our model described in § 2, we have clarified that the classical minimum values of the total energy just reproduce the mass formula for the compound particles (the bond character and the relation $V_B \simeq 2 m_B$).

On the other hand, if we would apply the quantum law to the relative motion* between the center-of-mass coordinates of the respective basic baryons within the compound particle, there could not exist the stable compound particles with masses given by the formula (1.1). The reason is that within the compound particle (the dimension is of order of l_0) the fluctuation energy due to the uncertainty principle becomes necessarily too large to yield the binding energy ($V_B \simeq 2 m_B$) with the use of $V_B^{12}(\mathbf{R}_1 - \mathbf{R}_2)$ (the depth $\gtrsim 2 m_B$, range $\sim l_0$ as shown in Eqs. (3.6) and (3.7)).

Therefore, in order to reproduce the mass formula based on the model of § 2, the uncertainty fluctuation energy (quantum effect) should be suppressed in the relative motion between the center-of-mass coordinates of the respective basic baryons within the compound particle.

Here, we make an attempt to determine a measure to what extent the uncertainty fluctuation energy is suppressed. We shall express the measure by the modification for \hbar appearing in the ordinary uncertainty relation. Putting the condition that the uncertainty fluctuation energy must be smaller than m_π in the compound particle π constituted from N, \bar{N} , we obtain (Appendix)

$$\hbar_{eff}^{L=0} \lesssim (1/10) \hbar, \quad (4.1)$$

as the upper limit for $\hbar_{eff}^{L=0}$, where L denotes the relative orbital angular momentum of N, \bar{N} .

[2] In our model the dimension of the compound particle is of the order of l_0 , so the effects due to the finite size ($\sim l_0$) of the basic baryon could not be neglected in the motion within the compound particle. And there it is expected that a new dynamical law in the sub-quantum-mechanical region would play an important role instead of the quantum law essentially based on the point model hypothesis.¹³⁾

[3] From the discussions [1] and [2] in this section and from the success of the derivation of the mass formula in § 3, we would be allowed to regard our model to be true and suppose that the new dynamical law governing the compound system has something like a classical feature and the structure of the basic baryon is not originated in the quantum mechanical law.

§ 5. Concluding Remarks

[1] *New boson-baryon family.* We have found that the B -matter has a charge-

* If the structure of the basic baryon were originated in the quantum mechanical law (and particles), the relative motion between the center-of-mass coordinates of the basic baryons should necessarily obey the law of quantum theory.

like character similar to the ε -charge. Here, if we further extend the similarity to the other properties between the B -matter and the ε -charge, we could expect also the existence of two degrees of freedom for the distribution of the B -matter similar to the ε -charge. Then another new boson-baryon family will be found besides the well-known family. The new baryons (p' , n' , Λ') will have fairly larger masses than those of (p , n , Λ) corresponding to the largeness of the μ -meson mass compared to the electron mass.

[2] *Photon*. In the same way that the compound systems are constructed by the B -charge interaction, new compound systems constructed by the ε -charge interaction can be expected in our model. On the other hand, in order to obtain the compound system constructed by the ε -charge interaction, the uncertainty fluctuation energy will have to be extremely suppressed as was shown in § 4. From this reason, it seems to be difficult to describe this new compound system by the same procedure as in § 3. However, Eq. (3.5) and the empirical relation such as

$$\frac{e^2}{f^2} \simeq \frac{m_e}{m_N} \quad (g \text{ is the } \pi\text{-}N \text{ interaction constant.}), \quad (5.1)^{14)}$$

may suggest that the photon is also a compound particle due to the ε -charge in a similar sense that π is due to the B -charge.

[3] The differences between e and μ , and also N and Λ will be the future important problems to be investigated in connection with the difference of the ε -charge distribution, the mass differences, the bond differences, the violation of the G.M.O.-symmetry⁷⁾ in the weak interaction, and so on.

Acknowledgement

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Appendix

π is considered to be constructed of N , \bar{N} according to Eq. (3.9). Then the condition that the uncertainty fluctuation energy should be smaller than m_π is expressed as follows.

$$\begin{aligned} \Delta p &\gtrsim \hbar_{\text{eff}}/r, \\ (\Delta p)^2/2(M/2) &< m_\pi, \\ 2M + V_B^{N\bar{N}}(r) &\lesssim m_\pi. \end{aligned} \quad (\text{A} \cdot 1)$$

From this, putting $l_0 \sim \hbar/M$, we obtain

$$\hbar_{\text{eff}} < \frac{2}{3} \left(\frac{m_\pi}{M} \right)^{3/2} \hbar \sim \frac{1}{20} \hbar. \quad (\text{A} \cdot 2)$$

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Electromagnetic Structure of the Nucleon. IV**—Charge Distribution of the Proton—*Kichiro HIIDA,[†] Noboru NAKANISHI,^{††} and Takanori SHIOZAKI^{††}[†]*Research Institute for Fundamental Physics
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It is suggested from experimental data and theoretical analyses that the charge distribution function of the proton will become negative in the inner region. The analysis of Stanford data on the electron-proton scattering is reexamined, and a lower bound for $\langle r^2 \rangle_{1,p}$ is given model-independently from some general assumptions. As examples, two trial proton models are considered which are consistent with the present Stanford data and in which the values of $\langle r^2 \rangle_{1,p}$ are rather smaller than the usually accepted ones.

§ 1. Introduction

It seems to be generally believed that the experiments of high-energy electron-proton and electron-deuteron scatterings^(1),2),3) and low-energy neutron-atom scattering indicate**

$$\langle r^2 \rangle_{1,n} \simeq 0, \quad (1.1)$$

$$\langle r^2 \rangle_{1,p} \simeq \langle r^2 \rangle_{2,p} \simeq (0.80 \pm 0.04)^2 Y^2 \quad (1.2)$$

$$\simeq \langle r^2 \rangle_{2,n} \quad (1.2')$$

for the mean square radii of charge and a.m.m. of the nucleons. We may say that these electron scattering experiments at Stanford are valuable and progressive ones in the following sense: They have shown experimentally that the electromagnetic interactions of the nucleons take place not at a point but over some finite regions which are extended over about $0.8Y$ from the centre of the nucleons.

From the experimental values we are confronted at once with a well-known difficult problem,³⁾ that is, why is only the charge radius of the neutron so vanishingly small in spite of the fact that the other radii are very large? It is worthwhile to examine whether or not this problem can be explained by the current meson theory, in order to see whether or not meson theory is valid in the region

* A preliminary report was published in Prog. Theor. Phys. **23** (1960), 192 (L). The notation is the same as those of I (see ref. 5)). throughout this paper, unless otherwise indicated.

** We always write the length in unit of $1Y (\equiv 1 \text{ yukawa}) = 10^{-13} \text{ cm}$ throughout this paper.

$1/M \sim 1/\mu$. It has been shown, on the other hand, by the meson theoretical analyses of the nuclear forces⁴⁾ and the P -wave pion scattering by nucleons that meson theory is valid in the region $r \gtrsim 1/\mu$. We have so far investigated the theoretical values of $\langle r^2 \rangle_{1,n}$, etc.⁵⁾ We may say that these theoretical values will not be inconsistent* with the experimental ones and meson theory will qualitatively be valid in the region $1/M \sim 1/\mu$, though we cannot definitely conclude so because of the approximation used.

Next we must begin to examine whether or not the experimental values (1.1) and (1.2) can be explained not only qualitatively but also quantitatively by meson theory. As a preparation to enter into this problem, in this paper, we shall check the reliability of the experimental values $\langle r^2 \rangle_{1,n}$, $\langle r^2 \rangle_{1,p}$ and $\langle r^2 \rangle_{2,p}$.

In this place we shall investigate the possible break-down effects of quantum electrodynamics (Q.E.D.) at small distances. If Q.E.D. breaks down at small distances and if its deviation from the present theory can be represented by form factors in some sense, we may write^{3), 6)}

$$\begin{aligned} F_{1,p}^{eep}(q^2) &= F_{1,p}(q^2) F_{ph}(q^2) F_e(q^2), \\ F_{2,p}^{eep}(q^2) &= F_{2,p}(q^2) F_{ph}(q^2) F_e(q^2), \end{aligned} \quad (1.3)$$

where $F_{ph}(q^2)$ and $F_e(q^2)$ stand for the form factor of bare photon propagator and that of bare electron-photon vertex respectively, and we have discriminated the experimental form factors, $F_{1,p}^{exp}$, etc., from the true ones of proton. From (1.3) we immediately have

$$\begin{aligned} \langle r^2 \rangle_{1,p}^{eep} &= \langle r^2 \rangle_{1,p} + \langle r^2 \rangle_{ph} + \langle r^2 \rangle_e, \\ \langle r^2 \rangle_{2,p}^{eep} &= \langle r^2 \rangle_{2,p} + \langle r^2 \rangle_{ph} + \langle r^2 \rangle_e. \end{aligned} \quad (1.4)$$

There is a possibility, therefore, that the experimental values of r.m.s. radii of the proton are different from the true ones of proton owing to the break-down of Q.E.D.

Next, it must be taken into account that the value (1.2) was determined by a form-dependent method in spite of the fact that the concept of r.m.s. radius is independent of the form of its distribution. Therefore there is no assurance that another model, which has not been used by Hofstadter *et al.*, never gives a value much different from (1.2).

Contrary to the proton case, (1.1) for the neutron is free from these ambiguities. The effective charge form factor, F , for low-energy electron-neutron scattering is given by

$$F(q^2) = [F_{1,n}(q^2) - (q^2/8M^2) \{F_{1,n}(q^2) + 2F_{2,n}(q^2)\}] F_{ph}(q^2) F_e(q^2), \quad (1.5)$$

and the corresponding mean square radius is

* That is, if we take an optimistic standpoint on the cutoff methods in the calculation of the 3π -state contribution.

$$\langle r^2 \rangle = \langle r^2 \rangle_{1,n} + (3/2M^2) F_{2,n}(0). \quad (1.6)$$

Therefore (1.1) can be regarded as what was determined independently of the shape of the charge distribution of neutron and of the possible break-down of Q.E.D. at small distances.

We may thus conclude that (1.1) is reliable, but (1.2) is not so reliable at least at the present stage because (1.2) is dependent on the assumption $F_{1,p} = F_{2,p}$ and the shapes of models (even apart from the possible break-down of Q.E.D.). In this paper, therefore, we investigate the possibility of modifying (1.2) by reanalysing Stanford data. Our standpoints in this semi-phenomenological analysis are as follows:

- i) We rely upon (1.1) and Stanford data for cross sections.
- ii) We believe consequences from the general theoretical framework, for example, we believe that the form factor has a spectral representation.
- iii) We rely upon the current meson theory in the outer region $r \gtrsim 1Y$.
- iv) We utilize information from theoretical investigations as much as possible.

In § 2, we suggest on the basis of our previous calculations⁵⁾ that the charge distribution function of the proton will become negative in the inner region. In § 3, model-independent considerations are developed. We present a lower bound for $\langle r^2 \rangle_{1,p}$ without using special models, and investigate the relations between positive definiteness properties of the form factor and the distribution function. In § 4, we propose two *trial* proton models which have such shapes as suggested in § 2. Using these models we show that the values of $\langle r^2 \rangle_{1,p}$ and $\langle r^2 \rangle_{2,p}$ become smaller than (1.2), despite that these models are consistent with all the present Stanford data.

§ 2. Charge distribution of the proton

In this section we shall conjecture a qualitative form of the charge distribution of the proton. For simplicity we assume that the bare-state probability is

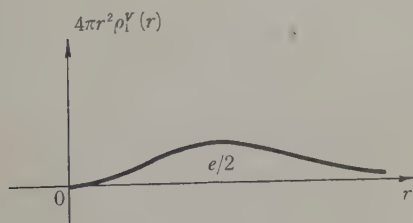


Fig. 1. (a).

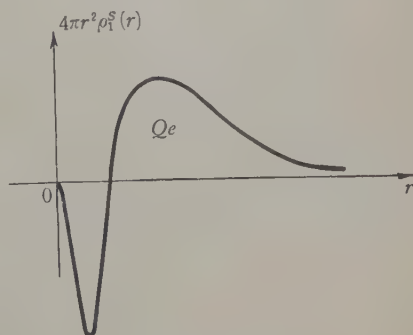


Fig. 1. (b).

zero, i.e. $Z_{2,p}=0$,* and that the charge distribution function of the isovector part, $\rho_1^V(r)$, is positive definite as shown in Fig. 1(a) (this is conjectured from perturbation and static theories⁵⁾). Furthermore the charge distribution function of the isoscalar part, $\rho_1^S(r)$, will be positive in the outer region and negative in the inner one, if we assume that our previous calculation of 3π -state contribution⁵⁾ is not drastically modified. But the inner part is naturally affected by higher-massive states so that the total charge become $e/2$ as shown in Fig. 1(b).

When we denote the amount of the outer positive charge of $\rho_1^S(r)$ by $Qe(Q>1/2)$, $\langle r^2 \rangle_1^S$ is given by

$$\begin{aligned} \langle r^2 \rangle_1^S &= Qr_s^2 - (Q-1/2)r_{s0}^2 \\ &\simeq Qr_s^2, \end{aligned} \quad (2.1)$$

where r_s and r_{s0} are the weighted mean distances of the positive and the negative charge from the origin, respectively. On account of $r_s^2 \gg r_{s0}^2$ and $Q > Q-1/2$, the second term of the right-hand side of (2.1) can be neglected. Likewise we have

$$\langle r^2 \rangle_1^V = (1/2)r_V^2. \quad (2.2)$$

Now we may assume that the ratio $r_s:r_V$ is approximately equal to $1/3:1/2$

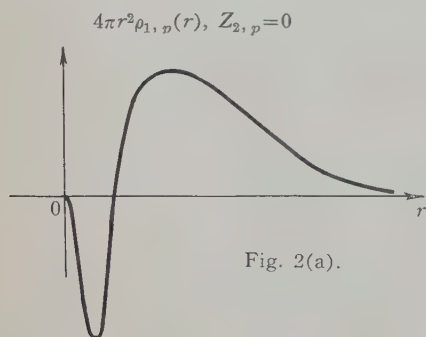


Fig. 2(a).

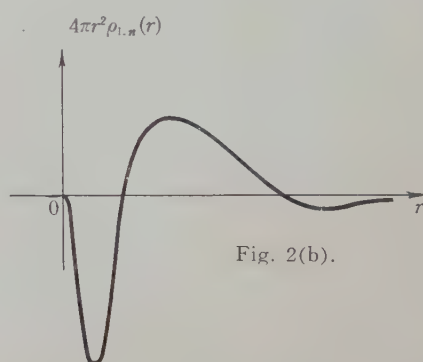


Fig. 2(b).

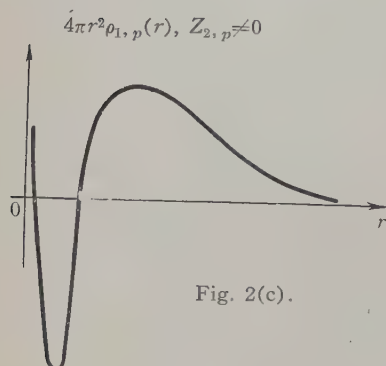


Fig. 2(c).

because the thresholds of $\alpha_1^S(m^2)$ and $\alpha_1^V(m^2)$ are respectively 3μ and 2μ , namely

$$r_s^2:r_V^2 \simeq (1/3)^2:(1/2)^2. \quad (2.3)$$

Then from (1.1), (2.1), (2.2) and (2.3), we can estimate the magnitude of Q , namely, we obtain

$$Q \simeq 1. \quad (2.4)$$

Therefore, when $Z_{2,p}=0$, the charge distributions of the proton and the neutron will

* Even if $Z_{2,p} \neq 0$, the following arguments will not be essentially modified.

probably have such forms as shown in Fig. 2(a) and 2(b), respectively. That is to say, the charge distribution of proton is positive in the outer region ($\sim 1.5e$) and negative in the inner region ($\sim -0.5e$). Katayama *et al.*⁷⁾ adopted such a form as Fig. 2(a) for the proton in order to explain the neutron-proton mass difference (assuming $F_{1,p}=F_{2,p}$). The charge distribution of neutron becomes negative-positive-negative as was pointed out by Schiff.⁸⁾ The outer part is negative because the isovector part is extended more outward than the isoscalar part as is shown in (2.3). If $Z_{2,p} \neq 0$, the charge distribution of proton has such a form as Fig. 2(c).

§ 3. Model-independent considerations

3-1. Lower bound for $\langle r^2 \rangle_{1,p}$

As was stressed in § 1, (1.2) is what was derived by assuming special models. Goldberger *et al.*⁹⁾ suggested that even

$$\langle r^2 \rangle_{1,p} \simeq 0 \quad (3.1)$$

might be possible. In this subsection we shall show that such a possibility will be excluded under some general assumptions.

Since experiments furnish cross sections only for the finite number of q^2 , we can always construct $F_{1,p}(q^2)$ so that it be consistent with all experimental data and give $\langle r^2 \rangle_{1,p} = -6F'_{1,p}(0) = 0$ (for example, take a function of $(q^2)^2$ alone). But if, as is deduced from the general theory, a spectral representation is possible for $F_{1,p}(q^2)$:

$$F_{1,p}(q^2) = \int_{(2\mu)^2}^{\infty} \frac{\alpha_{1,p}(m^2)}{q^2 + m^2} dm^2 \quad (3.2)$$

with

$$\int_{(2\mu)^2}^{\infty} \frac{\alpha_{1,p}(m^2)}{m^2} dm^2 = 1, \quad (3.3)$$

and if the spectral function $\alpha_{1,p}(m^2)$ is not much oscillating (more precise conditions will be stated later), then $F_{1,p}(q^2)$ has no inflexion point for small q^2 , hence we can safely extrapolate it to $q^2=0$ as is usually done in order to obtain $\langle r^2 \rangle_{1,p}$.

For simplicity, we first consider the case in which $\alpha_{1,p}(m^2)$ has at most only one zero-point for $m^2 > (2\mu)^2$. Since $\alpha_{1,p}(m^2)$ is of course positive near threshold because it is owing to π^+ -cloud, we can put

$$\begin{aligned} \alpha_{1,p}(m^2) &> 0 & \text{for } (2\mu)^2 < m^2 < \kappa^2, \\ \alpha_{1,p}(m^2) &< 0 & \text{for } \kappa^2 < m^2, \end{aligned} \quad (3.4)$$

where κ^2 may or may not be infinite. If there is an inflexion point, there holds that

$$F_{1,p}''(q^2) = 2 \int_{(2\mu)^2}^{\infty} \frac{\alpha_{1,p}(m^2)}{(q^2 + m^2)^3} dm^2 = 0, \quad (3.5)$$

i. e.

$$\int_{(2\mu)^2}^{\kappa^2} \frac{m^2}{(q^2 + m^2)^3} \cdot \frac{\alpha_{1,p}(m^2)}{m^2} dm^2 = \int_{\kappa^2}^{\infty} \frac{m^2}{(q^2 + m^2)^3} \cdot \frac{|\alpha_{1,p}(m^2)|}{m^2} dm^2. \quad (3.6)$$

Since the function $m^2/(q^2 + m^2)^3$ for $q^2 < 8\mu^2$ is monotonously decreasing with respect to $m^2 (\geq 4\mu^2)$, we have

$$\int_{(2\mu)^2}^{\kappa^2} \frac{\alpha_{1,p}(m^2)}{m^2} dm^2 \leq \int_{\kappa^2}^{\infty} \frac{\alpha_{1,p}(m^2)}{m^2} dm^2. \quad (3.7)$$

On the other hand, (3.3) can be rewritten as

$$\int_{\kappa^2}^{\infty} \frac{|\alpha_{1,p}(m^2)|}{m^2} dm^2 = \int_{(2\mu)^2}^{\kappa^2} \frac{\alpha_{1,p}(m^2)}{m^2} dm^2 - 1 < \int_{(2\mu)^2}^{\kappa^2} \frac{\alpha_{1,p}(m^2)}{m^2} dm^2, \quad (3.8)$$

which is inconsistent with (3.7). Therefore, when $q^2 < 8\mu^2$, $F_{1,p}(q^2)$ cannot have an inflexion point. The above arguments hold as it stands in the case in which $\alpha_{1,p}(m^2)$ again changes its sign at $m^2 = \kappa'^2 (> \kappa^2)$ if the total charge for $m^2 > \kappa'^2$ is less than unity.

We may thus conclude that $F_{1,p}(q^2)$ has no inflexion point for $0 \leq q^2 < 8\mu^2 = 16Y^{-2}$ as far as $\alpha_{1,p}(m^2)$ is not too curious. Stanford data indicate that $F_{1,p}(q^2)$ is concave for $q^2 = 0.5 \sim 17Y^{-2}$. We can therefore give a lower bound for $\langle r^2 \rangle_{1,p}$ from experimental data.

For small q^2 we may identify $F_{1,p}^2(q^2)$ with the ratio of the experimental cross section to the Mott cross section. We denote the experimental values of $F_{1,p}(q^2)$ for $q^2 = q_1^2$ and q_2^2 by F_1 and F_2 , respectively, and write $F_{1,p}(0) = F_0$. The absence of the inflexion point leads to

$$F_0 \geq A \equiv \frac{q_2^2 F_1 - q_1^2 F_2}{q_2^2 - q_1^2}. \quad (3.9)$$

If $F_{1,p}(q^2)$ were correctly normalized, there should hold $F_0 = 1$. But as the absolute cross section of Stanford experiments has large ambiguity, we must normalize it. Therefore the normalized negative tangent at $q^2 = 0$ is

$$\frac{1}{6} \langle r^2 \rangle_{1,p} = -F_{1,p}'(0)/F_0 \geq \frac{(F_0 - F_1)/F_0}{q_1^2}. \quad (3.10)$$

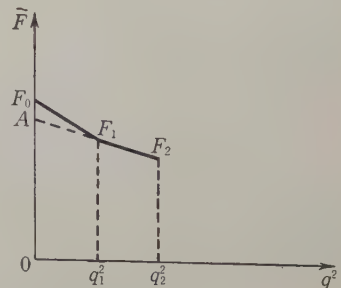


Fig. 3.

Since the right-hand side of (3.10) is monotonously increasing with respect to F_0 , from (3.9) we obtain

$$\langle r^2 \rangle_{1,p} \geq 6 \cdot \frac{F_1 - F_2}{q_2^2 F_1 - q_1^2 F_2}. \quad (3.11)$$

As an example, we use the Stanford 300 Mev data,¹⁾ which seems to be most effective among the existing data. Substituting

$$\begin{aligned} q_1^2 &= 0.59, \quad q_2^2 = 1.24, \\ F_1 &= 1.00, \quad F_2 = 0.85, \quad (\text{mean values}) \end{aligned} \quad (3.12)$$

in the right-hand side of (3.11), we obtain $1.2Y^2$, but of course this value is not reliable. When we take into account an experimental error of $\sim 10\%$ for cross sections, a lower bound for $\langle r^2 \rangle_{1,p}$ becomes

$$\langle r^2 \rangle_{1,p} \gtrsim (0.4 \sim 0.5) Y^2. \quad (3.13)$$

This value will be rather reliable since it can be smoothly extrapolated to larger q^2 without inflexion point within experimental error. Since this lower bound is rather strongly dependent on experimental errors, more accurate measurement for small q^2 is highly desirable.

3-2. Positive definiteness of Fourier transform

In § 2 we pointed out the possibility that $\rho_{1,p}(r)$ becomes negative in the inner region. But its rigorous experimental verification will be rather difficult, because

- i) There are ambiguities in the determination of $F_{1,p}(q^2)$ from experimental cross sections.
- ii) Fourier transform does not preserve the positive-definiteness property.

The first problem will in principle be soluble by more accurate measurements. In this subsection we shall investigate the second problem. We must not of course assume any special form for $\rho_{1,p}(r)$.

According to Bochner's theorem,¹⁰⁾ the necessary and sufficient condition for the positive definiteness of Fourier transform is that the function in question is of "positive type". Its implication and proof are presented in the Appendix. We write here only the main results deduced from Bochner's theorem:

- i) If $F_{1,p}(q^2) \geq 0$ for $q^2 \geq 0$, then there holds for arbitrary r that

$$|\rho_{1,p}(r)| \leq \rho_{1,p}(0), \quad (3.14)$$

that is to say, $\rho_{1,p}(r)$ has the maximum at the origin. Therefore $F_{1,p}(q^2)$ must become negative for some q^2 for Hofstadter's cutoff model²⁾ or our model in § 2.

- ii) If there exists a point $q^2 (\geq 0)$ at which

$$F_{1,p}(q^2) < -1/3, \quad (3.15)$$

$\rho_{1,p}(r)$ cannot be positive definite.

iii) If for some $a > 0$ it happens that

$$\int_0^{\infty} \rho_{1,p}(r) j_1(ar) r dr < 0, \quad (3.16)$$

where j_1 is a spherical Bessel function, then $F_{1,p}(q^2)$ cannot be positive definite in the region $0 \leq q^2 \leq a^2$.

Making use of these properties, our proposal in § 2 will be checked by experiments in near future.

§ 4. Trial models for the proton*

In this section we shall propose two trial proton models which have such shapes as were suggested in § 2 and which are consistent with all the present Stanford data. Our aim is not to claim physical reality of these trial models, but to point out concretely the following respects:

- i) It is possible to explain the present data of electron-proton scatterings by using the proton charge distribution deduced from (1.1) and meson theory.
- ii) There is no reason why one should exclude the proton models of $F_{1,p} \neq F_{2,p}$.
- iii) The values of $\langle r^2 \rangle_{1,p}$ and $\langle r^2 \rangle_{2,p}$ are not determined so accurately as shown in (1.2).

In constructing our trial models, we have taken into account the following properties which are conjectured from meson theory.**

- i) The charge distribution is negative in the inner region, and the amount of the outer positive charge is about $1.5e$.
- ii) The a.m.m. distribution is almost positive definite. ("Almost" means that we do not exclude possible small negative parts in comparison with the negative charge of $\rho_{1,p}(r)$.)

But we are indifferent to the asymptotic form of $\rho_{1,p}(r)$, of which the current meson theory predicts to be proportional to $^{2),5),11)}$

$$(\mu r)^{-n} e^{-2\mu r}, \quad (n > 0). \quad (4.1)$$

Because (4.1) is meaningful only in the outermost regions, namely, of the very low momentum transfers which have not been measured in the Stanford experiments, we may forget about (4.1) in analysing the present data. If necessary, one may connect (4.1) to our models at some large distances, since the simple extrapolation of the asymptotic form to the inward region is meaningless.

* We will discriminate between two phrases "proton model" and "trial (proton) model". The reader should be careful about the terminology "trial" by which we mean special cases of our proton model.

** We call the proton model having these properties "our proton model".

Trial Model I.

$$F_{1,p}(q^2) = 28.5(1 + q^2/45)^{-1} - 27.5(1 + q^2/50)^{-1}$$

$$F_{2,p}(q^2) = (1 + q^2/24)^{-2} \quad (4.2)$$

with

$$\langle r^2 \rangle_{1,p} = \langle r^2 \rangle_{2,p} = 0.5 Y^2.$$

Namely, $\rho_{2,p}(r)$ is of an exponential type, and $\rho_{1,p}(r)$ is a difference of two Yukawa-type functions whose approximate forms are displayed in Fig. 4. The position of the zero point, r_0 , the total amount of the positive charge, Q , and the charge more outward than $0.5Y$, Q' , are as follows:

$$r_0 = 0.192 Y,$$

$$Q = 1.31 e, \quad (4.3)$$

$$Q' = 0.70 e.$$

Next, we will demonstrate that this model is consistent with Stanford data.¹⁾ To see this, we define the quantity

$$\tilde{F}_p^2(q^2, E) = \frac{F_{1,p}^2(q^2) + (q^2/4M^2)[2\{F_{1,p}(q^2) + \mu_p F_{2,p}(q^2)\}^2 \tan^2 \theta/2 + \mu_p^2 F_{2,p}^2(q^2)]}{1 + (q^2/4M^2)[2(1 + \mu_p)^2 \tan^2 \theta/2 + \mu_p^2]}, \quad (4.4)$$

where E and θ stand respectively for the incident energy and the scattering angle

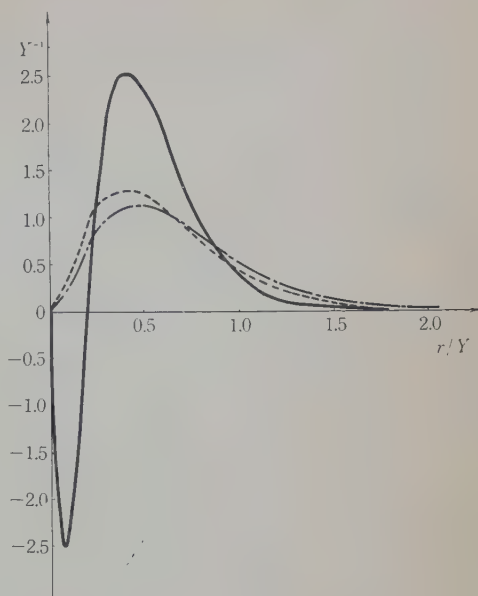


Fig. 4.

$$\left. \begin{array}{l} \text{—} 4\pi r^2 \rho_{1,p}(r) \\ \text{---} 4\pi r^2 \rho_{2,p}(r) \\ \text{--}\cdot\text{--} 4\pi r^2 \rho^{\text{exp}}(r), \langle r^2 \rangle = 0.64 Y^2 \end{array} \right\} \text{ Trial Model I}$$

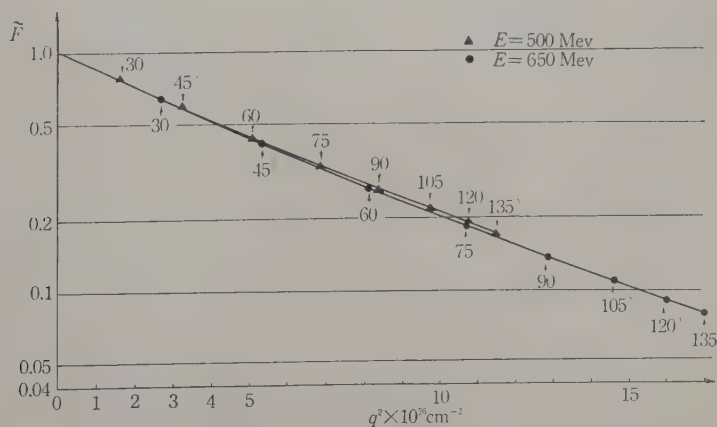


Fig. 5.

of the electron in the laboratory system. Since θ is a function of E and q^2 , \widetilde{F}_p^2 depends on both q^2 and E . Fig. 5 displays $\widetilde{F}_p^2(q^2, E)$ for $E=500$ Mev and $E=650$ Mev. From this graph we see that $\widetilde{F}_p^2(q^2, E)$ is almost independent of E , i.e. we may put

$$\widetilde{F}_p^2(q^2, E) \simeq F_p^2(q^2), \tag{4.5}$$

which holds also for the Stanford experiments of other incident energies, $E=200, 300, 400, 500, 550, 600, 650$ Mev. Thus one will see that Fig. 5 is consistent with the data of $E=200\sim 650$ Mev, $\theta=30^\circ\sim 135^\circ$.

One of the most reliable data of the Stanford experiments will be the ratio of the cross sections for $\theta=75^\circ$ to that for $\theta=135^\circ$:

$$R(E) = \sigma(E, 75^\circ) / \sigma(E, 135^\circ). \tag{4.6}$$

Table I shows the values of $R(E)$ given by Trial Model I, which are consistent with the experimental data as well as in the case of the exponential model¹⁾ ($F_{1,p}=F_{2,p}$, $\langle r^2 \rangle_{1,p}=\langle r^2 \rangle_{2,p}=0.64Y^2$) or the Clementel-Villi model¹²⁾ ($F_{1,p}=F_{2,p}$, $\eta=1.2$, $\langle r^2 \rangle_{1,p}=\langle r^2 \rangle_{2,p}=0.64Y^2$).

Table I

E (Mev)	200	300	400	500	550	600	650
$R(E)$	10.55	8.90	8.60	8.92	9.24	9.64	10.15

Trial Model II.

$$\begin{aligned} F_{1,p}(q^2) &= 1.7[1 - 0.01657 \cdot q^2] \\ &\times e^{-0.02485 \cdot q^2} - 0.7, \\ F_{2,p}(q^2) &= [1 + 0.09375 \cdot q^2]^{-1} \end{aligned} \tag{4.7}$$

with

$$\begin{aligned} \langle r^2 \rangle_{1,p} &= (0.65)^2 Y^2, \\ \langle r^2 \rangle_{2,p} &= (0.75)^2 Y^2, \end{aligned}$$

and

$$\begin{aligned} r_0 &= 0, \\ Q &= 1.70, \\ Q' &= 0.69. \end{aligned} \tag{4.8}$$

$F_{1,p}(q^2)$ in this model is not expressible in terms of spectral function, but it will be possible to write it in such a form approximately in a finite range of q^2 (its inflexion point is

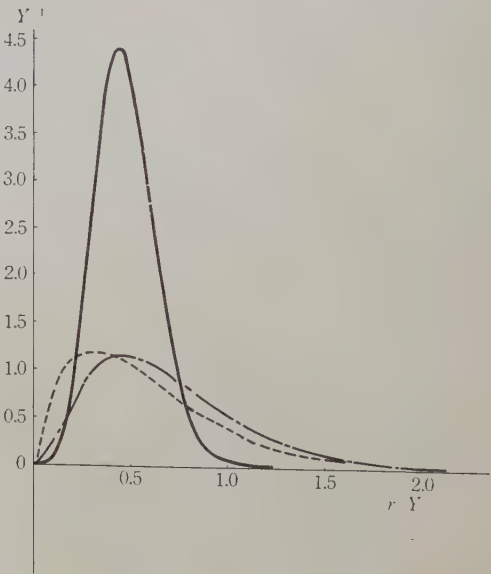


Fig. 6.

$$\begin{aligned} \text{—} & 4\pi r^2 \rho_{1,p}(r) \\ \text{---} & 4\pi r^2 \rho_{2,p}(r) \\ \text{- - -} & 4\pi r^2 \rho^{\text{exp}}(r), \langle r^2 \rangle = 0.64 Y^2 \end{aligned} \quad \text{Trial Model II}$$

$q^2 \simeq 141 \text{ Y}^{-2}$). Moreover, we have used a δ -function instead of the inner extended negative charge,* since detailed forms in the region $r \lesssim (0.4 \sim 0.5) \text{ Y}$ have no effect on the experiments up to $E \leq 650 \text{ Mev}$.

This model also has the property (4.5) like Trial Model I, and gives $\widehat{F}^2 = 0.0724$ for $E = 650 \text{ Mev}$, $\theta = 135^\circ$. Table II shows the values of $R(E)$ given by Trial Model II.

Table II

$E \text{ (Mev)}$	200	300	400	500	550	600	650
$R(E)$	10.64	9.00	8.65	8.90	9.21	9.65	10.25

The above two models are naturally nothing but very special examples among various possibilities. It will not be difficult to construct still better proton models.

Finally, we consider how the characteristics of our proton model are revealed on the $F_{1,p}/F_{2,p}$ ratio. Generally speaking, the larger the momentum transfer q^2 becomes, the nearer the electron passes by the proton in a probabilistic sense. When the electron passes through remote points from the proton, the former will feel only the positive definite charge and a.m.m. distributions of the latter. But when the electron runs more inward, it will become aware of the inner negative charge, and moreover the electron which runs near the proton center will pass through the negative charge sea, while it will still feel the positive a.m.m. distribution. Thus when q^2 becomes large, $F_{1,p}/F_{2,p}$ will rapidly approach zero and further become negative. On the other hand, it is evident that $F_{1,p}/F_{2,p} \cong 1$ for small q^2 corresponds to $\langle r^2 \rangle_{1,p} \cong \langle r^2 \rangle_{2,p}$. The present data seem to indicate¹³⁾

$$\langle r^2 \rangle_{1,p} \lesssim \langle r^2 \rangle_{2,p}, \quad (4.9)$$

while meson theory will probably predict

$$\langle r^2 \rangle_{1,p} \gtrsim \langle r^2 \rangle_{2,p} \quad (4.10)$$

according to our perturbational calculations. Fig. 6 displays the $F_{1,p}/F_{2,p}$ ratios for Trial Models I and II in which $\langle r^2 \rangle_{1,p} = \langle r^2 \rangle_{2,p}$ and $\langle r^2 \rangle_{1,p} < \langle r^2 \rangle_{2,p}$, respectively. If (4.9) is valid, such a curve as shown by the dotted line III is expected.

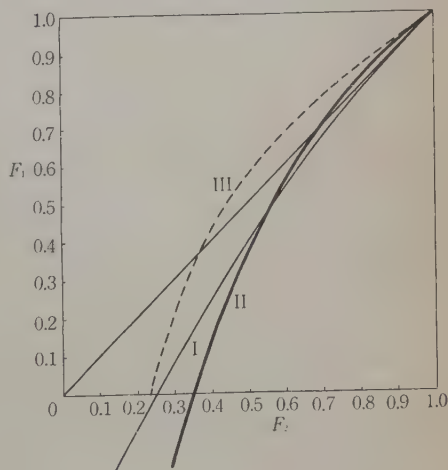


Fig. 7.

* Of course, this replacement does not mean that we have assumed $Z_{2,p} < 0$.

§ 5. Discussions and summary

5-1. On the lower bound for $\langle r^2 \rangle_{1,p}$

According to the definition

$$\langle r^2 \rangle_{1,p} = -6 F'_{1,p}(0),$$

$\langle r^2 \rangle_{1,p}$ should be determined from the derivative at $q^2=0$ on the $F_{1,p}-q^2$ diagram independently of the forms of the proton charge and a.m.m. distributions. But in the present Stanford data the experimental error for small q^2 is about 10% as that for large q^2 . Therefore, since \tilde{F} is not distinguishable from the unity within the present experimental error, one has so far used the data for large q^2 in order to determine $\langle r^2 \rangle_{1,p}$, namely it is determined by the method which depends on the special forms of the proton charge and a.m.m. distributions.

We have shown in § 3-1 that $F_{1,p}(q^2)$ has no inflexion point for small q^2 under rather loose general assumptions (this conclusion is valid under much looser ones). Theoretical results (at least so far obtained) seem to satisfy these assumptions. We have stated the method by which one determines a lower bound for $\langle r^2 \rangle_{1,p}$, using this property, independently of the special forms of the proton charge and a.m.m. distributions and of the absolute cross sections of electron-proton scattering. (3.13) is a lower bound for $\langle r^2 \rangle_{1,p}$ obtained from analysing the existing data, but unfortunately this value is rather sensitive to the experimental accuracy. Therefore we hope that the experimental cross sections for at least two points of small q^2 at the same energy are measured with high precision.

5-2. On the reliability of (1.2)

As was stated in the introduction and the previous subsection, (1.2) is what was determined dependently on the forms of the proton charge and a.m.m. distributions. One may therefore take the standpoint from which one doubts about (1.2).

Indeed, in § 4 we have proposed two trial proton models, which have rather smaller $\langle r^2 \rangle_{1,p}$ and $\langle r^2 \rangle_{2,p}$ than those in (1.2) and are consistent with the present Stanford data. Therefore it does not seem that $\langle r^2 \rangle_{1,p}$ is determined as precisely as shown in (1.2). But we wish to emphasize again that we assert *not* to claim physical reality of our trial models *but* to point out the possible large ambiguity for (1.2).

Next, we will comment on $\langle r^2 \rangle_{2,p}$. It seems to be difficult to determine $\langle r^2 \rangle_{2,p}$ as accurately as $\langle r^2 \rangle_{1,p}$. As was shown by Hofstadter¹⁴⁾ and Karplus¹³⁾ at the Kiev Conference, $F_{1,p}$ and $F_{2,p}$ can be determined by means of (4.4) from the absolute cross sections at two points (E_1, θ_1) and (E_2, θ_2) for a common q^2 . Thus if $F_{2,p}$ is obtained rather accurately, a lower bound for $\langle r^2 \rangle_{2,p}$ will be presented as was done for $\langle r^2 \rangle_{1,p}$ in § 3-1. This is in principle possible, but since

1) $F_{2,p}$ for small q^2 is necessary though (4.4) hardly depends on $F_{2,p}$ in the region of small q^2 , and

2) Absolute cross sections must be measured at least at four points, it will be very difficult as an actual problem to determine $\langle r^2 \rangle_{2,p}$ form-independently.

5-3. On our proton model

In § 2 we have presumed our proton model on the basis of the reliable experimental value (1.1) and information from meson theory. When $F_{1,p}$ and $F_{2,p}$ are independently determined in a wide region of q^2 by Hofstadter¹⁴⁾ and Karplus¹³⁾ method stated in § 5-2, if our conjecture is valid, $F_{1,p}-F_{2,p}$ diagram will show such tendencies as

1) lines II, III in Fig. 7 or Karplus' diagram when $\langle r^2 \rangle_{1,p} \lesssim \langle r^2 \rangle_{2,p}$, or

2) line I in Fig. 7 when $\langle r^2 \rangle_{1,p} \gtrsim \langle r^2 \rangle_{2,p}$.

Thus it may be possible to check our proton model experimentally. As examples, consider Trial Models I and II, for which the sign change of $F_{1,p}$ takes place at $q^2 \simeq 25Y^{-2}$ and at $q^2 \simeq 20Y^{-2}$, respectively. Therefore the validity of our proton model may be checked in the region of such momentum transfers.

In order to confirm the existence of a negative part in the proton charge distribution model-independently, it is sufficient to show experimentally the existence of such a q^2 that $F_{1,p}(q^2) < -1/3$, as is proved in the Appendix. For instance, in Trial Model II this occurs for $q^2 \gtrsim 32Y^{-2}$, and for Trial Model I a still larger q^2 is required. Therefore the experimental confirmation of this problem may be very difficult.

Acknowledgements

The authors would like to express their sincere thanks to Prof. H. Yukawa for valuable discussions and to Prof. R. Hofstadter for sending his experimental data.

Appendix

Bochner's theorem and its applications

Let $f(x)$ be a function of an n -dimensional Euclidean vector x , and denote its Fourier transform by affixing the symbol \sim :

$$f(x) = \frac{1}{(2\pi)^n} \int \tilde{f}(k) e^{ikx} d^n k. \quad (\text{A} \cdot 1)$$

The necessary and sufficient condition for $\tilde{f}(k) \geq 0$ is that

$$\int \tilde{f}(k) \tilde{g}(k) d^n k \geq 0 \quad (\text{A} \cdot 2)$$

for any arbitrary $\tilde{g}(k) \geq 0$. On account of Parseval's equality (A.2) is equivalent to

$$\int f(x) g(-x) d^n x \geq 0. \quad (\text{A} \cdot 3)$$

Take an arbitrary complex function $\tilde{\varphi}(k)$ and put

$$|\tilde{\varphi}(k)|^2 = \tilde{g}(k), \quad (\text{A} \cdot 4)$$

then we have

$$g(-x) = \int \varphi(t) \varphi^*(t+x) d^n t. \quad (\text{A} \cdot 5)$$

Substituting (A·5) in (A·3) and transforming integration variables, we obtain

$$\iint f(x-t) \varphi(x) \varphi^*(t) d^n x d^n t \geq 0. \quad (\text{A} \cdot 6)$$

Since it suffices to adopt discrete functions

$$\varphi(x) = \sum_j z_j \cdot \delta(x - x_j) \quad (\text{A} \cdot 7)$$

where z_j 's are arbitrary complex numbers, we get the following theorem.

Bochner's theorem:¹⁰⁾

The necessary and sufficient condition for the positive definiteness* of Fourier transform of $f(x)$ is that $f(x)$ is a function of positive type, that is to say, for any arbitrary set of arbitrary n -dimensional vectors x_j and arbitrary complex numbers z_j ($j=1, 2, \dots, l$) there always holds that

$$\sum_{j,k=1}^l f(x_j - x_k) z_j z_k^* \geq 0. \quad (\text{A} \cdot 8)$$

Namely, the matrix $\{f(x_j - x_k)\}$ is hermitian (i.e. $f(-x) = f^*(x)$) and its eigenvalues are all positive definite. Therefore there holds that

$$\det[f(x_j - x_k)] \geq 0 \quad (\text{A} \cdot 9)$$

for any order l . In particular we have

$$f(0) \geq 0 \quad \text{for } l=1, \quad (\text{A} \cdot 10)$$

$$\begin{vmatrix} f(0) & f^*(x) \\ f(x) & f(0) \end{vmatrix} = f^2(0) - |f(x)|^2 \geq 0 \quad \text{for } l=2. \quad (\text{A} \cdot 11)$$

(A·10) and (A·11) imply

$$|f(x)| \leq f(0), \quad (\text{A} \cdot 12)$$

which gives (3·14).

Now, consider the case in which $f(x)$ is real and a function of $|x|$ alone. Since in the n -dimensional space there exist $n+1$ points such that the distance between any two of them is always equal to r , we can take them as x_j ($l=n+1$); then

* More strictly speaking, "non-negative definiteness".

$$\begin{vmatrix} f(0) f(r) \cdots f(r) \\ f(r) f(0) \cdots f(r) \\ \vdots \\ f(r) f(r) \cdots f(0) \end{vmatrix} \geq 0, \quad (\text{A} \cdot 13)$$

$$\text{i. e.} \quad \{f(0) - f(r)\}^n \{nf(r) + f(0)\} \geq 0. \quad (\text{A} \cdot 14)$$

Therefore we obtain

$$f(r) \geq -f(0)/n \quad (\text{A} \cdot 15)$$

with the aid of (A·12). Especially, when we take $F_{1,p}(q^2)$ (with $q_0=0$) as $f(x)$, on account of $F_{1,p}(0)=1$ and $n=3$ it follows that

$$F_{1,p}(q^2) \geq -1/3 \quad (\text{A} \cdot 16)$$

if $\rho_{1,p}(r) \geq 0$.

Finally, (3·16) immediately follows from (A·3) by putting $\tilde{f}(q) \equiv F_{1,p}(q^2)$, $\tilde{g}(q) \equiv \theta(q^2 - a^2)$.

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Letters to the Editor

The opinions expressed in these columns do not necessarily reflect those of the Board of Editors. Communications should be submitted in duplicate and should be held to within 100 lines (pica type) on standard size letter paper (approx. 21×30 cm.) so that each letter may be arranged into two pages when printed. Do not forget to count in enough space for formulas, figures or tables.

Form Factor and Structure of Particles

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February 20, 1960

In order to simplify the discussion in this note we shall use the model in which the original particle with mass M virtually dissociates into particles with masses m and m' .

In the relativistic field theory, there exists a difference between the particle density (like charge density or energy density) and the probability density (as that expressed by the square of the absolute value of the statevector), and what has a physical meaning is only the former. If the effect of pairs of particles and antiparticles can be neglected, the former agrees with the latter and only in this case the latter has the physical meaning.

Consider the m -particle density $F_m(\mathbf{x})$ in M -particle state. For the inner behaviour of $F_m(\mathbf{x})$, of course, the effect of the pair of m -particle is not negligible, accordingly for the inner region the probability has no meaning. In the peripheral region, however, $F_m(\mathbf{x})$ takes different behaviours corresponding to the

following two cases. The case A : Even in this region the effect of pair gives the large contribution to $F_m(\mathbf{x})$, consequently the probability density loses its meaning. The case B : The effect of pair is negligible. In this case in the peripheral region the probability density gets the physical meaning.

We wish to emphasize that the difference of structure in the peripheral region in the above two cases seems to be of use for the definition of the elementary particle, namely, our definition of the elementary particle is that if the M -particle belongs to the case A (or B) then the M -particle is elementary (or composite). This definition of the elementary particle is identical with Oehme's one¹⁾ as we shall see in the following discussion.

As is well known, $\widetilde{F}_m(\mathbf{q}^2) = \int d\mathbf{x} \exp(-i\mathbf{q} \cdot \mathbf{x}) F_m(\mathbf{x})$ can be represented as

$$\widetilde{F}_m(\mathbf{q}^2) = \int_{\mu^2}^{\infty} d\kappa^2 \sigma(\kappa^2) (\kappa^2 + \mathbf{q}^2 - i\epsilon)^{-1}. \quad (1)$$

Using the perturbation theory, several authors¹⁾⁻³⁾ have investigated the magnitude of the lower limit μ^2 in the integral of (1) and have concluded that

$$\mu^2 = \begin{cases} 4m^2; & M^2 < m^2 + m'^2 \\ 4m^2 - (M^2 - m^2 - m'^2)/m'^2; & M^2 > m^2 + m'^2. \end{cases} \quad (2)$$

$$\mu^2 = \begin{cases} 4m^2; & M^2 < m^2 + m'^2 \\ 4m^2 - (M^2 - m^2 - m'^2)/m'^2; & M^2 > m^2 + m'^2. \end{cases} \quad (3)$$

Next we shall investigate the wave function of the bound state of m - and m' -particle $\psi(\mathbf{x}, \mathbf{y}) = \int d\mathbf{z} \chi(\mathbf{x} - \mathbf{z}, \mathbf{y} - \mathbf{z}) \varphi(\mathbf{z})$, where

$$\chi(\mathbf{x}, \mathbf{y}) = \int d\mathbf{p} d\mathbf{q} \exp i(\mathbf{p} \cdot \mathbf{x} + \mathbf{q} \cdot \mathbf{y}) \times \frac{f(\mathbf{p}, \mathbf{q})}{E_m(\mathbf{p}) + E_{m'}(\mathbf{q}) - E_M(\mathbf{p} + \mathbf{q})}. \quad (4)$$

Here $\varphi(\mathbf{z})$ is the wave function describing the packet of M -particle and $f(\mathbf{p}, \mathbf{q})$ is some function having no singularity. For large $|\mathbf{x}|$ in (4) the probability of the m -particle can be determined from complex vector \mathbf{p} for which the denominator of the integrand of (4) vanishes. When the mass condition (2) is satisfied, the probability density is small compared with the m -particle density obtained from the effect of pair. On the other hand, if the mass condition (3) is satisfied, for the large $|\mathbf{x}|$ the effect of pair is negligible, the probability density has the meaning. We can easily see that in this case the behaviour of the probability density for large $|\mathbf{x}|$ exactly agrees with the result calculated from (3). Therefore, we conclude that the difference between (2) and (3) corresponds to the difference of structure in cases A and B respectively.

The author would like to express his sincere thanks to Prof. R. Utiyama for his continual encouragement and discussions.

On the Inelastically Scattered Proton at High Energy

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March 4, 1960

In an earlier note¹⁾ the high energy protons inelastically scattered from the levels of the dipole giant γ -resonance regions in C^{12} and O^{16} were studied as functions of the cross sections of dipole γ -absorption and the ratios of spin flip to non spin flip parts of the nucleons in the target nuclei. The good agreements of the calculated results with the observed ones encourage us to investigate the protons inelastically scattered from other levels of nuclei. In the present note we report some results about the protons inelastically scattered from the levels of magnetic dipole excitation, and suggest that the observation of the proton gives a possibility to fix the spin and isotopic spin for the level.

Within the framework of the impulse approximation, the amplitude of the proton scattered from the nucleus is expressed by the product of terms related to the two-body scattering amplitudes and the nuclear form factor. In the inelastic scattering the nuclear form factor is a function of the initial and final state of the nucleus and a momentum transfer of the incident and scattered proton. In very small scattering angular region, since $kR < 1$ (R is the nuclear radius) one can expand $\exp[-k\mathbf{r}_i]$ in the power series in terms

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of kr_i , which is included in the nuclear form factor, where \mathbf{k} is the momentum transfer of the proton and \mathbf{r}_i is coordinate of the i -th nucleon in the nucleus. The first term in the expansion gives the matrix element corresponding to the magnetic dipole transition.

Now, we replace the matrix element related to the magnetic dipole transition with the experimental value, i.e. the radiation width.* Therefore we can calculate the angular distribution of the proton inelastically scattered from the level without any knowledge on the wave functions of the nucleus.

Fig. 1 shows the result for the angular distribution of the proton inelastically scattered from the 15.11 Mev level of C^{12} .** Here we use the two-body scattering phase shifts of 180 Mev derived from the Gammel-Thaler potential²⁾ to calculate the terms determined by the two-body scattering amplitudes, and the experimental value measured by Garwin³⁾ for the radiation width to the ground state. The experimental points are measured by Tyrén and Maris.⁴⁾ The agreement between the result and the experimental value is well in the interesting angular region. Here we use 0.5 for the value of ρ . The value of ρ has been introduced in order to fix the polarization of the proton inelastically scattered from the level magnetic exci-

ation in the nucleus.⁵⁾ For the sake of evidence, the polarization of the proton is shown in Fig. 2. The experimental points are measured by Hillman et. al.⁶⁾ The result in Fig. 2 is calculated by the use of $\rho=0.5$.

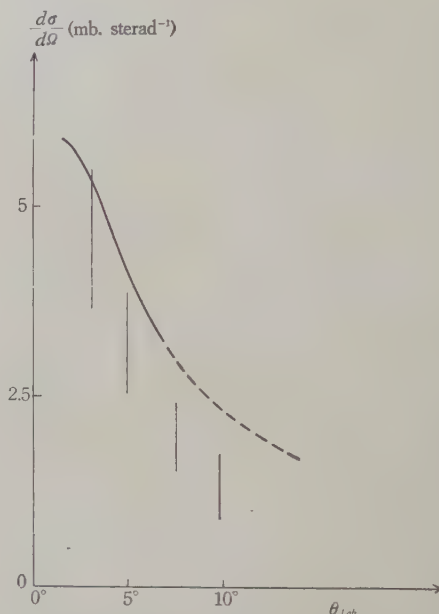


Fig. 1. Angular distribution of the proton inelastically scattered from 15.11 Mev level of C^{12} .

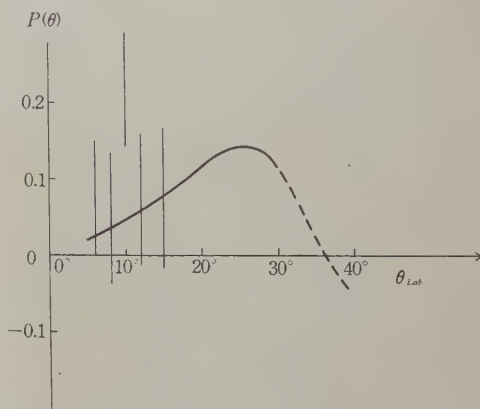


Fig. 2. Polarization of the proton inelastically scattered from 15.11 Mev level.

* This was brought to our attention by Dr. Y. Nishida in connection with his work (to be published).

** After completing the calculation, the author was informed by Dr. M. Kawai that he had calculated the energy distribution of the high energy proton scattered from 15.11 Mev level of C^{12} , independently.

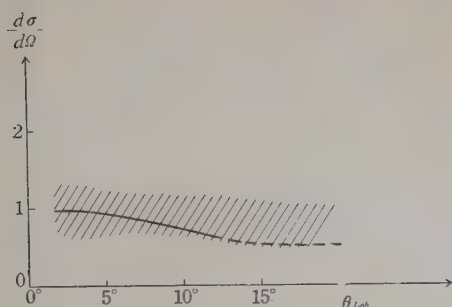


Fig. 3. Angular distribution of the proton inelastically scattered from 12.73 Mev level of C^{12} . The shaded area represents the experimental data.⁴⁾

Fig. 3 shows the angular distribution of the proton scattered from the 12.73 Mev level of C^{12} . The angular distribution of the proton scattered from 12.73 Mev level of C^{12} is observed by Tyrén and Maris.⁴⁾ Unfortunately we have no experimental data for the radiation width corresponding to the matrix element, then we do not show the agreement between the calculated result and the experimental one in their absolute values. However the pattern of the calculated result agrees with the one of the experimental data. If we have the experimental data for the polarization of the proton, we could fix the value of ρ , and furthermore might conjecture the magnitude of the radiation width.

15.11 Mev level of C^{12} is $J=1^+$, $T=1$, while 12.73 Mev level of C^{12} is $J=1^+$, $T=0$.⁷⁾ In the forward direction the angular distribution of the scattered proton which gives rise to the isotopic spin flip of the residual nucleus is sharper than the one of proton which does not give rise to the isotopic spin flip of the nucleus. (See Figs. 1 and 3.) Then, it seems that the observations of

the protons inelastically scattered from the levels of the nucleus are helpful to fix the isotopic spin of the levels in the nucleus on the basis of the angular distributions of them for the isotopic spin flip and non spin-flip of the residual nucleus.

The author is grateful for the co-operation of Dr. Y. Nishida in making his work available in advance of publication. He is greatly indebted to Prof. M. Kobayasi for his kind encouragement.

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Anomalous Creation of μ -Mesons Originating in Weak Interactions

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March 14, 1960

It has been anticipated by many authors¹⁾ that weak interactions might become stronger in the super high energy regions. This seems to have a connection with anomalous creation of

μ -mesons which has recently been observed experimentally in the super high energy collisions above $10^{15}\text{ev.}^{2)}$ This seems to have some intimate connection also with our multipole model³⁾ of elementary particles.

In this paper, we carry out a simple estimation about μ -meson production caused by the weak interaction in the super high energy regions. For simplicity, we will consider a collision between a super high energy proton and a fixed field of force V . The cross section of the process (Fig. 1a) creating one μ -meson through the weak interaction ($\hbar=c=1$) is given by

$$-\mathcal{L} = \varepsilon r_0^2 [\bar{n}\gamma_\mu(1+\gamma_5)p][\bar{\mu}\gamma_\mu(1+\gamma_5)\nu]. \quad (1)$$

Here the coupling constant εr_0^2 is estimated from the experiments on μ -meson capture. If we take $r_0 = M_\pi^{-1} \approx 10^{-13}\text{cm}$, the weak interactions are characterized by a single constant $\varepsilon \approx 10^{-7}$. We now calculate the energy above which this cross section reaches a comparable order to that of the process (Fig. 1b) producing one π -meson through the strong interaction.

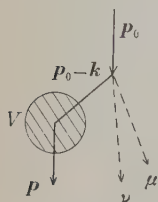


Fig. 1a.

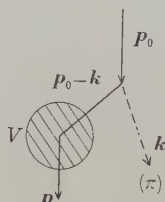


Fig. 1b.

In order to simplify the calculations, we approximate the transition matrix element of Eq. (1) by

$$\langle \mu^+ \nu n | \bar{T}_w | p \rangle \approx \frac{\varepsilon r_0^2}{\Omega} \quad (2)$$

where Ω is the volume of normalization. Then the cross section to the process of Fig. 1a turns out to be

$$\frac{d\sigma_w}{dE_p} \approx \frac{2\pi}{v_0} \int \left| \frac{V(\mathbf{p}_0 - \mathbf{p} - \mathbf{k}) \varepsilon r_0^2}{(E_p - E_{p_0-k}) \Omega} \right|^2 \Omega^2 \times \frac{d\mathbf{k} d\nu}{(2\pi)^6} \delta(E_{p_0} - E_p - E_\mu - \nu). \quad (3)$$

Analogously, using the strong interaction $-\mathcal{L}_s = ig\bar{n}\gamma_5 p\pi^+$ and approximating the transition matrix element by

$$\langle n\pi^+ | \bar{T}_s | p \rangle \approx \frac{g}{(2\Omega E_\pi)^{1/2}}, \quad (4)$$

we obtain the following cross section for the process of Fig. 1b

$$\frac{d\sigma_s}{dE_p} \approx \frac{2\pi}{v_0} \int \left| \frac{V(\mathbf{p}_0 - \mathbf{p} - \mathbf{k}) g}{(E_p - E_{p_0-k}) (2\Omega E_\pi)^{1/2}} \right|^2 \Omega^2 \times \frac{d\mathbf{k}}{(2\pi)^3} \delta(E_{p_0} - E_p - E_\pi). \quad (5)$$

For brevity, we approximate the common part $\left| \frac{V(\mathbf{p}_0 - \mathbf{p} - \mathbf{k})}{E_p - E_{p_0-k}} \right|^2$ which appeared in both Eq. (3) and Eq. (4) by its average values, and we put these parts out of the integration. Then the ratio of these two cross sections turns out to be

$$\frac{(d\sigma_w/dE_p)}{(d\sigma_s/dE_p)} \approx \frac{2\Delta E \varepsilon^2 r_0^4}{g^2} \cdot \frac{J_w}{J_s}, \quad (6)$$

$(\Delta E \equiv E_{p_0} - E_p),$

where J_w and J_s are the phase volumes of the final states of each process. They are given by

$$J_w = \frac{1}{(2\pi)^6} \int d\mathbf{k} d\mu d\nu \delta(\mathbf{k} - \mu - \nu) \times \delta(\Delta E - \mu - \nu) \approx \frac{(\Delta E)^5}{12(2\pi)^4}$$

and

$$J_s = \frac{1}{(2\pi)^3} \int d\mathbf{k} \delta(\Delta E - E_\pi) \approx \frac{(\Delta E)^2}{(2\pi^2)} \quad (7)$$

Then, we have

$$\frac{(d\sigma_w/dE_p)}{(d\sigma_s/dE_p)} \approx \frac{\varepsilon^2(r_0 \Delta E)^4}{48\pi^2 g^2} \quad (8)$$

Now by using the values $g^2/4\pi \approx 15$, $r_0^{-1} = M_\pi \approx 1.4 \times 10^8 \text{ ev}$, we estimate the transferred energy ΔE at which the ratio of Eq. (8) becomes of the order of unity. Then we have $(\Delta E/M_\pi)^4 \approx 3(4\pi)^3 (g^2/4\pi) \varepsilon^{-2} \approx 9 \times 10^{18}$, i.e. $\Delta E/M_\pi \approx 5.5 \times 10^4$. Therefore, it is expected that the weak interaction and the strong one become of comparable order to each other when the transferred energy reaches

$$\Delta E = E_{p_0} - E_p \approx 8 \times 10^{12} \text{ ev.} \quad (9)$$

It is not clear whether this estimation was carried out in the center of mass system or in the laboratory one, as we have approximated the target by an external field. If this corresponds to the calculation in the center of mass system, the corresponding value in the laboratory system should become $\Delta E \approx 10^{16} \text{ ev}$.

As we have made too many simplifications, it may be premature to compare the above estimation with the anomalous μ -meson production experimentally observed in about 10^{15} ev energy regions.

However, it seems very interesting that some anomalies are observed experimentally, in the energy region expected by the above simple consideration. If these phenomena are due to such a mechanism, these produced particles are expected to be extremely collimated, as very large transfer of energy is required in order that the weak interaction behaves as strong one. As there remain many other possibilities of interpreting these experimental results, here we will be satisfied only with pointing out the possibility mentioned above.

In conclusion, the authors express their cordial thanks to Professor Y. Ono, Physics Institute, Hokkaido University, and to Professor I. Miura, the Research Institute for Nuclear study, Tokyo University, for their kind direction.

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Phenomenological Model of Elementary Particle Interactions

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March 18, 1960

In this short note, it is pointed out that almost all interactions of the elementary particles, not only strong but also weak ones, are comprised in the following rather simple and symmetric form of the interaction Lagrangian:

$$-\mathcal{L}_{int} = \sum_{\alpha, \beta}^2 G_{\alpha} [\bar{F}_{\beta}^{\alpha} O^{\alpha} F_{\beta}^{\prime \alpha} B_{\gamma}^{\alpha} + \varepsilon r_0 \bar{F}_{\alpha}^{\beta} O^{\beta} F_{\alpha}^{\prime \beta} B_{\mu}^{\gamma}] + \text{H. C.} \quad (1)$$

The matrices F_{β}^{α} and B_{β}^{α} are given by

$$\begin{bmatrix} F_1^1 \\ F_1^2 \\ F_2^1 \\ F_2^2 \end{bmatrix} = \begin{bmatrix} B_1 & B_2 \\ L_1 & L_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} B_1^1 & B_2^1 \\ B_1^2 & B_2^2 \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 \\ \phi_1^L & \phi_2^L \end{bmatrix}. \quad (2)$$

Here B_{α} , L_{α} , ϕ_{α} and ϕ_{α}^L are baryons, leptons, bosons and the bilinear form of lepton fields, respectively. The lower indices of F_{β}^{α} discriminate between the "helicities". G_{α} and $G_{\alpha} \varepsilon r_0$ ($\approx 10^{-20} \text{ cm}$ in $\hbar=c=1$ unit, where $r_0 \approx 10^{-13} \text{ cm}$ and $\varepsilon \approx 10^{-7}$) denote the strong and weak coupling constants, respectively.

By using the expression (2), the Lagrangian (1) turns out to be

$$\begin{aligned} -\mathcal{L}_{int} = & G_1 [\bar{B}_1 O^B B'_1 + \bar{B}_2 O^B B'_2] [\phi_1 + \phi_2] \\ & + G_2 [\bar{L}_1 O^L L'_1 + \bar{L}_2 O^L L'_2] [\phi_1^L + \phi_2^L] \\ & + G_1 \varepsilon r_0 [\bar{B}_1 O_{\mu}^B B'_1 + \bar{L}_1 O_{\mu}^L L'_1] \\ & \times [\phi_{1,\mu} + \phi_{1,\mu}^L] \\ & + G_2 \varepsilon r_0 [\bar{B}_2 O_{\mu}^B B'_2 + \bar{L}_2 O_{\mu}^L L'_2] \\ & \times [\phi_{2,\mu} + \phi_{2,\mu}^L] + \text{H. C.} \end{aligned} \quad (3)$$

If we express the helicity more explicitly as

$$B_{1(2)} = 1/2 \{1 + (-)\gamma_5\} B,$$

$$L_{1(2)} = 1/2 \{1 + (-)\gamma_5\} L$$

$$\text{and} \quad \phi_{1(2)} = 1/2 \{\phi_{ps} + (-)\phi_s\}$$

(where ϕ_{ps} , ϕ_s represent pseudoscalar and scalar bosons, respectively), our Lagrangian becomes

$$\begin{aligned} -\mathcal{L}_{int} = & G_1 (\bar{B} O^B B') \phi_{ps} + G_2 (\bar{L} O^L L') \\ & \times (\phi_1^L + \phi_2^L) \\ & + \frac{1}{4} G_1 \varepsilon r_0 [\bar{B} O_{\mu}^B (1 + \gamma_5) B' \\ & + \bar{L} O_{\mu}^L (1 + \gamma_5) L'] \\ & \times [\partial_{\mu} (\phi_{ps} + \phi_s) + r_0 \bar{L} O_{\mu}^L (1 + \gamma_5) L] \\ & + \frac{1}{4} G_2 \varepsilon r_0 [\bar{B} O_{\mu}^B (1 - \gamma_5) B' \\ & + \bar{L} O_{\mu}^L (1 - \gamma_5) L'] \\ & \times [\partial_{\mu} (\phi_{ps} - \phi_s) + r_0 \bar{L} O_{\mu}^L (1 - \gamma_5) L] \\ & + \text{H. C.} \end{aligned} \quad (4)$$

Here we have used the definition of ϕ_{μ}^L ,

$$\phi_{1,\mu}^L \equiv r_0 \bar{L} O_{\mu}^L \frac{1 + \gamma_5}{2} L'$$

$$\text{and} \quad \phi_{2,\mu}^L \equiv r_0 \bar{L} O_{\mu}^L \frac{1 - \gamma_5}{2} L'.$$

In order to make our Lagrangian correspond to the physical realities, we should take such limiting values of the coupling constants G_1 and G_2 as

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$G_1 \rightarrow G$ (coupling constant of the strong interaction) and $G_2 \rightarrow 0$. With this choice, leptons disappear from the strong interactions, and simultaneously the weak interactions with $(1-\gamma_5)$ are completely eliminated. Thus we obtain

$$\begin{aligned}
 -\mathcal{L}_{int} = & G(\bar{B}O^B B')\phi_{ps} + \frac{1}{4}G\epsilon r_0 \\
 & \times [\bar{B}O_\mu^B(1+\gamma_5)B' + \bar{L}O_\mu^L(1+\gamma_5)L'] \\
 & \times [\partial_\mu(\phi_{ps} + \phi_s) + r_0\bar{L}O^L(1+\gamma_5)L'] \\
 & + \text{H. C.} \quad (5)
 \end{aligned}$$

To be more definite, we should specify the species and charges of B, L, ϕ_s 's and ϕ_{ps} 's, and we have to give explicit forms of the operators O 's. For example,

$$\begin{aligned}
 & \partial_\mu(\phi_{ps} + \phi_s) + r_0\bar{L}O_\mu^L(1+\gamma_5)L' \\
 & \quad \partial_\mu(\phi_{ps}^{(+)} + \phi_s^{(-)}) \\
 & \quad + r_0(\bar{e} + \mu)\gamma_\mu(1+\gamma_5)\nu, \\
 = & \left\{ \begin{aligned} & \partial_\mu(\phi_{ps}^{(0)} + \phi_s^{(0)}), \\ & \partial_\mu(\phi_{ps}^{(-)} + \phi_s^{(-)}) \\ & - r_0\bar{\nu}\gamma_\mu(1+\gamma_5)(e + \mu), \end{aligned} \right. \quad (6)
 \end{aligned}$$

where the upper indices of ϕ 's refer to the charge states of bosons. All the details of this short note and discussions on the theoretical implications of our Lagrangian will be given in a later issue of this journal.

Virial Expansion Formulae for the Microfield and Micropotential Distribution Functions and Their Application to a High Temperature Plasma

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March 26, 1960

When an atom radiates under the influence of the instantaneous microfield or micropotential, the spectral line widths will broaden due to the fluctuation of the microfield or micropotential (statistical broadening).¹⁾ In a gas mixture of N molecules in a volume V ,* the microfield \mathbf{F} or micropotential \mathcal{V} is usually of the form $\mathbf{F} = \sum_{i=1}^N \mathbf{f}_{\nu_i}(\mathbf{r}_i)$ or $\mathcal{V} = \sum_{i=1}^N \phi_{\nu_i}(\mathbf{r}_i)$ and its distribution function in thermal equilibrium, at temperature $kT = 1/\beta$, is defined by

$$\begin{aligned}
 P(\mathbf{F}) = & \frac{1}{Z_N} \int d\mathbf{r}^N \delta(\mathbf{F} - \sum_{i=1}^N \mathbf{f}_{\nu_i}(\mathbf{r}_i)) \\
 & \times \exp\left\{-\beta \sum_{i>j} \phi_{\nu_i \nu_j}(\mathbf{r}_{ij})\right\} \quad (1)
 \end{aligned}$$

or

$$\begin{aligned}
 P(\mathcal{V}) = & \frac{1}{Z_N} \int d\mathbf{r}^N \delta(\mathcal{V} - \sum_{i=1}^N \phi_{\nu_i}(\mathbf{r}_i)) \\
 & \times \exp\left\{-\beta \sum_{i>j} \phi_{\nu_i \nu_j}(\mathbf{r}_{ij})\right\} \quad (2)
 \end{aligned}$$

where $Z_N = \int d\mathbf{r}^N \exp\left\{-\beta \sum_{i>j} \phi_{\nu_i \nu_j}(\mathbf{r}_{ij})\right\}$. With the help of Fourier transformation,

* The gas mixture is assumed to consist of molecules of species $\nu = \alpha, \beta, \dots, \tau$, the number of molecules of species ν to be $N_\nu \equiv \rho_\nu V$ ($N = \sum_{\nu=\alpha}^\tau N_\nu$) and the intermolecular potential energy to be $\phi_{\nu_i \nu_j}(\mathbf{r}_{ij})$; where $\nu_1 = \nu_2 = \dots = \nu_{N_\alpha} = \alpha$, $\nu_{N_\alpha+1} = \dots = \nu_{N_\alpha+N_\beta} = \beta$, \dots , $\nu_{N_\alpha+\dots+N_\sigma+1} = \dots = \nu_{N_\alpha+\dots+N_\sigma+N_\tau} = \tau$.

we can rewrite the above distribution functions as²⁾

$$P(\mathbf{F}) = \frac{1}{(2\pi)^3} \int d\mathbf{q} e^{i\mathbf{q} \cdot \mathbf{F}} P(\mathbf{q}) \quad (3)$$

where

$$P(\mathbf{q}) = \frac{1}{Z_N} \int d\mathbf{r}^N \exp \left\{ -i\mathbf{q} \cdot \sum_{i=1}^N \mathbf{f}_{\nu_i}(\mathbf{r}_i) - \beta \sum_{i>j} \phi_{\nu_i \nu_j}(\mathbf{r}_{ij}) \right\}, \quad (3')$$

and

$$P(\mathcal{F}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dq e^{iq\mathcal{F}} P(q) \quad (4)$$

where

$$P(q) = \frac{1}{Z_N} \int d\mathbf{r}^N \exp \left\{ -iq \sum_{i=1}^N \psi_{\nu_i}(\mathbf{r}_i) - \beta \sum_{i>j} \phi_{\nu_i \nu_j}(\mathbf{r}_{ij}) \right\}. \quad (4')$$

It is noticed that these Fourier transforms of the distribution functions, (3') and (4'), are similar in their forms to the definition of the interaction part of the chemical potential, $\mu_{\nu'}^{(3)}$

$$e^{-\mu_{\nu'}^{(3)}/kT} = \frac{Z_{N+1}}{Z_N V} = \frac{1}{Z_N} \int d\mathbf{r}^N \times \exp \left\{ -\beta \sum_{i=1}^N \phi_{\nu \nu_i}(\mathbf{r}_{0i}) - \beta \sum_{N \geq i > j \geq 1} \phi_{\nu_i \nu_j}(\mathbf{r}_{ij}) \right\} \quad (5)$$

where the extra particle present in Z_{N+1} but not in Z_N is labelled as 0th particle, the species of which is ν . Then we notice that the formalisms which are developed for calculating the chemical potential may be applicable to the calculation of the Fourier transforms of the distribution functions (3') and (4'). Especially we shall here notice the virial expansion method. The virial expansion formula of the chemical potential is^{3,4)}

$$-\frac{\mu_{\nu'}^{(3)}}{kT} = \sum_{\nu'=0}^{\tau} \rho_{\nu'} \int d\mathbf{r} b_{\nu \nu'}(\mathbf{r}) + \sum_{\substack{n_{\alpha}=0 \\ n_{\alpha}+...+n_{\tau} \geq 2}}^{\infty} \cdots \sum_{\substack{n_{\tau}=0 \\ n_{\tau}+...+n_{\alpha} \geq 2}}^{\infty} \times \prod_{\nu'=0}^{\tau} \frac{\rho_{\nu'}^{n_{\nu'}}}{n_{\nu'}!} \cdot \int \cdots \int d\mathbf{r}_1 \cdots d\mathbf{r}_n \sum_{n \geq i > j \geq 0}^{(M)} \prod b_{ij} \quad (6)$$

where $b_{ij} = b_{\nu_i \nu_j}(\mathbf{r}_{ij}) = \exp \{ -\beta \phi_{\nu_i \nu_j}(\mathbf{r}_{ij}) \} - 1$ for $1 \leq i < j \leq N$ as well as for $i=0$ and $1 \leq j \leq N$, and $\sum^{(M)}$ denotes that the sum is to be taken over all products which are more than singly connected, including 0th particle. The argument leading (5) to (6) can be applied to (3') and (4'), giving the virial expansion formulae for $\ln P(\mathbf{q})$ and $\ln P(q)$ in the form identical with the right-hand side of (6), where b_{ij} for $1 \leq i < j \leq N$ is taken to be the same as in (6), while b_{0i} is to be replaced by

$$b_{0i} = b_{\nu_i}(\mathbf{r}_i) = \exp \{ -i\mathbf{q} \cdot \mathbf{f}_{\nu_i}(\mathbf{r}_i) \} - 1 \quad \text{for } \ln P(\mathbf{q}) \quad (7)$$

and

$$b_{0i} = b_{\nu_i}(\mathbf{r}_i) = \exp \{ -iq \psi_{\nu_i}(\mathbf{r}_i) \} - 1 \quad \text{for } \ln P(q). \quad (8)$$

For low density gases where (6) is useful, these expansion formulae for $\ln P(\mathbf{q})$ and $\ln P(q)$ are expected to be useful. Furthermore, the general scheme for calculating the chemical potential by starting with (6),^{4),5)} will be applicable to calculation of the distribution functions.

As an example, we consider a plasma where $\phi_{\nu_i \nu_j}(\mathbf{r}_{ij}) = e_{\nu_i} e_{\nu_j} / r_{ij}$, $\psi_{\nu_i}(\mathbf{r}_i) = e_{\nu_i} / r_i$, $\mathbf{f}_{\nu_i}(\mathbf{r}_i) = -e_{\nu_i} \mathbf{r}_i / r_i^3$, e_{ν} denoting the charge of a molecule of species ν . If the micropotential distribution function is calculated in the *ring approximation*, the following result can be obtained:

$$\ln P(q) = -kT\kappa q^2/2, \quad (9)$$

hence

$$p(F) = (2\pi kT\kappa)^{-1/2} \exp(-F^2/2kT\kappa) \quad (10)$$

where $kT\kappa = \sqrt{4\pi kT \sum \rho_\nu e_\nu^2}$, $1/\kappa$ being the Debye shielding length. The terms which have been neglected in (9) for $\ln P(q)$ are of the order of ρ , as can be confirmed by an analysis similar to that made for the case of the free energy.⁶⁾ On the other hand, the lowest order of the microfield distribution function comes from the first term of (6) which corresponds to the 2nd virial coefficient. That is,

$$\ln P(q) = -\frac{4}{15} (2\pi q)^{3/2} \sum \rho_\nu |e_\nu|^{3/2} \quad (11)$$

$$P(F) = \frac{1}{2\pi^2} \cdot \frac{1}{F} \int_0^\infty dq q \sin(qF) \\ \times \exp \left\{ -\frac{4}{15} (2\pi q)^{3/2} \sum \rho_\nu |e_\nu|^{3/2} \right\}. \quad (12)$$

This expression is nothing but the Holtmark distribution function.⁷⁾ The correction to (11) is shown to be of order $O(\rho^2)$.

It is noted that (9) and (11) express the distribution functions to the lowest order in ρ , for the respective cases.

It is shown that the situation is similar for the distribution functions of the microfield and micropotential either on a particular particle or at a point being separated from a particular particle by distance r or at a pair of points separated from each other by distance r , The situation is similar also for the distribution functions of other micro quantities and for the simultaneous distribution functions of several micro

quantities. Their virial expansion formulae are found to be of the form of the chemical potential, pair distribution function, and so on.

The details of this note and the calculations to the higher approximations for a high temperature plasma will be published shortly. The following paper will also include a comparison with Baranger and Mozer's work^{8)*} in which an attempt to apply the cluster expansion method to (3) with (3') is proposed.

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Polarization in Heavy Particle Stripping

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Expressions for polarization of outgoing particles in deuteron stripping reactions have been obtained by various authors by relaxing some of the stripping approximations. Under the assumption of a purely central distorting force, the expression for the polarization is¹⁾

$$P = \frac{1}{3} \left\{ \frac{\theta_{l+1/2}^2}{(l+1)} - \frac{\theta_{l-1/2}^2}{l} \right\} \frac{\langle m \rangle}{\theta_{l+1/2}^2 + \theta_{l-1/2}^2} \quad (1)$$

It can be seen that the maximum polarization predicted is

$$P = \begin{cases} (1/3) (l/(l+1)) & \text{for } j=l+1/2 \\ 1/3 & \text{for } j=l-1/2. \end{cases} \quad (2)$$

Recently,²⁾ it has been recognized that one has to invoke the heavy particle stripping mechanism in addition to the direct stripping in order to explain the angular distribution in stripping reactions completely. In view of this fact, it is of interest to study the polarization one can obtain from the heavy particle stripping mechanism. In the case of the direct stripping, it is assumed that one of the constituents of the deuteron is stripped off and captured by the target nucleus, while the other particle escapes. The interaction causing the stripping could be chosen from either of the binding interactions, namely, that of the deuteron, or that the captured particle in the final nucleus.

In a similar manner, it is assumed in the case of heavy particle stripping, that the core (containing $A-1$ particles) is stripped off the target nucleus and captured by the deuteron, while the other particle escapes. Here again, the interaction causing the reaction is either the binding interaction of the outgoing particle in the target nucleus, or that of the deuteron in the final nucleus.

Assuming the above described mechanism, we write the target state $|J_i M_i\rangle$, where J_i and M_i are the target spin and its z -component respectively, as

$$\begin{aligned} |J_i M_i\rangle &= \sum_{\substack{M_c M_p \\ m_p \mu_p}} \beta_{j_p} \begin{bmatrix} J_c & j_p & J_i \\ M_c & M_p & M_i \end{bmatrix} \\ &\times \begin{bmatrix} l_p & 1/2 & j_p \\ m_p & \mu_p & M_p \end{bmatrix} |J_c M_c\rangle \left| \frac{1}{2} \mu_p \right\rangle \\ &\times U_{l_p}(r_p) Y_{l_p m_p}(\Omega_p) \end{aligned} \quad (3)$$

where β_{j_p} is the "reduced width amplitude", defined in the same way as θ of Eq. (1). $\begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}$ are vector-addition coefficients, J_c and M_c are the total angular momentum and its z -component, respectively, of the core, and $U_{l_p}(r_p)$ is the wave function describing the motion of the exchangeable particle relative to the center-of-mass of the core, with an orbital angular momentum l_p . According to our model, the final nucleus is composed of the core and a deuteron, and we write for the state of the final nucleus $|J_f M_f\rangle$, where J_f and M_f are the total angular momentum and its z -component,

$$\begin{aligned} |J_f M_f\rangle &= \sum_{M_c \mu_d} (4\pi)^{-1/2} \beta_1 \begin{bmatrix} J_c & 1 & J_f \\ M_c & \mu_d & M_f \end{bmatrix} \\ &\times (-)^{j_n + j_p - 1} \left[\frac{(2j_n + 1)}{(2l_p + 1)} \right]^{1/2} \end{aligned}$$

$$\begin{aligned} & \times U(l_{p\frac{1}{2}} j_n 1; j_{p\frac{1}{2}}) \\ & \times |J_c M_c\rangle |\mu_a\rangle U_0(r_a) \end{aligned} \quad (4)$$

In Eq. (4), it has been assumed that the deuteron moves in an s -orbit relative to the core. β_1 is a measure of the probability that the final nucleus has been formed by the capture of the deuteron by the core in an s -orbit. $U(l_{p\frac{1}{2}} j_n 1; j_{p\frac{1}{2}})$ is the modified Racah coefficient as defined by Jahn.³⁾ l_p is the angular momentum of the orbit occupied by the proton of the deuteron in the final nucleus, and j_n and j_p are the total angular momenta of the neutron and proton in the final nucleus.

The heavy particle stripping amplitude is given by the expression

$$\begin{aligned} E = & \langle \mathbf{k}_f; \mu_p | \langle J_f M_f | \sum_{i=1}^{A-1} (V_{pi} + V_{ni}) \\ & \times |J_i M_i\rangle | \mathbf{k}_i; \mu_a \rangle \end{aligned} \quad (5)$$

$$P_E = \frac{2[U^2(l_{p\frac{1}{2}} j_n 1; j_{p\frac{1}{2}}^+) \beta_{j_p^+}^2 - U^2(l_{p\frac{1}{2}} j_n 1; j_{p\frac{1}{2}}^-) \beta_{j_p^-}^2]}{\sum_{j_p} (2j_p + 1) U^2(l_{p\frac{1}{2}} j_n 1; j_{p\frac{1}{2}}) \beta_{j_p}^2} \frac{\sum_{m_p} m_p |E_{l_p m_p}|^2}{\sum_{m_p} |E_{l_p m_p}|^2}$$

where \mathbf{k}_i and \mathbf{k}_f are the momenta of the incident deuteron and the outgoing particle, respectively. $\sum_{i=1}^{A-1} V_{ni}$ and $\sum_{i=1}^{A-1} V_{pi}$ are the interactions of the neutron and proton of the deuteron with the nucleons of the core. We write the differential cross-section as

$$\sigma(\mathbf{k}_f; \mathbf{k}_i) = G_E^2 |E_{l_p m_p}|^2 \quad (6)$$

with

$$\begin{aligned} E_{l_p m_p} = & \int \langle \mathbf{k}_f | U_0(r_{dc}) V_{dc} U_{l_p}(r_{pc}) \\ & \times Y_{l_p m_p}(\varrho_{pc}) | \mathbf{k}_i \rangle d\mathbf{r}_{pc} d\mathbf{r}_{dc} \end{aligned} \quad (7)$$

where $\langle \mathbf{k}_f |$ denotes the wave function of the outgoing particle relative to the final nucleus, and $| \mathbf{k}_i \rangle$ that of the in-

cident deuteron relative to the target nucleus. All the details of the nuclear wave functions are contained in G_E^2 . V_{dc} is defined as

$$V_{dc} = \langle J_c M_c | \sum_{i=1}^{A-1} (V_{pi} + V_{ni}) | J_c M_c \rangle. \quad (8)$$

G_E^2 is given by the expression

$$\begin{aligned} G_E^2 = & \frac{(2J_i + 1)(2J_f + 1)(2j_n + 1)}{2(2J_c + 1)(2j_p + 1)(2l_p + 1)} \\ & \times U^2(l_{p\frac{1}{2}} j_n 1; j_{p\frac{1}{2}}) \\ & \times \sum_{m_p \mu_p} \left[\begin{matrix} l_p & 1/2 & j_p \\ m_p & \mu_p & M_p \end{matrix} \right]^2 \beta_{j_p}^2 \beta_1^2 \end{aligned} \quad (9)$$

The expression for the polarization is easily obtained from Eq. (9). The number of particles with spin up and down are obtained by substituting $\mu_p = +1/2$ and $-1/2$, respectively. Finally, we have

We see that, in this case also, the polarization vanishes under plane wave approximation. The maximum polarization that can be obtained is

$$P = \begin{cases} l_p/(l_p + 1) & \text{for } j_p = j_p^+ = l_p + \frac{1}{2} \\ 1 & \text{for } j_p = j_p^- = l_p - \frac{1}{2} \end{cases} \quad (11)$$

Thus, we see that, one can expect a large polarization in the heavy particle stripping mechanism. To conclude, it has been believed that in order to account for polarizations larger than $1/3$, observed in experiments, spin-dependent forces have to be invoked. Since it is known that, at least in a few reactions, heavy particle stripping plays an important

part, it is necessary to calculate its contribution to the polarization, and the relative importance of these effects can only be judged after evaluating the radial integrals.

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The Effect of π^2 Term in the S-Wave π -N Scattering

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As an effective Hamiltonian, several authors¹⁾ have assumed the form:

$$H_{eff.} = H_{\pi} + \lambda_1 \bar{\varphi}_i \bar{\varphi}_i + \lambda_2 \varepsilon_{ijk} (\psi^+ \tau_i \psi) \bar{\pi}_j \bar{\varphi}_k,$$

which is obtained from the ps - ps Hamiltonian by the Tani-Foldy transformation,²⁾ where the notation is the usual one, e.g. $\bar{\varphi}_i = \int \rho(x) \varphi_i(x) d^3x$ and $\rho(x)$ is a spherically symmetric nucleon form factor, normalized by the condition $\int \rho(x) d^3x = 1$. But, if one eliminates the odd operators up to the order $1/M$ from the ps - ps Hamiltonian and retains only even ones up to the second order in the coupling constant g , another term π^2 is to be added to the above

Hamiltonian $H_{eff.}$ In the present letter, some remarks of the π^2 effect are given by using Lomon and Kobayashi-Klein's approximation methods.

We start with the following effective Hamiltonian H_s :

$$H_s = H_{\pi} + \lambda_1 \bar{\varphi}_i \bar{\varphi}_i + \lambda_2 \varepsilon_{ijk} (\psi^+ \tau_i \psi) \bar{\pi}_j \bar{\varphi}_k + \lambda_3 \bar{\pi}_i \bar{\pi}_i. \quad (1)$$

From the Hamiltonian H_s , there follows the equation of motion for $\varphi_i(x, t)$:

$$\begin{aligned} (-\square + \mu^2) \varphi_i(x, t) = & -(1 + 2\lambda_3 V)^{-1} \\ & \times [(2\lambda_1 + 4\lambda_1 \lambda_3 V - 2\lambda_2^2 V) \bar{\varphi}_i \rho(x) \\ & - 2\lambda_3 \ddot{\bar{\varphi}}_i \rho(x) + \lambda_2 \varepsilon_{ijk} (\psi^+ \tau_j \psi) \\ & \times \{i\lambda_2 V \bar{\varphi}_k + 2\dot{\bar{\varphi}}_k\} \rho(x) + A_i], \quad (2) \end{aligned}$$

where $A_i = i\lambda_2 \varepsilon_{ijk} [H_s, \psi^+ \tau_j \psi] \bar{\varphi}_i(x)$, $V = \int |\rho(x)|^2 d^3x$. This term A_i is neglected in Kobayashi-Klein's treatment. By calculating commutator $[H_s, \psi^+ \tau_j \psi]$, A_i can be rewritten as follows:

$$A_i = \{2i\lambda_2^2 V \varepsilon_{ijk} (\psi^+ \tau_j \psi) \bar{\varphi}_k + 2\lambda_2^2 (\psi^+ \tau_i \psi) \bar{\varphi}_i \varepsilon_{ijk} \bar{\varphi}_j \bar{\pi}_k\} \rho(x). \quad (3)$$

We introduce the wave function defined by $T_{ki'}(\mathbf{k}, x) = \langle N | \varphi_i(x, t) | \mathbf{k}, i \rangle$, where $|N\rangle$ is the one-nucleon state and $|\mathbf{k}, i\rangle$ represents the scattering state of a pion with momentum \mathbf{k} and isospin i' . Then the equation for $T_{ki'}(\mathbf{k}, x)$ reads:

$$\begin{aligned} (-\nabla^2 + \mu^2 - \omega^2) T_{ki'}(\mathbf{k}, x) = & -(1 + 2\lambda_3 V)^{-1} \{A(k) \bar{T}_{ki'}(\mathbf{k}) \rho(x) \\ & + i\varepsilon_{ijk} (\lambda_2^2 V - 2\lambda_2 \omega) \tau_j \bar{T}_{ki'}(\mathbf{k}) \rho(x) \\ & + \langle N | A_i | \mathbf{k}, i' \rangle \}, \quad (4) \end{aligned}$$

where $\bar{T}_{ki'}(\mathbf{k}) = \int \rho(x) T_{ki'}(\mathbf{k}, x) d^3x$, $A(k) = (2\lambda_1 + 4\lambda_1 \lambda_3 V - 2\lambda_2^2 V + 2\lambda_3 \omega^2)$ and we have treated ψ as an approximately free field in the remote past by replacing

$\langle N | (\psi^\dagger \tau_i \psi) \bar{\varphi}_i | \mathbf{k}, i' \rangle$ with $\tau_i \langle N | \bar{\varphi}_i | \mathbf{k}, i' \rangle$. In Eq. (4) we have to estimate such a term as $\langle N | \bar{\varphi}_i \bar{\varphi}_m \bar{\pi}_n | \mathbf{k}, i' \rangle$ which contributes to the equation for $T_{ii'}(\mathbf{k}, x)$ as a non-linear term. We linearize this term by using a perturbational technique and obtain an approximate expression for $\langle N | A_i | \mathbf{k}, i' \rangle$:

$$\begin{aligned} \langle N | \bar{\varphi}_i \bar{\varphi}_m \bar{\pi}_n | \mathbf{k}, i' \rangle &\simeq \delta_{im} W \langle N | \pi_n | \mathbf{k}, i' \rangle \\ &+ (iV/2) \{ \delta_{mn} \langle N | \bar{\varphi}_i | \mathbf{k}, i' \rangle \\ &+ \delta_{in} \langle N | \bar{\varphi}_m | \mathbf{k}, i' \rangle \}, \quad (5) \\ \langle N | A_i | \mathbf{k}, i' \rangle &\simeq i\lambda_2^2 V \varepsilon_{ijk} \tau_j T_{ki'}(\mathbf{k}) \\ &+ 2\lambda_2^2 W \varepsilon_{ijk} \tau_j \langle N | \bar{\pi}_k | \mathbf{k}, i' \rangle \rho(x), \quad (6) \end{aligned}$$

where $W = \int d^3k |\rho(k)|^2 / 2\omega$, $\rho(k) = (2\pi)^{-3/2} \int \rho(x) e^{-ikx} d^3x$, $\omega^2 = k^2 + \mu^2$. Consistently with this treatment, $\bar{\pi}_k$ in Eq. (6) may be replaced by $(1 + 2\lambda_3 V)^{-1} \bar{\varphi}_k$ which is free from the τ -dependent interaction, or by completely free φ_k . Decomposing $T_{ii'}(\mathbf{k}, x)$ into the eigenstates of total isospin 1/2 and 3/2, we obtain the final expression of the equation for $T_{ii'}^{(I)}(\mathbf{k}, x)$:

$$\begin{aligned} (-\nabla^2 + \mu^2 - \omega^2) T_{ii'}^{(I)}(\mathbf{k}, x) \\ = -2(1 + 2\lambda_3 V)^{-1} \{ A(k) + J^{(I)} \lambda_2 \Gamma \omega \\ - \lambda_2^2 V (J^{(I)} + 1) \} \bar{T}_{ii'}^{(I)}(\mathbf{k}) \rho(x), \quad (7) \end{aligned}$$

where $A(k) = \lambda_1 + 2\lambda_1 \lambda_3 V + \lambda_3 \omega^2$, $\Gamma = 1 + W\lambda_2(1 + 2\lambda_3 V)^{-1}$, $J^{(1)} = 2$, $J^{(3)} = -1$ and $\bar{T}_{ii'}^{(I)}(\mathbf{k}, x)$ represents the component of the isospin $I/2$ state. Using the Fourier transform of Eq. (7) and integrating it, we immediately obtain the scattering phase shifts $\delta_{(I)}(k)$:

$$\begin{aligned} \tan \delta_I(k) &= -2\pi^2 k |\rho(k)|^2 \\ &\times \{ 2(1 + 2\lambda_3 V)^{-1} \Sigma^{(I)} \} \\ &\times \{ 1 - 2(1 + 2\lambda_3 V)^{-1} \Sigma^{(I)} \} \end{aligned}$$

$$\times \int d^3k' |\rho(k')|^2 (k^2 - k'^2)^{-1} \}^{-1}, \quad (8)$$

where $\Sigma^{(I)} = A(k) + J^{(I)} \lambda_2 \Gamma \omega - \lambda_2^2 V (J^{(I)} + 1)$. If one puts $W \rightarrow 0$, $\lambda_3 \rightarrow 0$ and $J+1 \rightarrow (J+2)/2$ in Eq. (8), the resultant expression for $\delta_{(I)}(k)$ is the same as Kobayashi-Klein's. Using Lomon's diagonalization method, one gets exactly the same result as that of Kobayashi-Klein's in the case $A_i = 0^*$.

In regard to the coupling constant λ_1 , λ_2 and λ_3 , there are some complexities¹⁾²⁾³⁾ of the renormalization and the higher order effect in the Tani-Foldy transformation. But, in order to see the π^2 effect on the behaviour of $\delta_{(I)}(k)$, we tentatively use $\lambda_1 = g^2/2M$, $\lambda_2 = -g^2/4M_2$ and $\lambda_3 = g^2/8M^3$ as are derived by the Tani-Foldy transformation and assume the Yukawa type for the form factor, $\rho(x) = (\alpha^2/4\pi)(1/r) \exp(-\alpha r)$, $\int \rho(x) d^3x = 1$.

We examine two cases: case i) $g^2/4\pi = 15$, cutoff momentum $\alpha = 4.7\mu c$, $\lambda_3 = 0$ and case ii) $g^2/4\pi = 15$, cutoff momentum $\alpha = 6.8\mu c \sim 7.0\mu c$, $\lambda_3 = g^2/8M^3$. Both case i) and case ii) can reproduce the main features of the low energy phase shift, but the momentum (k)-dependence of $\delta_1(k)$ in case i) is stronger than that in case ii). Furthermore, the ratio $|\delta_3(k)|/|\delta_1(k)|$ in the former case is larger than that in the latter, although even in the latter it is larger than unity and is not in good agreement with the experimental data. Thus it should be remarked that if we assume $g^2/4\pi \simeq 15$ and the cutoff momentum $\simeq MC$, the π^2 term contributing to Eq. (8) as $(1 + 2\lambda_3 V)^{-1}$ factor is effective in improving the agreement between theory and experiment.

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